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Conditional distributions which do not satisfy the Chapman-Kolmogorov equation

By Masaru Iizuka, Miyuki Maeno and Matsuyo Tomisaki

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Abstract. We consider one-dimensional generalized diffusion processes (ODGDPs for brief), where both boundary points are accessible or asymptotically accessible. For such ODGDPs we consider stochastic processes induced by conditioning on hitting or asymptotical hitting the right boundary point before hitting or asymptotical hitting the left boundary point. The induced stochastic processes are again ODGDPs when the right boundary point is either accessible with the absorbing boundary condition or asymptotically accessible. However the probability distributions of the induced stochastic processes do not satisfy the Chapman-Kolmogorov equation when the right boundary point is accessible with the reflecting or elastic boundary condition.

1. Introduction.

For some one-dimensional diffusion processes on the interval [0,1] that are related to diffusion models in population genetics, Ewens [**3**] considered induced stochastic processes by conditioning on hitting the boundary point 1 before hitting the other boundary point 0. The boundary points 0 and 1 are accessible and absorbing boundaries for the diffusion processes that he considered and the induced stochastic processes are again diffusion processes. Then the induced stochastic processes are referred to as the conditional diffusion processes by Ewens [**3**] (see also [**7**]). The boundary points of one-dimensional diffusion processes in population genetics, however, can be other kinds of boundaries such as regular boundary in general (see [**4**]). If a boundary point is regular boundary the reflecting boundary condition has been posed usually in population genetics though other boundary conditions may be possible (see [**1**], [**4**], [**11**] and [**14**]).

In this paper, we are concerned with one-dimensional generalized diffusion processes (ODGDPs for brief) on an open interval (l_1, l_2). We consider stochastic processes induced by conditioning on hitting or asymptotical hitting the right boundary point l_2 before hitting or asymptotical hitting the left boundary point l_1. The right boundary point l_2 is assumed to be absorbing or reflecting or elastic boundary for the original ODGDPs when it is accessible. We will show that the induced stochastic processes are again ODGDPs when the boundary point l_2 is either accessible with the absorbing boundary condition, or asymptotically accessible. Next we will consider the case where the boundary point l_2 is accessible with the reflecting or elastic boundary condition. It will be shown that the probability distributions of the induced stochastic processes do not satisfy the Chapman-Kolmogorov equation. Hence the induced

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stochastic processes can not be Markov processes in this case. In Section 2 we state our results more precisely. Section 3 is devoted to their proofs.

2. Main results.

Let $S=(l_1,l_2)$ be an open interval, where $-\infty \le l_1 < l_2 \le \infty$, and $s(x)$ and $m(x)$ be real valued functions on S such that $s(x)$ is continuous and increasing, and $m(x)$ is right continuous and increasing. We denote by $ds(x)$ and $dm(x)$ the induced measures. Given a function $u(x)$ on S, we set $u(l_i)=\lim_{x\to l_i, x\in S} u(x)$, $i=1,2$, and $u^+(x)=\lim_{\varepsilon\downarrow 0}\{u(x+\varepsilon)-u(x)\}/\{s(x+\varepsilon)-s(x)\}$, if there exist the limits.

Let $\boldsymbol{D}=[X(t): t\ge 0, P_x : x\in S^*]$ be an ODGDP with the generator $\mathscr{G}=\frac{d}{dm}\frac{d}{ds}$, where $S^*=S\cup\{l_i : |s(l_i)|+|m(l_i)|<\infty, i=1,2\}$. If $|s(l_i)|+|m(l_i)|<\infty$, then we set one of the following boundary conditions (2.1) and (2.2) at l_i.

$$u(l_i)=0. \tag{2.1}$$

$$\theta_i u(l_i)+(-1)^i u^+(l_i)=0, \tag{2.2}$$

where θ_i, $i=1,2$, are nonnegative numbers. When (2.1) [resp. (2.2)] is posed at l_i, it is called to be absorbing [resp. elastic]. Note that the condition (2.2) with $\theta_i=0$ is reduced to the reflecting boundary condition, that is, $u^+(l_i)=0$. Let σ_a be the first hitting time at a, that is, $\sigma_a=\inf\{t>0 : X(t)=a\}$. It is known that

$$P_x(\sigma_a<\sigma_b)=\frac{s(x)-s(b)}{s(a)-s(b)}, \quad a\wedge b\le x\le a\vee b, \tag{2.3}$$

for $x\in S^*, a,b\in[l_1,l_2]$, $a\ne b$ (cf. [**6**]), where $a\wedge b=\min\{a,b\}$, $a\vee b=\max\{a,b\}$. It is also known that there exists the transition probability density $p(t,x,y)$ with respect to $dm(y)$, that is,

$$P_x(X(t)\in\Lambda)=\int_\Lambda p(t,x,y)\,dm(y), \quad t>0,\ x\in S^*,\ \Lambda\in\mathscr{B},$$

where $\mathscr{B}$ is the set of all Borel measurable subset of S (cf. [**6**], [**12**]). We note that $p(t,x,y)$ is positive and continuous on $(0,\infty)\times S\times S$ (cf. [**6**], [**12**]). We fix $c\in S$ arbitrarily and set

$$I(x)=\int_{(c,x]} ds(y)\int_{(c,y]} dm(z), \quad J(x)=\int_{(c,x]} dm(y)\int_{(c,y]} ds(z), \quad x\in S,$$

where the integral $\int_{(a,b]}$ is read as $-\int_{(b,a]}$ if $a>b$. Following [**5**], we call the boundary l_i to be

$$\begin{array}{llll}
(s,m)\text{-regular} & \text{if} \quad I(l_i)<\infty & \text{and} & J(l_i)<\infty,\\
(s,m)\text{-exit} & \text{if} \quad I(l_i)<\infty & \text{and} & J(l_i)=\infty,\\
(s,m)\text{-entrance} & \text{if} \quad I(l_i)=\infty & \text{and} & J(l_i)<\infty,\\
(s,m)\text{-natural} & \text{if} \quad I(l_i)=\infty & \text{and} & J(l_i)=\infty.
\end{array}$$

Note that

$$\begin{aligned}
&|s(l_i)| < \infty \quad \text{and} \quad |m(l_i)| < \infty \quad \text{if } l_i \text{ is } (s,m)\text{-regular},\\
&|s(l_i)| < \infty \quad \text{and} \quad |m(l_i)| = \infty \quad \text{if } l_i \text{ is } (s,m)\text{-exit},\\
&|s(l_i)| = \infty \quad \text{and} \quad |m(l_i)| < \infty \quad \text{if } l_i \text{ is } (s,m)\text{-entrance},\\
&|s(l_i)| = \infty \quad \text{or} \quad |m(l_i)| = \infty \quad \text{if } l_i \text{ is } (s,m)\text{-natural}.
\end{aligned}$$

The boundary l_i is accessible [resp. asymptotically accessible] if it is (s,m)-regular or exit [resp. natural with $|s(l_i)| < \infty$]. Throughout this paper we assume that

$$s(l_2) < \infty. \tag{2.4}$$

This assumption implies that l_2 is accessible or asymptotically accessible. We set

$$h(x) = \begin{cases} \dfrac{s(x) - s(l_1)}{s(l_2) - s(l_1)}, & \text{if } s(l_1) > -\infty, \\ 1, & \text{if } s(l_1) = -\infty, \end{cases} \quad l_1 \le x \le l_2. \tag{2.5}$$

Let $\boldsymbol{D}_\bullet$ and $\boldsymbol{D}_\diamond$ be the stopped processes of $\boldsymbol{D}$ such that $\boldsymbol{D}_\bullet = [X(t \wedge \sigma_{l_1} \wedge \sigma_{l_2}) : t \ge 0, P_x : x \in S^*]$ and $\boldsymbol{D}_\diamond = [X(t \wedge \sigma_{l_1}) : t \ge 0, P_x : x \in S^*]$. Let denote by $p_\bullet(t,x,y)$ and $p_\diamond(t,x,y)$ the transition probability densities of $\boldsymbol{D}_\bullet$ and $\boldsymbol{D}_\diamond$ with respect to m, respectively. Then we see that

$$P_x(X(t) \in \Lambda,\ t < \sigma_{l_1} \wedge \sigma_{l_2}) = \int_\Lambda p_\bullet(t,x,y)\, dm(y),$$
$$P_x(X(t) \in \Lambda,\ t < \sigma_{l_1}) = \int_\Lambda p_\diamond(t,x,y)\, dm(y),$$

for $t > 0,\ x \in S,\ \Lambda \in \mathscr{B}$. We note that $p_\bullet(t,x,y)$ and $p_\diamond(t,x,y)$ are positive and continuous on $(0,\infty) \times S \times S$. Let $\boldsymbol{D}^h$ be the h-transform of $\boldsymbol{D}_\bullet$ with h given by (2.5). Namely, $\boldsymbol{D}^h$ is an ODGDP with the generator $\mathscr{G}^h = \frac{d}{dm^h}\frac{d}{ds^h}$, where $dm^h(x) = h(x)^2\, dm(x)$ and $ds^h(x) = h(x)^{-2}\, ds(x)$. It is known that the transition probability density $p^h(t,x,y)$ of $\boldsymbol{D}^h$ with respect to m^h is given by

$$p^h(t,x,y) = p_\bullet(t,x,y)/h(x)h(y), \quad t > 0,\ x, y \in S, \tag{2.6}$$

(see [**9**], [**10**]). We define a nonnegative function $q(t,x,y)$ as follows: For $t > 0$ and $x, y \in S$,

$$q(t,x,y) = \frac{1}{h(x)} \{p_\bullet(t,x,y)h(y) + q_1(t,x,y) + q_2(t,x,y)\}, \tag{2.7}$$

where

$$q_1(t,x,y) = \begin{cases} \displaystyle\int_{(0,t)} P_x(\sigma_{l_2} \in du,\ \sigma_{l_2} < \sigma_{l_1}) p_\diamond(t-u, l_2, y) \\ \qquad\qquad \text{if } l_2 \text{ is } (s,m)\text{-regular and elastic,} \\ 0 \qquad\qquad \text{otherwise,} \end{cases} \tag{2.8}$$

$$\begin{aligned} &q_2(t,x,y) \\ &= \begin{cases} \displaystyle\int_{(0,t)} P_x(\sigma_{l_2} \in du,\ \sigma_{l_2} < \sigma_{l_1}) \\ \displaystyle\times \int_{(0,t-u)} P_{l_2}(\sigma_{l_1} \in dv) p(t-u-v, l_1, y) \\ \qquad\qquad \text{if both of } l_1 \text{ and } l_2 \text{ are } (s,m)\text{-regular and elastic,} \\ 0 \qquad\qquad \text{otherwise.} \end{cases} \end{aligned} \tag{2.9}$$

Let $\{\xi_n\}_n$ and $\{\eta_n\}_n$ be the sequences satisfying

$$\begin{aligned} &\xi_n = l_1 \ (n \in \boldsymbol{N}) \quad \text{if } l_1 \text{ is } (s,m)\text{-regular or exit,} \\ &\xi_n \in S \ (n \in \boldsymbol{N}), \quad \xi_n \downarrow l_1 \quad \text{if } l_1 \text{ is } (s,m)\text{-entrance or natural,} \\ &\eta_n = l_2 \ (n \in \boldsymbol{N}) \quad \text{if } l_2 \text{ is } (s,m)\text{-regular or exit,} \\ &\eta_n \in S \ (n \in \boldsymbol{N}), \quad \eta_n \uparrow l_2 \quad \text{if } l_2 \text{ is } (s,m)\text{-natural.} \end{aligned}$$

By virtue of (2.3) and (2.4),

$$h(x) = \lim_{n\to\infty} P_x(\sigma_{\eta_n} < \sigma_{\xi_n}), \quad x \in S^*.$$

Let us consider the following function $Q(t,x,\Lambda)$.

$$Q(t,x,\Lambda) = \lim_{n\to\infty} P_x(X(t) \in \Lambda | \sigma_{\eta_n} < \sigma_{\xi_n}), \tag{2.10}$$

for $t > 0$, $x \in S$ and $\Lambda \in \mathscr{B}$. Following the same argument as in the proof of Theorem 2.1 of [**9**], we easily see that there exists the limit in the right-hand side of (2.10), and

$$Q(t,x,\Lambda) = \int_\Lambda q(t,x,y)\, dm(y), \quad t > 0,\ x \in S,\ \Lambda \in \mathscr{B}. \tag{2.11}$$

If $s(l_1) = -\infty$, then (2.11) is obviously reduced to

$$Q(t,x,\Lambda) = P_x(X(t) \in \Lambda).$$

We are interested in the case $s(l_1) > -\infty$ which implies that l_1 is accessible or asymptotically accessible. The aim of the present paper is to show the following theorems.

Theorem 2.1. *Assume that $s(l_1) > -\infty$ and l_2 is (s,m)-regular with the absorbing boundary condition or exit or natural. Then Q defined by* (2.10) *is the transition* *probability of the ODGDP $\boldsymbol{D}^h$. If the boundary l_1 is (s,m)-regular or exit, then it is (s^h,m^h)-entrance. If l_1 is (s,m)-natural, then it is (s^h,m^h)-natural. The boundary l_2 is (s^h,m^h)-regular or exit or natural according to (s,m)-regular or exit or natural. Especially l_2 is absorbing if it is (s^h,m^h)-regular.*

For the special case of one-dimensional diffusion processes, the assertions of Theorem 2.1 and their application to population genetics models were obtained by Maeno [**9**].

As in Remark 3.1 below, Theorem 2.1 can be extended to the case that $S(m) \neq \emptyset$, where $S(m)$ is the support of the measure dm. Hence the assertions hold true for birth and death processes, too.

THEOREM 2.2. *Assume that $s(l_1) > -\infty$ and l_2 is (s,m)-regular with the elastic boundary condition. Then Q defined by* (2.10) *does not satisfy the Chapman-Kolmogorov equation.*

3. Proofs.

3.1. Proof of Theorem 2.1.

We assume that $s(l_1) > -\infty$ and l_2 is (s,m)-regular with the absorbing boundary condition or exit or natural. Then (2.11) is reduced to

$$Q(t,x,\Lambda) = \frac{1}{h(x)} \int_\Lambda p_\bullet(t,x,y) h(y)\, dm(y). \tag{3.1}$$

By means of (2.6) and (3.1),

$$Q(t,x,\Lambda) = \int_\Lambda p^h(t,x,y)\, dm^h(y), \tag{3.2}$$

which shows that Q is the transition probability of the ODGDP $\boldsymbol{D}^h$. The rest of the theorem follows from (3.2) and the results of [**10**]. □

REMARK 3.1. Suppose that $m(x)$ is nondecreasing, and hence $S(m)$ is not necessarily identical with S. We assume that $S(m) \neq \emptyset$. Then (2.11) holds true. Since the results of [**10**] are valid under the assumption $S(m) \neq \emptyset$, we also obtain (3.2). Thus Theorem 2.1 can be extended to the case that $S(m) \neq \emptyset$.

3.2. Proof of Theorem 2.2.

We assume that $s(l_1) > -\infty$ and l_2 is (s,m)-regular and elastic, that is, (2.2) is satisfied for $i = 2$.

First we note the following.

LEMMA 3.2.

(1) *For $t > 0$ and $x \in S$, $q(t,x,y)$ is continuous in $y \in S$.*
(2) *For $t > 0$ and $l_1 < a < b < l_2$, $\sup_{x \in S,\, a \le y \le b} q(t,x,y) < \infty$.*

Proof. Let $l_1 < a < b < l_2$. We have that $\sup_{t>0} p(t,a,b) < \infty$. Combining this with Theorem 4.2 of [**12**], we obtain that

$$\sup_{t>0,\ l_1<x\le a,\ b\le y<l_2} p(t,x,y) < \infty. \tag{3.3}$$

Note that (3.3) holds true replacing $p(\cdot,\cdot,\cdot)$ by $p_\bullet(\cdot,\cdot,\cdot)$ or $p_\diamond(\cdot,\cdot,\cdot)$ or $p^h(\cdot,\cdot,\cdot)$. Since the transition probability densities $p(t,x,y)$, $p_\bullet(t,x,y)$, and $p_\diamond(t,x,y)$ are continuous in (t,x,y), the first assertion follows from (2.7), (2.8), (2.9) and (3.3).

We will show the second assertion. By means of (2.3), (2.5), (2.8), (2.9), and (3.3), we see that

$$\sup_{x\in S,\ a\le y\le b} q_i(t,x,y)/h(x) < \infty, \quad t>0,\ i=1,2. \tag{3.4}$$

It is obvious that

$$\sup_{x,y\in S} p_\bullet(t,x,y) < \infty, \quad t>0.$$

Combining this with (3.3) replacing $p(\cdot,\cdot,\cdot)$ by $p^h(\cdot,\cdot,\cdot)$, we see that

$$\sup_{x\in S,\ a\le y\le b} p_\bullet(t,x,y)h(y)/h(x) < \infty, \quad t>0. \tag{3.5}$$

The second assertion follows from (3.4) and (3.5). □

For $\alpha>0$ and $i=1,2$, let $g_i(\cdot,\alpha)$ be a positive and continuous function on S satisfying the following properties.

$$g_1(\cdot,\alpha) \text{ is nondecreasing and } g_2(\cdot,\alpha) \text{ is nonincreasing on } S. \tag{3.6}$$

$$g_i(l_i,\alpha)=0 \quad \text{if } l_i \text{ is } (s,m)\text{-regular with the boundary condition (2.1) or exit or natural.} \tag{3.7}$$

$$\theta_i g_i(l_i,\alpha) + (-1)^i g_i^+(l_i,\alpha) = 0 \quad \text{if } l_i \text{ is } (s,m)\text{-regular with the boundary condition (2.2).} \tag{3.8}$$

$$\begin{aligned} g_i(x,\alpha) &= g_i(c,\alpha) + g_i^+(c,\alpha)\{s(x)-s(c)\} \\ &\quad + \alpha\int_{(c,x]} \{s(x)-s(y)\} g_i(y,\alpha)\,dm(y), \quad x\in S, \end{aligned} \tag{3.9}$$

where $c\in S$ is fixed arbitrarily and $g_i^+(x,\alpha) = \lim_{\varepsilon\downarrow 0}\{g_i(x+\varepsilon,\alpha)-g_i(x,\alpha)\}/\{s(x+\varepsilon)-s(x)\}$. It is known that there exist such functions $g_i(\cdot,\alpha)$, $i=1,2$ (see [**6**]). We set $W(\alpha) = g_1^+(x,\alpha)g_2(x,\alpha) - g_1(x,\alpha)g_2^+(x,\alpha)$. Note that $W(\alpha)$ is a positive number independent of $x\in S$. We put

$$G(\alpha,x,y) = G(\alpha,y,x) = W(\alpha)^{-1} g_1(x,\alpha) g_2(y,\alpha),$$

for $\alpha>0$ and $l_1 < x \le y < l_2$, which is the Green function corresponding to the generator $\mathscr{G}$. It is known that

$$G(\alpha, x, y) = \int_0^\infty e^{-\alpha t} p(t, x, y)\, dt, \tag{3.10}$$

for $\alpha > 0$ and $x, y \in S$ (see [**6**]). We denote by $\mathscr{G}_\bullet$ and $\mathscr{G}_\diamond$ the generators of $\boldsymbol{D}_\bullet$ and $\boldsymbol{D}_\diamond$, respectively. We denote by $G_\bullet(\alpha, x, y)$, $g_{\bullet,i}(x, \alpha)$ [resp. $G_\diamond(\alpha, x, y)$, $g_{\diamond,i}(x, \alpha)$] the quantities corresponding to $\mathscr{G}_\bullet$ [resp. $\mathscr{G}_\diamond$]. For $G_\bullet(\alpha, x, y)$ [resp. $G_\diamond(\alpha, x, y)$], (3.10) with $p(t, x, y)$ replaced by $p_\bullet(t, x, y)$ [resp. $p_\diamond(t, x, y)$] holds true. We set

$$\begin{aligned} H(\alpha, x, y) &= \int_0^\infty e^{-\alpha t} q(t, x, y)\, dt, \\ H_i(\alpha, x, y) &= \int_0^\infty e^{-\alpha t} q_i(t, x, y)\, dt, \end{aligned}$$

for $\alpha > 0,\ x, y \in S$ and $i = 1, 2$. It is easy to see that

$$\begin{aligned} H(\alpha, x, y) &= \frac{1}{h(x)} \{G_\bullet(\alpha, x, y) h(y) + H_1(\alpha, x, y) + H_2(\alpha, x, y)\}, \\ H_1(\alpha, x, y) &= E_x[e^{-\alpha\sigma_{l_2}}; \sigma_{l_2} < \sigma_{l_1}] G_\diamond(\alpha, l_2, y), \\ H_2(\alpha, x, y) &= \begin{cases} E_x[e^{-\alpha\sigma_{l_2}}; \sigma_{l_2} < \sigma_{l_1}] E_{l_2}[e^{-\alpha\sigma_{l_1}}] G(\alpha, l_1, y) \\ \qquad \text{if } l_1 \text{ is } (s, m)\text{-regular and elastic}, \\ 0 \qquad \text{otherwise}, \end{cases} \end{aligned}$$

for $\alpha > 0$ and $x, y \in S$, where $E_x[e^{-\alpha\sigma_{l_2}}; \sigma_{l_2} < \sigma_{l_1}] = \int_{(0,\infty)} e^{-\alpha u}\, P_x(\sigma_{l_2} \in du,\ \sigma_{l_2} < \sigma_{l_1})$.

Now we assume that Q satisfies the Chapman-Kolmogorov equation, that is,

$$Q(s + t, x, \Lambda) = \int_S Q(s, x, dz) Q(t, z, \Lambda),$$

for $s, t > 0,\ x \in S$ and $\Lambda \in \mathscr{B}$. By virtue of (2.11) and Lemma 3.2, we obtain that

$$q(s + t, x, y) = \int_S q(s, x, z) q(t, z, y)\, dm(z), \quad s, t > 0,\ x, y \in S. \tag{3.11}$$

Integrating (3.11) by the measure $e^{-\alpha(s+2t)}\, ds\, dt$ on $(0, \infty) \times (0, \infty)$, we see that

$$\frac{1}{\alpha} \{H(\alpha, x, y) - H(2\alpha, x, y)\} = \int_S H(\alpha, x, z) H(2\alpha, z, y)\, dm(z), \tag{3.12}$$

for $\alpha > 0$ and $x, y \in S$.

Suppose that l_1 is (s, m)-regular with the absorbing boundary condition or exit or natural. Letting $x \uparrow l_2$ and $y \uparrow l_2$ in (3.12), by means of the monotone convergence theorem we see that

$$\frac{1}{\alpha}\{H_1(\alpha,l_2,l_2)-H_1(2\alpha,l_2,l_2)\}=\int_S H_1(\alpha,l_2,z)h(z)^{-1}H_1(2\alpha,z,l_2)\,dm(z),$$

and hence

$$\begin{aligned}&\frac{1}{\alpha}\{G_\diamond(\alpha,l_2,l_2)-G_\diamond(2\alpha,l_2,l_2)\}\\&\quad=\int_S G_\diamond(\alpha,l_2,z)h(z)^{-1}E_z\big[e^{-2\alpha\sigma_{l_2}};\sigma_{l_2}<\sigma_{l_1}\big]G_\diamond(2\alpha,l_2,l_2)\,dm(z).\end{aligned}$$

Since l_2 is (s,m)-regular, the left-hand side (and hence the right-hand side) is finite. We note the resolvent equation for the Green function, that is,

$$G(\alpha,x,y)-G(\beta,x,y)+(\alpha-\beta)\int_S G(\alpha,x,z)G(\beta,z,y)\,dm(z)=0, \tag{3.13}$$

for $\alpha,\ \beta>0$ and $x,y\in S^*$. This holds true replacing $G(\cdot,\cdot,\cdot)$ by $G_\diamond(\cdot,\cdot,\cdot)$. Therefore

$$\begin{aligned}&\int_S G_\diamond(\alpha,l_2,z)G_\diamond(2\alpha,z,l_2)\,dm(z)\\&\quad=\int_S G_\diamond(\alpha,l_2,z)h(z)^{-1}E_z\big[e^{-2\alpha\sigma_{l_2}};\sigma_{l_2}<\sigma_{l_1}\big]G_\diamond(2\alpha,l_2,l_2)\,dm(z).\end{aligned} \tag{3.14}$$

We may take $g_{\diamond,1}(x,\alpha)=E_x[e^{-\alpha\sigma_{l_2}};\sigma_{l_2}<\sigma_{l_1}]$, $x\in S$ (see [**6**]). Therefore (3.14) is reduced to

$$\int_S g_{\diamond,1}(z,\alpha)g_{\diamond,1}(z,2\alpha)\,dm(z)=\int_S g_{\diamond,1}(z,\alpha)h(z)^{-1}g_{\diamond,1}(z,2\alpha)\,dm(z). \tag{3.15}$$

However (3.15) does not hold true since $0<h(z)<1$, $z\in S$. Thus it does not happen that l_1 is (s,m)-regular with the absorbing boundary condition or exit or natural.

Next we assume that l_1 is (s,m)-regular and elastic, that is, (2.2) is satisfied for $i=1$. Letting $x\uparrow l_2$ in (3.12), we see that

$$\begin{aligned}&\frac{1}{\alpha}\Big\{G_\diamond(\alpha,l_2,y)+E_{l_2}[e^{-\alpha\sigma_{l_1}}]G(\alpha,l_1,y)\\&\qquad-G_\diamond(2\alpha,l_2,y)-E_{l_2}\big[e^{-2\alpha\sigma_{l_1}}\big]G(2\alpha,l_1,y)\Big\}\\&\quad=\int_S\Big\{G_\diamond(\alpha,l_2,z)+E_{l_2}[e^{-\alpha\sigma_{l_1}}]G(\alpha,l_1,z)\Big\}\\&\qquad\times h(z)^{-1}\Big\{G_\bullet(2\alpha,z,y)h(y)+E_z\big[e^{-2\alpha\sigma_{l_2}};\sigma_{l_2}<\sigma_{l_1}\big]G_\diamond(2\alpha,l_2,y)\\&\qquad+E_z\big[e^{-2\alpha\sigma_{l_2}};\sigma_{l_2}<\sigma_{l_1}\big]E_{l_2}\big[e^{-2\alpha\sigma_{l_1}}\big]G(2\alpha,l_1,y)\Big\}\,dm(z).\end{aligned} \tag{3.16}$$

We note that there exist finite limits $G_\bullet(x,y) = \lim_{\alpha\downarrow 0} G_\bullet(\alpha,x,y)$ and $G_\diamond(x,y) = \lim_{\alpha\downarrow 0} G_\diamond(\alpha,x,y)$. We may take that

$$g_{\bullet,1}(x) = g_{\diamond,1}(x) = s(x) - s(l_1), \quad g_{\bullet,2}(x) = s(l_2) - s(x), \tag{3.17}$$

where $g_{\bullet,i}(x) = \lim_{\alpha\downarrow 0} g_{\bullet,i}(x,\alpha)$ and $g_{\diamond,i}(x) = \lim_{\alpha\downarrow 0} g_{\diamond,i}(x,\alpha)$, $i = 1,2$ (see [**6**]).

First we consider the case $\theta_1 + \theta_2 > 0$. Then there exists a finite limit $G(x,y) = \lim_{\alpha\downarrow 0} G(\alpha,x,y)$ and we may take

$$g_1(x) = \theta_1\{s(x) - s(l_1)\} + 1, \tag{3.18}$$

$$g_2(x) = g_{\diamond,2}(x) = \theta_2\{s(l_2) - s(x)\} + 1, \tag{3.19}$$

where $g_i(x) = \lim_{\alpha\downarrow 0} g_i(x,\alpha)$, $i = 1,2$ (see [**6**]). By virtue of (3.13) for $G(\cdot,\cdot,\cdot)$ or $G_\diamond(\cdot,\cdot,\cdot)$, we see that

$$\begin{aligned}
&\text{the left-hand side of (3.16)}\\
&\quad= \int_S G_\diamond(\alpha,l_2,z)G_\diamond(2\alpha,z,y)\,dm(z)\\
&\qquad+ \alpha^{-1}E_{l_2}[e^{-\alpha\sigma_{l_1}} - 1; \sigma_{l_1} < \infty]G(\alpha,l_1,y)\\
&\qquad- \alpha^{-1}E_{l_2}\big[e^{-2\alpha\sigma_{l_1}} - 1; \sigma_{l_1} < \infty\big]G(2\alpha,l_1,y)\\
&\qquad+ P_{l_2}(\sigma_{l_1} < \infty)\int_S G(\alpha,l_1,z)G(2\alpha,z,y)\,dm(z).
\end{aligned}$$

Therefore letting $\alpha \downarrow 0$ in (3.16) leads us to the following.

$$\begin{aligned}
&\int_S G_\diamond(l_2,z)G_\diamond(z,y)\,dm(z) + E_{l_2}[\sigma_{l_1}; \sigma_{l_1} < \infty]G(l_1,y)\\
&\qquad+ P_{l_2}(\sigma_{l_1} < \infty)\int_S G(l_1,z)G(z,y)\,dm(z)\\
&\quad= \int_S \Big\{G_\diamond(l_2,z) + P_{l_2}(\sigma_{l_1} < \infty)G(l_1,z)\Big\}\\
&\qquad\times h(z)^{-1}\Big\{G_\bullet(z,y)h(y) + h(z)G_\diamond(l_2,y)\\
&\qquad+ h(z)P_{l_2}(\sigma_{l_1} < \infty)G(l_1,y)\Big\}\,dm(z).
\end{aligned} \tag{3.20}$$

It is known that $P_{l_2}(\sigma_{l_1} < \infty) = g_2(l_2)/g_2(l_1)$ (see [**6**]). We note that

$$E_{l_2}[\sigma_{l_1}; \sigma_{l_1} < \infty] = g_2(l_1)^{-2}\int_S g_{\diamond,1}(z)g_2(z)^2\,dm(z). \tag{3.21}$$

We will give the proof of (3.21) in Appendix. Assume $\theta_2 > 0$. Letting $y \downarrow l_1$ in (3.20) leads us to the following.

$$\begin{aligned}&\{\theta_1 g_2(l_1)+\theta_2\}\int_S g_{\diamond,1}(z)g_2(z)^2\,dm(z)+\int_S g_2(z)^2\,dm(z)\\&\quad=\{\theta_1 g_2(l_1)+\theta_2\}\int_S g_{\diamond,1}(z)\,dm(z)+\int_S g_2(z)\,dm(z).\end{aligned}$$

This identity is not true since $g_2(z)>1$ for $z\in S$. Thus $\theta_2=0$, so that $\theta_1>0$. Letting $y\uparrow l_2$ in (3.20), we find that

$$\begin{aligned}&\theta_1^2\int_S g_{\diamond,1}(z)^2\,dm(z)+\theta_1\int_S g_{\diamond,1}(z)\,dm(z)+\int_S g_1(z)\,dm(z)\\&\quad=\bigl\{\theta_1 g_{\diamond,1}(l_2)+1\bigr\}\int_S\bigl\{\theta_1 g_{\diamond,1}(z)+1\bigr\}\,dm(z),\end{aligned}$$

from which

$$\begin{aligned}0&=\theta_1^2\int_S g_{\diamond,1}(z)^2\,dm(z)+\theta_1\int_S g_{\diamond,1}(z)\,dm(z)+\int_S g_1(z)\,dm(z)\\&\quad-\bigl\{\theta_1 g_{\diamond,1}(l_2)+1\bigr\}\int_S g_1(z)\,dm(z)\\&<\theta_1^2\{s(l_2)-s(l_1)\}\int_S\{s(z)-s(l_1)\}\,dm(z)+\theta_1\int_S\{s(z)-s(l_1)\}\,dm(z)\\&\quad-\theta_1\{s(l_2)-s(l_1)\}\int_S\{\theta_1(s(z)-s(l_1))+1\}\,dm(z)\\&=\theta_1\int_S\{s(z)-s(l_1)\}\,dm(z)-\theta_1\{s(l_2)-s(l_1)\}\{m(l_2)-m(l_1)\}.\end{aligned}$$

This contradicts the fact that the last term is negative. Thus it does not happen that $\theta_1+\theta_2>0$.

We consider the case $\theta_1=\theta_2=0$. In this case we have that

$$\lim_{\alpha\downarrow 0}\alpha G(\alpha,x,y)=M^{-1},$$

uniformly in $x,y\in S^*$, where $M=m(l_2)-m(l_1)$ (cf. [**8**], [**13**], [**15**]). We denote by ∂_y^+ the right derivative in $s(y)$, that is, $\partial_y^+G(\alpha,x,y)=\lim_{\varepsilon\downarrow 0}\{G(\alpha,x,y+\varepsilon)-G(\alpha,x,y)\}/\{s(y+\varepsilon)-s(y)\}$. By means of $\theta_2=0$ and (3.8),

$$\partial_y^+G(\alpha,l_1,y)=-\alpha W(\alpha)^{-1}g_1(l_1,\alpha)\int_{(y,l_2)}g_2(z,\alpha)\,dm(z),$$

and hence

$$\lim_{\alpha\downarrow 0}\partial_y^+G(\alpha,l_1,y)=-M^{-1}\{m(l_2)-m(y)\}.$$

It is easy to see that

$$\lim_{\alpha\downarrow 0}\partial_y^+ G_\diamond(\alpha, l_2, y) = 1,$$
$$\lim_{\alpha\downarrow 0}\partial_y^+ G_\bullet(\alpha, z, y) = -h(z)1_{(l_1,y]}(z) + \{1-h(z)\}1_{(y,l_2)}(z),$$

where $1_\Lambda(z) = 1$ if $z \in \Lambda$, and $= 0$ if $z \notin \Lambda$. Multiply (3.16) by α, operate ∂_y^+ and let $\alpha \downarrow 0$. Then we obtain the following.

$$0 = \int_S M^{-1}h(z)^{-1}\Big\{-1_{(l_1,y]}(z)h(z)h(y) + 1_{(y,l_2)}(z)(1-h(z))h(y) + G_\bullet(z,y)(s(l_2)-s(l_1))^{-1} + h(z) - h(z)M^{-1}(m(l_2)-m(y))\Big\}\,dm(z).$$

By virtue of (3.17),

$$G_\bullet(z,y)\{s(l_2)-s(l_1)\}^{-1} = -h(z)h(y) + h(z)1_{(l_1,y]}(z) + h(y)1_{(y,l_2)}(z).$$

By using this and $h(z) < 1$,

$$\begin{aligned} 0 &= \int_S \Big\{2h(y)h(z)^{-1}1_{(y,l_2)}(z) - 2h(y) + 1_{(l_1,y]}(z) + 1 \\ &\quad - M^{-1}(m(l_2)-m(y))\Big\}\,dm(z) \\ &> 2h(y)\{m(l_2)-m(y)\} - 2h(y)M + m(y) - m(l_1) + M \\ &\quad - m(l_2) + m(y) \\ &= 2\{m(y)-m(l_1)\}\{1-h(y)\}, \end{aligned}$$

which is contradicting the fact that the last term is positive. Thus it does not happen that l_1 is (s,m)-regular and elastic.

Therefore Q does not satisfy the Chapman-Kolmogorov equation. □

Appendix

It is known that the ODGDP $\boldsymbol{D}$ is identical in law with a time changed process of the Brownian motion. By using this fact we show (3.21). Assume that both of l_1 and l_2 are regular and the boundary condition (2.2) is satisfied for $i = 1, 2$. We may assume that the scale function is natural, that is, $s(x) = x,\ x \in S$, without loss of generality. We set $\widetilde{l_1} = l_1 - \theta_1^{-1}$ and $\widetilde{l_2} = l_2 + \theta_2^{-1}$, where $1/0 = \infty$. Further we set

$$\widetilde{m}(x) = \begin{cases} -\infty, & x \le \widetilde{l_1}, \\ m(l_1), & \widetilde{l_1} < x < l_1, \\ m(x), & l_1 \le x < l_2, \\ m(l_2), & l_2 \le x < \widetilde{l_2}, \\ \infty, & \widetilde{l_2} \le x. \end{cases}$$

Let $\boldsymbol{B} = [B(t) : t \geq 0,\ P_x^{\boldsymbol{B}} : x \in \boldsymbol{R}]$ be the Brownian motion and $\mathfrak{t}(t,\xi),\ t \geq 0$, be the local time at ξ. We set $\mathfrak{f}(t) = \int_{(\tilde{l}_1,\tilde{l}_2)} \mathfrak{t}(t,\xi)\, d\tilde{m}(\xi),\ t \geq 0$, and denote by $\mathfrak{f}^{-1}(t)$ the inverse function. Then the time changed process $[B(\mathfrak{f}^{-1}(t)) : t \geq 0,\ P_x^{\boldsymbol{B}} : x \in S^*]$ is identical in law with $\boldsymbol{D}$ (see [**6**], [**8**], [**15**]). Denoting by $\sigma_a^{\boldsymbol{B}}$ the hitting time at a of the Brownian motion, we obtain that

$$\begin{aligned} E_{l_2}[\sigma_{l_1}; \sigma_{l_1} < \infty] &= E_{l_2}^{\boldsymbol{B}}\Big[\mathfrak{f}(\sigma_{l_1}^{\boldsymbol{B}}); \sigma_{l_1}^{\boldsymbol{B}} < \sigma_{\tilde{l}_2}^{\boldsymbol{B}}\Big] \\ &= \int_{(\tilde{l}_1,\tilde{l}_2)} E_{l_2}^{\boldsymbol{B}}\Big[\mathfrak{t}(\sigma_{l_1}^{\boldsymbol{B}}, \xi); \sigma_{l_1}^{\boldsymbol{B}} < \sigma_{\tilde{l}_2}^{\boldsymbol{B}}\Big]\, d\tilde{m}(\xi). \end{aligned} \tag{A.1}$$

Let $a, b > 0$. By using $E_0^{\boldsymbol{B}}[e^{-\alpha\mathfrak{t}(\sigma_a^{\boldsymbol{B}},0)}] = (1+\alpha a)^{-1}$ (cf. [**6**]) and the strong Markov property of the Brownian motion, we obtain that

$$E_0^{\boldsymbol{B}}\Big[e^{-\alpha\mathfrak{t}(\sigma_a^{\boldsymbol{B}},0)}; \sigma_a^{\boldsymbol{B}} < \sigma_{-b}^{\boldsymbol{B}}\Big] = b(\alpha ab + a + b)^{-1},$$

from which

$$E_0^{\boldsymbol{B}}\big[\mathfrak{t}(\sigma_a^{\boldsymbol{B}}, 0); \sigma_a^{\boldsymbol{B}} < \sigma_{-b}^{\boldsymbol{B}}\big] = ab^2(a+b)^{-2}.$$

Therefore

$$\begin{aligned} &\int_{(\tilde{l}_1,\tilde{l}_2)} E_{l_2}^{\boldsymbol{B}}\Big[\mathfrak{t}(\sigma_{l_1}^{\boldsymbol{B}}, \xi); \sigma_{l_1}^{\boldsymbol{B}} < \sigma_{\tilde{l}_2}^{\boldsymbol{B}}\Big]\, d\tilde{m}(\xi) \\ &\quad = \int_S E_{\xi}^{\boldsymbol{B}}\Big[\mathfrak{t}(\sigma_{l_1}^{\boldsymbol{B}}, \xi); \sigma_{l_1}^{\boldsymbol{B}} < \sigma_{\tilde{l}_2}^{\boldsymbol{B}}\Big]\, dm(\xi) \\ &\quad = \int_S E_0^{\boldsymbol{B}}\Big[\mathfrak{t}(\sigma_{\xi-l_1}^{\boldsymbol{B}}, 0); \sigma_{\xi-l_1}^{\boldsymbol{B}} < \sigma_{\xi-\tilde{l}_2}^{\boldsymbol{B}}\Big]\, dm(\xi) \\ &\quad = \int_S (\xi - l_1)(\tilde{l}_2 - \xi)^2(\tilde{l}_2 - l_1)^{-2}\, dm(\xi) \\ &\quad = g_2(l_1)^{-2} \int_S g_{\diamond,1}(\xi) g_2(\xi)^2\, dm(\xi), \end{aligned}$$

where we used (3.17) and (3.19). Combining this with (A.1), we obtain (3.21). □

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Masaru IIZUKA
Division of Mathematics
Kyushu Dental College
Kitakyushu 803-8580, Japan

Miyuki MAENO
School of Interdisciplinary Research
of Scientific Phenomena and Information
Graduate School of Humanities Sciences
Nara Women's University
Nara 630-8506, Japan

Current address:
Postal Life Insurance Business Department
Financial Business Headquarters
Japan Post
Tokyo 100-8798, Japan

Matsuyo TOMISAKI
Department of Mathematics
Faculty of Science
Nara Women's University
Nara 630-8506, Japan