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# Non-smooth points set of fibres of a semialgebraic mapping

By Satoshi Koike and Masahiro Shiota

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**Abstract.** For a semialgebraic mapping between semialgebraic sets, we consider the set of points at which the fibre is not smooth. In this paper we discuss whether the singular set is itself semialgebraic, when it has codimension bigger than or equal to 2 in the domain of $f$ and whether the mapping is semialgebraically trivial along the smooth part of the fibre, giving several examples which show optimality of those results. In addition, we give an example of a polynomial function $f$ such that even the $(a_f)$ condition in the weak sense fails in a neighbourhood of a smooth fibre, but $f$ is semialgebraically trivial along it.

## Introduction

A semialgebraic set of $\boldsymbol{R}^n$ is a finite union of sets of the form

$$\{x \in \boldsymbol{R}^n \mid f_1(x) = \cdots = f_k(x) = 0,\ g_1(x) > 0, \cdots, g_s(x) > 0\}$$

where $f_1, \cdots, f_k, g_1, \cdots, g_s$ are polynomial functions on $\boldsymbol{R}^n$. Let $r = 1, 2, \cdots, \infty, \omega$. A semialgebraic set $M \subset \boldsymbol{R}^m$ is called a $C^r$ Nash manifold, if it is a $C^r$ (regular) submanifold of $\boldsymbol{R}^m$. Let $M \subset \boldsymbol{R}^m$ and $N \subset \boldsymbol{R}^n$ be $C^r$ Nash manifolds. A $C^s$ mapping $f : M \to N$, $s \leq r$, is called a $C^s$ Nash mapping, if the graph of $f$ is semialgebraic in $\boldsymbol{R}^m \times \boldsymbol{R}^n$. A mapping $f : M \to N$, where $M \subset \boldsymbol{R}^m$ and $N \subset \boldsymbol{R}^n$ are semialgebraic sets, is called a semialgebraic mapping, if the graph of $f$ is semialgebraic in $\boldsymbol{R}^m \times \boldsymbol{R}^n$. (We don't assume the continuity of $f$.)

Here we make one remark on $C^\infty$ Nash and $C^\omega$ Nash.

Remark 0.1 (B. Malgrange [**10**]).
(1) A $C^\infty$ Nash manifold is a $C^\omega$ Nash manifold.
(2) A $C^\infty$ Nash mapping between $C^\omega$ Nash manifolds is a $C^\omega$ Nash mapping.

After this, we call a $C^\infty$ Nash manifold a Nash manifold and a $C^\infty$ Nash mapping a Nash mapping. Now we observe the fibres of a $C^r$ Nash mapping.

Observation 0.2. Let $r < \infty$. Then it is easy to construct a $C^r$ Nash function $f$ from the 2 dimensional sphere $S^2$ to $\boldsymbol{R}$ such that
(i) the fibre of the star point $\star$ is $C^r$ Nash diffeomorphic to a closed disk,

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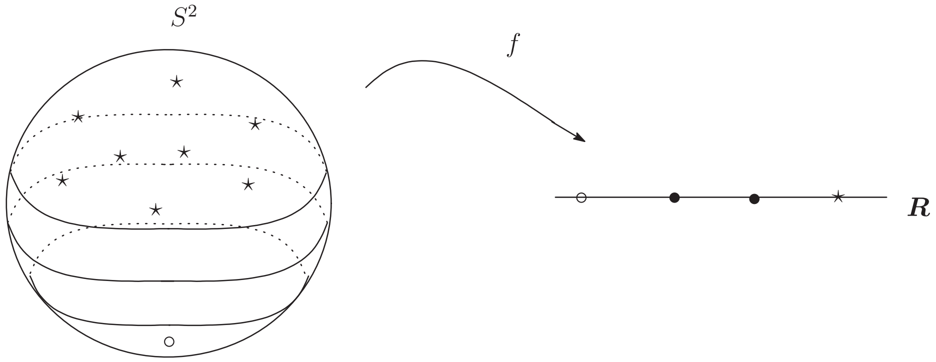
(ii) the fibre of each point • is $C^r$ Nash diffeomorphic to a circle, and
(iii) the fibre of the circle point ∘ is a point.




In fact, we can construct such a function as follows. Define $g : \boldsymbol{R}^2 \to \boldsymbol{R}$ by

$$g(x,y) = \begin{cases} -(x^2+y^2-1)^{2r} & (x^2+y^2 \leq 1) \\ 0 & (x^2+y^2 \geq 1). \end{cases}$$

Let $p : S^2 - \{N\} \to \boldsymbol{R}$ be the stereographic projection from the north pole $N$ of $S^2$. Define $f : S^2 - \{N\} \to \boldsymbol{R}$ by $f = g \circ p$ and $f(N) = 0$.

Observe that for $a \in \boldsymbol{R}$, $\dim f^{-1}(a) = 2, 1, 0, -1$ where $\dim \emptyset = -1$. We next consider the set of points at which the fibre is not smooth. In this case it is a circle which is the boundary of the closed disk. This example suggests such a set is a semialgebraic subset of $M$ of codimension $\geq 1$.

Suppose we are given a semialgebraic mapping between semialgebraic sets. In this paper we give some results on semialgebraicity, codimension 2 property and closedness of the set of points $x$ at which the fibres $f^{-1}(f(x))$ are not $C^k$ Nash manifolds. We discuss also semialgebraic triviality of a $C^1$ Nash mapping along the smooth part of the fibre in the last section.

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## 1. Semialgebraic properties.

We first recall some important properties of semialgebraicity.

THEOREM 1.1 (Tarski-Seidenberg Theorem [**15**]). *Let $A$ be a semialgebraic set in $\boldsymbol{R}^k$, and let $f : \boldsymbol{R}^k \to \boldsymbol{R}^m$ be a semialgebraic mapping. Then $f(A)$ is semialgebraic in $\boldsymbol{R}^m$.*

THEOREM 1.2 (Lojasiewicz's Semialgebraic Triangulation Theorem [**8**],[**9**]). *Given a finite system of bounded semialgebraic sets $X_\alpha$ in $\boldsymbol{R}^n$, there exist a simplicial decomposition $\boldsymbol{R}^n = \cup_a C_a$ by open simplexes and a semialgebraic automorphism $\tau$ of $\boldsymbol{R}^n$ such that*

(1) *each $X_\alpha$ is a finite union of some of the $\tau(C_a)$,*

(2) *$\tau(C_a)$ is a Nash manifold in $\boldsymbol{R}^n$ and $\tau$ induces a Nash diffeomorphism $C_a \to \tau(C_a)$, for every $a$.*

REMARK 1.3. There is a Nash embedding of $\boldsymbol{R}^n$ into $\boldsymbol{R}^{n+1}$ via $\boldsymbol{R}^n \subset S^n$. Then every semialgebraic set in $\boldsymbol{R}^n$ can be considered as a bounded semialgebraic set in $\boldsymbol{R}^{n+1}$.

THEOREM 1.4 (Hardt's Semialgebraic Triviality Theorem [**5**]). *Let $B$ be a semialgebraic set, and let $\Pi : \boldsymbol{R}^m \times B \to B$ be the projection. For any semialgebraic subset $X$ of $\boldsymbol{R}^m \times B$, there is a finite partition of $B$ into semialgebraic sets $N_i$, and for any $i$, there are a semialgebraic set $F_i \subset \boldsymbol{R}^m$ and a semialgebraic homeomorphism*

$$h_i : F_i \times N_i \to X \cap \Pi^{-1}(N_i)$$

*compatible with the projection onto $N_i$.*

REMARK 1.5. By the Semialgebraic Triangulation Theorem, we can assume that the $N_i$'s are Nash manifolds taking a finite subdivision again if necessary.

Let $M \subset \boldsymbol{R}^m$ and $N \subset \boldsymbol{R}^n$ be semialgebraic sets, let $A \subset M$ be a semialgebraic subset and let $f : M \to N$ be a semialgebraic mapping. For $Q \in f(A)$, set

$$\Sigma_k(f^{-1}(Q)) = \{x \in f^{-1}(Q) \mid f^{-1}(Q) \text{ is not a } C^k \text{ Nash manifold in } \boldsymbol{R}^m \text{ at } x\},$$

$k = 1, 2, \cdots, \infty$ $(\omega)$. Then we can easily see the following lemma by the Semialgebraic Triangulation Theorem.

LEMMA 1.6. *For $Q \in f(A)$, $\dim \Sigma_k(f^{-1}(Q)) < \dim f^{-1}(Q)$.*

Using the Semialgebraic Triviality Theorem, we can show the following:

LEMMA 1.7. *Let $f : M \to N$ be a continuous, semialgebraic mapping, and let $b \in \boldsymbol{N}$. If $\dim A \cap f^{-1}(Q) + b \leq \dim f^{-1}(Q)$ for any $Q \in f(A)$, then $\dim A + b \leq \dim M$.*

PROOF OF LEMMA 1.7. Let $\Pi : \boldsymbol{R}^m \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ be the canonical projection. Then we can identify $M$, $A$, and $f$ with graph$f$, graph$f|_A$, and $\Pi|_{graph\, f}$ : graph$f \to \boldsymbol{R}^n$, respectively. By the Semialgebraic Triviality Theorem, there is a finite partition of $f(M)$ into semialgebraic sets $R_i$, and for any $i$ there are a semialgebraic set $D_i \subset \boldsymbol{R}^m$ and a semialgebraic homeomorphism $\phi_i : D_i \times R_i \to f^{-1}(R_i)$ compatible with the projection onto $R_i$. In addition, there is a finite partition of $f(A)$ into semialgebraic sets $S_j$, and for any $j$ there are a semialgebraic set $E_j \subset \boldsymbol{R}^m$ and a semialgebraic homeomorphism $\psi_i$ : $E_j \times S_j \to A \cap f^{-1}(S_j)$ compatible with the projection onto $S_j$. By the Semialgebraic Triangulation Theorem, there is a finite partition of $f(A)$ into Nash manifolds $N_k$ which is compatible with $R_i$'s and $S_j$'s, namely, $N_k \subset R_{i(k)}$ and $N_k \subset S_{j(k)}$ for some $i(k)$ and $j(k)$. For any $k$,

$$\phi_{i(k)}|_{D_{i(k)} \times N_k} : D_{i(k)} \times N_k \to f^{-1}(N_k), \quad \psi_{j(k)}|_{E_{j(k)} \times N_k} : E_{j(k)} \times N_k \to A \cap f^{-1}(N_k)$$

are semialgebraic homeomorphisms compatible with the projections. Then it follows that

$$\dim f^{-1}(Q) + \dim N_k = \dim D_{i(k)} + \dim N_k = \dim f^{-1}(N_k) \leq \dim M, \quad Q \in N_k.$$

In addition, there is $k_0$ such that

$$\begin{aligned}\dim A \cap f^{-1}(Q) + \dim N_{k_0} &= \dim E_{j(k_0)} + \dim N_{k_0} \\ &= \dim A \cap f^{-1}(N_{k_0}) = \dim A, \quad Q \in N_{k_0}.\end{aligned}$$

Assume that $\dim A + b > \dim M$. Then

$$\dim A \cap f^{-1}(Q) + \dim N_{k_0} + b > \dim f^{-1}(Q) + \dim N_{k_0}, \quad Q \in N_{k_0}.$$

Thus we have $\dim A \cap f^{-1}(Q) + b > \dim f^{-1}(Q)$, $Q \in N_{k_0}$. This contradicts the hypothesis. Therefore $\dim A + b \leq \dim M$. □

## 2. Main results.

Let $M \subset \boldsymbol{R}^m$ and $N \subset \boldsymbol{R}^n$ be semialgebraic sets, and let $f : M \to N$ be a semialgebraic mapping. For $k = 1, 2, \cdots, \infty$ $(\omega)$, set

$$\Sigma_k = \{x \in M \mid f^{-1}(f(x)) \text{ is not a } C^k \text{ submanifold of } \boldsymbol{R}^m \text{ at } x\}.$$

In case we need specify the mapping $f$ for $\Sigma_k$, we denote it by $\Sigma_k[f]$. By definition,

$$\Sigma_1 \subset \Sigma_2 \subset \cdots \subset \Sigma_\infty. \tag{2.1}$$

Since $f^{-1}(f(x)) - \Sigma_k$ is a regular submanifold of $\boldsymbol{R}^m$ and $f$ is a semialgebraic mapping, we can replace a $C^k$ submanifold of $\boldsymbol{R}^m$ in the definition of $\Sigma_k$ by a $C^k$ Nash manifold in $\boldsymbol{R}^m$ or a $C^k$ Nash submanifold of $M$ for $k > 0$. Then we can express $\Sigma_k(f^{-1}(Q))$ in the previous section as follows:

$$\Sigma_k(f^{-1}(Q)) = \Sigma_k \cap f^{-1}(Q), \quad k = 1, 2, \cdots, \infty.$$

THEOREM 2.1. *For $k = 1, 2, \cdots$, $\Sigma_k$ is semialgebraic.*

We shall give the proof of this theorem in the next section. In the case $k < \infty$, we can write down the conditions of $\Sigma_k$ using finitely many words related to semialgebraicity. How is the case $k = \infty$ ? If we try to write down the conditions in a similar way to the finite case, they must be infinitely many. Therefore it makes sense to consider the stabilisation property of (2.1).

THEOREM 2.2 (Stabilisation). *There is $k \in \boldsymbol{N}$ such that $\Sigma_k = \Sigma_{k+1} = \cdots = \Sigma_\infty$.*

EXAMPLE 2.3. Let $f : \boldsymbol{R}^2 \to \boldsymbol{R}$ be a polynomial function defined by $f(x, y) = x^3 - y^8$. Then $\Sigma_1 = \Sigma_2 = \emptyset$, $\Sigma_3 = \Sigma_4 = \cdots = \Sigma_\infty = \{(0, 0)\}$.

As a corollary of Theorems 2.1, 2.2 and Lemmas 1.6 and 1.7, we have

COROLLARY 2.4. *$\Sigma_k$, $k = 1, 2, \cdots, \infty$, are semialgebraic subsets of $M$ of codim $\geq 1$.*

By this corollary, we can see Observation 0.2 in the introduction. On the other hand, we have the following stronger result in the $C^\infty$ Nash case.

THEOREM 2.5. *Let $M$, $N$ be Nash manifolds with* $\dim M \geq 1$, *and let* $f: M \to N$ *be a Nash mapping. Then $\Sigma_k$ is a semialgebraic subset of $M$ of codimension* $\geq 2$.

In the $C^r$ Nash case where $r$ is finite, the codimension 2 property of $\Sigma_k$ does not hold in general as seen in Observation 0.2. But, using similar arguments to the proof of the theorem above, we can show the first statement in the following proposition. In addition, we can easily see that the second statement holds under the same assumption.

PROPOSITION 2.6. *Let $M$ and $N$ be $C^r$ Nash manifolds with* $\dim M \geq 1$ *and* $\dim N = 1$, *and let $f: M \to N$ be a $C^s$ Nash mapping* $(1 \leq s \leq r \leq \infty)$. *Assume that $k \leq s$. Then,*

(1) *if codim $f^{-1}(Q) \geq 1$ for any $Q \in f(\Sigma_k)$, then $\Sigma_k$ is a semialgebraic subset of $M$ of codim $\geq 2$, and*

(2) *$\Sigma_k$ is closed in $M$.*

In this proposition, the assumptions that $\dim N = 1$ and $k \leq s$ are essential for the codimension 2 property and closedness.

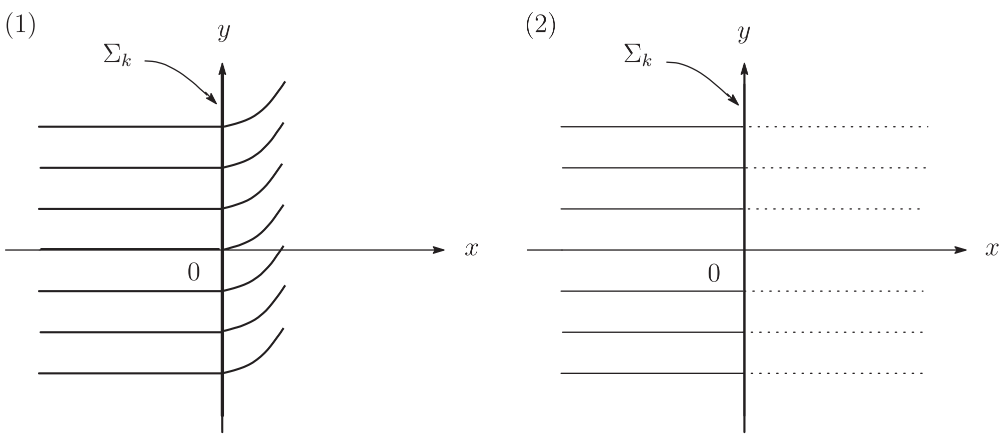
EXAMPLE 2.7. Let $g: \boldsymbol{R} \to \boldsymbol{R}$ be a $C^s$ function, $s \in \boldsymbol{N}$, defined by

$$g(x) = \begin{cases} x^{s+1} & (x \geq 0) \\ 0 & (x \leq 0) \end{cases}$$

(1) Define $f: \boldsymbol{R}^2 \to \boldsymbol{R}$ by $f(x,y) = y - g(x)$. In the case $k \leq s$, $\Sigma_k = \emptyset$. Therefore codim $\Sigma_k = 3$.

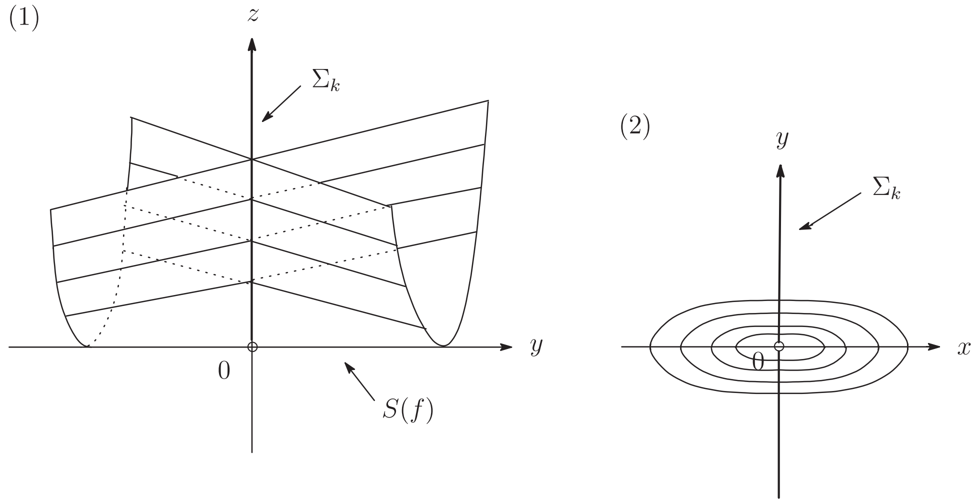
In the case $k > s$, $\Sigma_k = \{y\text{-axis}\}$. Therefore codim $\Sigma_k = 1$. Furthermore, for any $Q \in f(\Sigma_k) = \boldsymbol{R}$, codim $f^{-1}(Q) = 1$. Thus Proposition 2.6 (1) does not hold when $k > s$.

(2) Define $f: \boldsymbol{R}^2 \to \boldsymbol{R}^2$ by $f(x,y) = (y, y - g(x))$. For $k = 1, 2, \cdots, \infty$, $\Sigma_k = \{y\text{-axis}\}$. Therefore codim $\Sigma_k = 1$. Furthermore, for any $Q \in f(\Sigma_k)$, codim $f^{-1}(Q) = 1$. Thus Proposition 2.6 (1) does not hold in the case $\dim N \geq 2$.



EXAMPLE 2.8. (1) Let $f:\boldsymbol{R}^3 \to \boldsymbol{R}^2$ be a polynomial mapping defined by $f(x,y,z)=(x^2-zy^2,z)$. Then, for $k=1,2,\cdots,\infty$, $\Sigma_k=\{(x,y,z)\mid x=y=0,\ z>0\}$, which is not closed in $\boldsymbol{R}^3$. Thus Proposition 2.6 (2) does not hold in the case $\dim N \geq 2$.

(2) Let $f:\boldsymbol{R}^2\to\boldsymbol{R}$ be a $C^s$ function, $s\in\boldsymbol{N}$, defined by $f(x,y)=|x|^{\frac{3s+1}{3}}+y^2$. In the case $k\leq s$, $\Sigma_k=\emptyset$. But, in the case $k>s$, $\Sigma_k=\{y\text{-axis}\}-\{(0,0)\}$, which is not closed in $\boldsymbol{R}^2$. Thus Proposition 2.6 (2) does not hold when $k>s$.




## 3. Proofs of the main results.

### 3.1. Proof of Theorem 2.1.

Let $M\subset\boldsymbol{R}^m$ and $N\subset\boldsymbol{R}^n$ be semialgebraic sets and let $f:M\to N$ be a semialgebraic mapping. For a nonnegative integer $k$, let $A_k$ denote the set of points $x\in M$ at which $f^{-1}(f(x))$ is a $C^k$ submanifold of $\boldsymbol{R}^m$. Here a $C^0$ submanifold of $\boldsymbol{R}^m$ means a topological manifold with the relative topology induced from $\boldsymbol{R}^m$.

Before beginning the proof of Theorem 2.1, we recall the following well-known fact:

*A subset of $M$ which is expressed by a sentence consisting of a finite number of semialgebraic equalities, semialgebraic inequalities, implications (i.e., if $\cdots$ then $\cdots$) and logical symbols $\forall$, $\exists$ and $\neq$ is semialgebraic.*

For instance, the set $M_1$ of points in $M$ at which $f$ is continuous is semialgebraic because

$$M_1=\{x\in M\mid \forall\epsilon>0,\ \exists\delta>0\ \forall x'\in M \text{ if } |x-x'|<\delta \text{ then } |f(x)-f(x')|<\epsilon\}.$$

Other instances are the set of points in $M$ at which $f$ is locally injective,

$$\{x\in M\mid \exists\epsilon>0\ \forall x',x''\in M \text{ if } |x-x'|<\epsilon,\ |x-x''|<\epsilon \text{ and } x'\neq x'' \text{ then } f(x')\neq f(x'')\},$$

and the set of points in $M$ at which $f$ is locally surjective,

$$\left\{ x \in M \mid \begin{array}{l} \forall \epsilon > 0,\ \exists \delta > 0\ \forall y' \in N \text{ if } |y' - f(x)| < \delta \\ \text{then } \exists x' \in M \text{ s.t. } f(x') = y',\ |x - x'| < \epsilon \end{array} \right\}.$$

Now we start the proof. It is easy to see that $A_0$ is semialgebraic as follows. Let $\Pi : \boldsymbol{R}^m \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ be the canonical projection. We consider the map $F : M \to \boldsymbol{R}^m \times \boldsymbol{R}^n$ defined by $F(x) = (x, f(x))$. Since continuous is the restriction of $F$ to any level set of $f : M \to N$, the level set of $f$ is homeomorphic to the corresponding level set of $\Pi|_{graph\, f} : \text{graph} f \to N$. If we can show that semialgebraic is the set of points $(x, f(x)) \in \text{graph} f$ at which $\Pi|_{graph\, f}^{-1}(\Pi|_{graph\, f}(x, f(x))) = \Pi|_{graph\, f}^{-1}(f(x))$ is a $C^0$ submanifold of $\boldsymbol{R}^m \times \{f(x)\}$, then it follows from the Tarski-Seidenberg Theorem that $A_0$ is semialgebraic. Therefore it suffices to show the statement for $\Pi|_{graph\, f} : \text{graph} f \to N$. Let us replace $M$ and $f$ with graph $f$ and $\Pi|_{graph\, f}$ respectively. We then can assume that $f$ is continuous. By the Hardt Theorem we have a finite semialgebraic stratification $\{N_i\}$ of $N$ and a semialgebraic homeomorphism $\pi_i = (\pi'_i, f) : f^{-1}(N_i) \to f^{-1}(a_i) \times N_i$ for each $N_i$ and some $a_i \in N_i$. Hence for the proof that $A_0$ is semialgebraic we can suppose that $M = N_1 \times N$ for a semialgebraic set $N_1 \subset \boldsymbol{R}^m$ and $f : N_1 \times N \to N$ is the projection. Let us denote by $A_0^1$ the set of points of $N_1$ at which $N_1$ is a topological submanifold of $\boldsymbol{R}^m \subset S^m$. Then we only need to show that $A_0^1$ is semialgebraic. That is clear. Indeed, by the Semialgebraic Triangulation Theorem we can regard $(S^m, N_1)$ as $(|K|, \cup_{\sigma \in K'} \text{Int}(\sigma))$ for a finite simplicial complex $K$ and a subset $K'$ of $K$, where $|K|$ denotes the underlying polyhedron of $K$. Then $A_0^1$ is a union of some $\text{Int}(\sigma)$, $\sigma \in K$ because for $a_1,\ a_2 \in \text{Int}(\sigma)$, $a_1 \in A_0^1$ if and only if $a_2 \in A_0^1$. Thus $A_0$ is semialgebraic.

For $0 \le j \le m$, let $A_{k,j} \subset A_k$ denote the set of points $x \in M$ at which $p|_{f^{-1}(f(x))} : f^{-1}(f(x)) \to \boldsymbol{R}^j$ is a $C^k$ diffeomorphism (a homeomorphism if $k = 0$) locally at $x$, where $p : \boldsymbol{R}^m \to \boldsymbol{R}^j$ is a projection. Let $p_1, p_2, \dots$ denote all the projections $\boldsymbol{R}^m \to \boldsymbol{R}^j$ forgetting some factors. Assume $k > 0$. Since $\Sigma_k$ is the complement of the union of $A_{k,j}$'s for some $p = p_l$, $0 \le j \le m$ in $M$, it suffices to show that each $A_{k,j}$ is semialgebraic.

First consider $A_{0,j}$. Clearly, $A_{0,j}$ is the set of points $x \in M$ such that $p|_{f^{-1}(f(x))} : f^{-1}(f(x)) \to \boldsymbol{R}^j$ is injective and surjective locally at $x$ and $(p|_{f^{-1}(f(x))})^{-1}$ is of class $C^0$ at $p(x)$. Hence

$A_{0,j} = \{x \in M \mid \exists \epsilon > 0\ \forall x',\ x'' \in M \text{ if } |x - x'| < \epsilon,\ |x - x''| < \epsilon,\ x' \ne x'' \text{ and } f(x) = f(x') = f(x'') \text{ then } p(x') \ne p(x'');\ \forall \epsilon > 0\ \exists \delta > 0\ \forall a' \in \boldsymbol{R}^j \text{ if } |p(x) - a'| < \delta \text{ then } \exists x' \in M \text{ s.t. } |x - x'| < \epsilon,\ f(x) = f(x'),\ p(x') = a';\ \forall \epsilon > 0\ \exists \delta > 0\ \forall \epsilon' > 0\ \exists \delta' > 0\ \forall a',\ a'' \in \boldsymbol{R}^j \text{ if } |p(x) - a'| < \delta,\ |p(x) - a''| < \delta \text{ and } |a' - a''| < \delta' \text{ then } \exists x',\ x'' \in M \text{ s.t. } |x - x'| < \epsilon,\ |x - x''| < \epsilon,\ |x' - x''| < \epsilon',\ f(x) = f(x') = f(x''),\ p(x') = a',\ p(x'') = a''\}$.
Therefore $A_{0,j}$ is semialgebraic.

Next we simplify the claim that $A_{1,j}$ is semialgebraic. The sets $B = \{(x, x') \in A_{0,j}^2 \mid f(x) = f(x')\}$ and $B_x = B \cap \{x\} \times A_{0,j}$ for $x \in A_{0,j}$ are semialgebraic, and the map $B_x \ni (x, x') \to p(x') \in \boldsymbol{R}^j$ for each $x \in A_{0,j}$ is a local homeomorphism. Hence there exists a semialgebraic open neighbourhood $U$ of the diagonal of $A_{0,j}$ in $B$ such that the map $U \cap B_x \ni (x, x') \to p(x') \in \boldsymbol{R}^j$ is a homeomorphism onto an open set $V_x$ in $\boldsymbol{R}^j$. For each $x \in A_{0,j}$, let $q_x : V_x \to \boldsymbol{R}^m$ denote the composite of the inverse map : $V_x \to B_x$ and the projection $B_x \ni (x, x') \to x' \in A_{0,j} \subset \boldsymbol{R}^m$, and set $V = \bigcup_x \{x\} \times V_x \subset A_{0,j} \times \boldsymbol{R}^j$ where the union is taken over $A_{0,j}$ and $q(x, a) = q_x(a)$ for $(x, a) \in V$. Then $V$ and $q : V \to \boldsymbol{R}^m$ are semialgebraic, $q_x$ is a homeomorphism onto its image which contains $x$, and

$$A_{1,j} = \{x \in A_{0,j} \mid q_x \text{ is a } C^1 \text{ embedding at } p(x)\}.$$

Thus it suffices to show the following statement.

STATEMENT 1. *Let $C$ and $D \subset C \times \boldsymbol{R}^j$ be semialgebraic sets, and let $\phi : D \to \boldsymbol{R}^m$ be a semialgebraic map. Assume that for each $x \in C$, $D_x = D \cap \{x\} \times \boldsymbol{R}^j$ is open in $\{x\} \times \boldsymbol{R}^j$ and $\phi|_{D_x}$ is a homeomorphism onto its image. Then the set*

$$D^1 = \{(x, y) \in D \mid \phi|_{D_x} \text{ is a } C^1 \text{ embedding at } (x, y)\}$$

*is semialgebraic.*

Set

$$\begin{aligned}
\tilde{D} &= \{(x, y, y', t) \in D \times \boldsymbol{R}^j \times (0, 1] \mid \forall s \in [0, 1], (x, y + sy') \in D\},\\
\tilde{\phi}(x, y, y', t) &= (\phi(x, y + ty') - \phi(x, y))/t \text{ for } (x, y, y', t) \in \tilde{D},\\
G &= D \times \boldsymbol{R}^j \times \{0\} \times \boldsymbol{R}^m \cap \overline{\mathit{graph}\ \tilde{\phi}},\\
G_{x,y} &= \{(x, y)\} \times \boldsymbol{R}^j \times \{0\} \times \boldsymbol{R}^m \cap G \text{ for } (x, y) \in D,
\end{aligned}$$

and let $\rho_1 : G \to \boldsymbol{R}^j$ and $\rho_2 : G \to \boldsymbol{R}^m$ denote the projections. Then $\tilde{D}$, $\tilde{\phi} : \tilde{D} \to \boldsymbol{R}^m$, $G$, $G_{x,y}$, $\rho_1$ and $\rho_2$ are semialgebraic, and

$$D^1 = \{(x, y) \in D \mid \rho_1|_{G_{x,y}} \text{ and } \rho_2|_{G_{x,y}} \text{ are homeomorphisms onto } \boldsymbol{R}^j \text{ and } \rho_2(G_{x,y}) \text{ resp.}\}.$$

Hence we see $D^1$ is semialgebraic in the same way as the first arguments.

By the same reason as above, the claim that $A_{k,j}$, $k > 1$, is semialgebraic follows from Statement $k$, which is similarly defined by replacing $D^1$ in Statement 1 by

$$D^k = \{(x, y) \in D^{k-1} \mid \phi|_{D_x^{k-1}} \text{ is a } C^k \text{ embedding at } (x, y)\}.$$

Statement 2 is reduced to Statement 1 as follows. Set $E = D^1 \times \boldsymbol{R}^j$ and

$$\psi(x, y, y') = (\phi(x, y), d(\phi|_{D_x^1})_y y') \in \boldsymbol{R}^m \times \boldsymbol{R}^m \text{ for } (x, y, y') \in D \times \boldsymbol{R}^j,$$

where $d$ denotes differential operator. Then $E$ and $\psi : E \to \boldsymbol{R}^m \times \boldsymbol{R}^m$ are semialgebraic, and for each $x \in C$, $E_x = E \cap \{x\} \times \boldsymbol{R}^j \times \boldsymbol{R}^j$ is open in $\{x\} \times \boldsymbol{R}^j \times \boldsymbol{R}^j$ and $\psi|_{E_x}$ is a homeomorphism onto its image. Hence, by Statement 1, the set

$$E^1 = \{(x, y, y') \in E \mid \psi|_{E_x} \text{ is a } C^1 \text{ embedding at } (x, y, y')\}$$

is semialgebraic. On the other hand,

$$D^2 = \{(x, y) \in D^1 \mid \forall y' \in \boldsymbol{R}^j,\ \psi|_{E_x} \text{ is a } C^1 \text{ embedding at } (x, y, y')\}.$$

Therefore, it follows that $D^2$ is semialgebraic.

We can show Statement $k$ in the same way by induction on $k$.

### 3.2. Proof of Theorem 2.2.

We first recall some results of Ramanakoraisina on complexity of semialgebraicity and the Poly-Raby Theorem on characterisation of a $C^k$ submanifold of $\boldsymbol{R}^m$.

Let $U$ be an open semialgebraic subset of $\boldsymbol{R}^m$, and let $f : U \to \boldsymbol{R}$ be a semialgebraic function. In this subsection, a semialgebraic function means a continuous function whose graph is semialgebraic. Then there is a polynomial function $P : \boldsymbol{R}^m \times \boldsymbol{R} \to \boldsymbol{R}$ such that $P(X, f(X)) = 0$ for any $X \in U$. We call the minimum degree of such polynomials the complexity of $f$. Concerning this complexity, we have

THEOREM 3.1 (R. Ramanakoraisina [**13**], [**2**]). *Let $F : U \times T \to \boldsymbol{R}$ be a family of semialgebraic functions parametrised by a semialgebraic set $T$. Then there is a positive integer $d \in \boldsymbol{N}$ such that for any $t \in T$, $F_t = F|_{U\times\{t\}}$ has complexity $\leq d$.*

THEOREM 3.2 (R. Ramanakoraisina [**13**], [**2**]). *Let $f : U \to \boldsymbol{R}$ be a semialgebraic function with complexity $\leq d$. Then there is a positive integer $\gamma(d, m) \in \boldsymbol{N}$ such that if $f$ is of class $C^{\gamma(d,m)}$, then $f$ is of Nash class.*

REMARK 3.3. The integer $\gamma(d, m)$ depends only on the complexity $d$ and dimension $m$. It is independent of the choice of an open semialgebraic set $U$ of $\boldsymbol{R}^m$.

Let $Y$ be a closed set of $\boldsymbol{R}^m$. Define $g : \boldsymbol{R}^m \to \boldsymbol{R}$ by $g(x) = dist^2(x, Y)$. For $k = 1, 2, \cdots, \infty$, set

$$\Sigma_k(Y) = \{y \in Y \mid Y \text{ is not a } C^k \text{ submanifold of } \boldsymbol{R}^m \text{ at } y\},$$
$$S_k(g) = \{x \in \boldsymbol{R}^m \mid g \text{ is not of class } C^k \text{ at } x\}.$$

THEOREM 3.4 (J. B. Poly - G. Raby [**14**]). *For $k = 2, 3, \cdots, \infty$, $\Sigma_k(Y) = Y \cap S_k(g)$.*

Let us show Theorem 2.2 using the above theorems. Let $\Pi : \boldsymbol{R}^m \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ be the canonical projection. Similarly to the proof of Lemma 1.6, we identify $M$ and $f : M \to N$ with $\text{graph} f \subset \boldsymbol{R}^m \times \boldsymbol{R}^n$ and $\Pi|_{graph f} : \text{graph} f \to \boldsymbol{R}^n$, respectively.

For a semialgebraic subset $A \subset M$, set

$$\Gamma(A) = \{x \in A \mid A \text{ is not locally closed in } \boldsymbol{R}^m \text{ at } x\}.$$

Then, by the Semialgebraic Triangulation Theorem, we see that $\Gamma(A)$ is semialgebraic in $\boldsymbol{R}^m$, and for $Q \in f(M)$ and $k = 1, 2, \cdots, \Gamma(f^{-1}(Q)) \subset \Sigma_0(f^{-1}(Q)) \subset \Sigma_k(f^{-1}(Q))$.

Note that for $k = 1, 2, \cdots, \infty$,

$$\Sigma_k = \bigcup_{Q \in f(M)} \Sigma_k(f^{-1}(Q)).$$

By the Semialgebraic Triviality Theorem, there is a finite partition of $f(M)$ into Nash manifolds $\{N_i\}$ such that each $f|_{M\cap\Pi^{-1}(N_i)} : M \cap \Pi^{-1}(N_i) \to N_i$ is semialgebraically trivial over $N_i$. Thanks to the remark in this paragraph, we can regard $f|_{M\cap\Pi^{-1}(N_i)}$ as $f$ for the proof of Theorem 2.2. Therefore there is a semialgebraic homeomorphism $h : f^{-1}(Q) \times N \to M$ $(Q \in N)$ such that $P_N \circ h^{-1} = f$ where $P_N$ is the canonical projection onto $N$. (Here, $N$ is one of $N_i$.) In addition, we may assume that $\Gamma(M)$ is semialgebraically homeomorphic to $\Gamma(f^{-1}(Q)) \times N$ through $h^{-1}$. Then, for any $Q \in N$ and $k = 1, 2, \cdots$, we have

$$\Gamma(f^{-1}(Q)) \subset \Sigma_k(f^{-1}(Q)) \subset \Sigma_\infty(f^{-1}(Q)),\ \Gamma(M) \subset \Sigma_k \subset \Sigma_\infty.$$

Therefore, the removal of $\Gamma(M)$ from the domain of $f$ has no effect on the proof of Theorem 2.2. After this, we assume that $\Gamma(M) = \emptyset$ that is $M$ is locally closed in $\boldsymbol{R}^m$.

By the local closedness of $M$ and the Semialgebraic Triangulation Theorem, we see that there is a finite covering $\{U_i\}$ of $M$ by open semialgebraic sets in $\boldsymbol{R}^m$ such that for each $i$, there is a Nash diffeomorphism $\phi_i : U_i \to \boldsymbol{R}^m$ and $M \cap U_i$ is closed in $U_i$. Let us recall the above identifications $M = \text{graph}\, f$ and $f = \Pi|_{graph f}$ again. Then $M \cap (U_i \times \boldsymbol{R}^n)$ is closed in $U_i \times \boldsymbol{R}^n$ and for $y \in N$, $f^{-1}(y) \cap (U_i \times \{y\})$ is closed in $U_i \times \{y\}$.

For each $i$, we define $\Phi_i : U_i \times \boldsymbol{R}^n \to \boldsymbol{R}^m \times \boldsymbol{R}^n$ by $\Phi_i(x, y) = (\phi_i(x), y)$. Let $\Phi_{i,y} = \Phi|_{U_i \times \{y\}} : U_i \times \{y\} \to \boldsymbol{R}^m \times \{y\}$ for $y \in \boldsymbol{R}^n$. Then $\Phi_{i,y}$, $y \in \boldsymbol{R}^n$, and $\Phi$ are Nash diffeomorphisms. Define $F_i : \boldsymbol{R}^m \times f(M \cap (U_i \times \boldsymbol{R}^n)) \to \boldsymbol{R}$ by

$$F_i(x, y) = dist^2(x \times \{y\}, \Phi_{i,y}(f^{-1}(y) \cap (U_i \times \{y\}))),$$

and $F_{i,y} = F_i|_{\boldsymbol{R}^m \times \{y\}}$ for $y \in f(M \cap (U_i \times \boldsymbol{R}^n))$. By Theorem 3.4, for any $y \in f(M \cap (U_i \times \boldsymbol{R}^n))$ and $k = 2, 3, \cdots, \infty$,

$$S_k(F_{i,y}) \cap \Phi_{i,y}(f^{-1}(y) \cap (U_i \times \{y\})) = \Sigma_k(\Phi_{i,y}(f^{-1}(y) \cap (U_i \times \{y\}))).$$

Therefore we have

$$\begin{aligned}\bigcup_{y \in f(M\cap(U_i\times\boldsymbol{R}^n))} &(S_k(F_{i,y}) \cap \Phi_{i,y}(f^{-1}(y) \cap (U_i \times \{y\}))) = \Sigma_k[f \circ \Phi_i^{-1}|_{\Phi_i(M\cap(U_i\times\boldsymbol{R}^n))}] \\ &= \Phi_i(\Sigma_k[f|_{M\cap(U_i\times\boldsymbol{R}^n)}]).\end{aligned}$$

On the other hand, by Theorem 3.1, $\{F_{i,y}\}$ has a bounded complexity. Thus Theorem 2.2 follows from Theorem 3.2.

REMARK 3.5. In [**7**], K. Kurdyka proves that the regular points set of a subanalytic subset of an analytic manifold is also subanalytic, using a certain type of stabilisation property and the Poly-Raby Theorem.

### 3.3. Proof of Theorem 2.5.

We first prepare lemmas. Let $M$, $N$ be $C^r$ Nash manifolds and let $f : M \to N$ be a $C^s$ Nash mapping where $1 \leq s \leq r \leq \infty$. We denote by $S(f)$ the singular points set of $f$. Then we can easily see the following lemma by the implicit function theorem.

LEMMA 3.6. *For $k \leq s$, $\Sigma_k \subset S(f)$.*

In the function case, we have

LEMMA 3.7. *Let $\dim N = 1$. For $P \in S(f)$, there is an open semialgebraic neighbourhood $U$ of $P$ in $M$ such that $U \cap S(f) \subset f^{-1}(f(P))$.*

PROOF OF LEMMA 3.7. This lemma is obvious in the case where $f$ is constant over a neighbourhood of $P$ in $M$. Therefore it suffices to show the case where $f$ is nonconstant in any small neighbourhood of $P$ in $M$. Suppose that Lemma 3.7 does not hold. Then, by the curve selection lemma, there is an analytic curve $\lambda : [0, \delta) \to M$ such that

$$\lambda(0) = P \text{ and } \lambda(t) \in (M - f^{-1}(f(P))) \cap S(f) \text{ for } t > 0.$$

Then it is easy to see that $f$ is constant along $\lambda$. This is a contradiction. □

Now we start the proof of Theorem 2.5. Therefore, let $f : M \to N$ be a Nash mapping between Nash manifolds with $\dim M \geq 1$. By (2.1) and Corollary 2.4, it suffices to show that $\Sigma_\infty$ is a (semialgebraic) subset of $M$ of codimension $\geq 2$. In addition, it follows from Remark 0.1 that $\Sigma_\infty = \emptyset$ in the case where $\dim M = 1$. Therefore we assume that $\dim M \geq 2$.

Recall that $N$ is a subset of $\boldsymbol{R}^n$. Since we are considering a problem of fibres of a mapping, we may regard $f$ as a mapping from $M$ to $\boldsymbol{R}^n$. Let $f = (f_1, \cdots, f_n) : M \to \boldsymbol{R}^n$. For $1 \leq i \leq n$, define $F_i : M \to \boldsymbol{R}^i$ by $F_i = (f_1, \cdots, f_i)$. Then $F_1 = f_1$ and $F_n = f$. For each $F_i : M \to \boldsymbol{R}^i$, $1 \leq i \leq n$, set

$$\Sigma_\infty(F_i) = \{x \in M \mid F_i^{-1}(F_i(x)) \text{ is not a } C^\infty \text{ submanifold of } \boldsymbol{R}^m \text{ at } x\}.$$

By Corollary 2.4, $\Sigma_\infty(F_i)$, $1 \leq i \leq n$, are semialgebraic subsets of $M$. For $Q \in F_i(M)$, we define $\Sigma_\infty(F_i^{-1}(Q))$ similarly.

By induction on $i$, we show that codim $\Sigma_\infty(F_i) \geq 2$. We first consider the case $i = 1$. The analycity of $F_1$ implies that over each connected component of $M$, $\dim F_1^{-1}(Q) < \dim M$ or $F_1$ is constant. In the constant case, $\Sigma_\infty(F_1) = \emptyset$ on the connected component. We next consider the nonconstant case. By Lemma 1.6, $\dim \Sigma_\infty(F_1^{-1}(Q)) < \dim F_1^{-1}(Q)$ for $Q \in F_1(M)$. In addition, by Lemmas 3.6 and 3.7, $\Sigma_\infty(F_1)$ is contained locally in one fibre. It follows that codim $\Sigma_\infty(F_1) \geq 2$ on the connected component. Therefore the case $i = 1$ is shown.

Next suppose that $\Sigma_\infty(F_i)$ is a semialgebraic subset of $M$ of codimension $\geq 2$. Then $C_i = \overline{\Sigma_\infty(F_i)}$ is also a semialgebraic subset of $M$ of codimension $\geq 2$. Set $L_i = M - C_i$ and $A = \Sigma_\infty(F_{i+1}) \cap L_i$. Note that $A$ is a semialgebraic subset of $M$ and

$$A = \bigcup_{Q \in F_i(A)} \Sigma_\infty(f_{i+1}|_{F_i|_{L_i}^{-1}(Q)}).$$

If $\dim F_i|_{L_i}^{-1}(Q) \le 1$ for $Q \in F_i(L_i)$, then $\Sigma_\infty(f_{i+1}|_{F_i|_{L_i}^{-1}(Q)}) = \emptyset$. Therefore if $Q \in F_i(A)$, then $\dim F_i|_{L_i}^{-1}(Q) \ge 2$. Similarly to the case $i = 1$, it follows from Remark 0.1 and Lemmas 1.6, 3.6 and 3.7 that

$$\dim F_i|_{L_i}^{-1}(Q) \cap A + 2 \le \dim F_i|_{L_i}^{-1}(Q) \text{ for any } Q \in F_i(A).$$

By Lemma 1.7, we have codim $A \ge 2$ in $L_i$ and also in $M$. Since $\Sigma_\infty(F_{i+1}) \subset A \cup C_i$, we have codim $\Sigma_\infty(F_{i+1}) \ge 2$.

This completes the proof of Theorem 2.5.

## 4. Semialgebraic triviality along the fibre.

In this section, we discuss triviality of a $C^1$ Nash mapping along the smooth part of the fibre. Let $M \subset \boldsymbol{R}^m$ and $N \subset \boldsymbol{R}^n$ be $C^1$ Nash manifolds, and let $f : M \to N$ be a $C^1$ Nash mapping. Then we consider the following question:

Question 4.1. Let $P \in M - \overline{\Sigma_k}$ where $k = 1, 2, \cdots, \infty$. Is there an open semialgebraic neighbourhood $U$ of $P$ in $M$ with $U \subset M - \overline{\Sigma_k}$ such that a family of map-germs $\{ f : (M, Q) \to (N, F(P))\}_{Q \in f^{-1}(f(P)) \cap U}$ is semialgebraically trivial ?

The answer is No! There is a negative example in the mapping case to this question.

Example 4.2. Let $\boldsymbol{R}^3 \to \boldsymbol{R}^2$ be a polynomial mapping defined by

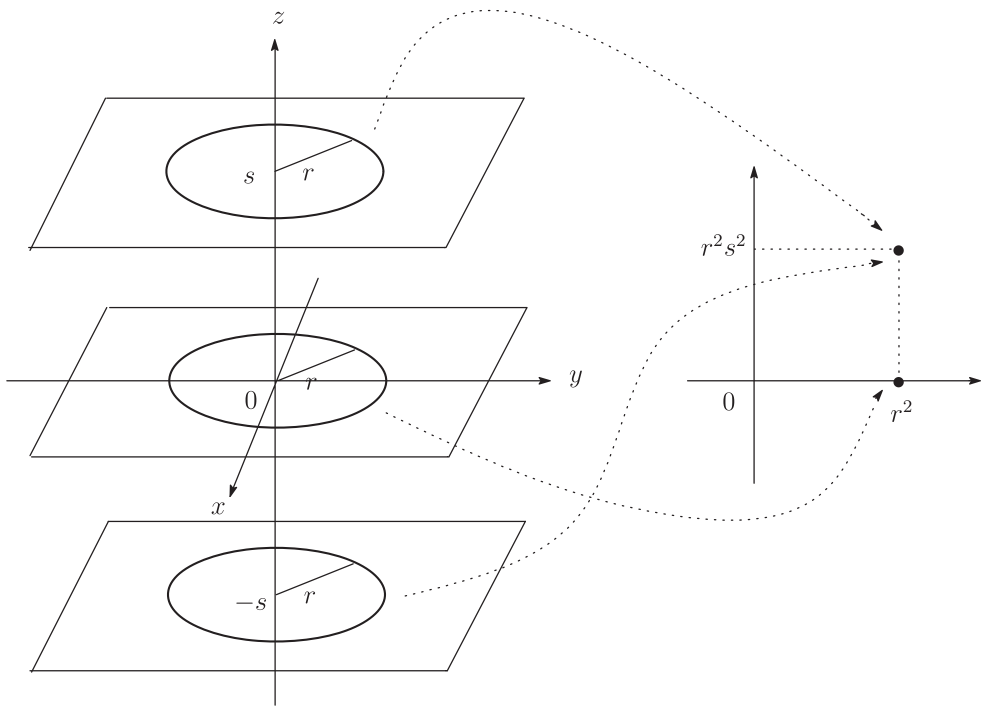
$$f(x, y, z) = (x^2 + y^2, (x^2 + y^2)z^2).$$

Then $\Sigma_k = \emptyset$. For $P = (x, y, z)$ with $(x, y) \neq (0, 0)$ and $z \neq 0$, $f^{-1}(f(P))$ consists of two circles. But, for $P = (x, y, z)$ with $(x, y) \neq (0, 0)$ and $z = 0$, $f^{-1}(f(P))$ consists of one circle. (See the figure in the next page.) Thus $f$ is not locally semialgebraically $\mathscr{R}\mathscr{L}$-trivial along $f^{-1}(f(O)) = \{z\text{-axis}\}$ at $O = (0, 0, 0)$.

Remark 4.3. There are negative examples also in polynomial mappings from $\boldsymbol{R}^2$ to $\boldsymbol{R}^2$. Actually, we can see $f(x, y) = (x^2, xy)$ is such an example.

On the other hand, we have the following positive answer in the function case to Question 4.1.

Theorem 4.4. *Let $M$ and $N$ be $C^1$ Nash manifolds with* $\dim N = 1$, *and let $f : M \to N$ be a semialgebraic mapping. Let $P \in M - \overline{\Sigma_k}$ where $k = 1, 2, \cdots, \infty$. Then $f$ is locally semialgebraically $\mathscr{R}$-trivial along $f^{-1}(f(P))$ around $P$.*




We recall the following fact on semialgebraic equivalence of semialgebraic functions.

LEMMA 4.5 ([**16**]). *Let $F \subset M \subset \boldsymbol{R}^m$ be semialgebraic sets, and let $f, g : M \to \boldsymbol{R}$ be semialgebraic functions such that*

$$f^{-1}(0) = g^{-1}(0) = F, \ \{f > 0\} = \{g > 0\} \text{ and } \{f < 0\} = \{g < 0\}.$$

*Then the germs of $f$ and $g$ at $F$ are semialgebraically $\mathscr{R}$-equivalent. Here we can choose the semialgebraic homeomorphism of the equivalence to be the identity map on $F$.*

PROOF OF THEOREM 4.4. We show this theorem using the above lemma. Since we consider the local problem around $P \in M - \overline{\Sigma_k}$, we can assume that $M = \boldsymbol{R}^q$, $N = \boldsymbol{R}$, $P = 0 \in \boldsymbol{R}^q$, $f(P) = 0 \in \boldsymbol{R}$, $F = f^{-1}(P) = \boldsymbol{R}^s \times \{0\} \subset \boldsymbol{R}^q$ and $f : \boldsymbol{R}^q \to \boldsymbol{R}$ is a semialgebraic function. The theorem is obvious in the case where $s = 0$ or $q$. Therefore, let $0 < s < q$. Then, by Lemma 4.5, $f$ is semialgebraically $\mathscr{R}$-equivalent to one of the following functions as germs at $F$:

(i) in the case $s = q - 1$, $x_q$, $\pm x_q^2$.
(ii) in the case $1 \le s \le q - 2$, $\pm(x_{s+1}^2 + \cdots + x_q^2)$.

We denote by $g$ such a function semialgebraically $\mathscr{R}$-equivalent to $f$. Remark that $g$ is independent of the variables $x_1, \cdots, x_s$.

Let $a$, $b$, $A$, $B : (\boldsymbol{R}^q \times F, F \times F) \to (\boldsymbol{R}, 0)$ be semialgebraic functions defined by

$$a(x,t) = f(x), b(x,t) = f(x+t),$$
$$A(x,t) = g(x), B(x,t) = g(x+t).$$

Then there are germs of $t$-level preserving semialgebraic homeomorphisms

$$\phi, \psi : (\boldsymbol{R}^q \times F, F \times F) \to (\boldsymbol{R}^q \times F, F \times F)$$

such that $A = a \circ \phi$ and $B = b \circ \psi$. In addition, $\phi|_{F\times F} = \psi|_{F\times F} = id|_{F\times F}$. For $r > 0$, let

$$C_r = \{(x_1, \cdots, x_q, t_1, \cdots, t_s, 0, \cdots, 0) \in \boldsymbol{R}^q \times F \mid |x_i| < r \ (1 \le i \le q),$$
$$|t_j| < r \ (1 \le j \le s)\},$$
$$D_r = \{(x_1, \cdots, x_q, t_1, \cdots, t_s, 0, \cdots, 0) \in \boldsymbol{R}^q \times F \mid |x_i - t_i| < r \ (1 \le i \le s),$$
$$|x_u| < r \ (s+1 \le u \le q), |t_j| < r \ (1 \le j \le s)\}.$$

Then there is a positive number $v > 0$ such that $C_v$ and $D_v$ are contained in the domains of $\phi$ and $\psi$, respectively. Since $g$ is independent of the variables $x_1, \cdots, x_s$, there is a $t$-level preserving semialgebraic homeomorphism

$$H : (C_v, C_v \cap F \times F, C_v \cap \{0\} \times F) \to (D_v, D_v \cap F \times F, D_v \cap \{0\} \times F)$$

such that $A|_{C_v} = B|_{D_v} \circ H$. Therefore $f$ is locally semialgebraically $\mathscr{R}$-trivial along $F$ around $0 \in F$. □

Remark 4.6. (1) In the above theorem, we can replace $C^1$ Nash manifolds $M \subset \boldsymbol{R}^m$ and $N \subset \boldsymbol{R}^n$ by semialgebraic regular submanifolds.

(2) At about the same time as we proved the above result several years ago, Karim Bekka announced that he has shown the corresponding result in the $o$-minimal structure.

To continue we consider the function case. Namely, $M$ and $N$ are $C^1$ Nash manifolds with $\dim N = 1$, and $f : M \to N$ is a $C^1$ Nash mapping. Let $P \in M - \overline{\Sigma_k}$ and $F = f^{-1}(f(P)) - \overline{\Sigma_k}$. As stated above, $f$ is locally semialgebraically $\mathscr{R}$-trivial along $F$ around $P$. From the viewpoint of stratification theory (e.g. [**18**], [**11**], [**4**], [**1**], [**6**], [**17**]), it is natural to ask if there are a local $C^1$ retraction $\Pi : M - \overline{\Sigma_k} \to F$ at $P$ and an open semialgebraic neighbourhood $U$ of $P$ in $M - \overline{\Sigma_k}$ such that for any $Q \in U - F$,

$$d\Pi_Q : \ker df(Q) \to T_{\Pi(Q)}F \tag{4.1}$$

is surjective. The answer to this question is also No.

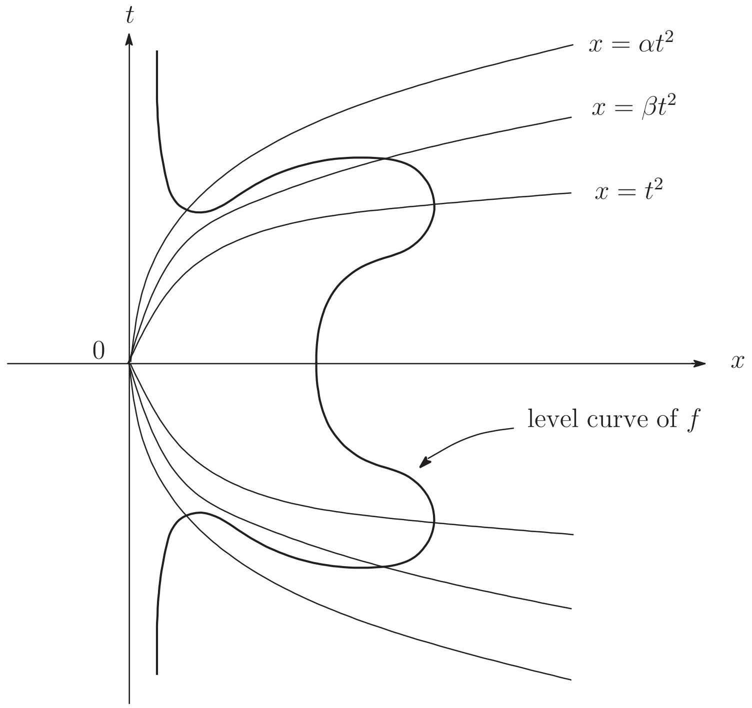
Example 4.7. Let $f : \boldsymbol{R}^2 \to \boldsymbol{R}$ be a polynomial function defined by

$$f(x,t) = x^2t^4 - 2x^3t^2 + \frac{21}{20}x^4.$$

Since $f(x,t) = x^2\{(t^2-x)^2 + \frac{1}{20}x^2\}$, $f^{-1}(0) = \{x=0\}$.

Now $\frac{\partial f}{\partial x} = 2xt^4 - 6x^2t^2 + \frac{21}{5}x^3 = \frac{21}{5}x(x-\alpha t^2)(x-\beta t^2)$ where $\alpha = \frac{15-\sqrt{15}}{21}$, $\beta = \frac{15+\sqrt{15}}{21}$, $\frac{\partial f}{\partial t} = 4x^2t^3 - 4x^3t = 4x^2t(t^2-x)$.

Therefore $S(f) = \{x=0\}$ and $\Sigma_k = \emptyset$.




Take $0 \in f^{-1}(f(0))$ and let $F = f^{-1}(f(0)) = \{x=0\}$. Since $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial t} \neq 0$ over $\{x=\alpha t^2\} \cup \{x = \beta t^2\}$ outside $F$, $d\varpi_Q : \ker df(Q) \to T_{\varpi(Q)}F$ is not surjective along $\{x = \alpha t^2\}$ or $\{x=\beta t^2\}$, where $\varpi : \boldsymbol{R}^2 \to F$ is the canonical projection.

Let $\Pi : \boldsymbol{R}^2 \to F$ be an arbitrary local $C^1$ retraction at $0 \in \boldsymbol{R}^2$. Then there is a local $C^1$ diffeomorphism $\sigma : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ such that $\Pi \circ \sigma^{-1} = \varpi$. Here we can assume that $\sigma$ has the following form:

$$\sigma^{-1}(x,t) = (x, \phi(x,t))$$

where $\phi(x,t) = ax + t + g(x,t)$ and $g$ is of class $C^1$ such that $j^1 g(0) = 0$. Then

$$f \circ \sigma^{-1}(x,t) = x^2(ax+t+g)^4 - 2x^3(ax+t+g)^2 + \tfrac{21}{20}x^4.$$

Therefore we have

$$\begin{aligned}\frac{\partial f\circ\sigma^{-1}}{\partial x} &= 2x(ax+t+g)^4 + 4x^2\Big(a+\frac{\partial g}{\partial x}\Big)(ax+t+g)^3\\ &\quad - 6x^2(ax+t+g)^2 - 4x^3\Big(a+\frac{\partial g}{\partial x}\Big)(ax+t+g) + \tfrac{21}{5}x^3\\ &= 2xh(x,t),\end{aligned}$$

$$\begin{aligned}\text{where } h(x,t) &= (ax+t+g)^4 + 2x\Big(a+\frac{\partial g}{\partial x}\Big)(ax+t+g)^3\\ &\quad - 3x(ax+t+g)^2 - 2x^2\Big(a+\frac{\partial g}{\partial x}\Big)(ax+t+g) + \tfrac{21}{10}x^2\\ &= t^4 - 3xt^2 + \tfrac{21}{10}x^2 + k(x,t).\end{aligned}$$

Let $v(x,t) = t^4 - 3xt^2 + \frac{21}{10}x^2$ which is a weighted homogeneous polynomial of type $\mu = (\frac{1}{2}, \frac{1}{4})$ with an isolated singularity. Set $|(x,t)|_\mu = (|x|^2 + |y|^4)^{\frac{1}{4}}$. Then $k$ is of class $C^0$ and

$$k(x,t) = O(|(x,t)|_\mu^4). \tag{4.2}$$

Since $v(x,t) = \frac{21}{10}(x-\alpha t^2)(x-\beta t^2)$, we have the following:

(i) If $0 < x < \alpha t^2$, then $v > 0$.
(ii) If $\alpha t^2 < x < \beta t^2$, then $v < 0$.
(iii) If $x > \beta t^2$, then $v > 0$.

It follows from (4.2) that there are positive numbers $C > 0$ and $K_1, K_2, K_3, K_4 > 0$ with $0 < K_1 < \alpha < K_2 < K_3 < \beta < K_4$ such that over any curve $|(x,t)|_\mu = s$ for $0 < s < C$,

(i) if $0 < x < K_1 t^2$, then $h > 0$,
(ii) if $K_2 t^2 < x < K_3 t^2$, then $h < 0$,
(iii) if $x > K_4 t^2$, then $h > 0$.

Therefore, over any curve $|(x,t)|_\mu = s$ for $0 < s < C$, there are at least two points in $\{x > 0\}$ at which $h = 0$. Thus there is an arbitrarily close point $Q$ in $\boldsymbol{R}^2 - F$ to the origin such that $h(Q) = 0$. This implies that $d\varpi_Q : \ker d(f\circ\sigma^{-1})(Q) \to T_{\varpi(Q)}F$ is not surjective at $Q$ arbitrarily close to the origin. It follows that $d\Pi_Q : \ker df(Q) \to T_{\Pi(Q)}F$ is not surjective at $Q$ arbitrarily close to the origin.

Remark 4.8. As seen in the above example, there is a real polynomial function such that for any $C^1$ retraction $\Pi : M - \overline{\Sigma_k} \to F$ at $P$, map (4.1) is not surjective in any neighbourhood of $P$. This phenomenon is in contrast to the complex situation. In the complex analytic function case, Whitney $(b)$-regularity implies Thom condition $(a_f)$ (see A. Parusiński [**12**] or J. Briançon, P. Maisonobe and M. Merle [**3**]). Therefore, for any $C^1$ retraction $\Pi$, map (4.1) is surjective in a neighbourhood of $P$.

## References


[1] K. Bekka, $C$-régularité et trivialité topologique, Singularity Theory and its Applications, Lecture Notes in Math., **1462** (1991), 42–62.
[2] J. Bochnak, M. Coste and M.-F. Roy, Real Algebraic Geometry, Ergeb. Math. Grenzgeb., **36** (1998).
[3] J. Briançon, P. Maisonobe and M. Merle, Localisation de systèmes differéntiels, stratifications de Whitney et condition de Thom, Invent. Math., **117** (1994), 531–550.
[4] C. G. Gibson, K. Wirthmüller, A. du Plessis and E. J. N. Looijenga, Topological stability of smooth mappings, Lecture Notes in Math., **552** (1976).
[5] R. M. Hardt, Semi-algebraic local triviality in semi-algebraic mappings, Amer. J. Math., **102** (1980), 291–302.
[6] S. Koike, On condition $(a_f)$ of a stratified mapping, Ann. Inst. Fourier, **33** (1983), 177–184.
[7] K. Kurdyka, Points reguliers d'un sous-analytique, Ann. Inst. Fourier, **38** (1988), 133–156.
[8] S. Lojasiewicz, Triangulation of semi-analytic sets, Ann. Scu. Norm. di Pisa, **18** (1964), 449–474.
[9] S. Lojasiewicz, Ensembles semi-analytiques, I.H.E.S. Lect. Note, 1965.
[10] B. Malgrange, Ideals of differentiable functions, Oxford Univ. Press, 1966.
[11] J. Mather, Notes on topological stability, Lecture Notes, Harvard University, 1970.
[12] A. Parusiński, Limits of tangent spaces to fibres and the $w_f$ condition, Duke Math. J., **72** (1993), 99–109.
[13] R. Ramanakoraisina, Complexité des fonctions de Nash, Communications in algebra, **17** (1989), 1395–1406.
[14] J. B. Poly and G. Raby, Fonction distance et singularités, Bull. Sci. Math., **108** (1984), 187–195.
[15] A. Seidenberg, A new decision method for elementary algebra, Ann. of Math., **60** (1954), 365–374.
[16] M. Shiota, Nash manifolds, Lecture Notes in Math., **1269** (1987).
[17] M. Shiota, Geometry of subanalytic and semialgebraic sets, Prog. Math., **150**, (1997).
[18] R. Thom, Local topological properties of differentiable mappings, Oxford Univ. Press, London, 1964, pp. 191–202.



Satoshi Koike

Department of Mathematics
Hyogo University of Teacher Education
942-1 Shimokume, Kato
Hyogo 673-1494, Japan
E-mail: koike@hyogo-u.ac.jp

Masahiro Shiota

Graduate School of Mathematics
Nagoya University
Furo-cho, Chigusa-ku
Nagoya 464-8602, Japan
E-mail: shiota@math.nagoya-u.ac.jp