J. Math. Soc. Japan
Vol. 59, No. 3 (2007) pp. 797–809

Hypersurfaces of E_s^4 with proper mean curvature vector

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(Received Nov. 16, 2006)

Abstract. Submanifolds satisfying $\Delta\vec{H} = \lambda\vec{H}$ are named by B. Y. Chen submanifolds with proper mean curvature vector. We prove that a hypersurface of the pseudo-Euclidean space E_s^4 with $\Delta\vec{H} = \lambda\vec{H}$ and diagonalizable shape operator, has constant mean curvature.

1. Introduction.

Let $x : M^m \to E_s^n$ be an isometric immersion of an n-dimensional connected submanifold of a pseudo-Euclidean space E_s^m. If we denote by $\vec{x}$, $\vec{H}$, and Δ the position vector field, the mean curvature vector field, and the Laplace operator respectively of M, with respect to the induced metric of M, then it is well known that (e.g. [**Ch1**])

$$\Delta\vec{x} = -n\vec{H}. \tag{1}$$

In particular, equation (1) shows that M is a minimal submanifold of E_s^n if and only if its coordinate functions are harmonic. We also observe that every minimal submanifold satisfies

$$\Delta\vec{H} = \vec{0}. \tag{2}$$

Submanifolds of E_s^n which satisfy condition (2) are said to have *harmonic mean curvature vector field*. These submanifolds are often called *biharmonic* since, in view of (1), condition (2) is equivalent to $\Delta^2\vec{x} = \vec{0}$. Equation (2) is a special case of the equation

$$\Delta\vec{H} = \lambda\vec{H}. \tag{3}$$

Submanifolds of E_s^m which satisfy condition (3) are said to have *proper mean curvature vector field*. Equations (2) and (3) can be related to the theory of harmonic and biharmonic maps as explained at the end of the present work (Section 3).

A conjecture of B. Y. Chen ([**Ch2**]) states that "the only biharmonic submanifolds of Euclidean spaces are the minimal submanifolds". For hypersurfaces in E^3 and E^4 the conjecture is supported by the work of several authors ([**Ch1**], [**Di1**], [**Di2**], [**Ha-Vl**], [**De1**]). However, it is not true in general for submanifolds in pseudo-Euclidean spaces E_s^m. Counterexamples were presented in [**Ch-Is1**] and [**Ch-Is2**]. In contrast, there is

2000 *Mathematics Subject Classification.* Primary 53C40; Secondary 53C50, 53A07.
Key Words and Phrases. biharmonic submanifold, minimal hypersurface, pseudo-Euclidean space.
The third author was partially supported by the Greek State Scholarships Foundation (I.K.Y.).

strong evidence that the conjecture is true for hypersurfaces in pseudo-Euclidean spaces. More precisely, in [**Ch-Is1**] is was shown that every biharmonic surface in E_s^3 is minimal, in [**De-Ka-Pa**] that every biharmonic hypersurface M_r^3 of E_s^4 whose shape operator is diagonal is minimal, and in [**Ar-De-Ka-Pa**] Chen's conjecture was proved for Lorentz hypersurfaces in E_1^4. Recently a "generalized Chen's conjecture" was posed in [**Ca-Mo-On**], stating that the only biharmonic submanifolds of a manifold with non-positive sectional curvature are the minimal ones.

Equation (3) was first appeared in [**Ch6**] where surfaces in E^3 satisfying (3) were classified. Also, in [**Ch7**] it was shown that a submanifold M of a Euclidean space satisfies (3) if and only if M is biharmonic or of 1-type or of null 2-type. Hypersurfaces in E^4 satisfying (3) with the additional condition of conformal flatness were classified in [**Ga**], and in [**De2**] it was proved that every hypersurface of E^4 satisfying (3) has constant mean curvature. For various other results about submanifolds satisfying (3) in Euclidean spaces, and more generally in space forms, contact, and Sasakian manifolds, we refer to [**Ch4**], [**Ch5**], [**Ek-Ya**], [**In1**], [**In2**].

The study of equation (3) for submanifolds in pseudo-Euclidean spaces was originally studied in [**Fe-Lu1**], where the authors classifed surfaces M_r^2 $(r = 0, 1)$ in the Lorenz-Minkowski space E_1^3. One of the possibilities for M_r^2 is that it is a submanifold of zero mean curvature H. The case of hypersurfaces M_r^{n-1} $(r = 0, 1)$ in E_1^n satisfying (3) and such that the minimal polynomial of the shape operator is at most of degree two, was stydied in [**Fe-Lu2**], showing that M_r^{n-1} has constant mean curvature. Also in [**Ch8**] various classification theorems for submanifolds in a Minkowski space-time were presented.

The results of the previous paragraph suggest a further study of hypersurfaces of E_s^n $(0 \leq s \leq n)$ satisfying equation (3). Towards this direction we prove the following:

THEOREM. *Let M_r^3 $(r = 0, 1, 2, 3)$ be a nondegenerate hypersurface of the pseudo-Euclidean space E_s^4 with diagonal shape operator. If the mean curvature vector field $\vec{H}$ of M_r^3 satisfies $\Delta\vec{H} = \lambda\vec{H}$, then M_r^3 has constant mean curvature.*

The idea of the proof is the following. Equation (3) reduces to the equations

$$\begin{aligned} S(\nabla H) &= -\varepsilon\frac{3H}{2}(\nabla H) \\ \Delta H + \varepsilon H \operatorname{tr} S^2 &= \lambda H. \end{aligned} \tag{*}$$

From the above equations together with Codazzi and Gauss equations we eliminate all derivatives. In this way we obtain a polynomial equation with constant coefficients which is satisfied by H, therefore H must be constant.

ACKNOWLEDGEMENTS. The authors wish to express their gratitute to Professor B. J. Papantoniou for several useful discussions.

2. Preliminaries.

Hypersurfaces in E_s^4.

Consider the 4-dimensional vector space $\boldsymbol{R}^4$ with the standard basis $\{e_1, e_2, e_3, e_4\}$. Let $\langle\ ,\ \rangle$ denote the indefinite inner product on $\boldsymbol{R}^4$ whose matrix with respect to the standard basis is a diagonal matrix of index $s \in \{0,1,2,3,4\}$. The space $\boldsymbol{R}^4$ with one of these metrics is called the 4-*dimensional pseudo-Euclidean space*, and is denoted by E_s^4.

A vector $X \in E_s^4$ is called *space-like*, *time-like*, or *light-like* if $\langle X, X\rangle$ is positive, negative, or zero respectively. Let $x : M_r^3 \to E_s^4$ be an isometric immersion of a hypersurface M_r^3 $(r = 0,1,2,3)$ in E_s^4 $(s = 0,1,2,3,4)$. The hypersurface M_r^3 can itself be endowed with a Riemannian or a pseudo-Riemannian metric structure, depending on whether the metric induced on M_r^3 from the pseudo-Riemannian metric on E_s^4, is positive-definite or indefinite.

Let $\vec{\xi}$ denote a unit normal vector field on M_r^3. Then $\langle\vec{\xi},\vec{\xi}\rangle = \varepsilon$, where $\varepsilon = -1$ (time-like) or $\varepsilon = +1$ (space-like). The mean curvature vector $\vec{H} = H\vec{\xi}$ with $H = \frac{1}{3\varepsilon}\operatorname{tr} S$ is a well-defined normal vector field of M_r^3 in E_s^4. Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M_r^3 and E_s^4 respectively. For any vector fields X, Y tangent to M_r^3, the Gauss formula is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\vec{\xi}, \tag{4}$$

where h is the second fundamental form. If S is the shape operator of M_r^3 associated to $\vec{\xi}$, then the Weingarten formula is given by

$$\tilde{\nabla}_X \vec{\xi} = -S(X), \tag{5}$$

where $\langle S(X), Y\rangle = \varepsilon h(X,Y)$. The Codazzi equation is given by

$$(\nabla_X S)Y = (\nabla_Y S)X, \tag{6}$$

and the Gauss equation by (cf. [**ON**])

$$R(X,Y)Z = \langle S(Y), Z\rangle S(X) - \langle S(X), Z\rangle S(Y). \tag{7}$$

Our convention for the curvature tensor is

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

The equation $\Delta\vec{H} = \lambda\vec{H}$.

We now consider a hypersurface M_r^3 of E_s^4 satisfying the condition

$$\Delta\vec{H} = \lambda\vec{H}, \qquad \lambda \in \boldsymbol{R}\setminus\{0\}. \tag{8}$$

Here the Laplace operator Δ acting on a vector-valued function $\vec{V}$ is given by

$$\Delta \vec{V} = \sum_{i=1}^{3} \big(\tilde{\nabla}_{\nabla_{e_i} e_i} \vec{V} - \tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} \vec{V} \big),$$

with respect to a local orthonormal frame $\{e_i\}_{i=1}^{3}$.

The shape operator S of a Riemannian hypersurface of E_s^4 is always diagonalizable, but for pseudo-Riemannian hypersurfaces there may be other forms for S as well (e.g. [**Ma**]). In the present work we assume that the shape operator of the hypersurface M_r^3 in E_s^4 is diagonalizable, i.e. $S = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3\}$. Here the $\lambda_i' s$ are known as the *principal curvature functions* of M_r^3. The other possibilities for the shape operator need a separate investigation. The following proposition can be found in various forms (e.g. [**Ch1**], [**Ch-Is1**], [**Fe-Lu2**]), but we give a proof here adjusted to our problem.

PROPOSITION 1. *For an isometric immersion $x : M_r^3 \to E_s^4$ with diagonal shape operator, the following formula holds*:

$$\Delta \vec{H} = \{2S(\nabla H) + 3\varepsilon H(\nabla H)\} + \{\Delta H + \varepsilon H \operatorname{tr} S^2\}\vec{\xi}.$$

PROOF. Let $\{X_1, X_2, X_3\}$ be an orthonormal frame such that $\nabla_{X_i} X_j(P) = 0$, at some point $P \in M_r^3$. From the relation

$$\tilde{\nabla}_{X_i} \tilde{\nabla}_{X_i} \vec{H} = X_i X_i(H)\vec{\xi} - 2X_i(H)SX_i - H(\nabla_{X_i} S)X_i - Hh(SX_i, X_i)\vec{\xi}$$

and summing with respect to i, we obtain that

$$\Delta \vec{H} = \{2S\,(\nabla H) + H tr \nabla S\} + \{\Delta H + \varepsilon H tr S^2\}\vec{\xi}.$$

We need to find an expression for $tr \nabla S$. If $\{e_i\}$, $i = 1, 2, 3$ be an orthonormal basis of eigenvectors of the shape operator S such that $Se_i = \lambda_i e_i$, then

$$\begin{aligned}
\operatorname{tr} \nabla S &= \sum_{i=1}^{3} \epsilon_i (\nabla_{e_i} S) e_i \\
&= \big[\epsilon_1 e_1(\lambda_1) + \epsilon_2(\lambda_2 - \lambda_1)\omega_{22}^1 + \epsilon_3(\lambda_3 - \lambda_1)\omega_{33}^1\big] e_1 \\
&\quad + \big[\epsilon_2 e_2(\lambda_2) + \epsilon_1(\lambda_1 - \lambda_2)\omega_{11}^2 + \epsilon_3(\lambda_3 - \lambda_2)\omega_{33}^2\big] e_2 \\
&\quad + \big[\epsilon_3 e_3(\lambda_3) + \epsilon_1(\lambda_1 - \lambda_3)\omega_{11}^3 + \epsilon_2(\lambda_2 - \lambda_3)\omega_{22}^3\big] e_3,
\end{aligned}$$

where $\epsilon_i = \langle e_i, e_i \rangle = \pm 1$. From the Codazzi equation $(\nabla_{e_1} S)e_2 = (\nabla_{e_2} S)e_1$ it follows that

$$\epsilon_i(\lambda_i - \lambda_j)\omega_{ii}^j = \epsilon_j e_j(\lambda_i)$$

with $i, j = 1, 2, 3$. Therefore,

$$\begin{aligned}\operatorname{tr}\nabla S &= \epsilon_1 e_1(\lambda_1+\lambda_2+\lambda_3)e_1 + \epsilon_2 e_2(\lambda_1+\lambda_2+\lambda_3)e_2\\ &\quad + \epsilon_3 e_3(\lambda_1+\lambda_2+\lambda_3)e_3 = 3\varepsilon\nabla H,\end{aligned}$$

and this completes the proof. □

Due to Proposition 1 condition (8) is equivalent to

$$\{2S(\nabla H) + 3\varepsilon H(\nabla H)\} + \{\Delta H + \varepsilon H \operatorname{tr} S^2\}\vec{\xi} = \lambda H\vec{\xi}. \tag{9}$$

Therefore we obtain the following necessary and sufficient conditions for a hypersurface M_r^3 of E_s^4 to satisfy $\Delta\vec{H} = \lambda\vec{H}$:

$$S(\nabla H) = -\varepsilon\frac{3H}{2}(\nabla H) \tag{10}$$

$$\Delta H + \varepsilon H \operatorname{tr} S^2 = \lambda H, \tag{11}$$

where the Laplace operator Δ acting on a scalar-valued function f is given by (e.g. [**Ch-Is1**])

$$\Delta f = -\sum_{i=1}^{3}\epsilon_i(e_i e_i f - \nabla_{e_i}e_i f). \tag{12}$$

Here $\{e_1, e_2, e_3\}$ is a local orthonormal frame of $T_pM_r^3$ with $\langle e_i, e_i\rangle = \epsilon_i = \pm 1$. Noting from equation (10) that ∇H is an eigenvector of the shape operator S, without loss of generality we can choose e_1 in the direction of ∇H, and therefore the shape operator of M_r^3 takes the form

$$S = \begin{pmatrix} -\varepsilon\frac{3H}{2} & & \\ & \lambda_2 & \\ & & \lambda_3\end{pmatrix}. \tag{13}$$

In the special case in which all principal curvatures are equal, using the relation $\operatorname{tr} S = 3\varepsilon H$, it follows immediately that $H = 0$. Therefore it suffices to examine the case where all principal curvatures are different, and the case when the two principal curvatures are equal.

3. Proof of the theorem.

Three mutually different principal curvatures.

We need to show that if $\Delta\vec{H} = \lambda\vec{H}$, and the shape operator has three mutually different principal curvatures, then H is constant. Suppose on the contrary that M_r^3 $(r = 0,1,2,3)$ does not have constant mean curvature H. Then $\nabla H \neq \vec{0}$, and (10) shows that ∇H is an eigenvector of S with corresponding eigenvalue $\lambda_1 = -\frac{3\varepsilon H}{2}$. Expressing

∇H as $\nabla H = e_1(H)e_1 + e_2(H)e_2 + e_3(H)e_3$, and since e_1 is in the direction of ∇H it follows that

$$e_1(H) \neq 0,\ e_2(H) = e_3(H) = 0. \tag{14}$$

By assumption, we have that $\varepsilon\frac{3H}{2} + \lambda_2 \neq 0$, $\varepsilon\frac{3H}{2} + \lambda_3 \neq 0$, $\lambda_3 - \lambda_2 \neq 0$.

We write $\nabla_{e_i} e_j = \sum_{k=1}^3 \omega_{ij}^k e_k$, we take into account the action of S on the basis $\{e_1, e_2, e_3\}$, and use the Codazzi equation (6). Then the relations

$$\begin{aligned}
&\langle(\nabla_{e_1} S)e_2, e_1\rangle = \langle(\nabla_{e_2} S)e_1, e_1\rangle \qquad &&\langle(\nabla_{e_2} S)e_3, e_3\rangle = \langle(\nabla_{e_3} S)e_2, e_3\rangle\\
&\langle(\nabla_{e_1} S)e_3, e_3\rangle = \langle(\nabla_{e_3} S)e_1, e_3\rangle \qquad &&\langle(\nabla_{e_2} S)e_3, e_2\rangle = \langle(\nabla_{e_3} S)e_2, e_2\rangle\\
&\langle(\nabla_{e_1} S)e_2, e_2\rangle = \langle(\nabla_{e_2} S)e_1, e_2\rangle \qquad &&\langle(\nabla_{e_1} S)e_3, e_3\rangle = \langle(\nabla_{e_3} S)e_1, e_3\rangle
\end{aligned}$$

imply that $\omega_{12}^1 = \omega_{13}^1 = 0$, and that

$$\omega_{21}^1 = \frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2} - \lambda_2},\quad \omega_{31}^3 = \frac{e_1(\lambda_3)}{-\varepsilon\frac{3H}{2} - \lambda_3},\quad \omega_{23}^2 = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2},\quad \omega_{32}^3 = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}. \tag{15}$$

Also, in view of (14) it follows that $\nabla_{e_2} e_3(H) - \nabla_{e_3} e_2(H) = [e_2, e_3](H) = 0$. Thus, together with the Codazzi equations for $\langle(\nabla_{e_1} S)e_2, e_3\rangle = \langle(\nabla_{e_2} S)e_1, e_3\rangle$, and $\langle(\nabla_{e_1} S)e_3, e_2\rangle = \langle(\nabla_{e_3} S)e_1, e_2\rangle$ we obtain that

$$\omega_{13}^2 = \omega_{21}^3 = \omega_{32}^1 = 0. \tag{16}$$

We use Gauss equation (7) and the definition of the curvature tensor for $\langle R(e_2, e_3)e_1, e_2\rangle$, $\langle R(e_2, e_3)e_1, e_3\rangle$, and $\langle R(e_3, e_1)e_2, e_3\rangle$, to obtain that

$$e_3\left(\frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2} - \lambda_2}\right) = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2}\left(\frac{e_1(\lambda_3)}{-\varepsilon\frac{3H}{2} - \lambda_3} - \frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2} - \lambda_2}\right) \tag{17}$$

$$e_2\left(\frac{e_1(\lambda_3)}{-\varepsilon\frac{3H}{2} - \lambda_3}\right) = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}\left(\frac{e_1(\lambda_3)}{-\varepsilon\frac{3H}{2} - \lambda_3} - \frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2} - \lambda_2}\right) \tag{18}$$

$$e_1\left(\frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}\right) = -\frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2} - \lambda_2}\frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}. \tag{19}$$

If we combine relations (12) and (14), then equation (11) takes the form

$$\epsilon_1 e_1 e_1(H) + \epsilon_1\left(\frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2} - \lambda_2} + \frac{e_1(\lambda_3)}{-\varepsilon\frac{3H}{2} - \lambda_3}\right)e_1(H) - \varepsilon H\left(\frac{45H^2}{2} - 2\lambda_2\lambda_3\right) + \lambda H = 0. \tag{20}$$

Acting on (20) with e_2 and e_3 succesively, and combining the results with (17), (18) it follows that

$$
\begin{aligned}
e_2\left(\frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2}-\lambda_2}\right) = & -\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2}-\lambda_2}-\frac{e_1(\lambda_3)}{-\varepsilon\frac{3H}{2}-\lambda_3}\right)\\
& -\frac{2\varepsilon H}{\epsilon_1 e_1(H)}(\lambda_2-\lambda_3)^2\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3} \qquad (21)\\
e_3\left(\frac{e_1(\lambda_3)}{-\varepsilon\frac{3H}{2}-\lambda_3}\right) = & -\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_3)}{-\varepsilon\frac{3H}{2}-\lambda_3}-\frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2}-\lambda_2}\right)\\
& -\frac{2\varepsilon H}{\epsilon_1 e_1(H)}(\lambda_3-\lambda_2)^2\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}. \qquad (22)
\end{aligned}
$$

Similarly, using Gauss equation for $\langle R(e_1,e_2)e_1,e_2\rangle$ and $\langle R(e_3,e_1)e_1,e_3\rangle$ we obtain that

$$
e_1\left(\frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2}-\lambda_2}\right)+\left(\frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2}-\lambda_2}\right)^2 = \epsilon_1\varepsilon\frac{3}{2}H\lambda_2 \qquad (23)
$$

$$
e_2\left(\frac{e_1(\lambda_3)}{-\varepsilon\frac{3H}{2}-\lambda_3}\right)+\left(\frac{e_1(\lambda_3)}{-\varepsilon\frac{3H}{2}-\lambda_3}\right)^2 = \epsilon_1\varepsilon\frac{3}{2}H\lambda_3. \qquad (24)
$$

We will need the following:

Lemma 2. *Let M_r^3 be a hypersurface of the pseudo-Euclidean space E_s^4 whose shape operator has the form* (13), *and three mutually different principal curvatures. Then* $e_2(\lambda_3)=e_3(\lambda_2)=0$

Proof. Relations (15) and (16) imply that

$$
[e_1,e_2] = \frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2}-\lambda_2}e_2. \qquad (25)
$$

Applying both sides of (25) on $\frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2}-\lambda_2}$, and using (21), (23), (24), and (19) we deduce that

$$
\begin{aligned}
&\left[\frac{\varepsilon H}{\epsilon_1 e_1(H)}\left(\left(3\frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2}-\lambda_2}-\frac{e_1(\lambda_3)}{-\varepsilon\frac{3H}{2}-\lambda_3}\right)(\lambda_2-\lambda_3)^2+2(\lambda_2-\lambda_3)e_1(\lambda_2-\lambda_3)\right)\right.\\
&\left.+\left(\frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2}-\lambda_2}-\frac{e_1(\lambda_3)}{-\varepsilon\frac{3H}{2}-\lambda_3}\right)^2+\frac{\varepsilon}{\epsilon_1}e_1\left(\frac{H}{e_1(H)}\right)(\lambda_2-\lambda_3)^2\right]\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}=0.
\end{aligned}
$$

We will show that if $e_2(\lambda_3)\neq 0$ we get a contradiction. Indeed, in that case we would have that

$$e_1\left(\frac{H}{e_1(H)}\right) = -\frac{H}{e_1(H)}\left(\left(3\frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2}-\lambda_2} - \frac{e_1(\lambda_3)}{-\varepsilon\frac{3H}{2}-\lambda_3}\right) + 2\frac{e_1(\lambda_2-\lambda_3)}{\lambda_2-\lambda_3}\right)$$
$$- \frac{\epsilon_1}{\varepsilon(\lambda_2-\lambda_3)^2}\left(\frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2}-\lambda_2} - \frac{e_1(\lambda_3)}{-\varepsilon\frac{3H}{2}-\lambda_3}\right)^2.$$

Acting with e_2 on both sides of the above equation, and in view of (18), (21), (25), we obtain that

$$2\left(\frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2}-\lambda_2} - \frac{e_1(\lambda_3)}{-\varepsilon\frac{3H}{2}-\lambda_3}\right) = -\frac{\varepsilon H}{\epsilon_1 e_1(H)}(\lambda_2-\lambda_3)^2. \tag{26}$$

We apply e_2 on (26) and obtain

$$\left(\frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2}-\lambda_2} - \frac{e_1(\lambda_3)}{-\varepsilon\frac{3H}{2}-\lambda_3}\right) = -2\frac{\varepsilon H}{\epsilon_1 e_1(H)}(\lambda_2-\lambda_3)^2.$$

From the above two equations it follows that $\lambda_2 = \lambda_3$, which is a contradiction. Hence, we conclude that $e_2(\lambda_3) = 0$. In an analogous manner, it can be shown that $e_3(\lambda_2) = 0$. $\square$

Coming back to the proof of the Theorem, we use Lemma 2 and Gauss equation for $\langle R(e_2, e_3)e_2, e_3\rangle$ to obtain that

$$-\epsilon_1\left(\frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2}-\lambda_2}\right)\left(\frac{e_1(\lambda_3)}{-\varepsilon\frac{3H}{2}-\lambda_3}\right) - \lambda_2\lambda_3 = 0. \tag{27}$$

Calculating $e_1e_1(H)$ from (23) and (24), and combining with (20) and (27) it follows that

$$\left(\frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2}-\lambda_2} + \frac{e_1(\lambda_3)}{-\varepsilon\frac{3H}{2}-\lambda_3}\right)e_1(H) = -\frac{54+135\varepsilon}{8\epsilon_1}H^3 + \frac{6+3\varepsilon}{2\epsilon_1}H\lambda_2\lambda_3 + \frac{3}{4\epsilon_1}\lambda H \tag{28}$$

$$e_1e_1(H) = \frac{54+315\varepsilon}{8\epsilon_1}H^3 - \frac{6+7\varepsilon}{2\epsilon_1}H\lambda_2\lambda_3 - \frac{7}{4\epsilon_1}\lambda H. \tag{29}$$

Acting with e_1 on both sides of (28) and using (23), (24), and (27) we deduce the expression

$$\left(\frac{e_1(\lambda_2)}{-\varepsilon\frac{3H}{2}-\lambda_2} + \frac{e_1(\lambda_3)}{-\varepsilon\frac{3H}{2}-\lambda_3}\right)\big(441H^2 - 26\lambda_2\lambda_3 + 10\lambda\big)H$$
$$= \big(432H^2 - 26\lambda_2\lambda_3 - 3\lambda\big)e_1(H). \tag{30}$$

If we apply e_1 on (30) and use (28), (29), (30) we obtain the following algebraic relation between H and $\lambda_2\lambda_3$ (notice that H and $\lambda_2\lambda_3$ are real functions in general):

$$
\begin{aligned}
0 &= a_{80}H^8 + a_{60}H^6 + a_{40}H^4 + a_{20}H^2 + a_{00} \\
&\quad + a_{61}H^6(\lambda_2\lambda_3) + a_{41}H^4(\lambda_2\lambda_3) + a_{21}H^2(\lambda_2\lambda_3) + a_{01}(\lambda_2\lambda_3) + a_{42}H^4(\lambda_2\lambda_3)^2 \\
&\quad + a_{22}H^2(\lambda_2\lambda_3)^2 + a_{02}(\lambda_2\lambda_3)^2 + a_{23}H^2(\lambda_2\lambda_3)^3 + a_{03}(\lambda_2\lambda_3)^3 + a_{04}(\lambda_2\lambda_3)^4 \\
&= f(H, \lambda_2\lambda_3), \qquad (31)
\end{aligned}
$$

where a_{ij} are known constants. Acting in (31) with e_1 twice, and using (28), (29), (30), we obtain another algebraic relation of H and $\lambda_2\lambda_3$ of the form

$$g(H, \lambda_2\lambda_3) = 0. \qquad (32)$$

Using a computer algebra program, we eliminate $\lambda_2\lambda_3$ between (31) and (32), so obtain an algebraic equation for H with constant coefficients. Thus, we have concluded that the real function H satisfies a polynomial equation $q(H) = 0$ with constant coefficients, therefore it must be a constant. This contradicts our original assumption, so the Theorem is proved in this case.

Two equal principal curvatures.

We need to show that if $\Delta\vec{H} = \lambda\vec{H}$, and the shape operator has two equal principal curvatures, then H is constant. Assume the contrary, and try to get a contradiction. As in the previous case, e_1 can be chosen in the direction of ∇H, yielding $\lambda_1 = -\frac{3\varepsilon H}{2}$ and

$$e_1(H) \neq 0, \ e_2(H) = e_3(H) = 0.$$

Then the shape operator of M_r^3 takes the form

$$S = \begin{pmatrix} -\varepsilon\frac{3H}{2} & & \\ & \mu & \\ & & \mu \end{pmatrix}$$

for some function μ. From $\operatorname{tr} S = 3\varepsilon H$ if follows that $\mu = \varepsilon\frac{9H}{4}$, and that $\operatorname{tr} S^2 = \frac{99H^2}{8}$. Applying the Codazzi equation (6) it follows that $\langle(\nabla_{e_1}S)e_2, e_2\rangle = \langle(\nabla_{e_2}S)e_1, e_2\rangle$ and that $\langle(\nabla_{e_1}S)e_3, e_3\rangle = \langle(\nabla_{e_3}S)e_1, e_3\rangle$, which in turn give that

$$\omega_{21}^2 = \omega_{31}^3 = -\frac{3}{5}\frac{e_1(H)}{H}. \qquad (33)$$

The Gauss equation (7) applied to $\langle R(e_1, e_2)e_1, e_2\rangle$ implies that

$$e_1(\omega_{21}^2) = \epsilon_1\frac{27H^2}{8} - (\omega_{21}^2)^2. \qquad (34)$$

Equation (11) then reduces to

$$-\epsilon_1 e_1 e_1(H) - 2\epsilon_1(\omega_{21}^2)^2 e_1(H) + \varepsilon\frac{99H^3}{8} = \lambda H. \tag{35}$$

We act on (33) with e_1 and use (34) to obtain that

$$e_1 e_1(H) = \frac{40}{9}H(\omega_{21}^2)^2 - \frac{45H^3}{8}\epsilon_1.$$

We substitute the above equation to (35) and get

$$H\left[\epsilon_1\frac{10}{9}(\omega_{21}^2)^2 + \lambda - \frac{45+99\varepsilon}{8}H^2\right] = 0,$$

and as $H \neq 0$ it follows that

$$\epsilon_1\frac{10}{9}(\omega_{21}^2)^2 + \lambda - \frac{45+99\varepsilon}{8}H^2 = 0.$$

Acting by e_1 in the above equation and using (33) and (34), it follows that

$$\epsilon_1\frac{10}{3}(\omega_{21}^2)^2 - \frac{945+165\varepsilon}{8}H^2 = 0.$$

If we eliminate the $(\omega_{21}^2)^2$ from the last two equations we obtain the relation

$$\lambda + \frac{810-132\varepsilon}{24}H^2 = 0,$$

that is H is constant, which contradicts our assumption.

4. Relation with biharmonic maps.

In this section we describe the relation of equation (3) to the theory of harmonic and biharmonic maps. For relative background we refer to [**Ca-Mo-On**], [**Ee-Le**], and [**Ur**]. Let (M^m, g) and (N^n, h) be Riemannian manifolds. A smooth map $\phi : M \to N$ is said to be *harmonic* if it is a critical point of the *energy* functional:

$$E_1(\phi) = \frac{1}{2}\int_M |d\phi|^2 dv_g.$$

Denote by ∇^ϕ the connection of the vector bundle ϕ^*TN induced from the Levi-Civita connection ∇^h of (N, h). The *second fundamental form* $\nabla d\phi$ is defined by

$$\nabla d\phi(X,Y) = \nabla^\phi_X d\phi(Y) - d\phi(\nabla_X Y), \qquad X, Y \in \Gamma(TM),$$

where ∇ is the Levi-Civita connection of (M, g). The *tension field* $\tau(\phi)$ is a section of ϕ^*TN defined by

$$\tau(\phi) = \mathrm{trace}(\nabla d\phi).$$

It is well known that the map ϕ is harmonic if and only if its tension field vanishes. Now assume that $\phi : M \to N$ is an isometric immersion with mean curvature vector field $\vec{H}$. Then $m\vec{H} = \tau(\phi)$ (cf. [**Ee-Sa**, p. 119]), therefore the immersion ϕ is a harmonic map if and only if M is a minimal submanifold of N.

A smooth map $\phi : (M, g) \to (N, h)$ is called *biharmonic* if it is a critical point of the *bienergy* functional:

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 dv_g.$$

This is a special case of a more general set-up suggested in [**Ee-Sa**] on studying *polyharmonic* maps. In [**Ji1**] and [**Ji2**] G. Y. Jiang derived the first variation formula of the bienergy showing that the Euler-Lagrange equation for E_2 is given by

$$\tau_2(\phi) = -J_\phi(\tau(\phi)) = 0.$$

Here J_ϕ is the *Jacobi operator* of ϕ acting on sections $V \in \Gamma(\phi^* TN)$. It is defined by

$$J_\phi(V) = \bar{\Delta}_\phi V - R_\phi(V),$$

$$\bar{\Delta}_\phi = -\sum_{i=1}^{m} \left(\nabla^\phi_{e_i}\nabla^\phi_{e_i} - \nabla^\phi_{\nabla_{e_i}e_i}\right), \quad R_\phi(V) = \sum_{i=1}^{m} R^N(V, d\phi(e_i))d\phi(e_i),$$

where R^N is the curvature tensor of N, and $\{e_i\}$ a local orthonormal frame field of M. If $x : (M^m, g) \to (E^n, \text{canonical})$ is an isometric immersion, then $\bar{\Delta}_x$ is simply the Laplace operator Δ of M with respect to the induced metric, thus

$$\tau_2(x) = \Delta\tau(x) = \Delta(m\vec{H}) = m\Delta\vec{H}.$$

Therefore, M^m is a biharmonic submanifold of the Euclidean space E^n with the canonical metric if and only if the immersion $x : M^m \to E^n$ is a biharmonic map. Finally, an isometric immersion $x : M \to N$ is called λ-*biharmonic* if it is a critical point of the functional

$$E_{2,\lambda}(x) = E_2(x) + \lambda E(x), \quad \lambda \in \boldsymbol{R}.$$

The Euler-Lagrange equation for λ-biharmonic immersions is

$$\tau_2(x) = \lambda\tau(x).$$

This is equivalent to the equation $\Delta\vec{H} = \lambda\vec{H}$, i.e. the submanifold M has proper mean curvature vector field.

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