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Open books supporting overtwisted contact structures and the Stallings twist

By Ryosuke YAMAMOTO

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Abstract. We study open books (or open book decompositions) of a closed oriented 3-manifold which support overtwisted contact structures. We focus on a simple closed curve along which one can perform Stallings twist, called "twisting loop". We show that the existence of a twisting loop on the fiber surface of an open book is equivalent up to positive stabilization to the existence of an overtwisted disk in the contact manifold supported by the open book. We also show a criterion for overtwistedness using a certain arc properly embedded in the fiber surface, which is an extension of Goodman's one.

1. Introduction.

Stallings [**15**] introduced two operations which create a new open book (or open book decomposition) of a closed oriented 3-manifold from another of the manifold. One of them is called a *Stallings twist*, which is a Dehn twist along a certain simple closed curve, called a *twisting loop*, on the fiber surface of an open book. The other is positive (resp. negative) stabilization of an open book, which is also known as plumbing of positive (resp. negative) Hopf band to the fiber surface of an open book (see Section 2). On S^3, Harer [**9**] showed that every open book can be obtained from the standard open book of S^3, i.e., the open book with a 2-disk as the fiber and the identity map as the monodromy map, by (de-)stabilizations and Stallings twists. Moreover he conjectured that Stallings twists can be omitted. This conjecture has been proved affirmatively by Giroux [**6**]. He showed a one-to-one correspondence between isotopy classes of contact structures on M and equivalence classes of open books on M modulo positive stabilization (see [**7**] for further information). We may say about this result that a contact structure leads to a topological property of open books via Giroux's one-to-one correspondence. In this paper we will deal with a study in the opposite direction. We show that a twisting loop on the fiber surface of an open book is related directly to an overtwisted disk in the contact structure which is corresponding to the open book via Giroux's one-to-one correspondence.

Let M be a closed oriented 3-manifold. Denote by (Σ, ϕ) an open-book of M, where Σ is a fiber surface embedded in M and ϕ is a monodromy map. We say that an contact structure ξ on M is *supported* by an open book (Σ, ϕ) if (Σ, ϕ) is a corresponding one to ξ on Giroux's one to one correspondence. We will investigate open books supporting an overtwisted contact structure. For simplicity, we call such open books *overtwisted open books*. We show the following:

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Theorem 1.1. *Let* (Σ, ϕ) *be an open book of a closed oriented* 3*-manifold. The following are equivalent*;

(1) (Σ, ϕ) *is overtwisted.*
(2) (Σ, ϕ) *is equivalent up to positive stabilization to an open book whose fiber surface has a twisting loop.*
(3) (Σ, ϕ) *is equivalent up to positive stabilization to an open book* (Σ', ϕ') *with an arc* a *properly embedded in* Σ' *such that* $i_\partial(a, \phi'(a)) \leq 0$.

The arc a in (3) is an extension of Goodman's *sobering arc* [**8**]. The boundary intersection number i_∂ of a and $\phi(a)$, introduced by Goodman, is defined in Section 3. We should mention that the equivalence between (1) and (3) has already been shown by Honda, Kazez and Matić [**11**, Theorem 1.1], but the proof given here differs from theirs in focusing a twisting loop to detect an overtwisted disk.

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2. Preliminaries.

Let M be a closed oriented 3-manifold. We denote by $E(X)$ the exterior of X in M and by $N(X)$ a regular neighbourhood of X in M, where X is a submanifold in M.

2.1. Open book.

Let K be a fibered knot or link in M, i.e., there is a fibration map $f : E(K) \to S^1 = \boldsymbol{R}/\boldsymbol{Z}$ such that f maps meridian of K to S^1 homeomorphically. We denote by Σ_t the fiber surfaces $f^{-1}(t)$ for each $t \in \boldsymbol{R}/\boldsymbol{Z}$, and by Σ the homeomorphism type of the fiber surface. We often identify the abstract Σ and Σ_0 embedded in M.

In this situation, M has a decomposition as follows;

$$M = \big(\Sigma \times [0,1]/(x,1) \sim (\phi(x),0)\big) \cup_g (D^2 \times \partial\Sigma),$$

where ϕ is an automorphism of Σ fixing $\partial\Sigma$ pointwise, and g is a gluing map between the boundary tori such that $g(\{p\} \times [0,1]/(p,1) \sim (p,0)) = \partial(D^2 \times \{p\})$ for $p \in \partial\Sigma$. We call this structure of M an *open book* of M and denote by a pair (Σ, ϕ). The automorphism ϕ is called a *monodromy map* of the open book.

Let c be a simple closed curve on Σ. We use notation $\mathrm{Fr}(c;\Sigma)$ for the framing of c determined by a curve parallel to c on Σ, and $D(c)$ (resp. $D(c)^{-1}$) for a positive (resp. negative) Dehn twist on Σ along c. We say that c is essential on Σ if c does not bound a disk region on Σ.

Definition 2.1. An essential simple closed curve c on Σ is a *twisting loop* if c bounds a disk D embedded in M and satisfies that $\mathrm{Fr}(c;\Sigma) = \mathrm{Fr}(c;D)$.

If an open book (Σ, ϕ) has a twisting loop on Σ, (± 1)-Dehn surgery along c yield a new open book (Σ', ϕ') of M. We call this operation a *Stallings twist* along a twisting loop c. Note that the surface Σ' is homeomorphic to Σ and its embedding into M is changed (see Figure 1), and the monodromy map $\phi' = D(c)^{\pm 1} \circ \phi$, where maps act Σ'

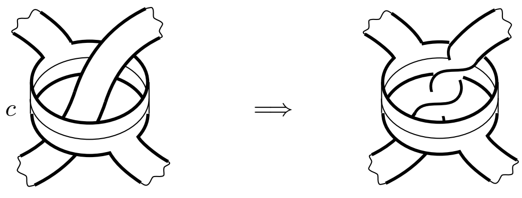

Figure 1. Stallings twist.

from the left.

A positive (resp. negative) stabilization of an open book (Σ, ϕ) of a closed oriented 3-manifold is an open book (Σ', ϕ') of M such that Σ' is a plumbing of a positive Hopf band H^+ (resp. negative Hopf band H^-) and Σ. The new monodromy map $\phi' = D(\gamma) \circ \phi$ (resp. $\phi' = D(\gamma)^{-1} \circ \phi$), where γ is the core curve of the Hopf band. We say that a stabilization along an arc properly embedded in Σ as a stabilization along a rectangle which is a regular neighbourhood of the arc in Σ.

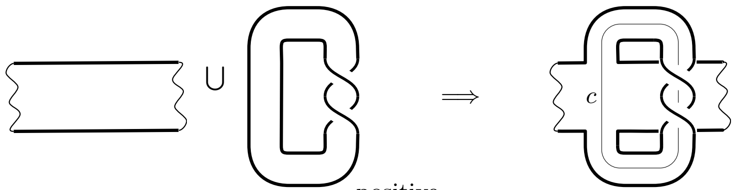

Figure 2. Positive stabilization of open book.

2.2. Contact structure.

A *contact form* on M is a smooth global non-vanishing 1-form α satisfying $\alpha \wedge d\alpha \neq 0$ at each point of M. A *contact structure* ξ on M is a 2-plane field defined by the kernel of α. The pair (M, ξ) is called a *contact* 3*-manifold*. We say that a contact structure $\xi = \ker \alpha$ is *positive* when $\alpha \wedge d\alpha > 0$. We assume that a contact structure is positive throughout this article.

We say that two contact structures on M, ξ_0 and ξ_1, are isotopic if there is a diffeomorphism $f : M \to M$ such that $\xi_1 = f_*(\xi_0)$.

A simple closed curve γ in contact 3-manifold (M, ξ) is *Legendrian* if γ is always tangent to ξ, i.e., for any point $x \in \gamma$, $T_x\gamma \subset \xi_x$. A Legendrian curve γ has a natural framing called the *Legendrian framing* denoted by $\mathrm{Fr}(\gamma; \xi)$, which is determined by a vector field on $\xi|_\gamma$ such that each vector is transverse to γ.

Let E be a disk embedded in a contact manifold (M, ξ). E is an *overtwisted disk* if ∂E is a Legendrian curve in (M, ξ) and $\mathrm{Fr}(\partial E; E) = \mathrm{Fr}(\partial E; \xi)$. A contact structure ξ on M is *overtwisted* if there is an overtwisted disk E in (M, ξ). A contact structure is called *tight* if it is not overtwisted.

2.3. Contact structures and open books.

A contact structure ξ on M is said to be *supported* by an open book (Σ, ϕ) if it is defined by a contact form α such that (1) on each fiber Σ_t, $\mathrm{d}\alpha|_{\Sigma_t} > 0$ and (2) on

$K = \partial\Sigma$, $\alpha(v_p) > 0$ for any point $p \in K$, where v_p is a positive tangent vector of K at p. W. Thurston and H. Winkelnkemper [**17**] showed that one can always construct a contact structure on M starting from a structure of an open book of M. The resulting contact structure is supported by the open book.

THEOREM 2.1 (Giroux 2000, [**6**]; Torisu 2000, [**16**]). *Contact structures supported by the same open book are isotopic.*

REMARK 2.2. It is known (e.g. [**8**]) that for an open book (Σ, ϕ) there is a contact structure supported by (Σ, ϕ) such that at any point $p \in \mathrm{Int}\,\Sigma$ the plane of ξ is arbitrary close to the tangent plane of Σ. By Theorem 2.1, we may assume that a contact structure supported by an open book always has the property.

As mentioned in Section 1, Giroux showed that there is a one-to-one correspondence between contact structures and open books.

THEOREM 2.2 (Giroux 2000, [**6**]). *Every contact structure of a closed oriented* 3*-manifold is supported by some open books. Moreover open books supporting the same contact structure are equivalent up to positive stabilization.*

In Section 3 we will need to show that a given simple closed curve on the fiber surface Σ of an open book (Σ, ϕ) can be realized as a Legendrian curve in the contact structure supported by the open book.

DEFINITION 2.3. A simple closed curve c on Σ is *isolated* if there is a connected component R of $\Sigma - c$ such that $R \cap \partial\Sigma = \varnothing$. We say that c is *non-isolated* if it is not isolated.

The following lemma is a variant of the Legendrian Realization Principle on the convex surface theory, due to Ko Honda [**10**].

LEMMA 2.3. *There is a contact structure ξ supported by (Σ, ϕ) such that a simple closed curve c on Σ_0 is Legendrian in (M, ξ) if and only if c is non-isolated on Σ_0.*

PROOF. Let (Σ, ϕ) be an open book of M, and α a contact form on M which define a contact structure $\xi_{(\Sigma,\phi)}$ supported by (Σ, ϕ).

'only if' part. Let c be a simple closed Legendrian curve in $\Sigma_0 \subset (M, \xi)$. Suppose that c is isolated in Σ_0, i.e., there is a subsurface $S \subset \Sigma_0$ with $\partial S = c$. By the fact that c is Legendrian and Stokes' theorem, we have that $0 = \int_c \alpha = \int_S d\alpha$. This contradicts the fact that ξ is supported by (Σ, ϕ), i.e., $d\alpha|_{\Sigma_t} > 0$. Thus we have done a proof of 'only if' part.

'if' part. Let c be a non-isolated simple closed curve on a fiber Σ_0 of an open book (Σ, ϕ). We will construct a contact structure ξ on M supported by (Σ, ϕ), setting c to be Legendrian.

J. M. Montesinos and H. R. Morton [**12**] showed that for any open book (Σ, ϕ) there are an open book (D^2, β) of S^3 and simple covering map $\pi : M \to S^3$ such that $\Sigma_t = \pi^{-1}(D_t)$ for each $t \in [0, 1)$ and $\pi \circ \phi = \beta \circ \pi$, where $\{D_t\}$ is a family of fiber surfaces of (D^2, β).

Since c is non-isolated, Σ has a handle decomposition such that c is decomposed into two core arcs of 1-handles and two arcs properly embedded in 0-handles. Then it is easy to see that we can choose the covering map π so that $\pi(c)$ covers an arc a on D_0.

Take a contact form α on S^3 which is supported by (D^2, β). We may assume that $\alpha|_{D_0} = x\mathrm{d}y - y\mathrm{d}x$ in coordinates (x, y) of D such that the arc a is contained in x-axis. Note that x-axis is Legendrian in $(S^3, \ker(\alpha))$ and so is a. Then the pullback $\pi^*\alpha$ is a contact form on M supported by (Σ, ϕ) and c is Legendrian in $(M, \ker(\pi^*\alpha))$. □

3. Overtwisted open books.

For the simplicity, we call an open book supporting an overtwisted contact structure an *overtwisted open book*. In this section, we first present two propositions which give Sufficient conditions of an open books to be overtwisted, and then we give a proof of Theorem 1.1 using the propositions.

3.1. Sufficient conditions for overtwistedness.

Let (Σ, ϕ) be an open book of a closed oriented 3-manifold M.

PROPOSITION 3.1. *(Σ, ϕ) is overtwisted if (Σ, ϕ) has a non-isolated twisting loop.*

PROOF. Let c be a non-isolating twisting loop on Σ, and let $\xi_{(\Sigma,\phi)}$ denote a contact structure supported by (Σ, ϕ). By the definition of twisting loop, c bounds a disk D in M such that

$$\mathrm{Fr}(c; D) = \mathrm{Fr}(c; \Sigma). \tag{1}$$

Since c is non-isolated in Σ_0, by Lemma 2.3 we may assume that c is Legendrian in $(M, \xi_{(\Sigma,\phi)})$.

On the interior of Σ, plains of $\xi_{(\Sigma,\phi)}$ are arbitrary close to tangent plains of Σ as mentioned Remark 2.2. So we have that

$$\mathrm{Fr}\left(c; \xi_{(\Sigma,\phi)}\right) = \mathrm{Fr}(c; \Sigma). \tag{2}$$

From the equations (1) and (2) we have that

$$\mathrm{Fr}\left(c; \xi_{(\Sigma,\phi)}\right) = \mathrm{Fr}(c; D).$$

This means that D is an overtwisted disk in $(M, \xi_{(\Sigma,\phi)})$. □

Next we focus on an arc properly embedded on the fiber surface of an open book and its image of the monodromy map, and show another criterion of overtwistedness of open books.

Let a be an arc properly embedded in Σ. We always assume that $\phi(a)$ is isotoped relative to the boundary so that the number of intersection points between a and $\phi(a)$ is minimized. We orient the closed curve $a \cup \phi(a)$. It does not matter which orientation is chosen. At a point p of $a \cap \phi(a)$ define i_p to be $+1$ if the oriented tangent to a at p followed by the oriented tangent to $\phi(a)$ at p is an oriented basis for Σ otherwise we set $i_p = -1$ (see

Figure 3). We define two kinds of intersection numbers of a and $\phi(a)$ as in Goodman's way [**8**]; The geometric intersection number, $i_{\text{geom}}(a, \phi(a)) = \sum_{a\cap\phi(a)\cap\text{Int}\,\Sigma} |i_p|$, is the number of intersection point of a and $\phi(a)$ in the interior of Σ. The boundary intersection number, $i_\partial(a, \phi(a)) = \frac{1}{2}\sum_{a\cap\phi(a)\cap\partial\Sigma} i_p$, is one-half the oriented sum over intersections at the boundaries of the arcs.

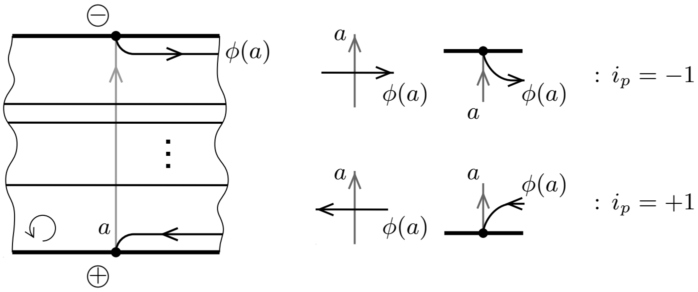

Figure 3.

Proposition 3.2. *(Σ, ϕ) is overtwisted if (Σ, ϕ) has a proper arc a such that a is not isotopic to $\phi(a)$ and satisfies*

$$i_{\text{geom}}(a, \phi(a)) = i_\partial(a, \phi(a)) = 0.$$

Proof. By the open book structure (Σ, ϕ) of M, we have a homeomorphism $h : E(\partial\Sigma) \to \Sigma \times [0,1]/(x,0) \sim (\phi(x),1)$ with $h(\Sigma \cap E(\partial\Sigma)) = \Sigma_0 = \Sigma_1$. Put $\Delta = h^{-1}(h(a) \times [0,1])$. Let B be a small regular neighbourhood of a on M, a' a parallel copy of a in $\Sigma \cap B$ such that $a' \cap \phi(a)$ is two points, say p and q, and α a sub-arc of a' connecting p and q (see Figure 4(1)).

Deform Δ as follows: Take a small regular neighbourhood B' of a' in B avoiding a, and let R be a closure of a connected component of $(B' \cap \Sigma) - \phi(a)$ which contains α. We identify R as $\alpha \times [-1,1]$ while identifying $\alpha \subset R$ as $\alpha \times \{0\}$. Let δ_p (resp. δ_q) be a connected component of $B' \cap \Delta$ which contains p (resp. q). Note that $\partial\delta_p$ (resp. $\partial\delta_q$) consists of the arc $p \times [-1,1]$ (resp. $q \times [-1,1]$) and an arc properly embedded in Δ, say β_p (resp. β_q). Consider a band $b = [0,1] \times [0,1]$ embedded in $\partial B'$ such that $\{0,1\} \times [0,1] = \{\beta_p, \beta_q\}$, $[0,1] \times \{0,1\} = \alpha \times \{-1,1\} \subset \partial R$, and $b \cap \partial\Sigma = \varnothing$. We then set $\Delta' = (\Delta - \delta_p \cup \delta_q) \cup b$ (Figure 4(2)).

Now we obtain an annulus Δ' such that one of the boundary components is a simple closed curve on Σ isotopic to $a \cup \phi(a)$ and another is consists of a and an arc a'' isotopic to a relative to ∂a. Slide Δ' on Σ so that a'' overlap with a. Finally we see that Δ' becomes a disk D embedded in M such that ∂D is a simple close curve on Σ and $D \cap \Sigma = a$ (Figure 4(3)).

It follows from the construction that there exists a collar neighbourhood of ∂D in D which does not intersect with Σ except in ∂D. It means that $\text{Fr}(\partial D; D) = \text{Fr}(\partial D; \Sigma)$, i.e., ∂D is a twisting loop on Σ.

If ∂D is non-isolated in Σ, by Proposition 3.1 we have that D is an overtwisted disk

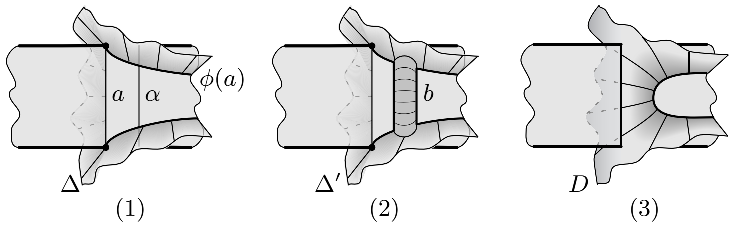

Figure 4.

in $(M, \xi_{(\Sigma,\phi)})$, where $\xi_{(\Sigma,\phi)}$ is a contact structure supported by (Σ, ϕ).

Suppose that ∂D is isolated. Note that ∂D is isotopic to $a \cup \phi(a)$. So we have a connected component S of $\Sigma - (a \cup \phi(a))$ such that $S \cap \partial\Sigma = \varnothing$. We will show that we can obtain a new open book (Σ', ϕ') by positive stabilizations such that a and $\phi'(a)$ satisfy the assumption of this proposition and $a \cup \phi'(a)$ is non-isolated in Σ'.

Note that S has the genus greater than 0, since a and $\phi(a)$ is not isotopic relative to the boundary. Let β_1 be an arc properly embedded in Σ as shown in Figure 5 (right) such that the arc $\beta_1 \cup S$ is not boundary-parallel and non-separating in S. By a positive stabilization along β_1 we have an open book (Σ'', ϕ'') and we can find an arc β_2 in Σ'' such that β_2 intersects with a and $\phi''(a)$ at just one point each as shown in Figure 5 (center). Then a positive stabilization along β_2 yields an open book (Σ', ϕ'), and we can easily see that $a \cup \phi'(a)$ is non-isolated on Σ' (Figure 5). $\square$

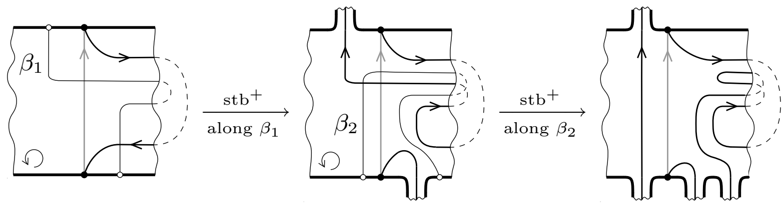

Figure 5.

3.2. Proof of Theorem 1.1.

<u>(1) ⇒ (2)</u> Let D be an overtwisted disk embedded in $(M, \xi_{(\Sigma,\phi)})$. By Corollary 4.23 in [**4**], we can construct an open book (Σ_0, ϕ_0) supporting $\xi_{(\Sigma,\phi)}$ such that ∂D is on Σ_0. It is easy to see that ∂D is a twisting loop on Σ_0. Since both (Σ, ϕ) and (Σ_0, ϕ_0) support the same contact structure, Giroux's one-to-one correspondence tells us that they are equivalent up to positive stabilization.

<u>(2) ⇒ (1)</u> This part immediately follows from Proposition 3.1 and Giroux's theorem.

<u>(1) ⇒ (3)</u> Goodman showed in [**8**, Theorem 5.1] that an overtwisted contact structure on a closed oriented 3-manifold has a supporting open book with a sobering arc, and a sobering arc has the boundary points satisfying $i_\partial \leq 0$.

<u>(3) ⇒ (1)</u> Let a be a properly embedded arc on Σ with $i_\partial(a, \phi(a)) \leq 0$. The arc a has at least one negative endpoint, say x_0. We orient a so that an oriented tangent vector of a at x_0 is outward from Σ. Put $g = i_{geom}(a, \phi(a))$. Stating from x_0, we assign $x_1, \ldots, x_{g+1}$ to the points of $a \cap \phi(a)$. Suppose that the another endpoint x_{g+1} is also negative, i.e.,

$i_\partial(a,\phi(a)) = -1$. Let β be a small properly embedded arc on Σ rounding x_{g+1}. By a positive stabilization along β we have a new open book with a new monodromy map, on which the point x_{g+1} is positive (see Figure 6). Thus we may assume that $i_\partial(a,\phi(a)) = 0$.

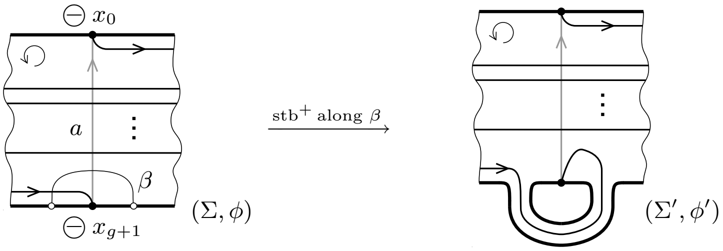

Figure 6. We may assume that $i_\partial = 0$.

We show this part by induction on g. In the case where $g = 0$, we have already proven as Proposition 3.2 that we can find an overtwisted disk in $(M, \xi_{(\Sigma,\phi)})$ with the boundary on Σ_0.

Suppose that $g > 0$. Let α_i be a sub-arc of a connecting x_{i-1} and x_i for $1 \le i \le g+1$ and γ_i a connected component of $N \cap \phi(a)$ containing the point x_i for $0 \le i \le g+1$. We denote by N_R a connected component of $N - a$ which has intersection with γ_0. Let R_0 be a connected region of $N_R - \gamma_0$ such that $R_0 \cap a = x_0$, and R_i a connected region of $N_R - \phi(a)$ such that $R_i \cap a = \alpha_i$ for $1 \le i \le g+1$ (see Figure 7). We denote by $\widehat{R}_k$ a connected region of $\Sigma - (a \cup \phi(a))$ containing R_k. Note that some regions of $\widehat{R}_0, \widehat{R}_1, \ldots, \widehat{R}_{g+1}$ might be the same one.

Tracing $\phi(a)$ along its orientation and picking up x_i's on the points of $a \cap \phi(a)$, we obtain a word $w = x_0^{-1} x_{p(1)}^{\epsilon_{p(1)}} x_{p(2)}^{\epsilon_{p(2)}} \ldots x_{p(g)}^{\epsilon_{p(g)}} x_{g+1}$, where p is a permutation of $\{1, 2, \ldots, g\}$ and ϵ_k is the sign of the point x_k for $1 \le k \le g$.

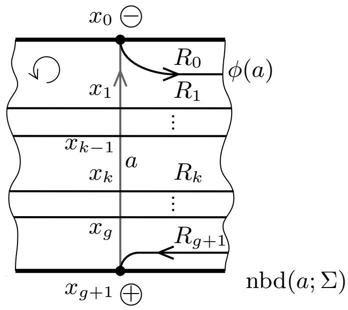

Figure 7.

Now we introduce two types of positive stabilization which reduces g, keeping $i_\partial = 0$.

First one is as follows: Suppose that there exists an integer k ($1 \le k \le g$) such that $\widehat{R}_k \cap \partial\Sigma \neq \varnothing$. Then there is an arc β properly embedded in Σ as shown in Figure 8 (left). We obtain an open book (Σ', ϕ') by positive stabilization along β such that

$$i_{geom}\big(a, \phi'(a)\big) = k - 1 < g.$$

We call this type of stabilization *reduction A*.

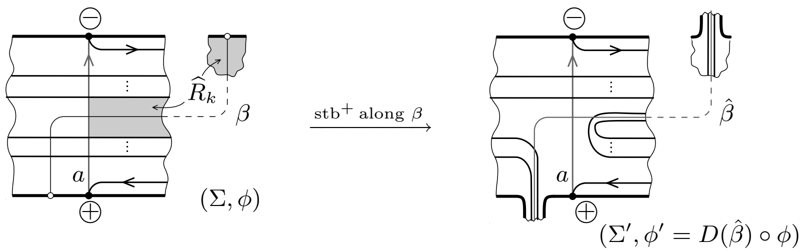

Figure 8. Reduction A.

Next, suppose that the word w contains two consecutive letters as $x_i^{-1}x_j^{-1}$ or x_jx_i $(1 \le i < j \le g)$. Then we can find an arc β properly embedded in Σ as shown in Figure 9 (left). By a positive stabilization along β, we obtain an open book (Σ', ϕ') such that

$$i_{geom}\big(a, \phi'(a)\big) = g - (j - i) < g.$$

We call this type of stabilization *reduction B*.

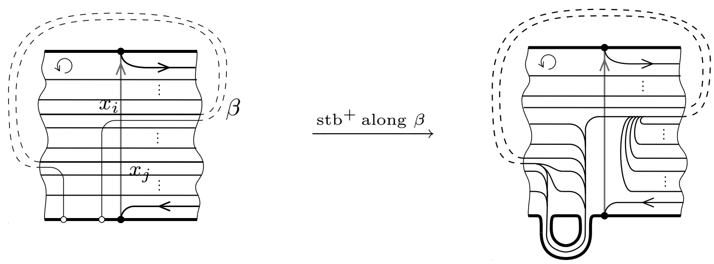

Figure 9. Reduction B.

We may assume that there are no proper arcs which admit reduction A or reduction B. Then the neighbourhood of a in Σ has the following properties;

(1) $\widehat{R}_k \cap \partial\Sigma = \varnothing$ for all $k = 1, 2, \ldots, g$,
(2) the word w contains no consecutive letters as $x_i^{-1}x_j^{-1}$ or x_jx_i, where $0 \le i < j \le g$.

Now we focus on the point $x_{p(1)}$. Suppose that $\epsilon_{p(1)} = -1$, i.e., the first two letters of w is $x_0^{-1}x_{p(1)}^{-1}$. This contradicts the property (2) above. Thus we have that $\epsilon_{p(1)} = +1$. Suppose that $p(1) < g$. We can easily see that $\widehat{R}_{p(1)+1} \cap \partial\Sigma \neq \varnothing$. This contradicts the property (1). Thus $p(1) = g$.

Case 1. $g > 1$. We look at the letter x_1 in the word w. By the property (2), we have that there is an integer k $(2 \le k \le g - 1)$ such that x_k is adjacent to x_1 in w and they appear as $x_1^{-1}x_k$ or $x_k^{-1}x_1$. Put $R = \bigcup_{i=1}^{g} R_i$. We can find an arc β properly embedded

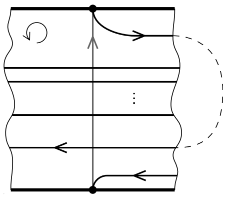

Figure 10. $w = x_0^{-1} x_g \cdots x_{g+1}$.

in Σ as shown in Figure 11 (left) such that $\beta \cap R$ is an arc not boundary-parallel in R and β does not intersect with $\phi(a)$ in R. The positive stabilization along β, which keeps the intersection number of a with its image of the monodromy map, yields a new open book, say (Σ', ϕ') (see Figure 11). Assign the names of regions $R_0, R_1, \ldots, R_{g+1}$ to the new regions of $N - (a \cup \phi'(a))$ in the same manner. It is easy to see that the region R_{g-k+1} (shaded in Figure 11 (right)) and R_0 are connected in Σ', i.e., $\widehat{R}_{g-k+1} = \widehat{R}_0$. Thus we can perform reduction A.

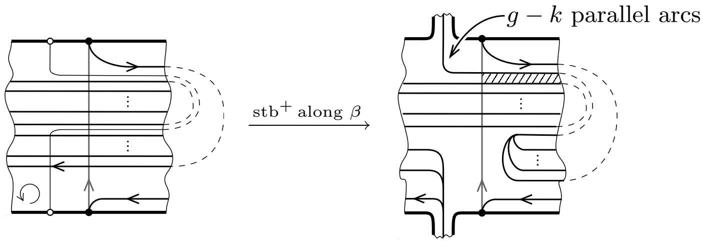

Figure 11.

<u>Case 2. $g = 1$.</u> Put $R = \widehat{R}_1$. Recall that R has no intersection with $\partial\Sigma$ and the genus of R is greater than 0. We have an arc β properly embedded in Σ as shown in Figure 12 such that $\beta \cap R$ is an arc not boundary-parallel, non-separating in R.

As in the previous case, the positive stabilization along β keeps the intersection number of a with its image of the monodromy map. Let (Σ', ϕ') be a resulting open book and reassign the names of regions R_0, R_1, R_2 to the new regions of $N - (a \cup \phi'(a))$ in the same manner. Since β is non-separating in R, the regions R_1 and R_2 are in the same connected component of $\Sigma - (a \cup \phi(a))$. Then R_1 and R_0 are also. Thus we can perform reduction A.

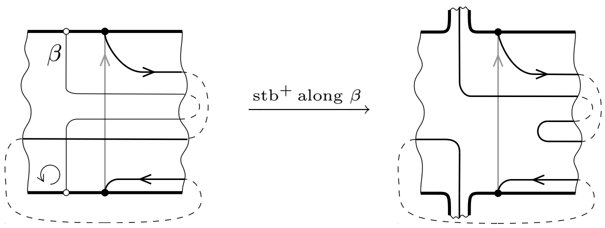

Figure 12.

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Ryosuke Yamamoto
Osaka City University
Advanced Mathematical Institute
3-3-138, Sugimoto, Sumiyoshi-ku
Osaka, 558-8585 Japan
E-mail: ryosuke@sci.osaka-cu.ac.jp