J. Math. Soc. Japan
Vol. 59, No. 2 (2007) pp. 583–601

Area type inequalities and integral means of harmonic functions on the unit ball

By Stevo Stević

(Received Aug. 25, 2005)
(Revised Jun. 6, 2006)

Abstract. In this paper we investigate the relationship among the following integrals

$$\int_B |u(x)|^{p-i} |\nabla u(x)|^i (1-|x|)^\alpha dV(x),$$

where $i \in \{0,1,2\}$, $1 < p < \infty$, $\alpha > 0$, and where u is an arbitrary harmonic function on the unit ball $B \subset \boldsymbol{R}^n$. Growth of the integral means of harmonic functions is also compared to the integral means of their gradient.

1. Introduction and auxiliary results.

Throughout this paper $B(a,r) = \{x \in \boldsymbol{R}^n \mid |x-a| < r\}$ denotes the open ball centered at a of radius r, where $|x|$ denotes the norm of $x \in \boldsymbol{R}^n$ and B is the open unit ball in $\boldsymbol{R}^n$, $rB = B(0,r)$, $S = \partial B = \{x \in \boldsymbol{R}^n \mid |x| = 1\}$ is the boundary of B. Let dV denote the Lebesgue measure on $\boldsymbol{R}^n$, $d\sigma$ the surface measure on S, σ_n the surface area of S, dV_N the normalized Lebesgue measure on B, $d\sigma_N$ the normalized surface measure on S.

Let $\mathscr{H}(B)$ denote the set of harmonic functions on B. Some basic facts on harmonic functions can be found, for example, in [**1**].

For $u \in \mathscr{H}(B)$ and $p \in (0,\infty)$, we denote the integral mean of u by

$$M_p^p(u,r) = \int_S |u(r\zeta)|^p d\sigma_N(\zeta), \quad r \in [0,1)$$

while

$$M_\infty(u,r) = \sup_{|x|<r} |u(x)|.$$

The Hardy harmonic space $\mathscr{H}^p(B)$, $p \in (0,\infty)$, consists of all $u \in \mathscr{H}(B)$ such that

$$\|u\|_{\mathscr{H}^p} = \sup_{0<r<1} M_p(u,r) < \infty.$$

A function $f \in C^1(B)$ is said to be a Bloch function if

2000 *Mathematics Subject Classification.* Primary 31B05.

Key Words and Phrases. weighted integrals, harmonic functions, area inequality, integral mean.

$$\|f\|_{\mathscr{B}} = \sup_{x\in B}(1-|x|)|\nabla f(x)| < +\infty$$

where $|\nabla f(x)| = \left(\sum_{i=1}^{n}\left|\frac{\partial f(x)}{\partial x_i}\right|^2\right)^{1/2}$. The space of Bloch functions is denoted by $\mathscr{B}(B)$.

Let $p > 0$. A Borel function f, locally integrable on B, is said to be a $BMO_p(B)$ function if

$$\|f\|_{BMO_p} = \sup_{B(a,r)\subset B}\left(\frac{1}{V(B(a,r))}\int_{B(a,r)}|f(x)-f_{B(a,r)}|^p dV(x)\right)^{1/p} < +\infty$$

where the supremum is taken over all balls $B(a,r)$ in B, and $f_{B(a,r)}$ is the mean value of f over $B(a,r)$. In [**8**] for $p \ge 1$, Muramoto proved that $\mathscr{B}(B)\cap\mathscr{H}(B)$ is isomorphic to $BMO_p(B)\cap\mathscr{H}(B)$ as Banach spaces, which inspired us to calculate exactly BMO_p norm for harmonic functions, which is theme of [**11**]. In the proof of the main result in [**11**], we essentially proved a generalization of Hardy-Stein identity, see, for example, [**6**]. This identity is included in the following lemma.

LEMMA 1. *Let $1 < p < +\infty$, $u \in \mathscr{H}(B)$, then for every $r \in (0,1)$ the following identity holds*

$$\int_S |u(r\zeta)|^p\, d\sigma_N(\zeta) = |u(0)|^p + \frac{p(p-1)}{n(n-2)}\int_{rB}|u(x)|^{p-2}|\nabla u(x)|^2(|x|^{2-n}-r^{2-n})dV_N(x), \quad n \ge 3. \tag{1}$$

It turns out that Lemma 1 is a very useful result. We already used this lemma in our investigations in [**13**] and [**14**], where among the other things we generalized some Yamashita's results in [**17**] and [**18**]. In this paper we present some new applications of the result. By differentiating formula (1) the following identity of Hardy-Stein type is obtained.

COROLLARY 1. *Let $1 < p < +\infty$, $u \in \mathscr{H}(B)$, $r \in (0,1)$, $n \ge 3$, then*

$$\frac{d}{dr}\int_S |u(r\zeta)|^p\, d\sigma_N(\zeta) = \frac{p(p-1)}{n}r^{1-n}\int_{rB}|u(x)|^{p-2}|\nabla u(x)|^2 dV_N(x). \tag{2}$$

For the case of holomorphic functions in $\boldsymbol{C}^n$, similar identity was proved in [**15**].

In the sequel we keep our attention to the case $n \ge 3$. Analogous results hold in the case $n = 2$. Formulations and proofs of the corresponding results we leave to the reader.

Multiplying (2) by r^{n-1}, using polar-coordinates in the right-hand side integral and then differentiating in r we obtain the next corollary:

COROLLARY 2. *Let $1 < p < +\infty$, $r \in (0,1)$, $n \ge 3$ and $u \in \mathscr{H}(B)$, then*

$$\frac{d}{dr}\left(r^{n-1}\frac{d}{dr}\big(M_p^p(u,r)\big)\right) = p(p-1)r^{n-1}\int_S |u(r\zeta)|^{p-2}|\nabla u(r\zeta)|^2 d\sigma_N(\zeta). \tag{3}$$

Let $p \in (1, \infty)$, $\alpha \in (-1, \infty)$ and

$$\mathscr{A}_{p,\alpha}(u) = p(p-1) \int_B |u(x)|^{p-2} |\nabla u(x)|^2 (1-|x|)^\alpha dV_N(x).$$

Corollary 3. *Let $1 < p < +\infty$, $\alpha > 0$ and $n \geq 3$, $u \in \mathscr{H}(B)$, then*

$$\mathscr{A}_{p,\alpha}(u) = n\alpha \int_0^1 \frac{d}{dr}\big(M_p^p(u,r)\big) r^{n-1} (1-r)^{\alpha-1} dr. \tag{4}$$

Proof. Multiplying (3) by $(1-r)^\alpha dr$, then integrating from 0 to 1 and using integration by parts it follows that:

$$\begin{aligned}
\mathscr{A}_{p,\alpha}(u) &= p(p-1)n \int_0^1 r^{n-1} \int_S |u(r\zeta)|^{p-2} |\nabla u(r\zeta)|^2 d\sigma_N(\zeta)(1-r)^\alpha dr \\
&= n \int_0^1 \frac{d}{dr}\left(r^{n-1} \frac{d}{dr}\big(M_p^p(u,r)\big)\right)(1-r)^\alpha dr \\
&= n\alpha \int_0^1 \frac{d}{dr}\big(M_p^p(u,r)\big) r^{n-1}(1-r)^{\alpha-1} dr,
\end{aligned}$$

as desired. □

For holomorphic functions on the unit disk U (we denote the set by $H(U)$), in [**16**] the authors considered the relationship between the following two integrals:

$$A(f,\alpha) = \int_U |f'(z)|^2 (1-|z|)^{1-\alpha} dm(z)$$

and

$$B(f,\alpha) = \int_U |f(z)|\,|f'(z)|(1-|z|)^{1-\alpha} dm(z).$$

They have proved the following result:

Theorem A. *Let $\alpha \in (-\infty, 2)$. Then the following statements are true.*

(a) *There is a constant $K_\alpha > 0$ depending only on α such that*

$$B(f,\alpha) \leq K_\alpha A(f, \alpha-1)$$

for all $f \in H(U)$ with $f(0) = 0$ and $A(f, \alpha-1) < \infty$, if and only if $\alpha \in (-\infty, 1)$. When $\alpha \in (-\infty, 1)$, K_α may be taken as $[(2-\alpha)(1-\alpha)/6]^{-1/2}$.

(b) *For all $f \in H(U)$ with $f(0) = 0$,*

$$A(f,\alpha-1) \le 2^{-1}(2-\alpha)B(f,\alpha),$$

where the constant $2^{-1}(2-\alpha)$ *is sharp.*

Theorem A motivated us to investigate the relationship between the integrals:

$$\int_U |f^{(n)}(z)|^p |f^{(k)}(z)|^q (1-|z|)^{np+kq+\alpha} dm \quad \text{and} \quad \int_U |f'|^{p+q}(1-|z|)^\alpha dm$$

where $0 < p, q < \infty$, $\alpha > -1$, $k, n \in \boldsymbol{N} \cup \{0\}$ and where f is an arbitrary analytic function on the unit disc U, see, [**11**]. As a consequence of Theorem 2.1 in [**9**] and Theorem 2 in [**12**], with the weight function $\omega(z) = (1-|z|)^\alpha$, we have that the next result holds:

Corollary 4. *Suppose* $0 < p, q < \infty$, $\alpha > -1$ *and* $k, n \in \boldsymbol{N} \cup \{0\}$. *Then there is a constant* $C = C(p, q, k, n, \alpha)$ *such that*

$$\int_U |f^{(n)}(z)|^p |f^{(k)}(z)|^q (1-|z|)^{np+kq+\alpha} dm \le C\left(|f(0)|^{p+q} + \int_U |f'(z)|^{p+q}(1-|z|)^{p+q+\alpha} dm\right)$$

for all $f \in H(U)$.

As a by-product we showed that the following quantities

$$A_{n,\alpha}(f) = |f(0)|^{n+1} + \int_U |f'(z)|^{n+1}(1-|z|)^{p+n+1} dm(z)$$

and

$$B_{n,\alpha}(f) = |f(0)|^{n+1} + \int_U |f(z)|^n \, |f'(z)|(1-|z|)^{p+1} dm(z),$$

are equivalent, for $n \in \boldsymbol{N}$ and $p > -1$.

The equivalence was motivated by Remark 1 in [**16**]. A natural question is whether any similar equivalence holds if n is replaced by a real parameter. Our aim is to obtain such results for the case of harmonic functions on the unit ball. Note that analytic functions are harmonic.

The paper is organized as follows. In Section 2 we give some auxiliary results which we use in the proofs of the main results of the paper. In Section 3 we investigate the relationship among the following integrals

$$\int_B |u(x)|^{p-i} |\nabla u(x)|^i (1-|x|)^\alpha dV(x),$$

where $i \in \{0, 1, 2\}$, $1 < p < \infty$, $\alpha > 0$, and where u is an arbitrary harmonic function on the unit ball $B \subset \boldsymbol{R}^n$. Motivated by paper [**5**] in Section 4 we prove some growth results concerning the integral means of harmonic functions and their gradients.

In what follows we shall be using the convention that C will denote a positive constant which is not necessarily the same at difference occurrences.

2. Auxiliary results.

In order to prove the main results of this paper we need several auxiliary results which are incorporated in the following lemmas.

LEMMA 2. *Suppose $0 < p < \infty$ and $u \in \mathscr{H}(B)$. Then*

$$\left|\frac{d}{dr}(|u(x)|^p)\right| \leq p|u(x)|^{p-1}|\nabla u(x)|, \tag{5}$$

for almost every $x = r\zeta \in B$.

PROOF. Since u is real analytic ([**1**]), the set of all zeros of u has Lebesgue measure 0. For $u \equiv 0$ the result is obvious. If $u \not\equiv 0$, at points x where u is not zero we have

$$\left|\frac{d}{dr}(|u(x)|^p)\right| = p|u(x)|^{p-1}\left|\left\langle \frac{u(r\zeta)}{|u(r\zeta)|}, \langle \nabla u(x), \zeta\rangle \right\rangle\right| \leq p|u(x)|^{p-1}|\nabla u(x)|, \tag{6}$$

where $x = r\zeta$ and where we interpret complex numbers $\frac{u(r\zeta)}{|u(r\zeta)|}$ and $\langle \nabla u(x), \zeta\rangle$ as vectors in $\boldsymbol{R}^2$. $\square$

LEMMA 3. *Suppose $1 \leq p < \infty$, $\alpha \in (-1, \infty)$ and $u \in \mathscr{H}(B)$. Then for every $r_0 \in (0,1)$, there is a positive constant C depending only on n, p, α and r_0 such that*

$$\int_B |u(x)|^p(1-|x|)^\alpha dV_N(x) \leq C\int_{B\setminus r_0B} |u(x)|^p(1-|x|)^\alpha dV_N(x). \tag{7}$$

PROOF. For each $r_0 \in (0,1)$ there is an $n_0 \in \boldsymbol{N}$ such that $(n_0-1)(1-r_0) \leq r_0 < n_0(1-r_0) \leq 1$. Hence $r_0B \subset n_0(1-r_0)B \subset B$, and consequently

$$\int_{r_0B} |u(x)|^p(1-|x|)^\alpha dV_N(x) \leq \int_{n_0(1-r_0)B} |u(x)|^p(1-|x|)^\alpha dV_N(x). \tag{8}$$

On the other hand, we have

$$\int_{n_0(1-r_0)B} |u(x)|^p(1-|x|)^\alpha dV_N(x) = \sum_{k=1}^{n_0} I_k, \tag{9}$$

where

$$I_k = \int_{k(1-r_0)B\setminus(k-1)(1-r_0)B} |u(x)|^p(1-|x|)^\alpha dV_N(x),$$

and $k \in \{1, \dots, n_0\}$.

Assume first that $\alpha \in (-1, 0]$. By the polar coordinates, and monotonicity of the function $M_p^p(u,r)(1-r)^\alpha r^{n-1}$ when $p \ge 1$ ([**7**]), we have

$$\begin{aligned}
I_k &= n\int_{(1-r_0)(k-1)}^{(1-r_0)k}\int_S |u(r\zeta)|^p d\sigma_N(\zeta)(1-r)^\alpha r^{n-1}dr \\
&\le n\int_{r_0}^{1}\int_S |u(r\zeta)|^p d\sigma_N(\zeta)(1-r)^\alpha r^{n-1}dr \\
&= \int_{B\setminus r_0B} |u(x)|^p(1-|x|)^\alpha dV_N(x). \qquad (10)
\end{aligned}$$

From (8), (9) and (10), we obtain (7) in this case.

Assume now that $\alpha \in (0, \infty)$. Then by the monotonicity of the function $M_p^p(u,r)r^{n-1}$, we have that for every $k \in \{1, \dots, n_0 - 1\}$

$$\begin{aligned}
I_k &\le nM_p^p(u,(1-r_0)k)[(1-r_0)k]^{n-1}\int_{(1-r_0)(k-1)}^{(1-r_0)k}(1-r)^\alpha dr \\
&= nM_p^p(u,(1-r_0)k)[(1-r_0)k]^{n-1}\frac{\int_{(1-r_0)(k-1)}^{(1-r_0)k}(1-r)^\alpha dr}{\int_{r_0}^1(1-r)^\alpha dr}\int_{r_0}^1(1-r)^\alpha dr \\
&\le nM_p^p(u,r_0)r_0^{n-1}\frac{\int_0^{1-r_0}(1-r)^\alpha dr}{\int_{r_0}^1(1-r)^\alpha dr}\int_{r_0}^1(1-r)^\alpha dr \\
&= nC(\alpha,r_0)M_p^p(u,r_0)r_0^{n-1}\int_{r_0}^1(1-r)^\alpha dr \\
&\le nC(\alpha,r_0)\int_{r_0}^1 M_p^p(u,r)(1-r)^\alpha r^{n-1}dr \\
&= C(\alpha,r_0)\int_{B\setminus r_0B}|u(x)|^p(1-|x|)^\alpha dV_N(x). \qquad (11)
\end{aligned}$$

Similarly, it can be proved that

$$\begin{aligned}
&\int_{(1-r_0)(n_0-1)}^{(1-r_0)n_0}\int_S |u(r\zeta)|^p d\sigma_N(\zeta)(1-r)^\alpha r^{n-1}dr \\
&\qquad \le C_1(\alpha,r_0)\int_{B\setminus r_0B}|u(x)|^p(1-|x|)^\alpha dV_N(x). \qquad (12)
\end{aligned}$$

From (9), (11) and (12) estimate (7) follows in the case $\alpha \in (0,\infty)$. $\square$

LEMMA 4. *Let $u \in C^{(1)}(B)$, $\alpha, \beta \in (0,\infty)$, $q \in [0,\alpha]$ and $\gamma > -1$, then*

$$
\begin{aligned}
J_{\alpha,\beta,\gamma}(u) &= \int_B |u(x)|^{\alpha}|\nabla u(x)|^{\beta}(1-|x|)^{\gamma}dV_N(x)\\
&\le \Big(\int_B |u(x)|^{\alpha-q}|\nabla u(x)|^{\beta+q}(1-|x|)^{\gamma+q}dV_N(x)\Big)^{\frac{\beta}{\beta+q}}\\
&\quad\times\Big(\int_B |u(x)|^{\alpha+\beta}(1-|x|)^{\gamma-\beta}dV_N(x)\Big)^{\frac{q}{\beta+q}}, \qquad (13)
\end{aligned}
$$

for a positive constant C independent of u.

Proof. When $q=0$ inequality (13) is obvious. Note that the integral $J_{\alpha,\beta,\gamma}(u)$ can be written in the following form

$$
J_{\alpha,\beta,\gamma}(u)=\int_B \big(|u(x)|^{\frac{\beta(\alpha-q)}{\beta+q}}|\nabla u(x)|^{\beta}(1-|x|)^{\frac{\beta(\gamma+q)}{\beta+q}}\big)\cdot\big(|u(x)|^{\frac{q(\alpha+\beta)}{\beta+q}}(1-|x|)^{\frac{q(\gamma-\beta)}{\beta+q}}\big)dV_N(x).
$$

Applying Hölder's inequality with exponents $\frac{\beta+q}{\beta}$ and $\frac{\beta+q}{q}$ to the last integral we obtain (13). □

The following lemma is well known and can be found, for example, in [**4**].

Lemma 5. *Let $p>\frac{n-1}{n}$, $x\in B$, $r=|x|$ and $\zeta\in S$, then*

$$
\int_S \frac{d\sigma(\zeta)}{|x-\zeta|^{np}}<\frac{c_{p,n}}{(1-r)^{np-n+1}},\qquad 0\le r<1 \qquad (14)
$$

for some positive constant $c_{p,n}$, depending only on p and n.

3. Area type inequalities.

In this section we investigate the relationship among some area types of integrals.

Theorem 1. *Let $u\in\mathscr{H}(B)$. Then the following statements are true:*
(a) *If $p>1$ and $\alpha>0$, then*

$$
\int_B |u(x)|^{p-2}|\nabla u(x)|^2(1-|x|)^{\alpha}dV_N(x)\le C\int_B |u(x)|^{p-1}|\nabla u(x)|(1-|x|)^{\alpha-1}dV_N(x),
$$

for some positive constant independent of u.
(b) *If $p>1$, $q\in(1,p]$, $\alpha>1$, then*

$$
\begin{aligned}
&\int_B |u(x)|^{p-1}|\nabla u(x)|(1-|x|)^{\alpha-1}dV_N(x)\\
&\quad\le C\int_B |u(x)|^{p-q}|\nabla u(x)|^{q}(1-|x|)^{\alpha+q-2}dV_N(x), \qquad (15)
\end{aligned}
$$

for some positive constant C depending only on n,p and α.

Proof. (a) From (4) and by Lemma 2, we have

$$\begin{aligned}
\mathscr{A}_{p,\alpha}(u) &= p(p-1)\int_B |u(x)|^{p-2}|\nabla u(x)|^2(1-|x|)^\alpha dV_N(x)\\
&= n\alpha\int_0^1 \frac{d}{dr}\big(M_p^p(u,r)\big)r^{n-1}(1-r)^{\alpha-1}dr\\
&\le np\alpha\int_0^1\int_S |u(r\zeta)|^{p-1}|\nabla u(r\zeta)|r^{n-1}(1-r)^{\alpha-1}dr\\
&\le p\alpha\int_B |u(x)|^{p-1}|\nabla u(x)|(1-|x|)^{\alpha-1}dV_N(x),
\end{aligned}$$

from which desired inequality follows. Note that since $M_p^p(u,r)$ is nondecreasing function it has the derivative for a.a. $r\in(0,1)$ and $\frac{d}{dr}M_p^p(u,r)=\int_S\frac{d}{dr}(|u(r\zeta)|^p)d\sigma_N(\zeta)$.

(b) Let

$$J_{p,\alpha}(u)=\int_B |u(x)|^{p-1}|\nabla u(x)|(1-|x|)^{\alpha-1}dV_N(x).$$

By Lemma 4 with $\alpha=p-1$, $\beta=1$, $\gamma=\alpha-1$ and with exponents q and $q'=q/(q-1)$, it follows that

$$\begin{aligned}
J_{p,\alpha}(u) &\le \bigg(\int_B |u(x)|^{p-q}|\nabla u(x)|^q(1-|x|)^{\alpha+q-2}dV_N(x)\bigg)^{1/q}\\
&\quad\times\bigg(\int_B |u(x)|^p(1-|x|)^{\alpha-2}dV_N(x)\bigg)^{1/q'}. \qquad (16)
\end{aligned}$$

Now we estimate the last integral by $J_{p,\alpha}(u)$. Applying polar-coordinates and integration by parts in r, we have

$$\begin{aligned}
J_{p,\alpha}(u) &= n\int_0^1\int_S |u(r\zeta)|^{p-1}|\nabla u(r\zeta)|d\sigma_N(\zeta)r^{n-1}(1-r)^{\alpha-1}dr\\
&\ge \frac{n}{p}\int_0^1\int_S \frac{d}{dr}(|u(r\zeta)|^p)d\sigma_N(\zeta)r^{n-1}(1-r)^{\alpha-1}dr \quad \text{(by Lemma 2)}\\
&\ge \frac{n}{p}\int_S\int_0^1 \frac{d}{dr}(|u(r\zeta)|^p)r^{n-1}(1-r)^{\alpha-1}dr\,d\sigma_N(\zeta)\\
&= \frac{n}{p}\int_S\int_0^1 |u(r\zeta)|^p((n+\alpha-2)r-n+1)r^{n-2}(1-r)^{\alpha-2}dr\,d\sigma_N(\zeta)\\
&\ge \frac{n(\alpha-1)}{2p}\int_{\frac{2n+\alpha-3}{2(n+\alpha-2)}}^1\int_S |u(r\zeta)|^p d\sigma_N(\zeta)(1-r)^{\alpha-2}r^{n-1}dr
\end{aligned}$$

$$= \frac{(\alpha-1)}{2p} \int_{B \setminus \frac{2n+\alpha-3}{2(n+\alpha-2)} B} |u(x)|^p (1-|x|)^{\alpha-2} dV_N(x)$$

$$\geq C_{p,n,\alpha} \int_B |u(x)|^p (1-|x|)^{\alpha-2} dV_N(x), \quad \text{(by Lemma 3).} \tag{17}$$

Replacing estimate (17) into inequality (16) and using the fact that q and q' are conjugate exponents, we obtain inequality (15). □

THEOREM 2. *Let $u \in \mathscr{H}(B)$. Then the following statements are true*:

(a) *If $2 \leq p < \infty$ and $\alpha > 1$, then*

$$\int_B |u(x)|^{p-1} |\nabla u(x)| (1-|x|)^{\alpha-1} dV_N(x) \leq C \int_B |u(x)|^p (1-|x|)^{\alpha-2} dV_N(x), \tag{18}$$

for some positive constant independent of u.

(b) *If $p > 1$ and $\alpha > 2$, then*

$$\int_B |u(x)|^p (1-|x|)^{\alpha-2} dV_N(x)$$
$$\leq C \bigg(M_p^p \Big(u, \frac{2n+\alpha-3}{2(n+\alpha-2)} \Big) + \int_B |u(x)|^{p-1} |\nabla u(x)| (1-|x|)^{\alpha-1} dV_N(x) \bigg), \tag{19}$$

for some positive constant C depending only on n, p and α.

PROOF. (a) Integrating (4) by parts, we get

$$\mathscr{A}_{p,\alpha}(u) = n\alpha \int_0^1 \frac{d}{dr} \big(M_p^p(u,r) \big) r^{n-1} (1-r)^{\alpha-1} dr$$
$$= n\alpha \int_0^1 M_p^p(u,r) [(\alpha-1) r - (n-1)(1-r)] r^{n-2} (1-r)^{\alpha-2} dr.$$

Hence,

$$\mathscr{A}_{p,\alpha}(u) \leq n\alpha(\alpha-1) \int_0^1 M_p^p(u,r) r^{n-1} (1-r)^{\alpha-2} dr$$
$$= \alpha(\alpha-1) \int_B |u(x)|^p (1-|x|)^{\alpha-2} dV_N(x). \tag{20}$$

From (15) with $q = 2$ and (20), (18) follows.

(b) As in the proof of inequality (17) and by Lemma 3, we have

$$\int_{\frac{2n+\alpha-3}{2(n+\alpha-2)}}^1 M_p^p(u,r) r^{n-1} (1-r)^{\alpha-2} dr$$

$$\leq \frac{2}{\alpha-1}\int_{\frac{2n+\alpha-3}{2(n+\alpha-2)}}^{1} M_p^p(u,r)[(\alpha-1)r-(n-1)(1-r)]r^{n-2}(1-r)^{\alpha-2}dr$$

$$= \frac{2}{\alpha-1}\int_0^{\frac{2n+\alpha-3}{2(n+\alpha-2)}} M_p^p(u,r)[n-1-(n+\alpha-2)r]r^{n-2}(1-r)^{\alpha-2}dr$$

$$+ \frac{2}{\alpha-1}\int_0^1 \frac{d}{dr}\big(M_p^p(u,r)\big)r^{n-1}(1-r)^{\alpha-1}dr$$

$$\leq CM_p^p\left(u, \frac{2n+\alpha-3}{2(n+\alpha-2)}\right) + \frac{2p}{n(\alpha-1)}\int_B |u(x)|^{p-1}|\nabla u(x)|(1-|x|)^{\alpha-1}dV_N(x). \quad (21)$$

On the other hand

$$\int_0^{\frac{2n+\alpha-3}{2(n+\alpha-2)}} M_p^p(u,r)r^{n-1}(1-r)^{\alpha-2}dr \leq CM_p^p\left(u, \frac{2n+\alpha-3}{2(n+\alpha-2)}\right), \quad (22)$$

for some positive C independent of u.

From (21) and (22) inequality (19) follows. □

The following statement is a consequence of Theorems 1 and 2.

Corollary 5. *Let $u \in \mathscr{H}(B)$, $u(0) = 0$, $p \geq 2$ and $\alpha > 1$. Then the following quantities*

$$\int_B |u(x)|^{p-2}|\nabla u(x)|^2(1-|x|)^{\alpha}dV_N(x),$$

and

$$\int_B |u(x)|^{p-1}|\nabla u(x)|(1-|x|)^{\alpha-1}dV_N(x)$$

are equivalent.

If $\alpha > 2$, then these quantities are equiconvergent with

$$\int_B |u(x)|^p(1-|x|)^{\alpha-2}dV_N(x).$$

Remark 1. If we multiply formula (3) by any $C^{(1)}$ radial weight function $\omega(1-|x|)$, such that $\omega(0) = 0$, and then integrate obtained equality by parts we obtain

$$\mathscr{A}_{p,\omega}(u) = \int_B |u(x)|^{p-2}|\nabla u(x)|^2\omega(1-|x|)dV_N(x)$$

$$= \frac{n}{p(p-1)}\int_0^1 \frac{d}{dr}\big(M_p^p(u,r)\big)\omega'(1-r)r^{n-1}dr.$$

If $\omega'(0)=0$ another integration by parts gives

$$\mathscr{A}_{p,\omega}(u)=\frac{n}{p(p-1)}\int_0^1 M_p^p(u,r)[r\omega''(1-r)-(n-1)\omega'(1-r)]r^{n-2}dr.$$

By these formulas, Lemma 2 and some simple calculations it follows that

$$\mathscr{A}_{p,\omega}(u)\leq\frac{1}{(p-1)}\int_B |u(x)|^{p-1}|\nabla u(x)|\omega'(1-|x|)dV_N(x)$$

and

$$\mathscr{A}_{p,\omega}(u)\leq\frac{1}{p(p-1)}\int_B |u(x)|^p\omega''(1-|x|)dV_N(x).$$

Imposing some additional conditions on the weight function ω one can obtain the converse inequalities.

4. Integral means of harmonic functions.

In this section we study the harmonic functions whose integral means satisfy the following growth condition

$$M_p(u,r)=O\bigg(\frac{1}{(1-r)^a}\bigg),\qquad \text{as}\quad r\to 1.$$

Our motivation stems from [**5**].

The following theorem is a well-known generalization of a result of Hardy and Littlewood for holomorphic functions on the unit disk:

THEOREM B. *Let $p\in[1,\infty]$, $a\in\boldsymbol{R}_+\setminus\{1\}$ and $u\in\mathscr{H}(B)$, then*

$$M_p(\nabla u,r)=O\bigg(\frac{1}{(1-r)^a}\bigg),\qquad \textit{as}\quad r\to 1 \tag{23}$$

if and only if

$$M_p(u,r)=O\bigg(\frac{1}{(1-r)^{a-1}}\bigg),\qquad \textit{as}\quad r\to 1. \tag{24}$$

We will sketch a proof of Theorem B for the benefit of the reader. In fact, if $p\geq 1$ then condition (24) implies (23) for every $a\in\boldsymbol{R}$. Indeed, by the Cauchy's estimate ([**1**]) and by the subharmonicity of the function $|u|^p$, we have

$$|\nabla u(x)|^p\leq\frac{C}{r}\sup_{y\in B(x,r/4)}|u(y)|^p\leq\frac{C}{r^{p+n}}\int_{x\in B(x,r/2)}|u(y)|^p dV(y). \tag{25}$$

Replacing r by $(1-|x|)/2$ in (25), x by Ux, where U is an arbitrary orthogonal transformation of $\boldsymbol{R}^n$, and then applying the change $t \to Ut$, we obtain

$$|\nabla u(Ux)|^p \le \frac{C}{(1-|x|)^{n+p}} \int_{B(x,(1-|x|)/2)} |u(Ut)|^p dV(t). \tag{26}$$

Integrating (26) with respect to the Haar measure on the orthogonal group and then applying Fubini's theorem, it follows that

$$M_p^p(\nabla u, r) \le \frac{C}{(1-|x|)^{n+p}} \int_{B(x,(1-|x|)/2)} M_p^p(u,|t|)dV(t). \tag{27}$$

From (27), using condition (24) and fact that

$$\frac{1}{2}(1-|x|) < 1-|t| < \frac{3}{2}(1-|x|) \quad \text{when} \quad t \in B(x,(1-|x|)/2),$$

we get (23).

On the other hand we have that

$$|u(x)| \le |u(0)| + \int_0^1 |\nabla u(xt)|dt.$$

From this and since $p \ge 1$, by Minkowski's inequality it follows that

$$M_p(u,r) \le |u(0)| + \int_0^1 M_p(\nabla u, rt)dt. \tag{28}$$

Using condition (23) in (28) we obtain that (24) holds.

REMARK 2. The proof of the implication (23) $\Rightarrow$ (24) holds also for the case $p \in (0,1)$. Namely, inequality (25) also holds in this case, since the function $|u|^p$ satisfied so called HL-property, see, for example [**3**] or [**11**].

REMARK 3. Note that if $a \in (0,1)$ and u satisfies (23), then $u \in \mathscr{H}^p(B)$.

From above mentioned we see that the case $a = 1$ is more interesting. Applying inequality (28) in the case $p \in [1,\infty)$, or the preceeding inequality, in the case $p = \infty$, it follows that:

If $p \ge 1$ and u is a harmonic function on B such that

$$M_p(\nabla u, r) = O\left(\frac{1}{1-r}\right), \qquad \text{as} \quad r \to 1 \tag{29}$$

then

$$M_p(u,r) = O\left(\ln\frac{1}{1-r}\right), \qquad \text{as} \quad r \to 1. \tag{30}$$

It is interesting that estimate (30) can be improved in the case $p \in (1,\infty)$ using methods described in [**5**]. We will formulate and prove a result corresponding to the main one in [**5**], although we will generalized it in Theorem 5, for its proof is interesting and relies on the fundamental identity (2). In what follows we use the notation $I_p(u,r) = M_p^p(u,r)$. The following result holds:

Theorem 3. *If $p \in [2,\infty)$ and u is a harmonic function on B which satisfies condition* (29), *then*

$$M_p(u,r) = O\left(\left(\ln\frac{1}{1-r}\right)^{\beta}\right), \qquad \textit{as} \quad r \to 1, \tag{31}$$

for all $\beta > 1/2$.

Proof. By Corollary 2 and using the fact that the integral means of harmonic functions are nondecreasing functions when $p > 1$, we have

$$I_p''(u,r) \le p(p-1)\int_S |u(r\zeta)|^{p-2}|\nabla u(r\zeta)|^2\,d\sigma_N(\zeta). \tag{32}$$

Set

$$A_1(r) = \left\{\zeta \in S \,:\, |\nabla u(r\zeta)| \le \frac{|u(r\zeta)|}{(1-r)\ln\frac{1}{1-r}}\right\},$$

and let $A_1^c(r)$ denote the complement of the set $A_1(r)$ with respect to the set S.

Since $p > 2$, from (32) we get

$$\begin{aligned} I_p''(u,r) &\le \frac{p(p-1)}{n}\left(\int_{A_1(r)} + \int_{A_1^c(r)}\right)|u(r\zeta)|^{p-2}|\nabla u(r\zeta)|^2 d\sigma(\zeta) \\ &\le \frac{p(p-1)}{n}\left(\frac{I_p(u,r)}{(1-r)^2\ln^2\frac{1}{1-r}} + \left((1-r)\ln\frac{1}{1-r}\right)^{p-2} I_p(\nabla u, r)\right). \end{aligned}$$

From (29) and (30), it follows that

$$I_p''(u,r) = O\left(\frac{1}{(1-r)^2}\ln^{p-2}\frac{1}{1-r}\right), \qquad \text{as} \quad r \to 1.$$

Integrating twice the last formula we obtain

$$I_p(u,r) = O\left(\ln^{p-1}\frac{1}{1-r}\right), \qquad \text{as} \quad r \to 1.$$

Repeating the procedure we can get that

$$I_p(u,r) = O\left(\ln^{pa_k}\frac{1}{1-r}\right), \qquad \text{as} \quad r \to 1,$$

where a_k satisfies the following difference equation

$$a_{k+1} = \frac{(p-2)a_k + 1}{p}, \quad a_1 = p.$$

Since $a_{k+1} - 1/2 = \frac{p-2}{p}(a_k - 1/2)$, it follows that the sequence a_k decreasingly converges to $1/2$, from which the result follows. □

Motivated by Theorem 3, we can expect that the constant β there can be replaced by $1/2$. In order to prove the result we need a consequence of [**13**, Theorem 3].

THEOREM 4. *Let $2 \le p < s+2$, $s \le p$, then*

$$\int_0^1 M^2_{\frac{2s}{s-p+2}}(\nabla u, \rho)(1-\rho)d\rho < \infty$$

implies $u \in \mathscr{H}^p(B)$, moreover

$$\|u\|^2_{\mathscr{H}^p} \le |u(0)|^2 + p(p-1)\int_0^1 M^2_{\frac{2s}{s-p+2}}(\nabla u, \rho)(1-\rho)d\rho. \tag{33}$$

PROOF. In [**13**, Theorem 3] we have proved that

$$M_p^2(u,r) \le |u(0)|^2 + p(p-1)\int_0^r M^2_{\frac{2s}{s-p+2}}(\nabla u, \rho)(1-\rho)d\rho. \tag{34}$$

Letting $r \to 1-0$ in (34) and applying the Monotone Convergence Theorem we get (33). □

We are now in a position to improve the estimate in Theorem 3.

THEOREM 5. *If $2 \le p < s+2$, $s \le p$, and $u \in \mathscr{H}(B)$ such that*

$$M_{\frac{2s}{s-p+2}}(\nabla u, r) = O\left(\frac{1}{1-r}\right), \qquad \text{as} \quad r \to 1 \tag{35}$$

then

$$M_p(u,r) = O\left(\left(\ln\frac{1}{1-r}\right)^{1/2}\right), \qquad \text{as} \quad r \to 1. \tag{36}$$

PROOF. By Theorem 4 applied to the (harmonic) functions $u(rx)$, $r \in (0,1)$, and condition (35), it follows that

$$\begin{aligned} M_p^2(u,r) &\le |u(0)|^2 + p(p-1)\int_0^1 M^2_{\frac{2s}{s-p+2}}(\nabla u, r\rho)(1-\rho)d\rho \\ &\le |u(0)|^2 + C\Big(-\frac{1}{r} + \frac{1}{r^2}\ln\frac{1}{1-r}\Big) \\ &= O\Big(\ln\frac{1}{1-r}\Big), \end{aligned} \tag{37}$$

as desired. □

If we chose $p = s$ in Theorem 5, we get the following corollary:

COROLLARY 6. *Let $p \in [2,\infty)$ and $u \in \mathscr{H}(B)$ such that*

$$M_p(\nabla u, r) = O\Big(\frac{1}{1-r}\Big), \qquad \text{as} \quad r \to 1.$$

Then

$$M_p(u,r) = O\Big(\Big(\ln\frac{1}{1-r}\Big)^{1/2}\Big), \qquad \text{as} \quad r \to 1.$$

The following theorem is a generalization of well-known Littlewood-Paley inequality for holomorphic functions on the unit disk:

THEOREM C. *Suppose $p \in (0,2]$ and $u \in \mathscr{H}(B)$. Then there is a constant $C = C(p,n)$ such that*

$$\sup_{0\le r<1}\int_S |u(r\zeta)|^p d\sigma(\zeta) \le C\Big(|u(0)|^p + \int_B |\nabla u(x)|^p(1-|x|)^{p-1}dV(x)\Big).$$

In particular, if $\int_B |\nabla u(x)|^p(1-|x|)^{p-1}dV(x) < \infty$, then $u \in \mathscr{H}^p(B)$.

For the case $p \in (0,2]$ we have the following result.

THEOREM 6. *If $p \in (0,2]$ and u is a harmonic function on B such that*

$$M_p(\nabla u, r) = O\Big(\frac{1}{1-r}\Big), \qquad \text{as} \quad r \to 1 \tag{38}$$

then

$$M_p(u,r) = O\Big(\Big(\ln\frac{1}{1-r}\Big)^{1/p}\Big), \qquad \text{as} \quad r \to 1. \tag{39}$$

PROOF. We may assume that $u(0)=0$. Applying Theorem C to the dilations $u_r(x)=u(rx)$, we obtain

$$\begin{aligned}
I_p(u,r) &\le C\int_B |\nabla u(rx)|^p(1-|x|)^{p-1}dV(x)\\
&= Cr^p\int_0^1 I_p(\nabla u, r\rho)(1-\rho)^{p-1}\rho^{n-1}d\rho\\
&\le C\int_0^1 \frac{(1-\rho)^{p-1}}{(1-r\rho)^p}d\rho\\
&= C\Big(\int_0^r \frac{(1-\rho)^{p-1}}{(1-r\rho)^p}d\rho+\int_r^1 \frac{(1-\rho)^{p-1}}{(1-r\rho)^p}d\rho\Big),
\end{aligned}$$

for $r\in(0,1)$.

Hence

$$\begin{aligned}
I_p(u,r) &\le C\Big(\int_0^r \frac{d\rho}{1-\rho}+\frac{1}{(1-r)^p}\int_r^1 (1-\rho)^{p-1}d\rho\Big),\\
&= O\Big(\ln\frac{1}{1-r}\Big), \qquad \text{as} \quad r\to 1,
\end{aligned}$$

finishing the proof of the result. □

REMARK 4. It is interesting that the proof of the fact that if a harmonic function u satisfies condition (23) then $u\in\mathscr{H}^p(B)$ for $a\in(0,1)$ can be proved as in the proof of Theorem 6. Indeed, we have

$$I_p(u,r)\le C\int_0^1 \frac{(1-\rho)^{p-1}}{(1-r\rho)^{ap}}d\rho \le \int_0^1 (1-\rho)^{p(1-a)-1}d\rho,$$

for every $r\in(0,1)$, from which it follows that $u\in\mathscr{H}^p(B)$.

The following result corresponds to Proposition 2 in [**5**].

THEOREM 7. *If $p\in(1,\infty)$, $u\in\mathscr{H}(B)$ satisfying condition (29), then*

$$M_\infty(u,r)=O\bigg(\Big(\frac{1}{1-r}\Big)^{(n-1)/p}\bigg), \qquad \textit{as} \quad r\to 1. \tag{40}$$

PROOF. By the Poisson integral formula and Hölder's inequality, we have

$$\begin{aligned}
|u(x)| &\le \rho^{n-2}\int_S \frac{\rho^2-|x|^2}{|\rho\zeta-x|^n}|u(\rho\zeta)|d\sigma_N(\zeta)\\
&\le \rho^{n-2}\bigg(\int_S \Big(\frac{\rho^2-|x|^2}{|\rho\zeta-x|^n}\Big)^{p'}d\sigma_N(\zeta)\bigg)^{1/p'} M_p(u,\rho),
\end{aligned}$$

for every $|x| < \rho < 1$.

Setting $\rho = (1+|x|)/2$, and applying Lemma 5, it follows that

$$
\begin{aligned}
|u(x)| &\le C(1-|x|)\Big(\int_S \frac{d\sigma_N(\zeta)}{|\rho\zeta - x|^{np'}}\Big)^{1/p'} M_p(u,(1+|x|)/2) \\
&\le \frac{C}{(1-|x|)^{\frac{n-1}{p}}} M_p(u,(1+|x|)/2). \qquad (41)
\end{aligned}
$$

On the other hand, we have

$$
|u(\rho\zeta) - u(0)| = \Big|\int_0^\rho \langle \nabla u(t\zeta), \zeta\rangle dt\Big| \le \int_0^\rho |\nabla u(t\zeta)| dt.
$$

Hence

$$
|u(\rho\zeta)| \le |u(0)| + \int_0^\rho |\nabla u(t\zeta)| dt.
$$

Using Minkowski's inequality in continuous form, it follows that

$$
\begin{aligned}
M_p(u,\rho) &\le |u(0)| + \Big(\int_S \Big(\int_0^\rho |\nabla u(t\zeta)| dt\Big)^p d\sigma_N(\zeta)\Big)^{1/p} \\
&\le |u(0)| + \int_0^\rho \Big(\int_S |\nabla u(t\zeta)|^p d\sigma_N(\zeta)\Big)^{1/p} dt \\
&\le |u(0)| + \int_0^\rho M_p(\nabla u, t) dt. \qquad (42)
\end{aligned}
$$

Combining (41) and (42) with $\rho = (1+r)/2$, using the change $p \to kp$, with $k > 1$, we obtain that

$$
\begin{aligned}
M_\infty(u,r) &\le C \frac{M_{kp}(u,(1+r)/2)}{(1-r)^{\frac{n-1}{kp}}} \\
&\le C\Big(\frac{|u(0)| + \int_0^{(1+r)/2} M_{kp}(\nabla u, s) ds}{(1-r)^{\frac{n-1}{kp}}}\Big). \qquad (43)
\end{aligned}
$$

Further we have

$$
\begin{aligned}
M_{kp}(\nabla u, s) &= \Big(\int_S |\nabla u(s\zeta)|^{kp-p} |\nabla u(s\zeta)|^p d\sigma_N(\zeta)\Big)^{1/kp} \\
&\le M_\infty(\nabla u, s)^{1-1/k} M_p(\nabla u, s)^{1/k}. \qquad (44)
\end{aligned}
$$

Since all partial derivatives of a harmonic function are harmonic, using (41) and an elementary inequality we have that

$$|\nabla u(x)| \le \frac{C}{(1-|x|)^{\frac{n-1}{p}}} M_p(\nabla u, (1+|x|)/2). \tag{45}$$

From (29), (44), (45) and the monotonicity of the integral mean $M_p(\nabla u, r)$ ($|\nabla u|$ is a subharmonic function, see [**15**]), it follows that

$$M_{kp}(\nabla u, s) \le C \frac{M_p(\nabla u, (1+s)/2)}{(1-s)^{\frac{n-1}{p}(1-\frac{1}{k})}} \le \frac{C}{(1-s)^{1+\frac{n-1}{p}(1-\frac{1}{k})}}. \tag{46}$$

From (43) and (46) it follows that

$$M_\infty(u, r) \le O\left(\frac{\int_0^{(1+r)/2}(1-s)^{-\left(1+\frac{n-1}{p}(1-\frac{1}{k})\right)}}{(1-r)^{\frac{n-1}{kp}}}\right) = O\left(\left(\frac{1}{1-r}\right)^{(n-1)/p}\right),$$

as $r \to 1$, finishing the proof. $\square$

References

[1] S. Axler, P. Bourdon and W. Ramey, Harmonic function theory, Springer-Verlag, New York 1992.

[2] P. L. Duren, Theory of H^p spaces, Academic Press, New York, 1970.

[3] C. Fefferman and E. M. Stein, H^p spaces of several variables, Acta Math., **129** (1972), 137–193.

[4] S. Gardiner, Growth properties of pth means of potentials in the unit ball, Proc. Amer. Math. Soc., **103** (1988), 861–869.

[5] D. Girela and J. A. Pelàez, Integral means of analytic functions, Ann. Acad. Sci. Fenn., **29** (2004), 459–469.

[6] W. Hayman, Multivalent functions, Cambridge Univ. Press, London, 1958.

[7] W. Hayman and P. Kennedy, Subharmonic functions, Vol. I, Academic Press, London, New York, San Francisco, 1976.

[8] K. Muramoto, Harmonic Bloch and BMO functions on the unit ball in several variables, Tokyo J. Math., **11** (1988), 381–386.

[9] Shi Ji-Huai, Inequalities for the integral means of holomorphic functions and their derivatives in the unit ball of $\boldsymbol{C}^n$, Trans. Amer. Math. Soc., **328** (1991), 619–637.

[10] E. Stein, Singular integrals and differentiability properties of functions, Princeton University Press, Princeton, New Jersey, 1970.

[11] S. Stević, An equivalent norm on BMO spaces, Acta Sci. Math., **66** (2000), 553–564.

[12] S. Stević, On an area inequality and weighted integrals of analytic functions, Result. Math., **41** (2002), 386–393.

[13] S. Stević, On harmonic Hardy and Bergman spaces, J. Math. Soc. Japan, **54** (2002), 983–996.

[14] S. Stević, On harmonic Hardy spaces and area integrals, J. Math. Soc. Japan, **56** (2004), 339–347.

[15] M. Stoll, A characterization of Hardy spaces on the unit ball of $\boldsymbol{C}^n$, J. London Math. Soc., **48** (1993), 126–136.

[16] K. J. Wirths and J. Xiao, An image-area inequality for some planar holomorphic maps, Results Math., **38** (2000), 172–179.

[17] S. Yamashita, Criteria for functions to be of Hardy class H^p, Proc. Amer. Math. Soc., **75** (1979), 69–72.

[18] S. Yamashita, Holomorphic functions and area integral, Bolettino. U. M. I., **6** 1-A (1982), 115–120.

Stevo Stević

Mathematical Institute of the Serbian Academy of Science,
Knez Mihailova 35/I, 11000 Beograd
Serbia
E-mail: sstevic@ptt.yu; sstevo@matf.bg.ac.yu