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Notes on operator categories

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Abstract. The bicategory of normal functors between W*-categories is shown to be monoidally equivalent to the bicategory of W*-bimodules.

Introduction.

Related to subfactor theory two kinds of tensor categories have been utilized in describing quantum symmetry ([**6**], [**9**]): Given a von Neumann algebra M, we have the tensor category ${}_M\mathscr{B}imod_M$ of M-M bimodules on the one hand and the tensor category $\mathscr{E}nd(M)$ of endomorphisms on the other hand. In spite of different appearances, they admit a close similarity: For an endomorphism ρ of M, assign the bimodule $L^2(M)\rho$, where $L^2(M)\rho$ is the standard Hilbert space of M with the right action modified by ρ. Then we have canonical isomorphisms $L^2(M)\rho \otimes_M L^2(M)\sigma \to L^2(M)(\rho \circ \sigma)$ for endomorphisms ρ and σ, which turn out to satisfy the condition of multiplicativity for monoidal functors and defines a fully faithful embedding of the opposite of $\mathscr{E}nd(M)$ into the tensor category ${}_M\mathscr{B}imod_M$.

If M is of infinite type, this monoidal embedding gives an equivalence of categories, i.e., they contain the same information as structural data. For a finite von Neumann algebra, however, the embedding is not surjective because of the independent sizes of left and right modules in that case.

Furthermore, the category $\mathscr{B}imod$ of bimodules is more flexible than $\mathscr{E}nd(M)$ in the point that we can work with categories which are closed under taking subobjects if we allow different algebras for left and right actions. The bimodules, together with the associative tensor products, then constitute a so-called bicategory (see Section 2 for explanations of the notion).

With these observations in mind, the author has anticipated the possibility of enlarging the tensor category $\mathscr{E}nd(M)$ in such a way that the above embedding can be extended to an equivalence of bicategories. The major purpose of the present article is to give an affirmative answer to this question in the framework of W*-categories: the bicategory of normal functors between W*-categories is shown to be monoidally equivalent to the bicategory $\mathscr{B}imod$ of W*-bimodules.

On the way of describing the above result, we shall also review some of basic facts on operator categories ([**11**], [**3**], [**7**]), where we have fully used tensor products of W*-modules (the relative tensor product) to obtain concise expressions.

Technically we have to rely on the modular theory in operator algebras on occasions,

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which is, however, algebraic (and formal) in nature rather than analytic in the present context if properly formulated (see [**16**]).

As backgrounds of the subject, we refer, for example, to [**15**] on operator algebras and to [**8**], [**10**] for background on category theory.

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1. W*-categories.

By a linear category, we shall mean an essentially small category for which hom-sets are vector spaces over the complex number field $\boldsymbol{C}$ and the operation of taking compositions is complex-linear in variables involved.

The restriction of essential smallness reflects our standing position that we shall not study operator algebras with the help of categorical languages but work with categories themselves of operator algebraic natures.

A functor between linear categories is said to be linear if the operation on morphisms is linear. Recall that a functor is said to be faithful if it is injective on hom-sets (no commitment to objects).

A linear category is called a ***-category** if it is furnished with conjugate-linear involutions on hom-sets satisfying (i) $f^* : Y \to X$ for $f : X \to Y$ and (ii) $(g \circ f)^* = f^* \circ g^*$ for $f : X \to Y$ and $g : Y \to Z$.

A *-category is called a **C*-category** if hom-sets are Banach spaces such that (i) $\|g \circ f\| \leq \|g\| \, \|f\|$ for morphisms of the form $f : X \to Y$ and $g : Y \to Z$, (ii) $\|f^* f\| = \|f\|^2$ and (iii) $f^* f \geq 0$ for any morphism f.

Note here that, by the conditions (i) and (ii), each $\mathrm{End}(X)$ is a C*-algebra and the meaning of positivity in (iii) is that for C*-algebras.

A C*-category $\mathscr{C}$ is called a **W*-category** if each Banach space $\mathrm{Hom}(X, Y)$ is the dual of a Banach space. 'Preduals' are uniquely determined by the C*-category $\mathscr{C}$ as can be checked easily (an analogue of Sakai's characterization, [**13**]).

A typical example of W*-categories is the category $\mathscr{R}ep(A)$ of *-representations of a C*-algebra A in Hilbert spaces of a specified class with hom-sets given by intertwiners of representations.

When A is separable (i.e., having a countable norm-dense subset), we can define the much smaller W*-category $\mathscr{S}\mathscr{R}ep(A)$ of *-representations of A in separable Hilbert spaces of a specified class with hom-sets given by intertwiners of representations. If $A = \boldsymbol{C}$, $\mathscr{R}ep(A)$ (resp. $\mathscr{S}\mathscr{R}ep(A)$) is the category of Hilbert spaces (resp. separable Hilbert spaces) of a specified class $\mathscr{H}ilb$ (resp. $\mathscr{S}\mathscr{H}ilb$) whose morphisms are bounded linear operators.

Another typical example of W*-category is the category $\mathscr{P}(M)$ of projections in a W*-algebra M: objects of $\mathscr{P}(M)$ are projections in M with hom-sets given by $\mathrm{Hom}(e, f) = fMe$ for projections e and f in M.

A typical example of C*-category is the category A-$\mathscr{M}od$ of Hilbert A-modules (again in a specified class) with A a C*-algebra.

Given *-categories $\mathscr{C}$ and $\mathscr{D}$, a functor $F : \mathscr{C} \to \mathscr{D}$ is called a ***-functor** if F is linear in morphisms and preserves the *-operation. If both of $\mathscr{C}$ and $\mathscr{D}$ are C*-categories, then a *-functor $F : \mathscr{C} \to \mathscr{D}$ is norm-decreasing:

$$\|F(f)\|^2 = \|F(f^*f)\| \leq \|f^*f\| = \|f\|^2$$

for a morphism $f : X \to Y$ in $\mathscr{C}$. The kernel of F (which is a C*-subcategory of $\mathscr{C}$) is then an analogue of closed *-ideals in C*-algebras and we have the exact sequence

$$0 \to \mathrm{Ker}_F(X,Y) \to \mathrm{Hom}(X,Y) \to F(\mathrm{Hom}(X,Y)) \to 0$$

of C*-categories.

Given *-categories $\mathscr{C}$ and $\mathscr{D}$, we define the category $\mathscr{H}om(\mathscr{C},\mathscr{D})$ with objects given by *-functors and morphisms consisting of natural transformations. Recall here that we have stuck to essentially small categories, which enables us to keep $\mathscr{H}om(\mathscr{C},\mathscr{D})$ essentially small. When $\mathscr{D}$ is a C*-category, morphisms are restricted to be bounded so that the category $\mathscr{H}om(\mathscr{C},\mathscr{D})$ of functors is again a C*-category.

Given an object X in a C*-category $\mathscr{C}$ and a positive linear functional φ on the C*-algebra $\mathrm{End}(X)$, we introduce the *-functor $F_\varphi : \mathscr{C} \to \mathscr{H}ilb$ by the GNS-construction: For an object Y in $\mathscr{C}$, the Hilbert space $F_\varphi(Y)$ is the completion of the vector space $\mathrm{Hom}(X,Y)$ with respect to the positive-semidefinite inner product

$$(y|y') = \varphi(y^*y') \quad \text{for } y, y' : X \to Y.$$

We shall often use the notation $y\varphi^{1/2}$ to distinguish the morphism y in $\mathscr{C}$ with the associated element in the Hilbert space $F_\varphi(Y)$.

For a morphism $f : Y \to Z$ in $\mathscr{C}$, $F_\varphi(f) : F_\varphi(Y) \to F_\varphi(Z)$ is the bounded linear map defined by

$$F_\varphi(f)(y\varphi^{1/2}) = (fy)\varphi^{1/2}.$$

These are well-defined by the positivity and the C*-norm assumption in the definition of C*-categories.

Note that, if $X \cong Y$ and positive linear functionals $\varphi : \mathrm{End}(X) \to \boldsymbol{C}$, $\psi : \mathrm{End}(Y) \to \boldsymbol{C}$ are related by a *-isomorphism between $\mathrm{End}(X)$ and $\mathrm{End}(Y)$, then the functors F_φ and F_ψ are unitarily equivalent.

Since (infinite) direct sums are permitted in the C*-category $\mathscr{H}ilb$, we can define the *-functor $F : \mathscr{C} \to \mathscr{H}ilb$ as the direct sum of $\{F_\varphi\}$, where $\varphi \in \mathrm{End}(X)^*_+$ with the family $\{X\}$ representing the isomorphism classes of objects in $\mathscr{C}$. Then F is faithful on each C*-algebra $\mathrm{End}(X)$ and we have

$$\|F(f)\|^2 = \|F(f^*f)\| = \|f^*f\| = \|f\|^2$$

for a morphism $f : X \to Y$, which implies that F is isometric on hom-sets. Thus, given a finite family $\{X_i\}_{1\leq i\leq n}$ of objects in the C*-category $\mathscr{C}$, the algebra of matrix form

$$\begin{pmatrix} F(\mathrm{Hom}(X_1,X_1)) & \cdots & F(\mathrm{Hom}(X_n,X_1)) \\ \vdots & \ddots & \vdots \\ F(\mathrm{Hom}(X_1,X_n)) & \cdots & F(\mathrm{Hom}(X_n,X_n)) \end{pmatrix}$$

is a C*-algebra on the Hilbert space

$$\bigoplus_{i=1}^{n} F(X_i),$$

which is *-isomorphic to the *-algebra

$$\begin{pmatrix} \mathrm{Hom}(X_1, X_1) & \cdots & \mathrm{Hom}(X_n, X_1) \\ \vdots & \ddots & \vdots \\ \mathrm{Hom}(X_1, X_n) & \cdots & \mathrm{Hom}(X_n, X_n) \end{pmatrix}.$$

PROPOSITION 1.1. *Given a C*-category $\mathscr{C}$ and a finite sequence $\{X_i\}_{1 \leq i \leq n}$ of objects in $\mathscr{C}$, the vector space*

$$\begin{pmatrix} \mathrm{Hom}(X_1, X_1) & \cdots & \mathrm{Hom}(X_n, X_1) \\ \vdots & \ddots & \vdots \\ \mathrm{Hom}(X_1, X_n) & \cdots & \mathrm{Hom}(X_n X_n) \end{pmatrix}$$

is a C-algebra with each matrix component isometrically identified with the Banach spaces* $\mathrm{Hom}(X_i, X_j)$.

The above proposition is used to enlarge a C*-category $\mathscr{C}$ so that it allows finite direct sums; consider the category $\widehat{\mathscr{C}}$ for which objects are finite sequences of objects in $\mathscr{C}$ and hom-sets are given by

$$\mathrm{Hom}(\{X_i\}, \{Y_j\}) = \begin{pmatrix} \mathrm{Hom}(X_1, Y_1) & \cdots & \mathrm{Hom}(X_m, Y_1) \\ \vdots & \ddots & \vdots \\ \mathrm{Hom}(X_1, Y_n) & \cdots & \mathrm{Hom}(X_m, Y_n) \end{pmatrix}$$

with the norm induced from a bigger C*-algebra of matrix form.

COROLLARY 1.2. *A C*-category is a W*-category if and only if*

$$M = \begin{pmatrix} \mathrm{End}(X) & \mathrm{Hom}(Y, X) \\ \mathrm{Hom}(X, Y) & \mathrm{End}(Y) \end{pmatrix}$$

is a W-algebra for any pair (X, Y) of objects in $\mathscr{C}$.*

Let $\mathscr{C}$ be a W*-category. Since $\mathrm{Hom}(X, Y)$ is realized as a corner of a W*-algebra, we can define their L^p-extensions (see [**4**] for L^p-theory on von Neumann algebras, cf. also [**16**], [**17**]): the Banach space $L^p(X, Y)$ is defined to be $qL^p(M)p$, where $L^p(M)$ is the L^p space associated to the W*-algebra M with $p : X \oplus Y \to X$ and $q : X \oplus Y \to Y$ the obvious projections in M. We then have the bounded bilinear map $L^q(X, Y) \times L^p(Y, Z) \to$

$L^r(X,Z)$ if $1/r = 1/p + 1/q$ and $L^\infty(X,Y)$ is identified with $\mathrm{Hom}(X,Y)$. We also have the duality $L^p(X,Y)^* = L^q(X,Y)$ with $1/p + 1/q = 1$ and the predual of $\mathrm{Hom}(X,Y)$ is identified with the Banach space $L^1(X,Y)$.

PROPOSITION 1.3. *Given a finite family $\{X_i\}$ of objects in a W*-category, the matrix algebra*

$$\begin{pmatrix} \mathrm{Hom}(X_1,X_1) & \cdots & \mathrm{Hom}(X_n,X_1) \\ \vdots & \ddots & \vdots \\ \mathrm{Hom}(X_1,X_n) & \cdots & \mathrm{Hom}(X_n,X_n) \end{pmatrix}$$

is a W-algebra. Moreover, any W*-category is extended (uniquely up to equivalence of *-categories) so that it admits finite direct sums.*

Let $\mathscr{C}$ be a *-category and $\mathscr{D}$ be a W*-category. Then $\mathscr{H}om(\mathscr{C},\mathscr{D})$ is again a W*-category ($\mathscr{C}$ being assumed to be essentially small and natural transformations being restricted to be bounded). For the identity functor $\mathrm{id}_\mathscr{C}$, $\mathrm{End}(\mathrm{id}_\mathscr{C})$ is then a commutative W*-algebra $Z(\mathscr{C})$ given by

$$\big\{t = \{t_X\}; t_X \in \mathrm{End}(X),\ f t_X = t_Y f \text{ for any } f \in \mathrm{Hom}(X,Y) \text{ in } \mathscr{C}\big\}.$$

We call $Z(\mathscr{C})$ the **center** of $\mathscr{C}$. If $\mathscr{C}$ is a full subcategory of M-$\mathscr{M}od$ including a faithful representation, then $Z(\mathscr{C})$ is naturally isomorphic to the center of M.

For a morphism $f : X \to Y$ in $\mathscr{C}$, the **central support** of f is defined to be the projection $c(f)$ in $Z(\mathscr{C})$ which is minimal among the projections c in $Z(\mathscr{C})$ satisfying $c_Y f = f = f c_X$. For a family $\{f_i\}$ of morphisms, we define its central support as the smallest projection in $Z(\mathscr{C})$ majorizing all $c(f_i)$'s: $c = \bigvee_i c(f_i)$. For a family $\{X_i\}$ of objects, its central support is defined to be the central support of the family $\{1_{X_i}\}$.

The central support c of f is calculated by

$$c_Z(f) = \bigvee_{g:Z\to X} s(fg),$$

where $s(fg)$ denotes the support projection of the element $g^* f^* f g$ in $\mathrm{End}(Z)$. In particular, if $\mathrm{Hom}(X,Y) = \{0\}$, i.e., X and Y are disjoint, then $c(1_X)c(1_Y) = 0$.

A family $\{U_i\}$ of objects in a *-category $\mathscr{C}$ is said to be **generating** if the associated family of hom-functors $\mathrm{Hom}(U_i,\cdot) : \mathscr{C} \to \mathscr{V}ec$ is faithful, i.e., the algebraic direct sum

$$\bigoplus_i \mathrm{Hom}(U_i,\cdot) : \mathscr{C} \to \mathscr{V}ec$$

is faithful. A single object U in $\mathscr{C}$ is called a **generator** if the one-object family $\{U\}$ is generating, i.e., the hom-functor $\mathrm{Hom}(U,\cdot) : \mathscr{C} \to \mathscr{V}ec$ is faithful.

EXAMPLE 1.4. Any family of objects representing all isomorphism classes is generating.

LEMMA 1.5. *Let $\{U_i\}$ be a family of objects in a W*-category $\mathscr{C}$. Then the following conditions are equivalent.*

(i) *The family $\{U_i\}$ is generating.*
(ii) *For any object X and any projection $0 \neq p \in \operatorname{End}(X)$, we can find a U_i such that $\operatorname{Hom}(U_i, pX) = p\operatorname{Hom}(U_i, X) \neq \{0\}$.*
(iii) *The central support of $\{U_i\}$ is equal to the family $\{1_X\}$ of identity morphisms.*
(iv) *For any object X in $\mathscr{C}$, we can find a family of partial isometries $\{u_{i,j} : U_i \to X\}$ such that*

$$\sum_{i,j} u_{i,j} u_{i,j}^* = 1_X.$$

PROOF. (i) $\Rightarrow$ (ii). If there exist an object X and a projection $0 \neq p \in \operatorname{End}(X)$ such that $\operatorname{Hom}(U_i, pX) = \{0\}$ for any i, then p can not be distinguished with 0 under the hom-functor $\operatorname{Hom}(U_i, \cdot)$, which contradicts with the condition (i).

(ii) $\Rightarrow$ (iii). If the central support $\{c_X\}$ of $\{U_i\}$ is different from $\{1_X\}$, we can find an object X such that $c_X \neq 1_X$, which satisfies $(1_X - c_X)f = 0$ for any i and any $f : U_i \to X$.

(iii) $\Rightarrow$ (iv). The formula for the central support together with polar decompositions shows that for any object X, we can find i and a non-zero partial isometry $u : U \to X$. Now the maximality argument gives the result.

(iv) $\Rightarrow$ (i) is obvious. □

LEMMA 1.6 ([**12**, Lemma 2.1]). *Let X, Y and Z be objects in $\mathscr{C}$. Let $T : L^2(X, Y) \to L^2(X, Z)$ be a bounded linear map satisfying $T(\xi x) = T(\xi)x$ for $\xi \in L^2(X, Y)$ and $x \in \operatorname{End}(X)$. Then we can find an element $y \in \operatorname{Hom}(Y, Z)$ such that $T(\xi) = y\xi$ for $\xi \in L^2(X, Y)$.*

PROOF. Let M be the W*-algebra associated to the family $\{X, Y, Z\}$ in Proposition 1.3 with e, f and g projections of M to the component X, Y and Z respectively. Consider a bounded operator T on $L^2(M)$ satisfying $T(\xi) = gT(f\xi e)e$ for $\xi \in L^2(M)$ and $T(\xi exe) = T(\xi)exe$ for $x \in M$.

Since the commutant of a reduction is the induction of the commutant, the second condition on T implies we can find $y \in M$ satisfying $T(\xi) = y\xi$. Now the restrictions on T for the domain and range reveal that we can replace y by the element gyf and we are done. □

PROPOSITION 1.7. *Given a generating family $\{U_i\}$ in a W*-category $\mathscr{C}$, let M be the opposite of the von Neumann algebra $\bigoplus_{i,j} \operatorname{Hom}(U_i, U_j)$. Then the functor $F : \mathscr{C} \to \mathscr{H}ilb$ defined by*

$$F(X) = \bigoplus_i L^2(U_i, X)$$

with the obvious right action of M° on $F(X)$ gives a fully faithful embedding of $\mathscr{C}$ into the W-category M-$\mathscr{M}od$.*

PROOF. The surjectivity of the functor on morphisms is a consequence of the previous lemma because solutions for finite indices give rise to a bounded net in $\mathrm{Hom}(X, Y)$ indexed by finite subsets of the index set, which admits a convergent cofinal subnet by the weak* compactness of the unit ball of $\mathrm{Hom}(X, Y)$.

On the other hand, if $f : X \to Y$ vanishes on the Hilbert space $F(X)$, then $f\mathrm{Hom}(U_i, X)\varphi_i^{1/2} = \{0\}$ for any i and any $\varphi_i \in \mathrm{End}(U_i)_*^+$, whence $f\mathrm{Hom}(U_i, X) = \{0\}$ for i. Since $\{U_i\}$ is generating, this implies $f = 0$. □

LEMMA 1.8. *For a von Neumann algebra M, the following are equivalent.*

(i) *M has the separable predual.*
(ii) *M has a faithful normal representation on a separable Hilbert space.*
(iii) *The standard space $L^2(M)$ is separable.*

PROOF. Non-trivial is (i) $\Rightarrow$ (iii).

Let $\{f_i\}_{i\geq 1}$ be a countable dense set of M_*. Then we can find a family $\{x_i \in M\}_{i\geq 1}$ such that $\|x_i\| \leq 1$ and $|f_i(x_i)| \geq \|f_i\|/2$, which turns out to be total in M with respect to the weak*-topology. In fact, if not, we can find an element $x \in M$ and a functional $f \in M_*$ satisfying $f(x) = 1$ and $f(x_i) = 0$ for $i \geq 1$. For each integer $n \geq 1$, choose $i_n \in \boldsymbol{N}$ so that $\|f_{i_n} - f\| \leq 1/n$. Then the inequality

$$\frac{1}{2}\|f_{i_n}\| \leq |f_{i_n}(x_{i_n})| = |f_{i_n}(x_{i_n}) - f(x_{i_n})| \leq \|f_{i_n} - f\| \leq \frac{1}{n}$$

implies $\|f_{i_n}\| \to 0$ and hence $f = \lim_n f_{i_n} = 0$, which contradicts the choice $f(x) = 1$.

The separability of the predual M_* also ensures that we can find a faithful positive functional φ in M_*. Now the set $\{x_i\varphi^{1/2}\}_{i\geq 1}$ is total in $L^2(M)$ by the Kaplansky's density theorem and we are done. □

REMARK.

(i) A similar argument shows that, if a Banach space X has the separable dual Banach space, then X itself is separable. Thus, the separability of $L^p(M)$ for some $1 < p < +\infty$ implies the separability of $L^q(M)$ with $1/p + 1/q = 1$ and then the predual is separable as a continuous image of $L^p(M) \times L^q(M)$.
(ii) The argument in the above proof reveals that a von Neumann algebra is countably generated if it has the separable predual. The converse implication is, however, not true; the dual Banach space of the separable C*-algebra $C[0, 1]$ is identified with the space $M[0, 1]$ of complex measures in the interval $[0, 1]$, which is not separable because $\|\delta_s - \delta_t\| = 2$ if $s \neq t$ in $[0, 1]$. Then the double dual von Neumann algebra $M[0, 1]^* = L^\infty[0, 1]$ is countably generated with the non-separable predual $M[0, 1]$.

DEFINITION 1.9. A W*-category is said to be **locally separable** if each $\mathrm{Hom}(X, Y)$ has the separable predual. A locally separable W*-category is said to be **separable** if it admits a countable generating family.

Given a W*-algebra M, we denote by $\mathscr{R}ep(M)$ the W*-category of normal *-representations of M on Hilbert spaces in a specified class. When M has the separable predual, we denote by $\mathscr{S}\mathscr{R}ep(M)$ the W*-category of normal *-representations of M on separable Hilbert spaces in a specified class.

Proposition 1.10.

(i) *A W*-category $\mathscr{C}$ is equivalent to $\mathscr{R}ep(M)$ for some W*-algebra M if and only if $\mathscr{C}$ admits subobjects* (*projections are associated to subobjects*) *and arbitrary direct sums.*

(ii) *A W*-category $\mathscr{C}$ is equivalent to $\mathscr{S}\mathscr{R}ep(M)$ for some W*-algebra M of separable predual if and only if $\mathscr{C}$ is separable and admits countable direct sums as well as subobjects.*

Proof. Clearly the condition is necessary. On the other hand, we have the fully faithful embedding $F : \mathscr{C} \to M\text{-}\mathscr{M}od$ by Proposition 1.7 and any M-module is isomorphic to $F(X)$ for some object X in $\mathscr{C}$ by the condition (cf. the structure theorem of normal representations [**1**], [**15**]). □

For functors between W*-categories, it is reasonable to restrict to normal ones: a *-functor $F : \mathscr{C} \to \mathscr{D}$ of W*-categories is said to be **normal** if F is weak*-continuous on each $\mathrm{Hom}(X, Y)$.

Lemma 1.11 ([**11**, Prop. 4.7]). *Let $\{U_i\}$ be a generating family of objects in $\mathscr{C}$. A *-functor $F : \mathscr{C} \to \mathscr{D}$ between W*-categories is normal if and only if it is normal on each W*-algebra* $\mathrm{End}(U_i)$.

Proof. Assume that F is normal on $\mathrm{End}(U_i)$'s.

We will first show that F is normal on each $\mathrm{End}(X)$: for any $\varphi \in \mathrm{End}(F(X))_*^+$, the functional $\mathrm{End}(X) \ni x \to \varphi(F(x))$ is weak*-continuous. To see this, it suffices to check the σ-strong continuity on the unit ball of $\mathrm{End}(X)$ by a well-known result ([**1**, Theorem I.3.1]). Let $x_\alpha \to x$ in the σ-strong topology with x_α and x in the unit ball. Here choose a family of partial isometries $\{u_{i,j} : U_i \to X\}$ satisfying $\sum u_{i,j}u_{i,j}^* = 1_X$.

Then $u_{i,j}(x_\alpha - x)^*(x_\alpha - x)u_{i,j}^* \in \mathrm{End}(U_i)$ converges to 0 in the weak*-topology and hence by the normality of F on $\mathrm{End}(U_i)$, we have

$$F(u_{i,j})^* F(x_\alpha - x)^* F(x_\alpha - x) F(u_{i,j}) = F\bigl(u_{i,j}^*(x_\alpha - x)(x_\alpha - x)u_{i,j}\bigr) \to 0$$

in the weak*-topology, i.e., $F(x_\alpha - x)F(u_{i,j})$ converges to 0 in the σ-strong topology for each index (i, j). Thus

$$\bigl(\varphi^{1/2}|F(x_\alpha - x)F(u_{i,j}u_{i,j}^*)\varphi^{1/2}\bigr) \to 0 \quad \text{as } \alpha \to \infty$$

for each (i, j).

Since $\|F(x_\alpha - x)\| \leq 2$ and

$$\sum \bigl(\varphi^{1/2}|(F(u_{i,j}u_{i,j}^*)\varphi^{1/2}\bigr) = (\varphi^{1/2}|\varphi^{1/2})$$

is finite, the usual argument shows that $\varphi(F(x_\alpha - x)) = (\varphi^{1/2}|F(x_\alpha - x)\varphi^{1/2})$ converges to 0, proving the normality of F on $\mathrm{End}(X)$.

Now let f_α be a net in $\mathrm{Hom}(X,Y)$ which converges to 0 in σ*-strong topology. Then $f_\alpha^* f_\alpha \to 0$ in weak*-topology and hence $F(f_\alpha)^* F(f_\alpha) = F(f_\alpha^* f_\alpha) \to 0$, i.e., $F(f_\alpha) \to 0$ in σ-strong topology. Thus the *-homomorphism

$$\begin{pmatrix} \mathrm{End}(X) & \mathrm{Hom}(Y,X) \\ \mathrm{Hom}(X,Y) & \mathrm{End}(Y) \end{pmatrix} \to \begin{pmatrix} \mathrm{End}(F(X)) & \mathrm{Hom}(F(Y),F(X)) \\ \mathrm{Hom}(F(X),F(Y)) & \mathrm{End}(F(Y)) \end{pmatrix}$$

between W*-algebras induced from F is σ-strongly continuous. Then it is weak*-continuous by the well-known result on topologies of W*-algebras ([**1**, Theorem I.3.1]) and we are done. $\square$

Corollary 1.12. *An equivalence of W*-categories is automatically normal.*

Proof. This follows from the automatical normality of *-isomorphisms between W*-algebras (the normality being described by a condition on the order structure of hermitian elements). $\square$

Let M and N be W*-algebras and H be an N-M bimodule. Then we can define a normal *-functor $\mathscr{R}ep(M) \to \mathscr{R}ep(N)$ by

$$F({}_M X) = {}_N H \otimes_M X$$

(see [**14**] further information on relative tensor products, cf. also [**16**], [**17**]).

Conversely we have the following reformulation of Rieffel's theorem on Morita equivalences.

Proposition 1.13. *Any normal *-functor $F : \mathscr{R}ep(M) \to \mathscr{R}ep(N)$ (resp. $F : \mathscr{S}\mathscr{R}ep(M) \to \mathscr{S}\mathscr{R}ep(N)$ with M and N having separable preduals) is unitarily equivalent to the one associated to an N-M bimodule H (resp. a separable N-M bimodule).*

*The bimodule H is uniquely determined up to unitary isomorphisms by the functor F: $H = {}_N F({}_M L^2(M))_M$ with the right action of $M = \mathrm{End}({}_M L^2(M))^\circ$ on H given by the normal *-homomorphism $\mathrm{End}({}_M L^2(M)) \to \mathrm{End}({}_N F({}_M L^2(M)))$ induced by the functor.*

Proof. Let ${}_M X$ be an M-module. By the structure theorem of normal *-homomorphisms ([**1**, Theorem I.4.3]), we can find an index set I and a projection in $\mathrm{Mat}_I(M)$ so that ${}_M X \cong {}_M L^2(M)^{\oplus I} p$. We shall then construct a unitary intertwiner

$${}_N F({}_M L^2(M)) \otimes_M X \to {}_N F({}_M X)$$

by the commutativity of the diagram

$$\begin{array}{ccc} H \otimes_M X & \longrightarrow & F(X) \\ \big\downarrow & & \big\downarrow \\ H \otimes_M L^2(M)^{\oplus I} p & \longrightarrow & F({}_M L^2(M)^{\oplus I} p). \end{array}$$

Here the bottom line is a unitary map given by the composition of natural identities

$$\begin{aligned}&{}_NF\big({}_ML^2(M)\big)\otimes_M L^2(M)^{\oplus I}p\\&\quad={}_NF\big({}_ML^2(M)\big)^{\oplus I}p={}_NF\big({}_ML^2(M)^{\oplus I}\big)F(p)={}_NF\big({}_ML^2(M)^{\oplus I}p\big).\end{aligned}$$

By the multiplicativity of F on morphisms, the unitary map $H\otimes_M X\to F(X)$ is independent of the choice of an isomorphism ${}_MX\to{}_ML^2(M)^{\oplus I}p$ and behaves naturally for intertwiners. □

A W*-category is said to be of **type I** if the matrix W*-algebra

$$\begin{pmatrix}\mathrm{End}(X) & \mathrm{Hom}(Y,X)\\ \mathrm{Hom}(X,Y) & \mathrm{End}(Y)\end{pmatrix}$$

is of type I (in the sense of Murray-von Neumann) for any pair (X,Y) of objects.

The following is an easy reformulation of the structure theorem on W*-algebras of type I.

Proposition 1.14 (cf. [**11**, Theorem 8.10]). *Assume that a W*-category $\mathscr{C}$ has subobjects. Then $\mathscr{C}$ is of type I if and only if we can find a generating family $\{U_i\}$ satisfying* (i) $\mathrm{End}(U_i)$ *is commutative and* (ii) $\mathrm{Hom}(U_i,U_j)=\{0\}$ *for $i\neq j$.*

2. W*-Tensor categories.

Recall that a **tensor category** is a linear category $\mathscr{T}$ together with a functor $\Phi:\mathscr{T}\times\mathscr{T}\to\mathscr{T}$ and a natural isomorphism (the associativity constraint) $a:\Phi(\Phi\times\mathrm{id}_{\mathscr{T}})\to\Phi(\mathrm{id}_{\mathscr{T}}\times\Phi)$ satisfying the so-called pentagonal condition. It is also assumed that $\mathscr{T}$ have a special object I (called the unit object) and two natural isomorphisms $l_X:I\otimes X\to X$, $r_X:X\otimes I\to X$ (left and right unit constraints respectively) satisfying the triangular condition. The tensor product notation is often used to denote the functor Φ: $\Phi(X,Y)=X\otimes Y$ and $\Phi(f,g)=f\otimes g$ for objects X, Y and morphisms f, g in $\mathscr{T}$.

When $\mathscr{T}$ is a *-category and Φ preserves the *-operation, i.e., $(f\otimes g)^*=f^*\otimes g^*$ for morphisms f, g in $\mathscr{T}$, $\mathscr{T}$ is called a ***-tensor category**. A *-tensor category is called a **C*-tensor category** if it is based on a C*-category and the associativity constraint is unitary.

A **W*-tensor category** is, by definition, a C*-tensor category $\mathscr{T}$ with the tensor product functor $\Phi:\mathscr{T}\times\mathscr{T}\to\mathscr{T}$ is **binormal** in the sense that it is separately normal on each variables.

Similar adjective definitions work for bicategories; we can talk about *-bicategories, C*-bicategories and W*-bicategories.

Recall that a bicategory consists of labels, $A,B,C,\dots$, categories $\mathscr{H}om(A,B)$ indexed by pairs of labels, functors

$$\Phi_{A,B,C}:\mathscr{H}om(B,A)\times\mathscr{H}om(C,B)\to\mathscr{H}om(C,A)$$

indexed by triples of labels together with natural isomorphisms

$$a_{A,B,C,D} : \Phi_{A,C,D}\big(\Phi_{A,B,C} \times \mathrm{id}_{\mathscr{H}om(D,C)}\big) \to \Phi_{A,B,D}\big(\mathrm{id}_{\mathscr{H}om(B,A)} \times \Phi_{B,C,D}\big)$$

indexed by quadruples of labels and satisfying the pentagonal relation.

The functor $\Phi_{A,B,C}$ is often denoted by the notation of composition, which reflects the view-point that a bicategory is a 'categorization' of hom-sets as hom-categories.

Here is another view-point from which we regard the hom-category $\mathscr{H}om(A,B)$ as an analogue of the category of B-A bimodules (if labels represent algebras) with the notation $\mathscr{H}om(A,B) = {}_B\mathscr{M}_A$. Then the functor $\Phi_{A,B,C}$ is consequently denoted by the tensor product notation: For objects X in ${}_A\mathscr{M}_B$ and Y in ${}_B\mathscr{M}_C$, $\Phi_{A,B,C}(X,Y)$ is denoted by $X \otimes_B Y$. Similarly for morphisms.

A typical example of W*-bicategory is provided by bimodules with normal actions of W*-algebras.

A bicategory is said to be strict if the natural isomorphisms $a_{A,B,C,D}$ are identities, i.e.,

$$\Phi_{A,C,D}\big(\Phi_{A,B,C} \times \mathrm{id}_{\mathscr{H}om(D,C)}\big) = \Phi_{A,B,D}\big(\mathrm{id}_{\mathscr{H}om(B,A)} \times \Phi_{B,C,D}\big)$$

and $a_{A,B,C,D}$ is the identity for each quadruplet.

A typical example of strict bicategory is provided by categories of functors: Let $F, F' : \mathscr{C} \to \mathscr{D}$ and $G, G' : \mathscr{D} \to \mathscr{E}$ be functors. Then, given natural transformations $s : F \to F'$ and $t : G \to G'$, we can associate the natural transformation $G \circ F \to G' \circ F'$ by the commutative diagram

$$\begin{array}{ccc} G(F(X)) & \xrightarrow{G(s_X)} & G(F'(X)) \\ {\scriptstyle t_{F(X)}}\big\downarrow & & \big\downarrow{\scriptstyle t_{F'(X)}} \\ G'(F(X)) & \xrightarrow[G'(s_X)]{} & G'(F'(X)) \,. \end{array}$$

Thus $F = {}_{\mathscr{D}}F_{\mathscr{C}}$, $G = {}_{\mathscr{E}}G_{\mathscr{D}}$ and ${}_{\mathscr{E}}G \otimes_{\mathscr{D}} F_{\mathscr{C}} = {}_{\mathscr{E}}(G \circ F)_{\mathscr{C}}$ with the unit objects given by identity functors. Note that the commutativity of the above diagram expresses the identity $(t \otimes_{\mathscr{D}} 1_{F'})(1_G \otimes_{\mathscr{D}} s) = (1_{G'} \otimes_{\mathscr{D}} s)(t \otimes_{\mathscr{D}} 1_F)$. The (strict) associativity for tensor products of morphisms is also immediate.

In particular, the category $\mathscr{E}nd(\mathscr{C})$ of functors from $\mathscr{C}$ into itself is a strict monoidal category.

EXAMPLE 2.1. If $\mathscr{C}$ is a one-object category, objects of $\mathscr{E}nd(\mathscr{C})$ are endomorphisms of the algebra $A = \mathrm{End}(\mathrm{pt})$ with hom-sets given by

$$\mathrm{Hom}(\rho,\sigma) = \{a \in A; a\rho(x) = \sigma(x)a,\ \forall x \in A\}.$$

The monoidal structure takes the form $\rho \otimes \sigma = \rho \circ \sigma$ for $\rho,\ \sigma \in \mathrm{End}(A)$ and $a \otimes b = a\rho(b) = \rho'(b)a$ for $a \in \mathrm{Hom}(\rho,\rho')$, $b \in \mathrm{Hom}(\sigma,\sigma')$.

When $\mathscr{C}$ is a W*-category, A is a W*-algebra and objects in $\mathscr{E}nd(\mathscr{C})$ are normal *-endomorphisms of A.

The tensor category $\mathscr{E}nd(\mathscr{C})$ is also denoted by $\mathscr{E}nd(A)$ (see [**6**] for more information on $\mathscr{E}nd(A)$).

Lemma 2.2 ([**3**, Theorem 7.13], [**2**, Lemma 2.1]). *Let $\mathscr{C}$, $\mathscr{D}$ be W*-categories and $\{U_i\}$ be a generating family in $\mathscr{C}$. Let $\mathscr{U}$ be the full subcategory of $\mathscr{C}$ consisting of objects in $\{U_i\}$.*

Then the restriction (*of functors and natural transformations*)

$$\mathscr{H}om(\mathscr{C},\mathscr{D}) \ni F \mapsto F|_{\mathscr{U}} \in \mathscr{H}om(\mathscr{U},\mathscr{D})$$

gives a fully faithful embedding of W-categories. Here $\mathscr{H}om(\mathscr{C},\mathscr{D})$ and $\mathscr{H}om(\mathscr{U},\mathscr{D})$ are W*-categories of normal *-functors and natural transformations.*

*More concretely, given a natural transformation $\{t_i : F(U_i) \to G(U_i)\}$ between normal *-functors $F|_{\mathscr{U}}$ and $G|_{\mathscr{U}}$, the natural transformation $t_X : F(X) \to G(X)$ is recovered by the formula*

$$t_X = \sum_{i,j} G(u_{i,j})t_iF(u_{i,j})^*,$$

where $\{u_{i,j} : U_i \to X\}$ is a family of partial isometries satisfying $\sum_{i,j} u_{i,j}u_{i,j}^* = 1_X$.

Proof. Let X be an object in $\mathscr{C}$. Since $\{U_i\}$ is a generating family, we can find a family $\{u_{i,j} : U_i \to X\}$ of partial isometries such that $\sum_{i,j} u_{i,j}u_{i,j}^* = 1_X$.

Given a natural transformation $t : F \to G$, we have $t_XF(u_{i,j}) = G(u_{i,j})t_i$ for any (i,j) and then

$$t_X = \sum_{i,j} t_XF\big(u_{i,j}u_{i,j}^*\big) = \sum_{i,j} G(u_{i,j})t_iF(u_{i,j})^*$$

by the normality of F. Thus the restriction is injective on natural transformations.

Conversely, given a natural transformation $\{t_i : F(U_i) \to G(U_i)\}$ between $F|_{\mathscr{U}}$ and $G|_{\mathscr{U}}$ and an object X in $\mathscr{C}$, write

$$t_X = \sum_{i,j} G(u_{i,j})t_iF(u_{i,j})^*,$$

which is a morphism in $\mathrm{Hom}(F(X),G(X))$.

If Y is another object in $\mathscr{C}$ with a family $\{v_{k,l} : U_k \to Y\}$ of partial isometries satisfying $\sum_{k,l} v_{k,l}v_{k,l}^* = 1_Y$, we associate another morphism $t_Y : F(Y) \to G(Y)$. Then, for any morphism $f : X \to Y$ in $\mathscr{C}$, we have

$$
\begin{aligned}
G(f)t_X &= \sum_{i,j} G(fu_{i,j})t_iF(u_{i,j})^* \\
&= \sum_{i,j}\sum_{k,l} G\big(v_{k,l}v_{k,l}^* fu_{i,j}\big)t_iF(u_{i,j})^* \\
&= \sum_{i,j}\sum_{k,l} G(v_{k,l})t_kF\big(v_{k,l}^* fu_{i,j}\big)F(u_{i,j})^* \\
&= t_YF(f)
\end{aligned}
$$

by the naturality of $\{t_i\}$ and the normality of F, G.

If we take $f = 1_X$ ($X = Y$ particularly), then the above formula means that the morphism t_X is well-defined. Thus, the natural transformation $\{t_X\}$ is recovered from $\{t_i\}$. □

If we restrict ourselves to W*-categories, normal *-functors and bounded natural transformations, then we obtain the strict W*-bicategory $\mathscr{F}unct$. Recall that the tensor category $\mathscr{E}nd(M)$ in Example 2.1 is a part of $\mathscr{F}unct$. In accordance with our separability notation, we denote by $\mathscr{S}\mathscr{F}unct$ the W*-bicategory of normal functors between separable W*-categories. Moreover we have the W*-bicategory $\mathscr{F}unct_I$ of normal functors between W*-categories of type I.

We shall now give a fully faithful embedding of $\mathscr{F}unct$ into the bicategory $\mathscr{B}imod$ of W*-bimodules.

To this end, we first enlarge the relevant W*-categories so that they admit generators; we shall take a generator $U_{\mathscr{A}}$ for each W*-category $\mathscr{A}$ and define the W*-algebra A by $A = \mathrm{End}(U_{\mathscr{A}})$ ($B = \mathrm{End}(U_{\mathscr{B}})$ and son on). Then, by the embedding theorem of W*-categories, we have fully faithful embedding $\Phi_{\mathscr{A}}$ of $\mathscr{A}$ into the category of right A-modules:

$$\Phi_{\mathscr{A}} : X \mapsto L^2(U_{\mathscr{A}}, X)_A.$$

To each functor $F : \mathscr{A} \to \mathscr{B}$, we associate the right B-module by

$$\Phi(F) = L^2(U_{\mathscr{B}}, F(U_{\mathscr{A}})).$$

Note here that the functor F has the unique extension to the enlargements of $\mathscr{A}$ and $\mathscr{B}$ (cf. Lemma 2.2). The Hilbert space $\Phi(F)$ admits the left action of A by

$$a\xi = F(a)\xi, \quad a \in A,\ \xi \in L^2(U_{\mathscr{B}}, F(U_{\mathscr{A}})),$$

which clearly commutes with the right action of B. Thus $\Phi(F)$ is an A-B module.

Moreover, given a natural transformation $t : F \to F'$ in $\mathscr{H}om(\mathscr{A}, \mathscr{B})$, the left multiplication of $t_{U_{\mathscr{A}}} \in A$ defines an A-B intertwiner between $\Phi(F)$ and $\Phi(F')$. In this way, we obtain a *-functor $\Phi : \mathscr{F}unct \to \mathscr{B}imod$. By the previous lemma, Φ is fully faithful.

Given a functor $F : \mathscr{A} \to \mathscr{B}$ and an object X in $\mathscr{A}$, we define a linear map $m_{X,F} : L^2(U_{\mathscr{A}}, X) \otimes_A \Phi(F) \to L^2(U_{\mathscr{B}}, F(X))$ by

$$\begin{aligned} L^2(U_{\mathscr{A}}, X) \otimes_A L^2(U_{\mathscr{B}}, F(U_{\mathscr{A}})) &\ni x\varphi^{1/2} \otimes_{\varphi^{-1/2}} f\psi^{1/2} \\ &\mapsto F(x) f\psi^{1/2} \in L^2(U_{\mathscr{B}}, F(U_{\mathscr{A}})), \end{aligned}$$

which is clearly B-linear and isometric because of

$$\big(x\varphi^{1/2} \otimes_{\varphi^{-1/2}} f\psi^{1/2} | x\varphi^{1/2} \otimes_{\varphi^{-1/2}} f\psi^{1/2}\big) = (f\psi^{1/2} | F(x^*x) f\psi^{1/2}) = \|F(x)f\psi^{1/2}\|^2,$$

where $\varphi \in A_*^+$ and $\psi \in B_*^+$.

To see the surjectivity of $m_{X,F}$, choose a family $\{x_i : U_{\mathscr{A}} \to X\}$ of partial isometries satisfying $\sum_i x_i x_i^* = 1_X$. Then, for $y : U_{\mathscr{B}} \to F(X)$, the normality of F shows that

$$y = F\bigg(\sum_i x_i x_i^*\bigg) y = \sum_i F(x_i) F(x_i^*) y = \sum_i F(x_i) f_i,$$

where $f_i = F(x_i^*)y$ is in $\mathrm{Hom}(U_{\mathscr{B}}, F(U_{\mathscr{A}}))$. Thus the image of $m_{X,F}$ is dense in $L^2(U_{\mathscr{B}}, F(X))$ and $m_{X,F}$ gives a unitary map.

Let $G : \mathscr{B} \to \mathscr{C}$ be another normal *-functor. We can then define a unitary map $m_{F,G} : \Phi(F) \otimes_B \Phi(G) \to \Phi(G \circ F)$ by

$$m_{F(U_{\mathscr{A}}),G} : L^2(U_{\mathscr{B}}, F(U_{\mathscr{A}})) \otimes_B L^2(U_{\mathscr{C}}, G(U_{\mathscr{B}})) \to L^2(U_{\mathscr{C}}, GF(U_{\mathscr{A}})),$$

which is A-C linear by the definition of $m_{X,F}$ and the left action of A. The explicit form of these multiplication maps also shows the identity

$$m_{F(X),G}(m_{X,F} \otimes 1_{\Phi(G)}) = m_{X,GF}(1_{\Phi(X)} \otimes m_{F,G}).$$

In other words, the following diagram commutes.

$$\begin{array}{ccc} \Phi(X) \otimes_A \Phi(F) \otimes_B \Phi(G) & \longrightarrow & \Phi(X) \otimes_A \Phi(GF) \\ \big\downarrow & & \big\downarrow \\ \Phi(F(X)) \otimes_B \Phi(G) & \longrightarrow & \Phi(GF(X)). \end{array}$$

Summarizing the argument so far, we obtain the following.

THEOREM 2.3. *The opposite of the bicategory $\mathscr{F}unct$ ($\mathscr{S}\mathscr{F}unct$ or $\mathscr{F}unct_I$ respectively) is monoidally equivalent to the bicategory $\mathscr{B}imod$ of W^*-bimodules ($\mathscr{S}\mathscr{B}imod$ of separable W^*-bimodules or $\mathscr{C}\mathscr{B}imod$ of W^*-bimodules with actions of commutative W^*-algebras respectively).*

Proof. We have a fully faithful monoidal embedding $\mathscr{F}unct \to \mathscr{B}imod$ by the above argument, whose image covers every bimodule up to unitary isomorphisms by Proposition 1.13. $\square$

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