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On a generalized resolvent estimate for the Stokes system with Robin boundary condition

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Abstract. We prove a generalized resolvent estimate of Stokes equations with nonhomogeneous Robin boundary condition and divergence condition in the L_q framework ($1 < q < \infty$) in a domain of $\boldsymbol{R}^n$ ($n \geqq 2$) that is a bounded domain or the exterior of a bounded domain. The Robin condition consists of two conditions: $\nu \cdot u = 0$ and $\alpha u + \beta(T(u,p)\nu - \langle T(u,p)\nu, \nu\rangle\nu) = h$ on the boundary of the domain with $\alpha, \beta \geqq 0$ and $\alpha + \beta = 1$, where u denotes a velocity vector, p a pressure, $T(u,p)$ the stress tensor for the Stokes flow, and ν the unit outer normal to the boundary of the domain. It presents the slip condition when $\beta = 1$ and the non-slip one when $\alpha = 1$, respectively.

1. Introduction.

Let Ω be a domain in $\boldsymbol{R}^n$ with boundary Γ that is a compact hypersurface. Given velocity vector $u = {}^t(u_1, \ldots, u_n)^*$ and pressure p, the stress tensor $T(u,p)$ of the Stokes flow is defined by the formula: $T(u,p) = D(u) - pI$, where $D(u)$ and I are $n \times n$ matrices whose (j,k) components $D(u)_{jk}$ and I_{jk} are given by the formulas:

$$D(u)_{jk} = \frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}, \quad I_{jk} = \delta_{jk} = 1\,(j = k) \text{ and } I_{jk} = \delta_{jk} = 0\,(j \neq k).$$

In this paper, we are interested in the L_q ($1 < q < \infty$) estimate of solutions u and p to the generalized Stokes resolvent problem in Ω with Robin boundary condition:

$$\begin{aligned} &\lambda u - \operatorname{Div} T(u,p) = f, \quad \operatorname{div} u = g && \text{in } \Omega \\ &\nu \cdot u = 0, \quad B_{\alpha,\beta}(u) = \alpha u + \beta(T(u,p)\nu - \langle T(u,p)\nu, \nu\rangle\nu) = h && \text{on } \Gamma \end{aligned} \tag{1.1}$$

where α and β are two constants such as α, $\beta \geqq 0$ and $\alpha + \beta = 1$; $\langle \cdot, \cdot \rangle$ denotes the inner product in $\boldsymbol{R}^n$, ν the unit outer normal to Γ, $f = {}^t(f_1, \ldots, f_n)$ the prescribed force for the motion, $h = {}^t(h_1, \ldots, h_n)$ the prescribed force on the boundary, and g the given divergence of the problem. Noting that $\langle \nu, \nu \rangle = 1$, we have

$$B_{\alpha,\beta}(u) = \alpha u + \beta(D(u)\nu - \langle D(u)\nu, \nu\rangle\nu) \tag{1.2}$$

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* ${}^tM$ denotes the transpose of M.

and therefore the Robin condition does not contain the pressure p, which is an essential difference from the pure Neumann condition: $T(u,p)\nu = h$ on Γ that was treated by Grubb and Solonnikov [**9**], Grubb [**10**], Shibata and Shimizu [**17**] and Steiger [**20**]. When $\beta = 0$, the boundary condition is the usual non-slip one, and this case has been studied by Farwig and Sohr [**5**]. Therefore, we shall study the case where $\beta > 0$ only in this paper.

The problem (1.1) with $\beta > 0$ was first studied by Giga [**8**] when Ω is bounded and $g = 0$. He actually considered more general boundary condition and gave some sufficent condition to obtain a resolvent estimate. Later on, Grubb and Solonnikov [**9**] and Grubb [**10**] proved the well-posedness of the non-stationary Stokes equation with general first order boundary condition. But, the arguments due to Giga [**8**] and also to Grubb and Solonnikov ([**9**], [**10**]) relied heavily on the calculus of pseudo-differential operators. Such arguments can be understood only by those who are quite familiar with the pseudo-differential operator techniques. However, when the boundary condition is the non-slip one ($u = 0$ on Γ), Farwig and Sohr [**5**] proved the resolvent estimate by using rather elementary method based on Fourier analysis and functional analysis. Our motivation of this paper is to study the generalized resolvent problem for the Stokes equation with Robin boundary condition by extending the method due to Farwig and Sohr, so that our argument in this paper is completely different from the argument due to [**8**], [**9**] and [**10**] but rather closed to that due to [**5**]. When Ω is a half space and $g = h = 0$, Saal [**15**] studied (1.1) and he proved not only the resolvent estimate but also H^∞ calculus. Miyakawa [**13**] and Akiyama, Kasai, Shibata and Tsutsumi [**1**] studied the Stokes resolvent problem with some first order boundary condition like $\nu \cdot u = 0$ and $(\mathrm{rot}\, u) \times \nu = 0$ on Γ which arises from the mathematical theory of the magnetohydrodynamics.

Concerning the non-stationary Navier-Stokes equation with Robin boundary condition, Itoh, Tanaka and Tani [**12**] proved a locally in time unique existence theorem in the Hölder space framework. When Ω is a bounded domain, Steiger [**20**] studied it in the L_q ($1 < q < \infty$) framework and he proved a locally in time unique existence theorem of solutions with very irregular initial data. He used Giga's result [**8**] to show the generation of the Stokes semigroup, so that he treated only the case where Ω is bounded. By using our result obtained in this paper, we can show the generation of Stokes semigroup even when Ω is an exterior domain, and therefore Steiger's result seems to hold when Ω is an exterior domain.

In order to state our main results, at this point we outline our notation. Given vector or matrix M, ${}^tM$ denotes the transpose of M. Given Banach space X with norm $\|\cdot\|_X$, we set

$$X^n = \{v = {}^t(v_1,\ldots,v_n) \mid v_j \in X\}, \quad \|v\|_X = \sum_{j=1}^n \|v_j\|_X.$$

To denote the inner-product in $\boldsymbol{R}^n$, we use the two symbols: $x \cdot y = \langle x, y\rangle = \sum_{j=1}^n x_j y_j$ for every $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n) \in \boldsymbol{R}^n$. $F = (F_{jk})$ means the $n\times n$ matrix whose (j,k) component is F_{jk}. For the differentiation of a scalar function p, an n-vector of functions $u = {}^t(u_1,\ldots,u_n)$ and an $n\times n$ matrix of functions $F = (F_{jk})$, we use the following symbols:

$$\partial_j p = \partial p/\partial x_j, \quad \nabla p = {}^t(\partial_1 p, \ldots, \partial_n p), \quad \nabla u = (\partial_j u_k),$$
$$\nabla^2 u = (\partial_j \partial_k u_\ell), \quad \operatorname{Div} F = {}^t\Bigl(\sum_{j=1}^n \partial_j F_{1j}, \ldots, \sum_{j=1}^n \partial_j F_{nj}\Bigr).$$

For any domain D in $\boldsymbol{R}^n$ with boundary ∂D, the inner products $(\cdot,\cdot)_D$ and $(\cdot,\cdot)_{\partial D}$ are defined by the formulas:

$$(u, v)_D = \int_D u(x) \cdot \overline{v(x)}\, dx, \quad (u, v)_{\partial D} = \int_{\partial D} u(x) \cdot \overline{v(x)}\, d\sigma$$

where $d\sigma$ denotes the surface element of ∂D and $\overline{v}$ the complex conjugate of v. $L_q(D)$ and $W_q^m(D)$ $(1 \leqq q \leqq \infty)$ denote the usual Lebesgue and Sobolev spaces of functions defined on D with norms $\|\cdot\|_{L_q(D)}$ and $\|\cdot\|_{W_q^m(D)}$, respectively. We denote the closure of $C_0^\infty(D)$ in $W_q^m(D)$ by $W_{q,0}^m(D)$ and the dual space of $W_{q',0}^1(D)$ by $W_q^{-1}(D)$ with $q' = q/(q-1)$. Set

$$B_L = \{x \in \boldsymbol{R}^n \mid |x| < L\}, \quad S_L = \{x \in \boldsymbol{R}^n \mid |x| = L\},$$
$$D_{L,L+1} = \{x \in \boldsymbol{R}^n \mid L \leqq |x| \leqq L+1\}.$$

Let R be a fixed positive number such as $B_{R-5} \supset \Gamma$. Set $\Omega_L = \Omega \cap B_L$ for $L > R-5$. If Ω is bounded, then $\Omega_L = \Omega$ when $L > R-5$.

As a space of pressure terms, we introduce the homogeneous space $\hat{W}_q^1(\Omega)$ that is defined by the formula:

$$\hat{W}_q^1(\Omega) = \bigl\{p \in L_{q,\mathrm{loc}}(\overline{\Omega}) \mid \nabla p \in L_q(\Omega)\bigr\}$$

where we have to identify two elements differing by a constant. When Ω is bounded, $L_{q,\mathrm{loc}}(\overline{\Omega})$ may be replaced by $L_q(\Omega)$. Moreover, we may fix a representative $u \in \hat{W}_q^1(\Omega)$ by $\int_\Omega u\, dx = 0$. Therefore, in view of Poincaré's inequality we may identify

$$\hat{W}_q^1(\Omega) = \Bigl\{p \in W_q^1(\Omega) \mid \int_\Omega p(x)\, dx = 0\Bigr\}$$

provided Ω is bounded. Let

$$\hat{W}_q^{-1}(\Omega) = \bigl[\hat{W}_{q'}^1(\Omega)\bigr]^*$$

be the dual space of $\hat{W}_{q'}^1(\Omega)$ ($q' = q/(q-1)$ and $1 < q < \infty$) endowed with the norm:

$$\|g\|_{\hat{W}_q^{-1}(\Omega)} = \sup_{0 \neq v \in \hat{W}_{q'}^1(\Omega)} |[g, v]| / \|\nabla v\|_{L_{q'}(\Omega)}$$

where $[\cdot,\cdot]$ denotes the duality of $\hat W_q^{-1}(\Omega)$ and $\hat W_{q'}^1(\Omega)$. When D is one of $\boldsymbol{R}^n$, $\boldsymbol{R}^n_+$ and a bent half space that will be defined in Section 4 below, $\hat W_q^1(D)$ and $\hat W_q^{-1}(D)$ are defined in the same manner as above. Here and hereafter, $\boldsymbol{R}^n_+$ denotes the half space that is defined by

$$\boldsymbol{R}^n_+ = \big\{x = (x', x_n) \in \boldsymbol{R}^n \mid x' = (x_1, \dots, x_{n-1}) \in \boldsymbol{R}^{n-1},\ x_n > 0\big\}.$$

Set

$$\dot W_q^1(\Omega_R) = \begin{cases} W_q^1(\Omega) & \text{if } \Omega \text{ is a bounded domain} \\ \big\{p \in W_q^1(\Omega_R) \mid p|_{S_R} = 0\big\} & \text{if } \Omega \text{ is an exterior domain} \end{cases}$$

and let $\dot W_q^{-1}(\Omega_R)$ be the dual space of $\dot W_{q'}^1(\Omega_R)$ endowed with the norm:

$$\|g\|_{\dot W_q^{-1}(\Omega_R)} = \sup_{0 \neq v \in \dot W_{q'}^1(\Omega_R)} |[g, v]| / \|v\|_{\dot W_{q'}^1(\Omega_R)}.$$

As a space of data for the divergence equation: $\operatorname{div} u = g$ in (1.1), we introduce the space $W_{q,\mathrm{div}}(\Omega)$ that is defined by

$$W_{q,\mathrm{div}}(\Omega) = \begin{cases} \big\{g \in W_q^1(\Omega) \mid \int_\Omega g(x)\,dx = 0 \mid\big\} & \text{if } \Omega \text{ is a bounded domain} \\ \big\{g = \operatorname{div} \tilde g \mid g \in W_q^1(\Omega),\ \tilde g \in L_q(\Omega)^n,\ \nu \cdot \tilde g|_\Gamma = 0\big\} & \\ & \text{if } \Omega \text{ is an exterior domain.} \end{cases}$$

We have

$$\|g\|_{\hat W_q^{-1}(\Omega)} \leqq \begin{cases} C\|g\|_{L_q(\Omega)} & \text{if } \Omega \text{ is a bounded domain} \\ \|\tilde g\|_{L_q(\Omega)} & \text{if } \Omega \text{ is an exterior domain} \end{cases} \tag{1.3}$$

for every $g \in W_{q,\mathrm{div}}(\Omega)$ where C is some positive constant arising from Poincaré's inequality. A useful characterization of the space of data for the divergence equation was given by Farwig-Sohr [**5**]. As a space of boundary forces, we introduce the space $W^j_{q,\partial}(\Omega)$ $(j = 1, 2)$ that is defined by

$$W^j_{q,\partial}(\Omega) = \big\{h \in W_q^j(\Omega)^n \mid \nu \cdot h|_\Gamma = 0\big\}.$$

To state the Helmholtz decomposition, we introduce the following spaces:

$$\begin{aligned} J_q(\Omega) &= \text{the closure of the space } C^\infty_{0,\sigma}(\Omega) \text{ in } L_q(\Omega)^n \\ C^\infty_{0,\sigma}(\Omega) &= \big\{u \in C_0^\infty(\Omega)^n \mid \operatorname{div} u = 0 \text{ in } \Omega\big\} \\ G_q(\Omega) &= \big\{\nabla p \mid p \in \hat W_q^1(\Omega)\big\}. \end{aligned}$$

Then, it is well-known that there holds the Helmholtz decomposition: $L_q(\Omega)^n = J_q(\Omega) \oplus G_q(\Omega)$, $\oplus$ being the direct sum (cf. [**6**], [**5**], [**7**], [**14**] and [**18**] and references therein). Namely, given any $u \in L_q(\Omega)^n$, there exist $v \in J_q(\Omega)$ and $p \in \hat{W}^1_q(\Omega)$ uniquely such that $u = v + \nabla p$. For the later use, we define the solenoidal projection $\mathscr{S}_q$ and the gradient projection $\mathscr{G}_q$ by the relations: $\mathscr{S}_q u = v$ and $\mathscr{G}_q u = \nabla p$, respectively. For the resolvent parameter λ, we introduce the set Σ_ϵ defined by the formula:

$$\Sigma_\epsilon = \{\lambda \in \boldsymbol{C} \setminus \{0\} \mid |\arg \lambda| \leqq \pi - \epsilon\}, \quad 0 < \epsilon < \pi.$$

For the notational simplicity, we set

$$\begin{aligned} \mathscr{I}_\lambda(u, p, D) &= |\lambda| \|u\|_{L_q(D)} + |\lambda|^{1/2} \|\nabla u\|_{L_q(D)} + \|\nabla^2 u\|_{L_q(D)} + \|\nabla p\|_{L_q(D)} \\ \mathscr{D}_\lambda(f, g, h, D) &= \|f\|_{L_q(D)} + |\lambda| \|g\|_{\hat{W}^{-1}_q(D)} + |\lambda|^{1/2} \|(g, h)\|_{L_q(D)} + \|\nabla(g, h)\|_{L_q(D)} \end{aligned}$$

which are used to state our generalized resolvent estimate. By C we denote a generic constant and $C_{a,b,\ldots}$ denotes the constant depending on the quantities $a, b, \ldots$. The constants C and $C_{a,b,\ldots}$ may change from line to line.

Throughout the paper we assume that the following two conditions hold:

Assumption 1.1. Assume that $\beta > 0$ and that Γ is a $C^{2,1}$ compact hypersurface.

Since $\lambda = 0$ is resolvent for the Stokes operator in any bounded domain with nonslip boundary condition (cf. [**5**]), in our case it is also natural to consider the solvability of the problem (1.1) when Ω is bounded and $\lambda = 0$. Concerning this topics, we need some geometrical assumption on Ω when $\alpha = 0$.

Definition 1.2. By a hyperline in $\boldsymbol{R}^n$ we mean that an affine subspace of codimension two.

> Ω is said to be *rotationally symmetric* with respect to a hyperline L if for any $a \in L$ the two-dimensional section $L^\perp_a \cap \Omega$ of Ω by $L^\perp_a$ is, if nonempty, symmetric with respect to a, where $L^\perp_a$ denotes the plane through a and orthogonal to L. Moreover, Ω is said to be *rotationally symmetric* if there exists a hyperline L such that Ω is rotationally symmetric with respect to L.

Two theorems which follow are our main results.

Theorem 1.3. *Let* $1 < q < \infty$.

(1) *Let* $0 < \epsilon < \pi/2$ *and* $\delta > 0$. *Then, for every* $\lambda \in \boldsymbol{C} \setminus (-\infty, 0]$, $f \in L_q(\Omega)^n$, $g \in W_{q,\mathrm{div}}(\Omega)$ *and* $h \in W^1_{q,\partial}(\Omega)$ *the problem* (1.1) *admits a unique solution* $(u, p) \in W^2_q(\Omega)^n \times \hat{W}^1_q(\Omega)$ *which satisfies the estimate*:

$$\mathscr{I}_\lambda(u, p, \Omega) \leqq C_{\epsilon,q,\delta} \mathscr{D}_\lambda(f, g, h, \Omega) \tag{1.4}$$

provided $\lambda \in \Sigma_\epsilon$ *and* $|\lambda| \geqq \delta$.

(2) *Let Ω be a bounded domain. Assume that Ω is not rotationally symmetric. Then, there exists a $\lambda_0 > 0$ depending only on Ω such that for every $\lambda \in \{\lambda \in \boldsymbol{C} \mid |\lambda| \leqq \lambda_0\}$, $f \in L_q(\Omega)^n$, $g \in W_{q,\mathrm{div}}(\Omega)$ and $h \in W^1_{q,\partial}(\Omega)$ the problem* (1.1) *admits a unique solution $(u,p) \in W^2_q(\Omega)^n \times \hat{W}^1_q(\Omega)$ which satisfies the estimate*:

$$\|u\|_{W^2_q(\Omega)} + \|p\|_{W^1_q(\Omega)} \leqq C_q\Big(\|f\|_{L_q(\Omega)} + \|(g,h)\|_{W^1_q(\Omega)}\Big) \tag{1.5}$$

provided $\lambda \in \boldsymbol{C}$ with $|\lambda| \leqq \lambda_0$.

Theorem 1.4. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Let us consider the equation* (1.1) *with $f \in J_q(\Omega)$, $g = 0$ and $h = 0$. If $(u,p) \in W^2_q(\Omega)^n \times \hat{W}^1_q(\Omega)$ is a solution to* (1.1), *then we have*

$$|\lambda|\|u\|_{W^{-1}_q(\Omega_R)} + \|p\|_{L_q(\Omega_R)} \leqq C_{\epsilon,q}|\lambda|^{-(1/2)(1-(1/q))}\|f\|_{L_q(\Omega)} \tag{1.6}$$

provided that $\lambda \in \Sigma_\epsilon$ and $|\lambda| \geqq 1$.

Remark 1.5.

(1) We call (1.1) the generalized resolvent problem when all of f, g and h are non-trivial, while we call (1.1) the resolvent problem when only f is non-trivial.

(2) When the boundary condition is the non-slip one ($u = 0$ on Γ), the theorem corresponding to Theorem 1.3 was proved by Farwig and Sohr [**5**] under only the assumption that $\Gamma \in C^{1,1}$. But, in our case we can not avoid the condition: $\Gamma \in C^{2,1}$ as far as we use the transformation (4.5) in Section 4 below that keeps the divergence condition and what the normal component of the velocity field vanishes on the boundary unchanged at the same time.

(3) When Ω is a bounded domain in $\boldsymbol{R}^3$, the assertion (2) of Theorem 1.3 was proved by Solonnikov-Ščadilov [**19**]. The point is Korn's first inequality. In our proof, we use some generalization of Korn's first inequality due to Ito [**11**].

(4) The estimate for the pressure term given in Theorem 1.4 also holds even if the boundary condition is the non-slip one, which Farwig and Sohr [**5**] did not mention. The estimate in Theorem 1.4 will play an important role to show uniform L_p-L_q decay estimates when Ω is an exterior domain, which will be shown elsewhere.

The organization of the paper is as follows: Using the Fourier multiplier theorem, we prove the estimates of $\mathscr{I}_\lambda(u,p,\boldsymbol{R}^n)$ and $\mathscr{I}_\lambda(u,p,\boldsymbol{R}^n_+)$ in Section 2 and Section 3, respectively. The half space problem has been studied by Saal [**15**] when $g = 0$ and $h = 0$. However, we have to study essentially the case where all of f, g and h are non-trivial, so that our solution formula is different from Saal's one. Therefore, we give a proof of estimates of solutions to the generalized resolvent problem in the half space. In Section 4, we solve the generalized resolvent problem in a bent half space by transforming the problem to the half space problem. Our assumption: $\Gamma \in C^{2,1}$ arises only from this transformation. In Section 5, first of all using the usual localization procedure and the results obtained in Section 2 and Section 4 we show the *a priori* estimate:

$$\mathscr{I}_\lambda(u,p,\Omega) \leqq C\Big\{\mathscr{D}_\lambda(f,g,h,\Omega) + \|p\|_{L_q(\Omega_R)} + |\lambda|\|u\|_{\dot W_q^{-1}(\Omega_R)} + \|\nabla u\|_{L_q(\Omega_R)} + |\lambda|^{1/2}\|u\|_{L_q(\Omega_R)}\Big\}. \tag{1.7}$$

In order to eliminate two perturbation terms: $\|p\|_{L_q(\Omega_R)}$ and $|\lambda|\|u\|_{\dot W_q^{-1}(\Omega_R)}$ in (1.7) for large $|\lambda|$, we use an estimate of solutions to the Neumann problem for the Laplace operator in Ω_R, while Farwig and Sohr used a compactness argument based on the uniqueness of the Helmholtz decomposition. In fact, we show that

$$\begin{aligned}\|p\|_{L_q(\Omega_R)} &\leqq C\Big\{\|\nabla u\|_{L_q(\Gamma)} + \|\nabla u\|_{L_q(\Omega)} + \|G_q f\|_{L_q(\Omega)} + |\lambda|\|g\|_{\hat W_q^{-1}(\Omega)}\Big\}\\ |\lambda|\|u\|_{\dot W_q^{-1}(\Omega_R)} &\leqq C\big\{\|(D(u),p)\|_{L_q(\Gamma)} + \|(D(u),p)\|_{L_q(\Omega_R)} + \|f\|_{L_q(\Omega)}\big\}.\end{aligned} \tag{1.8}$$

This is a key observation to eliminate the perturbation terms in the right hand side of (1.7), which seems to be new. Combining (1.7) and (1.8), we can eliminate the perturbation terms in (1.7) for large $|\lambda|$. As a by-product, we also have (1.6). When λ varies in any compact set of Σ_ϵ, we use a compactness argument based on the uniqueness of solutions to (1.1) in order to eliminate the perturbation terms in the right hand side of (1.7). In this way, we prove *a priori* estimates stated in (1.4), (1.5) and (1.6). In Section 6, we prove the unique existence of solutions to (1.1) under the assumption that Ω is bounded. We start with the unique existence of weak solutions to (1.1) in the L_2 framework. And then, by using the localization method and results obtained in Section 2 and Section 4, we show that such weak solutions actually belong to $W_q^2(\Omega)^n \times \hat W_q^1(\Omega)$. Finally, we show that such weak solutions satisfy the boundary condition. In the argument in Section 6 we essentially use the boundedness assumption of the domain Ω to show the existence of the pressure term by using the idea due to Solonnikov and Ščadilov [**19**]. As Farwig and Sohr did ([**5**, p. 630]), by the Riesz representation theorem and the de Rham theorem we can show the existence of $u \in W_2^1(\Omega)^n$ and $p \in \hat W_2^1(\Omega)$ such that $\lambda u - \operatorname{Div} T(u,p) = f$ and $\operatorname{div} u = 0$ in the distribution sense of Ω even when Ω is an exterior domain. This observation is useful to treat the case where $u = 0$ on Γ. But, the boundary condition is hidden in the weak formula of the equation in our case, because our boundary condition is of first order. Therefore, from what u and p satisfy the equation in the distribution sense we do not get any information whether u satisfies the boundary condition unlike the non-slip condition case. This is the reason why our discussion about the unique existence of solutions is different from that due to Farwig and Sohr. In Section 7, we prove the unique existence of solutions under the assumption that Ω is an exterior domain. We can not use the argument in Section 6, because the domain is unbounded. Therefore, in order to show the existence of solutions we construct a parametrix by combining the solutions of the whole space problem and those of the problem in Ω_R by a cut-off technique.

2. A generalized Stokes resolvent problem in the whole space.

In this section, we consider the generalized Stokes resolvent problem in $\boldsymbol{R}^n$:

$$\lambda u - \operatorname{Div} T(u,p) = f, \quad \operatorname{div} u = g \quad \text{in } \boldsymbol{R}^n. \tag{2.1}$$

By the Fourier transform and its inversion formula, the solutions u and p of (2.1) are given by the formulas:

$$u = \mathscr{F}_\xi^{-1}\left[\frac{\hat{F}(\xi)}{\lambda + |\xi|^2}\right], \quad F = f - \nabla p + \nabla g$$
$$p = -\mathscr{F}_\xi^{-1}\left[\frac{i\xi \cdot \hat{f}(\xi)}{|\xi|^2}\right] - \lambda \mathscr{F}_\xi^{-1}\left[\frac{\hat{g}(\xi)}{|\xi|^2}\right] + 2g \tag{2.2}$$

where $\hat{F}(\xi)$, $\hat{f}(\xi)$ and $\hat{g}(\xi)$ denote the Fourier transforms of F, f and g, respectively; $\mathscr{F}_\xi^{-1}$ denotes the Fourier inverse transform with respect to ξ variables.

The following theorem is a main result in this section.

Theorem 2.1. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Then, for every $\lambda \in \Sigma_\epsilon$, $f \in L_q(\boldsymbol{R}^n)^n$ and $g \in W_q^1(\boldsymbol{R}^n) \cap \hat{W}_q^{-1}(\boldsymbol{R}^n)$, the problem* (2.1) *admits a unique solution $(u, p) \in W_q^2(\boldsymbol{R}^n)^n \times \hat{W}_q^1(\boldsymbol{R}^n)$ which satisfies the estimate*:

$$\mathscr{I}_\lambda(u, p, \boldsymbol{R}^n) \leqq C\Big\{\|f\|_{L_q(\boldsymbol{R}^n)} + |\lambda|\|g\|_{\hat{W}_q^{-1}(\boldsymbol{R}^n)} + \|\nabla g\|_{L_q(\boldsymbol{R}^n)}\Big\}$$

for some constant C depending on ϵ, q and n only.

Proof. Noting that $C_0^\infty(\boldsymbol{R}^n)$ is dense in $\hat{W}_q^1(\boldsymbol{R}^n)$ (cf. Lemma 5.1 in Farwig and Sohr [**5**]), we have

$$(\lambda u - \operatorname{Div} T(u, p), v)_{\boldsymbol{R}^n} = (u, \bar{\lambda} v - \operatorname{Div} T(v, \theta))_{\boldsymbol{R}^n} \tag{2.3}$$

for every $\lambda \in \boldsymbol{C}$, $u \in W_q^2(\boldsymbol{R}^n)^n$ with $\operatorname{div} u = 0$, $v \in W_{q'}^2(\boldsymbol{R}^n)^n$ with $\operatorname{div} v = 0$, $p \in \hat{W}_q^1(\boldsymbol{R}^n)$ and $\theta \in \hat{W}_{q'}^1(\boldsymbol{R}^n)$, where $q' = q/(q-1)$. Therefore, the uniqueness follows from the existence of solutions to the dual problem. Since

$$|\lambda + |\xi|^2| \geqq (\sin(\epsilon/2))(|\lambda| + |\xi|^2) \tag{2.4}$$

for every $\lambda \in \Sigma_\epsilon$ and $\xi \in \boldsymbol{R}^n$, applying the Mikhlin Fourier multiplier theorem (cf. Triebel [**21**]) to the solution formula of u in (2.2), we have

$$\begin{aligned}|\lambda|\|u\|_{L_q(\boldsymbol{R}^n)} + |\lambda|^{1/2}\|\nabla u\|_{L_q(\boldsymbol{R}^n)} + \|\nabla^2 u\|_{L_q(\boldsymbol{R}^n)} &\leqq C_{\epsilon,q,n}\|F\|_{L_q(\boldsymbol{R}^n)} \\ &\leqq C_{\epsilon,q,n}\|(f, \nabla(p, g))\|_{L_q(\boldsymbol{R}^n)}\end{aligned}$$

for every $\lambda \in \Sigma_\epsilon$.

To estimate ∇p, we write

$$\nabla p = \mathscr{F}_\xi^{-1}\left[\frac{\xi(\xi \cdot \hat{f}(\xi))}{|\xi|^2}\right] - i\lambda \mathscr{F}_\xi^{-1}\left[\frac{\xi \hat{g}(\xi)}{|\xi|^2}\right] + 2\nabla g. \tag{2.5}$$

The L_q boundedness of the first term of the right side of (2.5) follows from the Mikhlin Fourier multiplier theorem immediately. To estimate the second term of the right side of (2.5), we take a test function $\varphi \in C_0^\infty(\boldsymbol{R}^n)$ and observe the following formulas:

$$\begin{aligned} \big|\big(\mathscr{F}_\xi^{-1}[\xi\hat{g}(\xi)|\xi|^{-2}],\varphi\big)_{\boldsymbol{R}^n}\big| &= \big|\big(g,\mathscr{F}_\xi^{-1}[\xi\hat{\varphi}(\xi)|\xi|^{-2}]\big)_{\boldsymbol{R}^n}\big| \\ &\leqq \|g\|_{\hat{W}_q^{-1}(\boldsymbol{R}^n)}\big\|\nabla\mathscr{F}_\xi^{-1}[\xi\hat{\varphi}(\xi)|\xi|^{-2}]\big\|_{L_{q'}(\boldsymbol{R}^n)}. \end{aligned} \tag{2.6}$$

Since

$$\big\|\nabla\mathscr{F}_\xi^{-1}[\xi\hat{\varphi}(\xi)|\xi|^{-2}]\big\|_{L_{q'}(\boldsymbol{R}^n)} \leqq C_{n,q'}\|\varphi\|_{L_{q'}(\boldsymbol{R}^n)} \tag{2.7}$$

as follows from the Mikhlin Fourier multiplier theorem, combining (2.6) and (2.7) we have

$$\big\|\mathscr{F}_\xi^{-1}[\xi\hat{g}(\xi)|\xi|^{-2}]\big\|_{L_q(\boldsymbol{R}^n)} \leqq C_{n,q}\|g\|_{\hat{W}_q^{-1}(\boldsymbol{R}^n)}.$$

Therefore, we have

$$\|\nabla p\|_{L_q(\boldsymbol{R}^n)} \leqq C_{\epsilon,q,n}\Big\{\|f\|_{L_q(\boldsymbol{R}^n)} + |\lambda|\|g\|_{\hat{W}_q^{-1}(\boldsymbol{R}^n)} + \|\nabla g\|_{L_q(\boldsymbol{R}^n)}\Big\}$$

which completes the proof of the theorem. $\square$

3. A generalized resolvent problem for the Stokes equation in the half space.

In this section, we consider the generalized resolvent problem in the half space $\boldsymbol{R}^n_+$:

$$\begin{aligned} \lambda u - \operatorname{Div} T(u,p) = f, \qquad \operatorname{div} u = g \quad &\text{in } \boldsymbol{R}^n_+ \\ \nu\cdot u = 0, \quad B_{\alpha,\beta}(u) = h \quad &\text{on } \partial\boldsymbol{R}^n_+ \end{aligned} \tag{3.1}$$

where $\partial\boldsymbol{R}^n_+ = \{(x',0) \mid x' \in \boldsymbol{R}^{n-1}\}$ and $B_{\alpha,\beta}(u) = \alpha u + \beta(D(u)\nu - \langle D(u)\nu,\nu\rangle\nu)$ with $\nu = {}^t(0,\ldots,0,-1)$. Since

$$D(u)\nu - \langle D(u)\nu,\nu\rangle\nu = -\begin{bmatrix} \partial_1 u_n + \partial_n u_1 \\ \vdots \\ \partial_{n-1}u_n + \partial_n u_{n-1} \\ 0 \end{bmatrix}, \quad (\partial_j = \partial/\partial x_j)$$

in the half space case, the boundary condition in (3.1) is written as follows:

$$u_n = 0,\ \alpha u_j - \beta\partial_n u_j = h_j\ (j = 1,\ldots,n-1) \text{ on } \partial\boldsymbol{R}^n_+. \tag{3.2}$$

As a compatibility condition, it is necessary to assume that $\nu \cdot h = -h_n = 0$ on $\partial \boldsymbol{R}^n_+$. Set $W^1_{q,\partial}(\boldsymbol{R}^n_+) = \{h \in W^1_q(\boldsymbol{R}^n_+)^n \mid \nu \cdot h = 0 \text{ on } \partial \boldsymbol{R}^n_+\}$. The following theorem is a main result in this section.

Theorem 3.1. *Let $1 < q < \infty$, $\delta > 0$ and $0 < \epsilon < \pi/2$. Let $\lambda \in \boldsymbol{C} \setminus (-\infty, 0]$, $f \in L_q(\boldsymbol{R}^n_+)^n$, $g \in W^1_q(\boldsymbol{R}^n_+) \cap \hat{W}^{-1}_q(\boldsymbol{R}^n_+)$ and $h \in W^1_{q,\partial}(\boldsymbol{R}^n_+)$ and assume that* $\operatorname{supp} g$ *is compact. Then, the problem* (3.1) *admits a unique solution $(u, p) \in W^2_q(\boldsymbol{R}^n_+)^n \times \hat{W}^1_q(\boldsymbol{R}^n_+)$ which satisfies the estimates*:

$$\mathscr{I}_\lambda(u, p, \boldsymbol{R}^n_+) \leqq C_{\epsilon,\delta}\Big\{\|f\|_{L_q(\boldsymbol{R}^n_+)} + \|\nabla(g, h)\|_{L_q(\boldsymbol{R}^n_+)} + |\lambda|^{1/2}\|(g, h)\|_{L_q(\boldsymbol{R}^n_+)} + |\lambda|\|g\|_{\hat{W}^{-1}_q(\boldsymbol{R}^n_+)}\Big\}$$

provided that $\lambda \in \Sigma_\epsilon$ and $|\lambda| \geqq \delta$, where $C_{\epsilon,\delta}$ is a positive constant depending on ϵ, δ, n and q.

In what follows, we shall show Theorem 3.1. We start with the following lemma, by which (3.1) will be reduced to the divergence free case.

Lemma 3.2. *Let $1 < q < \infty$ and $g \in W^1_q(\boldsymbol{R}^n_+) \cap \hat{W}^{-1}_q(\boldsymbol{R}^n_+)$. Assume that* $\operatorname{supp} g$ *is compact. Then, there exists a $v \in W^2_q(\boldsymbol{R}^n_+)^n$ such that* $\operatorname{div} v = g$ *in $\boldsymbol{R}^n_+$ and $v_n = 0$ on $\partial \boldsymbol{R}^n_+$.*

Moreover, v satisfies the following estimates:

$$\|v\|_{L_q(\boldsymbol{R}^n_+)} \leqq C_q\|g\|_{\hat{W}^{-1}_q(\boldsymbol{R}^n_+)}, \quad \|\nabla^{j+1} v\|_{L_q(\boldsymbol{R}^n_+)} \leqq C_q\|\nabla^j g\|_{L_q(\boldsymbol{R}^n_+)} \quad (j = 0, 1).$$

Proof. Let $E(x)$ be a fundamental solution of the Laplace operator Δ given by the formula: $E(x) = e(|x|)$, where $e(r)$ is a function such that $e(r) = c_n r^{-(n-2)}$ for $n \geqq 3$ and $c_2 \log r$ for $n = 2$ with some constant c_n depending on n. Since $\operatorname{supp} g$ is compact, setting

$$V(x) = \int_{\boldsymbol{R}^n_+} (E(x - y) + E(x - y*))g(y)dy = (E * g^e)(x)$$

where $y = (y', y_n)$ and $y^* = (y', -y_n)$; and g^e is the even extension of g to the whole space defined by the formula: $g^e(x) = g(x)$ for $x_n > 0$ and $g^e(x) = g(x', -x_n)$ for $x_n < 0$. We see easily that $V \in W^1_{q,\mathrm{loc}}(\overline{\boldsymbol{R}^n_+})$, $\|\nabla^2 V\|_{L_q(\boldsymbol{R}^n_+)} \leqq C\|g\|_{L_q(\boldsymbol{R}^n_+)}$, $\|\nabla^3 V\|_{L_q(\boldsymbol{R}^n_+)} \leqq C\|\nabla g\|_{L_q(\boldsymbol{R}^n_+)}$ and $\partial_n V = 0$ on $\partial \boldsymbol{R}^n_+$. To estimate $\|\nabla V\|_{L_q(\boldsymbol{R}^n_+)}$, we take $\varphi \in C^\infty_0(\boldsymbol{R}^n_+)$ arbitrarily and observe that $(\nabla V, \varphi)_{\boldsymbol{R}^n_+} = (g, (\nabla E) * \varphi^e)_{\boldsymbol{R}^n_+}$, where φ^e is the even extension of φ to the whole space. Therefore, we have

$$\big|(\nabla V, \varphi)_{\boldsymbol{R}^n_+}\big| \leqq \|g\|_{\hat{W}^{-1}_q(\boldsymbol{R}^n_+)}\|(\nabla^2 E) * \varphi^e\|_{L_{q'}(\boldsymbol{R}^n_+)} \leqq C\|g\|_{\hat{W}^{-1}_q(\boldsymbol{R}^n_+)}\|\varphi\|_{L_{q'}(\boldsymbol{R}^n_+)}$$

which implies that $\|\nabla V\|_{L_q(\boldsymbol{R}^n_+)} \leqq C\|g\|_{\hat{W}^{-1}_q(\boldsymbol{R}^n_+)}$. If we set $v = \nabla V = (\partial_1 V, \dots, \partial_n V)$, then v satisfies the required properties, which completes the proof of the lemma. $\square$

We set $u = v + w$ in (3.1) and (3.2), and then w and p satisfy the following equation:

$$\begin{aligned} &\lambda w - \operatorname{Div} T(w,p) = f - \lambda v + \operatorname{Div} D(v), \quad \operatorname{div} w = 0 && \text{in } \boldsymbol{R}^n_+ \\ &w_n = 0,\ \alpha w_j - \beta \partial_n w_j = h_j - (\alpha v_j - \beta \partial_n v_j)\ (j = 1, \ldots, n-1) && \text{on } \partial \boldsymbol{R}^n_+. \end{aligned} \tag{3.3}$$

Set $F = f - \lambda v + \operatorname{Div} D(v)$ and $F^*(x) = (F^e_1(x), \ldots, F^e_{n-1}(x), F^o_n(x))$, where F^o_n is the odd extension of F_n to the whole space defined by the formulas: $F^o_n(x) = F_n(x)$ for $x_n > 0$ and $F^o_n(x) = -F_n(x', x_n)$ for $x_n < 0$. Let (U, Φ) be a solution to the whole space problem:

$$\lambda U - \operatorname{Div} T(U, \Phi) = F^*, \quad \operatorname{div} U = 0 \quad \text{in } \boldsymbol{R}^n. \tag{3.4}$$

By Theorem 2.1 and Lemma 3.2, we have $U \in W^2_q(\boldsymbol{R}^n)^n$, $\Phi \in \hat{W}^1_q(\boldsymbol{R}^n)$ and

$$\begin{aligned} \mathscr{I}_\lambda(U, \Phi, \boldsymbol{R}^n) &\leqq C_{\epsilon,q,n} \|F^*\|_{L_q(\boldsymbol{R}^n)} \\ &\leqq C_{\epsilon,q,n} \Big\{ \|f\|_{L_q(\boldsymbol{R}^n_+)} + |\lambda| \|g\|_{\hat{W}^{-1}_q(\boldsymbol{R}^n_+)} + \|\nabla g\|_{L_q(\boldsymbol{R}^n_+)} \Big\}. \end{aligned} \tag{3.5}$$

Moreover, we see easilty that

$$U_n = 0 \quad \text{on } \partial \boldsymbol{R}^n_+. \tag{3.6}$$

We set $w = U + z$ and $p = \Phi + \theta$, and then it follows from (3.3), (3.4) and (3.6) that (z, θ) satisfies the equations:

$$\begin{aligned} &\lambda z - \operatorname{Div} T(z, \theta) = 0, \quad \operatorname{div} z = 0 && \text{in } \boldsymbol{R}^n_+ \\ &z_n = 0, \quad \alpha z_j - \beta \partial_n z_j = H_j\ (j = 1, \ldots, n-1) && \text{on } \partial \boldsymbol{R}^n_+ \end{aligned} \tag{3.7}$$

where we have set

$$H_j = h_j - (\alpha v_j - \beta \partial_n v_j) - (\alpha U_j - \beta \partial_n U_j).$$

By Lemma 3.2 and (3.5) we see that

$$\|\nabla H_j\|_{L_q(\boldsymbol{R}^n_+)} + |\lambda|^{1/2} \|H_j\|_{L_q(\boldsymbol{R}^n_+)} \leqq C_\delta \mathscr{D}_\lambda\big(f, g, h, \boldsymbol{R}^n_+\big) \tag{3.8}$$

provided that $\lambda \in \Sigma_\epsilon$ and $|\lambda| \geqq \delta$.

Now, we shall show the following theorem.

Theorem 3.3. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Then, the equation (3.7) admits a solution $(z, \theta) \in W^2_q(\boldsymbol{R}^n_+)^n \times \hat{W}^1_q(\boldsymbol{R}^n_+)$ which satisfies the estimate: $\mathscr{I}_\lambda(z, \theta, \boldsymbol{R}^n_+) \leqq C_\epsilon \mathscr{D}_\lambda(H)$, where we have set $\mathscr{D}_\lambda(H) = |\lambda|^{1/2} \|H\|_{L_q(\boldsymbol{R}^n_+)} + \|\nabla H\|_{L_q(\boldsymbol{R}^n_+)}$.*

Since the uniqueness follows from the existence theorem of the dual problem, from Theorem 3.3, (3.8), (3.5) and Lemma 3.2 we have Theorem 3.1. Therefore, we shall show Theorem 3.3 in what follows.

First of all, we shall derive the solution formula of (3.7). For this purpose, applying the partial Fourier transform with respect to x' to (3.7), we have a system of ordinary differential equations:

$$\begin{aligned}
&(\lambda + |\xi'|^2)Z_j(x_n) - \partial_n^2 Z_j(x_n) + i\xi_j \Theta(x_n) = 0 \quad && x_n > 0 \\
&(\lambda + |\xi'|^2)Z_n(x_n) - \partial_n^2 Z_n(x_n) + \partial_n \Theta(x_n) = 0 \quad && x_n > 0 \\
&\sum_{j=1}^{n-1} i\xi_j Z_j(x_n) + \partial_n Z_n(x_n) = 0 && x_n > 0 \\
&Z_n(0) = 0, \quad \alpha Z_j(0) - \beta(\partial_n Z_j)(0) = \hat{H}_j(\xi', 0)
\end{aligned} \tag{3.9}$$

for $j = 1, \ldots, n-1$, where we have set $Z_j(x_n) = \mathscr{F}_{x'}[z_j](\xi', x_n)$, $\Theta(x_n) = \mathscr{F}_{x'}[\theta](\xi', x_n)$, $\hat{H}_j(\xi', 0) = \mathscr{F}_{x'}[H_j](\xi', 0)$; and $\mathscr{F}_{x'}[k](\xi', x_n)$ denotes the partial Fourier transform of $k(x)$ with respect to x' that is defined by the formula:

$$\mathscr{F}_{x'}[k](\xi', x_n) = \int_{\boldsymbol{R}^{n-1}} e^{-ix'\cdot\xi'} k(x)dx', \ x' = (x_1, \ldots, x_{n-1}), \ \xi' = (\xi_1, \ldots, \xi_{n-1}). \tag{3.10}$$

To solve (3.9), we set $Z_j = P_j e^{-Ax_n} + Q_j e^{-Bx_n}$ and $\Theta = Re^{-Bx_n}$ with $A = \sqrt{\lambda + |\xi'|^2}$ and $B = |\xi'|$. Inserting these formulas into (3.9) implies that

$$\begin{gathered}
(A^2 - B^2)Q_j + i\xi_j R = 0, \quad (A^2 - B^2)Q_n - RB = 0 \\
\sum_{j=1}^{n-1} i\xi_j P_j - P_n A = 0, \quad \sum_{j=1}^{n-1} i\xi_j Q_j - Q_n B = 0 \\
P_n + Q_n = 0, \quad \alpha(P_j + Q_j) + \beta(AP_j + BQ_j) = \hat{H}_j(\xi', 0)
\end{gathered}$$

for $j = 1 \ldots, n-1$. Solving this system of linear equations, we have

$$\begin{aligned}
Z_j(x_n) &= \frac{e^{-Ax_n}}{\alpha + \beta A}\hat{H}_j(\xi', 0) + \frac{\alpha + \beta B}{\alpha + \beta A}\frac{\xi_j \sum_{k=1}^{n-1}\xi_k \hat{H}_k(\xi', 0)}{(A - B)(\alpha + \beta(A + B))B} e^{-Ax_n} \\
&\quad - \frac{\xi_j \sum_{k=1}^{n-1}\xi_k \hat{H}_k(\xi', 0)}{(A - B)(\alpha + \beta(A + B))B} e^{-Bx_n} \\
&= \frac{e^{-Ax_n}}{\alpha + \beta A}\hat{H}_j(\xi', 0) - \frac{\beta}{\alpha + \beta A}\frac{e^{-Ax_n}\xi_j}{(\alpha + \beta(A + B))B}\sum_{k=1}^{n-1}\xi_k \hat{H}_k(\xi', 0) \\
&\quad + \frac{1}{(\alpha + \beta(A + B))B}\frac{e^{-Ax_n} - e^{-Bx_n}}{A - B}\xi_j \sum_{k=1}^{n-1}\xi_k \hat{H}_k(\xi', 0)
\end{aligned}$$

$$Z_n(x_n) = \frac{i}{\alpha + \beta(A+B)} \frac{e^{-Ax_n} - e^{-Bx_n}}{A-B} \sum_{k=1}^{n-1} \xi_k \hat{H}_k(\xi', 0)$$

$$\Theta(x_n) = \frac{-i(A+B)}{(\alpha + \beta(A+B))B} e^{-Bx_n} \sum_{k=1}^{n-1} \xi_k \hat{H}_k(\xi', 0).$$

Set

$$M_\lambda(\xi', x_n) = \frac{e^{-Ax_n} - e^{-Bx_n}}{A-B}.$$

Using the identities:

$$\begin{aligned} a(x_n)b(0) &= -\int_0^\infty \frac{\partial}{\partial y_n}[a(x_n+y_n)b(y_n)]dy_n \\ &= -\int_0^\infty (\partial_n a)(x_n+y_n)b(y_n)dy_n - \int_0^\infty a(x_n+y_n)(\partial_n b)(y_n)dy_n; \end{aligned}$$

$$\partial_n M_\lambda(\xi', x_n) = -e^{-Ax_n} - BM_\lambda(\xi', x_n) \tag{3.11}$$

denoting the inverse partial Fourier transform with respect to ξ' by $\mathscr{F}^{-1}_{\xi'}$, writing $B_\lambda(\xi') = A = \sqrt{\lambda + |\xi'|^2}$ and recalling that $B = |\xi'|$, finally we arrive at the formulas:

$$\begin{aligned} z_j(x) &= \int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{e^{-B_\lambda(\xi')(x_n+y_n)}}{\alpha + \beta B_\lambda(\xi')}(B_\lambda(\xi') - \partial_n)\hat{H}_j(\xi', y_n)\right](x')dy_n \\ &\quad + \sum_{k=1}^{n-1}\int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{\xi_j\xi_k e^{-B_\lambda(\xi')(x_n+y_n)}}{(\alpha+\beta B_\lambda(\xi'))(\alpha+\beta(B_\lambda(\xi')+|\xi'|))|\xi'|}(\alpha+\beta\partial_n)\hat{H}_k(\xi', y_n)\right](x')dy_n \\ &\quad + \sum_{k=1}^{n-1}\int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{\xi_j\xi_k M_\lambda(\xi', x_n+y_n)}{(\alpha + \beta(B_\lambda(\xi') + |\xi'|))|\xi'|}(|\xi'| - \partial_n)\hat{H}_k(\xi', y_n)\right](x')dy_n \\ z_n(x) &= \sum_{k=1}^{n-1}\int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{i\xi_k e^{-B_\lambda(\xi')(x_n+y_n)}}{\alpha + \beta(B_\lambda(\xi') + |\xi'|)}\hat{H}_k(\xi', y_n)\right](x')dy_n \\ &\quad + \sum_{k=1}^{n-1}\int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{i\xi_k M_\lambda(\xi', x_n+y_n)}{\alpha + \beta(B_\lambda(\xi') + |\xi'|)}(|\xi'| - \partial_n)\hat{H}_k(\xi', y_n)\right](x')dy_n \\ \theta(x) &= \sum_{k=1}^{n-1}\int_0^\infty \mathscr{F}^{-1}_{\xi'}\left[\frac{i(|\xi'| + B_\lambda(\xi'))\xi_k e^{-|\xi'|(x_n+y_n)}}{(\alpha + \beta(B_\lambda(\xi') + |\xi'|))|\xi'|}(\partial_n - |\xi'|)\hat{H}_k(\xi', y_n)\right](x')dy_n. \end{aligned} \tag{3.12}$$

In order to estimate z_j and θ, we use the following two lemmas.

Lemma 3.4. *Let $0 < \epsilon < \pi/2$. Then, there exist constants c_1 and c_2 depending on ϵ such that*

$$c_1(|\lambda|^{1/2} + |\xi'|) \leqq \operatorname{Re} B_\lambda(\xi') \leqq c_2(|\lambda|^{1/2} + |\xi'|) \tag{3.13}$$

for every $\lambda \in \Sigma_\epsilon$ and $\xi' \in \boldsymbol{R}^{n-1}$.

Moreover, we have the following estimates:

$$\left|\partial_{\xi'}^{\alpha'} B_\lambda(\xi')^s\right| \leqq C_{\alpha',s,\epsilon}(|\lambda|^{1/2} + |\xi'|)^{s-|\alpha'|} \tag{3.14}$$

$$\left|\partial_{\xi'}^{\alpha'} |\xi'|^s\right| \leqq C_{\alpha',s}|\xi'|^{s-|\alpha'|} \tag{3.15}$$

$$\left|\partial_{\xi'}^{\alpha'} e^{-B_\lambda(\xi')x_n}\right| \leqq C_{\alpha',\epsilon}(|\lambda|^{1/2} + |\xi'|)^{-|\alpha'|}e^{-(c_1/2)(|\lambda|^{1/2}+|\xi'|)x_n} \tag{3.16}$$

$$\left|\partial_{\xi'}^{\alpha'} e^{-|\xi'|x_n}\right| \leqq C_{\alpha'}|\xi'|^{-|\alpha'|}e^{-(1/2)|\xi'|x_n} \tag{3.17}$$

$$\left|\partial_{\xi'}^{\alpha'} [|\xi'|M_\lambda(\xi', x_n)]\right| \leqq C_{\alpha',\epsilon}|\xi'|^{-|\alpha'|}e^{-d|\xi'|x_n} \tag{3.18}$$

$$\left|\partial_{\xi'}^{\alpha'} (\alpha + \beta B_\lambda(\xi'))^{-1}\right| \leqq C_{\alpha',\epsilon}(|\lambda|^{1/2} + |\xi'|)^{-1-|\alpha'|} \tag{3.19}$$

$$\left|\partial_{\xi'}^{\alpha'} (\alpha + \beta(B_\lambda(\xi') + |\xi'|))^{-1}\right| \leqq C_{\alpha',\epsilon}|\xi'|^{-1-|\alpha'|} \tag{3.20}$$

for every $\alpha' \in \boldsymbol{N}_0^{n-1}$, $x_n > 0$, $\xi' \in \boldsymbol{R}^{n-1}$, $\lambda \in \Sigma_\epsilon$ and $s \in \boldsymbol{R}$, where $C_{\alpha',s,\epsilon}$ and $C_{\alpha',\epsilon}$ are positive constants independent of x_n, ξ', and λ; and we have set $d = \min(1, c_1)/2$.

Lemma 3.5. *Let $0 < \epsilon < \pi/2$ and $1 < q < \infty$.* (1) *Let $m_1(\lambda, \xi')$ be a function defined on $\Sigma_\epsilon \times (\boldsymbol{R}^{n-1} \setminus \{0\})$ such that*

$$\left|\partial_{\xi'}^{\alpha'} m_1(\lambda, \xi')\right| \leqq C_{\alpha',\epsilon}|\lambda|^{1/2}(|\lambda|^{1/2} + |\xi'|)^{-|\alpha'|} \tag{3.21}$$

for any $\alpha' \in \boldsymbol{N}_0^{n-1}$, $\lambda \in \Sigma_\epsilon$ and $\xi' \in \boldsymbol{R}^{n-1}$. If we define the operator $K_1(\lambda)$ by the formula:

$$K_1(\lambda)[g](x) = \int_0^\infty \mathscr{F}_{\xi'}^{-1}\left[m_1(\lambda, \xi')e^{-B_\lambda(\xi')(x_n+y_n)}\hat{g}(\xi', y_n)\right](x')\,dy_n$$

then we have

$$\|K_1(\lambda)[g]\|_{L_q(\boldsymbol{R}_+^n)} \leqq C_\epsilon\|g\|_{L_q(\boldsymbol{R}_+^n)}$$

for any $\lambda \in \Sigma_\epsilon$ and $g \in L_q(\boldsymbol{R}_+^n)$. Here and hereafter, we write $\hat{g}(\xi', x_n) = \mathscr{F}_{x'}[g(\cdot, x_n)](\xi')$ (cf. (3.10)*).*

(2) *Let $m_2(\lambda, \xi')$ be a function defined on $\Sigma_\epsilon \times (\boldsymbol{R}^{n-1} \setminus \{0\})$ such that*

$$\left|\partial_{\xi'}^{\alpha'} m_2(\lambda, \xi')\right| \leqq C_{\alpha',\epsilon}|\xi'|^{1-|\alpha'|} \tag{3.22}$$

for any $\alpha' \in \boldsymbol{N}_0^{n-1}$, $\lambda \in \Sigma_\epsilon$ and $\xi' \in \boldsymbol{R}^{n-1}$. If we define the operators $K_2(\lambda)$, $K_3(\lambda)$ and

$K_4(\lambda)$ by the formulas:

$$K_2(\lambda)[g](x) = \int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[m_2(\lambda,\xi')e^{-B_\lambda(\xi')(x_n+y_n)}\hat{g}(\xi',y_n)\big](x')dy_n$$

$$K_3(\lambda)[g](x) = \int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[m_2(\lambda,\xi')e^{-|\xi'|(x_n+y_n)}\hat{g}(\xi',y_n)\big](x')dy_n$$

$$K_4(\lambda)[g](x) = \int_0^\infty \mathscr{F}_{\xi'}^{-1}\big[m_2(\lambda,\xi')|\xi'|M_\lambda(\xi',x_n+y_n)\hat{g}(\xi',y_n)\big](x')dy_n$$

then we have

$$\|K_j(\lambda)[g]\|_{L_q(\boldsymbol{R}^n_+)} \leqq C_\epsilon\|g\|_{L_q(\boldsymbol{R}^n_+)},\ j=2,3,4$$

for any $\lambda\in\Sigma_\epsilon$ and $g\in L_q(\boldsymbol{R}^n_+)$.

A PROOF OF LEMMA 3.4. Set $B_\lambda(\xi') = |\lambda+|\xi'|^2|^{1/2}e^{i\theta}$. Since $-\pi+\epsilon\leqq\arg(\lambda+|\xi'|^2)\leqq\pi-\epsilon$ for $\lambda\in\Sigma_\epsilon$, we see that $-(\pi-\epsilon)/2\leqq\theta\leqq(\pi-\epsilon)/2$, and therefore by (2.4) we have (3.13) immediately.

To prove (3.14), we set $f(t)=t^{s/2}$ and observe that

$$\big|\partial_{\xi'}^{\alpha'}B_\lambda(\xi')^s\big| \leqq C_{\alpha'}\sum_{\ell=1}^{|\alpha'|}\big|f^{(\ell)}(B_\lambda(\xi')^2)\big|\sum_{\substack{\alpha'_1+\cdots+\alpha'_\ell=\alpha'\\|\alpha'_i|\geqq1}}\big|\partial_{\xi'}^{\alpha'_1}B_\lambda(\xi')^2\big|\cdots\big|\partial_{\xi'}^{\alpha'_\ell}B_\lambda(\xi')^2\big|.$$

Since $\partial_{\xi'}^{\alpha'}B_\lambda(\xi')^2=\partial_{\xi'}^{\alpha'}|\xi'|^2$, by (3.13) we have

$$\begin{aligned}\big|\partial_{\xi'}^{\alpha'}B_\lambda(\xi')^s\big| &\leqq C_{\alpha',s}\sum_{\ell=1}^{|\alpha'|}(|\lambda|^{1/2}+|\xi'|)^{s-2\ell}\sum_{\substack{k+2(\ell-k)=|\alpha'|\\0\leqq k\leqq\ell}}|\xi'|^k\\ &= C_{\alpha',s}\sum_{\ell=1}^{|\alpha'|}(|\lambda|^{1/2}+|\xi'|)^{s-2\ell}(|\lambda|^{1/2}+|\xi'|)^{2\ell-|\alpha'|}\\ &\leqq C_{\alpha',s}(|\lambda|^{1/2}+|\xi'|)^{s-|\alpha'|}\end{aligned}$$

which shows (3.14). Analogously, we can show (3.15).

To show (3.16), we set $f(t)=e^{-tx_n}$ and observe that

$$\big|\partial_{\xi'}^{\alpha'}e^{-B_\lambda(\xi')x_n}\big| \leqq C_{\alpha'}\sum_{\ell=1}^{|\alpha'|}\big|f^{(\ell)}(B_\lambda(\xi'))\big|\sum_{\substack{\alpha'_1+\cdots+\alpha'_\ell=\alpha'\\|\alpha'_i|\geqq1}}\big|\partial_{\xi'}^{\alpha'_1}B_\lambda(\xi')\big|\cdots\big|\partial_{\xi'}^{\alpha'_\ell}B_\lambda(\xi')\big|$$

By (3.13) and (3.14) we have

$$\left|\partial_{\xi'}^{\alpha'} e^{-B_\lambda(\xi')x_n}\right| \leqq C_{\alpha'} \sum_{\ell=1}^{|\alpha'|} e^{-c_1(|\lambda|^{1/2}+|\xi'|)x_n} x_n^\ell (|\lambda|^{1/2}+|\xi'|)^{\ell-|\alpha'|}$$
$$\leqq C_{\alpha'} e^{-(c_1/2)(|\lambda|^{1/2}+|\xi'|)x_n}(|\lambda|^{1/2}+|\xi'|)^{-|\alpha'|}$$

which shows (3.16). Analogously, we have (3.17).

To show (3.18), by the Taylor formula we write

$$|\xi'|M_\lambda(\xi',x_n) = -|\xi'|x_n \int_0^1 e^{-((1-\theta)|\xi'|+\theta B_\lambda(\xi'))x_n} d\theta$$

Employing the same argument as in the proof of (3.16) and using (3.13), (3.14) and (3.15), we have

$$\begin{aligned}
&\left|\partial_{\xi'}^{\alpha'} e^{-((1-\theta)|\xi'|+\theta B_\lambda(\xi'))x_n}\right| \\
&\leqq C_{\alpha'} \sum_{\ell=1}^{|\alpha'|} x_n^\ell e^{-((1-\theta)|\xi'|+c_1\theta(|\lambda|^{1/2}+|\xi'|))x_n} \\
&\quad \times \sum_{\substack{\alpha'_1+\cdots+\alpha'_\ell=\alpha' \\ |\alpha'_i|\geqq 1}} \big((1-\theta)|\xi'|^{1-|\alpha'_1|}+\theta(|\lambda|^{1/2}+|\xi'|)^{1-|\alpha'_1|}\big) \\
&\qquad\qquad \cdots \big((1-\theta)|\xi'|^{1-|\alpha'_\ell|}+\theta(|\lambda|^{1/2}+|\xi'|)^{1-|\alpha'_\ell|}\big) \\
&\leqq C_{\alpha'} \sum_{\ell=1}^{|\alpha'|} x_n^\ell e^{-2d((1-\theta)|\xi'|+\theta(|\lambda|^{1/2}+|\xi'|))x_n} ((1-\theta)|\xi'|+\theta(|\lambda|^{1/2}+|\xi'|))^\ell |\xi'|^{-|\alpha'|} \\
&\leqq C_{\alpha'} e^{-(2d/3)((1-\theta)|\xi'|+\theta(|\lambda|^{1/2}+|\xi'|))x_n}|\xi'|^{-|\alpha'|} \leqq C_{\alpha'} e^{-(2d/3)|\xi'|x_n}|\xi'|^{-|\alpha'|}
\end{aligned}$$

where $2d = \min(1, c_1)$. Therefore, by the Leibniz formula we have (3.18).

To estimate (3.19), we set $f(t) = t^{-1}$ and observe that

$$\begin{aligned}
&\left|\partial_{\xi'}^{\alpha'} (\alpha+\beta B_\lambda(\xi'))^{-1}\right| \\
&\leqq C_{\alpha'} \sum_{\ell=1}^{|\alpha'|} |f^{(\ell)}(\alpha+\beta B_\lambda(\xi'))| \sum_{\substack{\alpha'_1+\cdots+\alpha'_\ell \\ |\alpha'_i|\geqq 1}} \left|\partial_{\xi'}^{\alpha'_1}(\alpha+\beta B_\lambda(\xi'))\right| \cdots \left|\partial_{\xi'}^{\alpha'_\ell}(\alpha+\beta B_\lambda(\xi'))\right| \\
&\leqq C_{\alpha',\epsilon} \sum_{\ell=1}^{|\alpha'|} (\beta c_1(|\lambda|^{1/2}+|\xi'|))^{-(\ell+1)} (\beta(|\lambda|^{1/2}+|\xi'|))^{\ell-|\alpha'|} \\
&\leqq C_{\alpha',\epsilon} (\beta(|\lambda|^{1/2}+|\xi'|))^{-1-|\alpha'|}
\end{aligned}$$

which shows (3.19). Since

$$\left|(\alpha+\beta(B_\lambda(\xi')+|\xi'|))^{-1}\right| \leqq (\beta c_1(|\lambda|^{1/2}+|\xi'|))^{-1}$$

as follows from the fact: $\alpha \geqq 0$ and (3.13), by (3.14) and (3.15) we have (3.20). This completes the proof of the lemma. $\square$

To prove Lemma 3.5 we shall use the following two lemmas.

LEMMA 3.6. *Let $0 < \epsilon < \pi/2$. (1) Let $m_1(\lambda, \xi')$ be the same function as in Lemma* 3.5 *and set*

$$k_1(\lambda, x) = \mathscr{F}_{\xi'}^{-1}\left[m_1(\lambda, \xi')e^{-B_\lambda(\xi')x_n}\right](x').$$

Then, we have

$$|k_1(\lambda, x)| \leqq C_\epsilon |x|^{-n} \tag{3.23}$$

for any $\lambda \in \Sigma_\epsilon$ and $x \in \boldsymbol{R}^n_+$.
(2) *Let $m_2(\lambda, \xi')$ be the same function as in Lemma* 3.5 *and set*

$$\begin{aligned} k_2(\lambda, x) &= \mathscr{F}_{\xi'}^{-1}\left[m_2(\lambda, \xi')e^{-B_\lambda(\xi')x_n}\right](x') \\ k_3(\lambda, x) &= \mathscr{F}_{\xi'}^{-1}\left[m_2(\lambda, \xi')e^{-|\xi'|x_n}\right](x') \\ k_4(\lambda, x) &= \mathscr{F}_{\xi'}^{-1}\left[m_2(\lambda, \xi')|\xi'|M_\lambda(\xi', x_n)\right](x'). \end{aligned}$$

Then, we have

$$|k_j(\lambda, x)| \leqq C_\epsilon |x|^{-n}, \quad j = 2, 3, 4 \tag{3.24}$$

for any $\lambda \in \Sigma_\epsilon$ and $x \in \boldsymbol{R}^n_+$.

LEMMA 3.7. *Let $0 < \epsilon < \pi/2$. Let $k(\lambda, x)$ be a function defined on $\Sigma_\epsilon \times \boldsymbol{R}^n_+$, which satisfies the estimate*:

$$|k(\lambda, x)| \leqq L|x|^{-n} \tag{3.25}$$

for any $x \in \boldsymbol{R}^n_+$ and $\lambda \in \Sigma_\epsilon$ with some constant L independent of x and λ. Let K_λ be the integral operator defined by the formula:

$$K_\lambda[g](x) = \int_{\boldsymbol{R}^n_+} k(\lambda, x' - y', x_n + y_n)g(y)dy, \quad x \in \boldsymbol{R}^n_+.$$

Then, we have

$$\|K_\lambda[g]\|_{L_q(\boldsymbol{R}^n_+)} \leqq C_{n,q} L \|g\|_{L_q(\boldsymbol{R}^n_+)} \tag{3.26}$$

for any $g \in L_q(\boldsymbol{R}^n_+)$.

A proof of Lemma 3.5. Let $k_j(\lambda, x)$ be functions defined in Lemma 3.6, and then the operators $K_j(\lambda)$ defined in Lemma 3.5 are written as follows:

$$K_j(\lambda)[g](x) = \int_{\boldsymbol{R}^n_+} k_j(\lambda, x' - y', x_n + y_n) g(y) dy.$$

Therefore, applying Lemma 3.7 implies Lemma 3.5 immediately. □

Therefore, we shall prove Lemmas 3.6 and 3.7 to complete the proof of Lemma 3.5.

A proof of Lemma 3.6. To show (3.23), we use the identity:

$$\sum_{j=1}^{n-1} \frac{x_j}{i|x'|^2} \frac{\partial}{\partial \xi_j} e^{ix'\cdot\xi'} = e^{ix'\cdot\xi'}. \tag{3.27}$$

Applying (3.27) n times and using (3.14) and (3.21), we have

$$\begin{aligned}
&\big|\mathscr{F}^{-1}_{\xi'}\big[m_1(\lambda, \xi') e^{-B_\lambda(\xi')x_n}\big](x')\big| \\
&\leqq C_n \sum_{|\alpha'|=n} \left|\left(\frac{x'}{i|x'|^2}\right)^{\alpha'}\right| \int_{\boldsymbol{R}^{n-1}} \big|\partial^{\alpha'}_{\xi'}\left[m_1(\lambda, \xi') e^{-B_\lambda(\xi')x_n}\right]\big| \, d\xi' \\
&\leqq \frac{C_{\alpha',n}}{|x'|^n} |\lambda|^{1/2} \int_{\boldsymbol{R}^{n-1}} (|\lambda|^{1/2} + |\xi'|)^{-n} d\xi'.
\end{aligned} \tag{3.28}$$

To proceed the estimate (3.28), we observe that

$$\begin{aligned}
\int_{\boldsymbol{R}^{n-1}} (|\lambda|^{1/2} + |\xi'|)^{-n} d\xi' &\leqq |\lambda|^{(1-n)/2} \int_{|\xi'|\leqq|\lambda|^{1/2}} d\xi' + |\lambda|^{1/2} \int_{|\xi'|\geqq|\lambda|^{1/2}} |\xi'|^{-n} d\xi' \\
&\leqq C_n
\end{aligned} \tag{3.29}$$

with some constant C_n independent of λ. Combining (3.28) and (3.29) we have

$$\big|\mathscr{F}^{-1}_{\xi'}\big[m_1(\lambda, \xi') e^{-B_\lambda(\xi')x_n}\big](x')\big| \leqq \frac{C_{\alpha',n}}{|x'|^n}. \tag{3.30}$$

On the other hand, by (3.13) and (3.28) we have

$$\begin{aligned}
\big|\mathscr{F}^{-1}_{\xi'}\big[m_1(\lambda, \xi') e^{-B_\lambda(\xi')x_n}\big](x')\big| &\leqq C_n |\lambda|^{1/2} \int_{\boldsymbol{R}^{n-1}} e^{-c_1(|\lambda|^{1/2} + |\xi'|)x_n} d\xi' \\
&\leqq \frac{C_n|\lambda|^{1/2}}{(x_n)^n} \int_{\boldsymbol{R}^{n-1}} (|\lambda|^{1/2} + |\xi'|)^{-n} d\xi' \leqq \frac{C_n}{(x_n)^n}
\end{aligned} \tag{3.31}$$

with some constant C_n independent of λ, which combined with (3.30) implies (3.23).

To show (3.24), we use the following lemma due to Shibata and Shimizu [**16**].

Lemma 3.8. *Let B be a Banach space and $|\cdot|_B$ its norm. Let α be a number $> -n$ and set $\alpha = N + \sigma - n$ where N is an integer and $0 < \sigma \leqq 1$. Let $f(\xi)$ be a function in $C^\infty(\boldsymbol{R}^n \setminus \{0\}, B)$ such that*

$$\partial_\xi^\gamma f(\xi) \in L_1(\boldsymbol{R}^n, B) \quad \textit{for } |\gamma| \leqq N$$
$$|\partial_\xi^\gamma f(\xi)|_B \leqq C_\gamma |\xi|^{\alpha - |\gamma|} \quad \textit{for every } \xi \neq 0 \textit{ and } \gamma \in \boldsymbol{N}_0^n$$

Set $g(x) = \int_{\boldsymbol{R}^n} e^{-ix\cdot\xi} f(\xi) d\xi$. Then, we have

$$|g(x)|_B \leqq C_{n,\alpha}\Big(\max_{|\gamma| \leqq N+2} C_\gamma\Big)|x|^{-(n+\alpha)}$$

for every $x \neq 0$, where $C_{n,\alpha}$ is a constant depending only on n and α.

By (3.16) and (3.22) we have

$$\left|\partial_{\xi'}^{\alpha'}\left[m_2(\lambda, \xi')e^{-B_\lambda(\xi')x_n}\right]\right| \leqq C_{\alpha',\epsilon}|\xi'|^{1-|\alpha'|}e^{-(c_1/2)|\xi'|x_n} \tag{3.32}$$

and therefore applying Lemma 3.8 with $\alpha = 1$ (replacing n by $n-1$), we have

$$\left|\mathscr{F}_{\xi'}^{-1}\left[m_2(\lambda, \xi')e^{-B_\lambda(\xi')x_n}\right](x')\right| \leqq C_\epsilon |x'|^{-n}.$$

On the other hand, by (3.13) and the change of variable: $\xi' x_n = \eta'$, we have

$$\begin{aligned}&\left|\mathscr{F}_{\xi'}^{-1}\left[m_2(\lambda, \xi')e^{-B_\lambda(\xi')x_n}\right](x')\right| \\ &\quad \leqq C_n \int_{\boldsymbol{R}^{n-1}} |\xi'|e^{-c_1|\xi'|x_n} d\xi' = \frac{C_n}{(x_n)^n}\int_{\boldsymbol{R}^{n-1}} |\eta'|e^{-c_1|\eta'|}\,d\eta'.\end{aligned}$$

Combining these estimates implies that $|k_2(\lambda, x)| \leqq C_\epsilon |x|^{-n}$. Analogously, by (3.22), (3.15), (3.18) and Lemma 3.8 we see that (3.24) holds for $j = 3, 4$, which completes the proof of the lemma.

A proof of Lemma 3.7. By Minkowski's inequality for integral, Young's inequality and (3.25), we have

$$\begin{aligned}&\|K_\lambda[g](\cdot, x_n)\|_{L_q(\boldsymbol{R}^{n-1})} \\ &\quad \leqq L\int_0^\infty \left[\int_{\boldsymbol{R}^{n-1}}\left|\int_{\boldsymbol{R}^{n-1}} \frac{|g(y', y_n)|}{(|x'-y'|^2 + (x_n+y_n)^2)^{n/2}}dy'\right|^q dx'\right]^{1/q} dy_n \\ &\quad = L\int_{\boldsymbol{R}^{n-1}} \frac{dy'}{(1+|y'|^2)^{n/2}}\int_0^\infty \frac{\|g(\cdot, y_n)\|_{L_q(\boldsymbol{R}^{n-1})}}{x_n+y_n}dy_n.\end{aligned} \tag{3.33}$$

To proceed the estimate, we apply the Marchinkiewitz interpolation inequality to the integral operator:

$$G[h](x_n) = \int_0^\infty \frac{h(y_n)}{x_n + y_n} dy_n.$$

In fact, by Hölder's inequality we have

$$\begin{aligned} |G[h](x_n)| &\leqq \left(\int_0^\infty \frac{dy_n}{(x_n + y_n)^{q'}} \right)^{1/q'} \|h\|_{L_q(\boldsymbol{R}_+)} \\ &= \left(\int_0^\infty \frac{dt}{(1+t)^{q'}} \right)^{1/q'} (x_n)^{-1/q} \|h\|_{L_q(\boldsymbol{R}_+)} \end{aligned}$$

which implies that

$$\sup_{R \geqq 0} R\, \mu(\{x_n > 0 \mid |G[h](x_n)| \geqq R\})^{1/q} \leqq (q-1)^{1/q'} \|h\|_{L_q(\boldsymbol{R}_+)}$$

where μ denotes the Lebesgue measure on $\boldsymbol{R}$. Therefore, by the Marcinkiewitz interpolation inequality G becomes a bounded linear operator on $L_q(\boldsymbol{R}_+)$ for any $1 < q < \infty$, which applied to (3.33) implies (3.26). This completes the proof of Lemma 3.7. □

Now, by Lemmas 3.4 and 3.5 we shall show Theorem 3.3. First of all, we consider the term:

$$w(x) = \int_0^\infty \mathscr{F}_{\xi'}^{-1} \left[\frac{\xi_k M_\lambda(\xi', x_n + y_n)}{\alpha + \beta(B_\lambda(\xi') + |\xi'|)} \hat{g}(\xi', y_n) \right](x') dy_n, \quad k = 1, \ldots, n-1$$

where $\hat{g}(\xi', x_n)$ stands for one of $\partial_n \hat{H}_j(\xi', x_n)$, $B_\lambda(\xi') \hat{H}_j(\xi', x_n)$ and $|\xi'| \hat{H}_j(\xi', x_n)$. To show that

$$|\lambda| \|w\|_{L_q(\boldsymbol{R}^n_+)} \leqq C_\epsilon \|g\|_{L_q(\boldsymbol{R}^n_+)} \tag{3.34}$$

using the identity: $\lambda M_\lambda(\xi', x_n) = (B_\lambda(\xi') + |\xi'|)(e^{-B_\lambda(\xi') x_n} - e^{-|\xi'| x_n})$ we write

$$\lambda w(x) = \int_0^\infty \mathscr{F}_{\xi'}^{-1} \left[\xi_k A(\xi', \lambda)(e^{-B_\lambda(\xi')(x_n + y_n)} - e^{-|\xi'|(x_n + y_n)}) \hat{g}(\xi', y_n) \right](x') dy_n$$

where we have set $A(\xi', \lambda) = (B_\lambda(\xi') + |\xi'|)(\alpha + \beta(B_\lambda(\xi') + |\xi'|))^{-1}$. Since

$$\left| \partial_{\xi'}^{\alpha'} [\xi_k A(\xi', \lambda)] \right| \leqq C_{\alpha', \epsilon} |\xi'|^{1 - |\alpha'|}$$

for every $\alpha' \in \boldsymbol{N}_0^{n-1}$, $\lambda \in \Sigma_\epsilon$ and $\xi' \in \boldsymbol{R}^{n-1}$ as follows from Lemma 3.4, by Lemma 3.5 we have (3.34).

To show that

$$|\lambda|^{1/2}\|\nabla w\|_{L_q(\boldsymbol{R}^n_+)} + \|\nabla^2 w\|_{L_q(\boldsymbol{R}^n_+)} \leqq C_\epsilon \|g\|_{L_q(\boldsymbol{R}^n_+)} \tag{3.35}$$

we write

$$\begin{aligned}
|\lambda|^{1/2}\partial_\ell w(x) &= \int_0^\infty \mathscr{F}^{-1}_{\xi'}\Big[\frac{|\lambda|^{1/2} i\xi_k\xi_\ell|\xi'|M_\lambda(\xi', x_n+y_n)}{(\alpha+\beta(B_\lambda(\xi')+|\xi'|))|\xi'|}\hat g(\xi', y_n)\Big](x')dy_n \\
|\lambda|^{1/2}\partial_n w(x) &= -\int_0^\infty \mathscr{F}^{-1}_{\xi'}\Big[\frac{|\lambda|^{1/2}\xi_k(e^{-B_\lambda(\xi')(x_n+y_n)}+|\xi'|M_\lambda(\xi', x_n+y_n))}{\alpha+\beta(B_\lambda(\xi')+|\xi'|)}\hat g(\xi', y_n)\Big](x')dy_n \\
\partial_\ell\partial_m w(x) &= -\int_0^\infty \mathscr{F}^{-1}_{\xi'}\Big[\frac{\xi_k\xi_\ell\xi_m|\xi'|M_\lambda(\xi', x_n+y_n)}{(\alpha+\beta(B_\lambda(\xi')+|\xi'|))|\xi'|}\hat g(\xi', y_n)\Big](x')dy_n \\
\partial_\ell\partial_n w(x) &= -i\int_0^\infty \mathscr{F}^{-1}_{\xi'}\Big[\frac{\xi_k\xi_\ell(e^{-B_\lambda(\xi')(x_n+y_n)}+|\xi'|M_\lambda(\xi', x_n+y_n))}{\alpha+\beta(B_\lambda(\xi')+|\xi'|)}\hat g(\xi', y_n)\Big](x')dy_n \\
\partial_n^2 w(x) &= \int_0^\infty \mathscr{F}^{-1}_{\xi'}\Big[\frac{\xi_k(B_\lambda(\xi')+|\xi'|)e^{-B_\lambda(\xi')(x_n+y_n)}}{\alpha+\beta(B_\lambda(\xi')+|\xi'|)}\hat g(\xi', y_n)\Big](x')dy_n - \sum_{\ell=1}^{n-1}\partial_\ell^2 w(x)
\end{aligned}$$

where ℓ and m range from 1 to $n-1$ and we have used (3.11). Since

$$\begin{aligned}
\big|\partial_{\xi'}^{\alpha'}\big[|\lambda|^{1/2}\xi_k\xi_\ell\{(\alpha+\beta(B_\lambda(\xi')+|\xi'|))|\xi'|\}^{-1}\big]\big| &\leqq C_{\alpha',\epsilon}|\xi'|^{1-|\alpha'|} \\
\big|\partial_{\xi'}^{\alpha'}\big[|\lambda|^{1/2}\xi_k(\alpha+\beta(B_\lambda(\xi')+|\xi'|))^{-1}\big]\big| &\leqq C_{\alpha',\epsilon}|\xi'|^{1-|\alpha'|} \\
\big|\partial_{\xi'}^{\alpha'}\big[\xi_k\xi_\ell\xi_m\{(\alpha+\beta(B_\lambda(\xi')+|\xi'|))|\xi'|\}^{-1}\big]\big| &\leqq C_{\alpha',\epsilon}|\xi'|^{1-|\alpha'|} \\
\big|\partial_{\xi'}^{\alpha'}\big[\xi_k\xi_\ell(\alpha+\beta(B_\lambda(\xi')+|\xi'|))^{-1}\big]\big| &\leqq C_{\alpha',\epsilon}|\xi'|^{1-|\alpha'|} \\
\big|\partial_{\xi'}^{\alpha'}\big[\xi_k(B_\lambda(\xi')+|\xi'|)(\alpha+\beta(B_\lambda(\xi')+|\xi'|))^{-1}\big]\big| &\leqq C_{\alpha',\epsilon}|\xi'|^{1-|\alpha'|}
\end{aligned}$$

for every $\alpha' \in \boldsymbol{N}_0^{n-1}$, $\lambda \in \Sigma_\epsilon$ and $\xi' \in \boldsymbol{R}^{n-1}$ as follows from Lemma 3.4, by Lemma 3.5 we have (3.35).

Write

$$\begin{aligned}
B_\lambda(\xi')\hat H_j(\xi', x_n) &= \lambda(B_\lambda(\xi')|\lambda|^{1/2})^{-1}\widehat{|\lambda|^{1/2}H_j}(\xi', x_n) - i\sum_{\ell=1}^{n-1}\xi_\ell B_\lambda(\xi')^{-1}\widehat{\partial_\ell H_j}(\xi', x_n) \\
|\xi'|\hat H_j(\xi', x_n) &= -i\sum_{\ell=1}^{n-1}\xi_\ell|\xi'|^{-1}\widehat{\partial_\ell H_j}(\xi', x_n).
\end{aligned}$$

Since

$$\big|\partial_{\xi'}^{\alpha'}\big[\lambda(B_\lambda(\xi')|\lambda|^{1/2})^{-1}\big]\big| \leqq C_{\alpha',\epsilon}|\xi'|^{-|\alpha'|}, \quad \big|\partial_{\xi'}^{\alpha'}\big[\xi_\ell B_\lambda(\xi')^{-1}\big]\big| \leqq C_{\alpha',\epsilon}|\xi'|^{-|\alpha'|}$$
$$\big|\partial_{\xi'}^{\alpha'}\big[\xi'_\ell|\xi'|^{-1}\big]\big| \leqq C_{\alpha',\epsilon}|\xi'|^{-|\alpha'|}$$

for every $\alpha' \in \boldsymbol{N}_0^{n-1}$, $\lambda \in \Sigma_\epsilon$ and $\xi' \in \boldsymbol{R}^{n-1}$ as follows from Lemma 3.4, by Fourier multiplier theorem (cf. Triebel [**21**]) we have

$$\|g\|_{L_q(\boldsymbol{R}^n_+)} \leqq C_\epsilon\Big(|\lambda|^{1/2}\|H_j\|_{L_q(\boldsymbol{R}^n_+)} + \|\nabla H_j\|_{L_q(\boldsymbol{R}^n_+)}\Big)$$

where $g(x) = \mathscr{F}^{-1}_{\xi'}[B_\lambda(\xi')\hat{H}_j(\xi', x_n)](x')$ or $\mathscr{F}^{-1}_{\xi'}[|\xi'|\hat{H}_j(\xi', x_n)](x')$. Therefore, we have

$$\begin{aligned}|\lambda|\|w\|_{L_q(\boldsymbol{R}^n_+)} &+ |\lambda|^{1/2}\|\nabla w\|_{L_q(\boldsymbol{R}^n_+)} + \|\nabla^2 w\|_{L_q(\boldsymbol{R}^n_+)}\\ &\leqq C_\epsilon\Big(|\lambda|^{1/2}\|H_j\|_{L_q(\boldsymbol{R}^n_+)} + \|\nabla H_j\|_{L_q(\boldsymbol{R}^n_+)}\Big)\end{aligned}$$

for every $\lambda \in \Sigma_\epsilon$.

Employing the same arguments as above, by Lemmas 3.4 and 3.5 we can estimate other terms in (3.12) and therefore we may omit the detailed proof of Theorem 3.3.

4. Resolvent problem of the Stokes system in a bent half space.

Let $\omega : \boldsymbol{R}^{n-1} \to \boldsymbol{R}$ be a bounded function in $C^{2,1}$ class whose derivatives up to third order are all essentially bounded on $\boldsymbol{R}^{n-1}$. Let us define a bent half space H by the formula:

$$H = \{x = (x', x_n) \in \boldsymbol{R}^n \mid x_n > w(x')\}.$$

The boundary ∂H and the unit outer normal $\nu = \nu(x')$ to ∂H are given by the formulas: $\partial H = \{x = (x', \omega(x')) \mid x' \in \boldsymbol{R}^{n-1}\}$ and $\nu(x') = (\nabla'\omega, -1)/\sqrt{1 + |\nabla'\omega|^2}$, respectively. Here and hereafter, we set $\nabla'\omega = (\partial_1\omega, \ldots, \partial_{n-1}\omega)$. In this section, we consider the following generalized resolvent problem of the Stokes system in H:

$$\begin{aligned}\lambda u - \operatorname{Div} T(u,p) = f, \qquad \operatorname{div} u = g \quad &\text{in } H\\ \nu\cdot u = 0, \quad B_{\alpha,\beta}(u) = h \quad &\text{on } \partial H.\end{aligned} \tag{4.1}$$

The following theorem is a main result of this section.

Theorem 4.1. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Let $\lambda \in \Sigma_\epsilon$, $f \in L_q(H)^n$, $g \in \hat{W}^1_q(H) \cap \hat{W}^{-1}_q(H)$ and $h \in W^1_q(H)^n$. Assume that* $\operatorname{supp} g$ *is compact and that $\nu\cdot h = 0$ on ∂H. Then, there exist constants $\lambda_0 = \lambda_0(q, \epsilon, \|\nabla'\omega\|_{\mathscr{B}^2(\boldsymbol{R}^{n-1})}) \geqq 1$ and $K_0 = K_0(q,\epsilon)$ with $0 < K_0 \leqq 1$ such that if $\|\nabla'\omega\|_{L_\infty(\boldsymbol{R}^{n-1})} \leqq K_0$ and $|\lambda| \geqq \lambda_0$, then the problem* (4.1) *admits a unique solution $(u,p) \in W^2_q(H)^n \times \hat{W}^1_q(H)$ which satisfies the estimate: $\mathscr{I}_\lambda(u,p,H) \leqq C\mathscr{D}_\lambda(f,g,h,H)$ with some constant $C = C(q,\epsilon,\|\nabla'\omega\|_{\mathscr{B}^2(\boldsymbol{R}^{n-1})}) > 0$. Here and hereafter, we set*

$$\|\nabla'\omega\|_{\mathscr{B}^k(\boldsymbol{R}^{n-1})} = \sum_{1\leqq|\alpha'|\leqq k+1} \big\|\partial^{\alpha'}_{x'}\omega\big\|_{L_\infty(\boldsymbol{R}^{n-1})}, \quad k = 1,2$$

PROOF. First of all, we shall make one remark concerning the boundary condition. If u satisfies n boundary conditions:

$$\nu\cdot u=0,\ \alpha u_k+\beta\sum_{\ell=1}^n D_{k\ell}(u)\nu_\ell-\langle D(u)\nu,\nu\rangle\nu_k=h_k,\ (k=1,\ldots,n-1) \tag{4.2}$$

on ∂H, then u satisfies the following condition automatically:

$$\alpha u_n+\beta\sum_{\ell=1}^n D_{n\ell}(u)\nu_\ell-\langle D(u)\nu,\nu\rangle\nu_n=h_n \tag{4.3}$$

on ∂H provided that $\nu\cdot h=0$ on ∂H. In fact, by (4.2) and the facts that $\nu\cdot h=0$ and $\nu\cdot\nu=1$ on ∂H, we have

$$\begin{aligned}0&=\big\langle \alpha u+\beta(D(u)\nu-\langle D(u)\nu,\nu\rangle\nu),\nu\big\rangle\\&=\sum_{k=1}^{n-1}\nu_kh_k+\nu_n\bigg\{\alpha u_n+\beta\bigg(\sum_{\ell=1}^n D_{n\ell}(u)\nu_\ell-\bigg(\sum_{\ell,m=1}^n D_{\ell m}(u)\nu_\ell\nu_m\bigg)\nu_n\bigg)\bigg\}\\&=\nu_n\bigg\{-h_n+\alpha u_n+\beta\bigg(\sum_{\ell=1}^n D_{n\ell}(u)\nu_\ell-\bigg(\sum_{\ell,m=1}^n D_{\ell m}(u)\nu_\ell\nu_m\bigg)\nu_n\bigg)\bigg\}.\end{aligned} \tag{4.4}$$

Since $\nu_n=-(1+|\nabla'\omega|^2)^{-1/2}\neq 0$, (4.4) implies (4.3).

Now, we shall prove the theorem. For this purpose, first of all we reduce the problem (4.1) with (4.2) to the half space problem. Let us introduce the transformation $\varphi: H\to \boldsymbol{R}$ defined by $y=\varphi(x)=(x',x_n-\omega(x'))$. Obviously, φ is a bijection whose Jacobian is equal to 1. For a function or a vector field w defined on H, we define $\tilde w(y)$ by the formula: $\tilde w(y)=w(x)$. Note that

$$\frac{\partial}{\partial x_j}=\frac{\partial}{\partial y_j}-\frac{\partial\omega}{\partial y_j}\frac{\partial}{\partial y_n}\ (j=1,\ldots,n-1),\quad \frac{\partial}{\partial x_n}=\frac{\partial}{\partial y_n}.$$

Below, for the notational simplicity we write $K_0=\|\nabla'\omega\|_{L_\infty(\boldsymbol{R}^{n-1})}$ and $K_i=\|\nabla'\omega\|_{\mathscr{B}^i(\boldsymbol{R}^{n-1})}$ $(i=1,2)$. Since we shall choose K_0 small enough, we may assume that $0<K_0\leqq 1$ from the beginning.

Let (u,p) solve (4.1). If we set

$$v_j(y)=\tilde u_j(y)\ (j=1,\ldots,n-1),\quad v_n(y)=\tilde u_n(y)-\sum_{\ell=1}^{n-1}\frac{\partial\omega}{\partial y_\ell}(y')\tilde u_\ell(u),\quad \theta(y)=\tilde p(y) \tag{4.5}$$

we see that the problem (4.1) is reduced to the equation:

$$\begin{aligned}&\lambda v-\operatorname{Div}T(v,\theta)=\tilde f+S(\lambda,v,\theta),\quad \operatorname{div}v=\tilde g &&\text{in } \boldsymbol{R}^n_+\\&v_n=0,\quad \alpha v_j-\beta\partial_n v_j=(1+|\nabla'\omega|^2)^{-1/2}\tilde h_j+S^\alpha_\partial(v) &&\text{on } \partial\boldsymbol{R}^n_+\end{aligned} \tag{4.6}$$

for $j=1,\dots,n-1$, where $S(\lambda,v,\theta)$ and $S^\alpha_\partial(v)$ are suitable functions which are linear with respect to v, ∇v, $\nabla^2 v$ and $\nabla\theta$, and satisfy the following estimates:

$$
\begin{aligned}
\|S(\lambda,v,\theta)\|_{L_q(\boldsymbol{R}^n_+)} &\leqq C\Big\{K_0\big(|\lambda|\|v\|_{L_q(\boldsymbol{R}^n_+)}+\|\nabla^2 v\|_{L_q(\boldsymbol{R}^n_+)}+\|\nabla\theta\|_{L_q(\boldsymbol{R}^n_+)}\big)\\
&\qquad + K_1\|\nabla v\|_{L_q(\boldsymbol{R}^n_+)}+K_2\|v\|_{L_q(\boldsymbol{R}^n_+)}\Big\}\\
\big\|S^\alpha_\partial(v)\big\|_{L_q(\boldsymbol{R}^n_+)} &\leqq C\Big\{K_0\|\nabla v\|_{L_q(\boldsymbol{R}^n_+)}+(K_1+\alpha)\|v\|_{L_q(\boldsymbol{R}^n_+)}\Big\}\\
\big\|\nabla S^\alpha_\partial(v)\big\|_{L_q(\boldsymbol{R}^n_+)} &\leqq C\Big\{K_0\|\nabla^2 v\|_{L_q(\boldsymbol{R}^n_+)}+K_1\big(\|\nabla v\|_{L_q(\boldsymbol{R}^n_+)}+\alpha\|v\|_{L_q(\boldsymbol{R}^n_+)}\big)\\
&\qquad + K_2\|v\|_{L_q(\boldsymbol{R}^n_+)}\Big\}.
\end{aligned}
\tag{4.7}
$$

Given $(v,\theta)\in W^2_q(\boldsymbol{R}^n_+)^n\times \hat W^1_q(\boldsymbol{R}^n_+)$, let (w,κ) be a solution to the equation:

$$
\begin{aligned}
&\lambda w-\operatorname{Div}T(w,\kappa)=\tilde f+S(\lambda,v,\theta),\quad \operatorname{div}v=\tilde g && \text{in } \boldsymbol{R}^n_+\\
&w_n=0,\quad \alpha w_j-\beta\partial_n w_j=(1+|\nabla'\omega|^2)^{-1/2}\tilde h_j+S^\alpha_\partial(v) && \text{on } \partial\boldsymbol{R}^n_+
\end{aligned}
$$

for $j=1,\dots,n-1$. Applying Theorem 3.1 and using (4.7), we see that (w,κ) exists uniquely in $W^2_q(\boldsymbol{R}^n_+)^n\times\hat W^1_q(\boldsymbol{R}^n_+)$ and satisfies the estimate:

$$
\begin{aligned}
\mathscr{I}_\lambda\big(w,\kappa,\boldsymbol{R}^n_+\big) &\leqq C\Big\{\mathscr{D}_\lambda\big(\tilde f,\tilde g,\tilde h,\boldsymbol{R}^n_+\big)+K_0\mathscr{I}_\lambda\big(v,\theta,\boldsymbol{R}^n_+\big)\\
&\qquad + K_1\|\nabla v\|_{L_q(\boldsymbol{R}^n_+)}+((\alpha+K_1)|\lambda|^{1/2}+K_2)\|v\|_{L_q(\boldsymbol{R}^n_+)}\Big\}
\end{aligned}
\tag{4.8}
$$

provided that $|\lambda|\geqq 1$. Let us define the map G by the formula: $G(v,\theta)=(w,\kappa)$, and then G is a linear map from $W^2_q(\boldsymbol{R}^n_+)^n\times\hat W^1_q(\boldsymbol{R}^n_+)$ into itself. If we choose K_0 and λ_0 in such a way that $C(K_0+(2K_1+\alpha)\lambda_0^{-1/2}+\lambda_0^{-1}K_2)\leqq 1/2$, then by (4.8) G becomes a contraction map on $W^2_q(\boldsymbol{R}^n_+)^n\times\hat W^{-1}_q(\boldsymbol{R}^n_+)$ provided that $\lambda\in\Sigma_\epsilon$ and $|\lambda|\geqq\lambda_0$, which shows the unique existence of solutions (v,θ) of the equation (4.6). Moreover, by (4.8) we see that $\mathscr{I}_\lambda(v,\theta,\boldsymbol{R}^n_+)\leqq 2C\mathscr{D}_\lambda(\tilde f,\tilde g,\tilde h,\boldsymbol{R}^n_+)$, which completes the proof of Theorem 4.1. □

5. A priori estimate.

In this section, we shall show the following three theorems and Theorem 1.4.

Theorem 5.1. *Let $1<q<\infty$ and $0<\epsilon<\pi/2$. Let $\lambda\in\Sigma_\epsilon$, $f\in L_q(\Omega)^n$, $g\in W^1_q(\Omega)\cap\hat W^{-1}_q(\Omega)$ and $h\in W^1_{q,\partial}(\Omega)$. Let $(u,p)\in W^2_q(\Omega)^n\times\hat W^1_q(\Omega)$ solve the equation* (1.1). *Then, there exists a constant $\lambda_1\geqq 1$ depending on ϵ, q and Ω such that $\mathscr{I}_\lambda(u,p,\Omega)\leqq C_{q,\Omega,\epsilon}\mathscr{D}_\lambda(f,g,h,\Omega)$ provided that $|\lambda|\geqq\lambda_1$.*

Theorem 5.2. *Let $1<q<\infty$ and $0<\epsilon<\pi/2$. Assume that Ω is an exterior* *domain. Let K be a compact set in Σ_ϵ. Let $f \in L_q(\Omega)^n$, $g \in W^1_q(\Omega) \cap \hat{W}^{-1}_q(\Omega)$ and $h \in W^1_{q,\partial}(\Omega)$. Assume that for $\lambda \in K$ the uniqueness of solutions to* (1.1) *holds. If $(u,p) \in W^2_q(\Omega)^n \times \hat{W}^1_q(\Omega)$ solves the equation* (1.1) *for $\lambda \in K$, then we have*

$$\|u\|_{W^2_q(\Omega)} + \|p\|_{W^1_q(\Omega)} \leqq C\Big(\|f\|_{L_q(\Omega)} + \|(g,h)\|_{W^1_q(\Omega)} + \|g\|_{\hat{W}^{-1}_q(\Omega)}\Big)$$

where C is a constant that depends on q, ϵ, Ω and K.

THEOREM 5.3. *Let $1 < q < \infty$ and $0 < \epsilon < \pi/2$. Assume that Ω is a bounded domain. Let K be a compact set in $\boldsymbol{C}$. Let $f \in L_q(\Omega)^n$, $g \in W_{q,\mathrm{div}}(\Omega)$ and $h \in W^1_{q,\partial}(\Omega)$. Assume that for $\lambda \in K$ the uniqueness of solutions to* (1.1) *holds. If $(u,p) \in W^2_q(\Omega)^n \times \hat{W}^1_q(\Omega)$ solves the equation* (1.1) *for $\lambda \in K$, then we have*

$$\|u\|_{W^2_q(\Omega)} + \|p\|_{W^1_q(\Omega)} \leqq C\Big(\|f\|_{L_q(\Omega)} + \|(g,h)\|_{W^1_q(\Omega)}\Big)$$

where C is a constant that depends on q, ϵ, Ω and K.

In order to prove Theorems 5.1, 5.2 and 5.3, we start with the following theorem.

THEOREM 5.4. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ and $\lambda_0 > 0$. Let $\lambda \in \Sigma_\epsilon$, $f \in L_q(\Omega)^n$, $g \in W^1_q(\Omega) \cap \hat{W}^{-1}_q(\Omega)$ and $h \in W^1_{q,\partial}(\Omega)$. Let $(u,p) \in W^2_q(\Omega)^n \times \hat{W}^1_q(\Omega)$ solve the equation* (1.1). *Then, there exists a constant C depending on λ_0, ϵ, q and Ω such that*

$$\begin{aligned}\mathscr{I}_\lambda(u,p,\Omega) \leqq C\Big(&\mathscr{D}_\lambda(f,g,h,\Omega) + \|\nabla u\|_{L_q(\Omega_R)} \\ &+ |\lambda|^{1/2}\|u\|_{L_q(\Omega_R)} + |\lambda|\|u\|_{\dot{W}^{-1}_q(\Omega_R)} + \|p\|_{L_q(\Omega_R)}\Big)\end{aligned} \tag{5.1}$$

provided that $|\lambda| \geqq \lambda_0$.

PROOF. Let $(u,p) \in W^2_q(\Omega)^n \times \hat{W}^1_q(\Omega)$ solve the equation (1.1). Given cut-off function $\varphi \in C^\infty(\boldsymbol{R}^n)$, we have

$$\begin{aligned}\lambda(\varphi u) - \mathrm{Div}\, T(\varphi u, \varphi p) = f_\varphi, \quad \mathrm{div}\,(\varphi u) = g_\varphi \quad \text{in } \Omega \\ \nu\cdot(\varphi u) = 0, \quad B_{\alpha,\beta}(\varphi u) = h_\varphi \quad \text{on } \Gamma\end{aligned} \tag{5.2}$$

where we have set

$$(f_\varphi)_i = \varphi f_i - \sum_{j=1}^n \frac{\partial\varphi}{\partial x_j} D_{ij}(u) - \sum_{j=1}^n \frac{\partial}{\partial x_j}\left(\frac{\partial\varphi}{\partial x_j}u_i + \frac{\partial\varphi}{\partial x_i}u_j\right) + \frac{\partial\varphi}{\partial x_i}p;$$

$$(h_\varphi)_i = \varphi h_i + \beta\sum_{j=1}^n\left(\frac{\partial\varphi}{\partial x_j}u_i + \frac{\partial\varphi}{\partial x_i}u_j\right)\nu_j - \beta\bigg(\sum_{j,k=1}^n\left(\frac{\partial\varphi}{\partial x_k}u_j + \frac{\partial\varphi}{\partial x_j}u_k\right)\nu_j\nu_k\bigg)\nu_i;$$

$$g_\varphi = \mathrm{div}\,(\varphi u) = \varphi g + (\nabla\varphi)\cdot u$$

and k_i denotes the i-th component of n vector k. Here and hereafter, $\nu = {}^t(\nu_1, \dots, \nu_n)$ is suitably extended into $\boldsymbol{R}^n$ as a vector of functions in $C^{2,1}(\boldsymbol{R}^n)$ having compact support. First of all, we shall derive an estimate near the boundary. Pick $x_0 \in \Gamma$ up and consider a small neighborhood of $B_\sigma(x_0) = \{x \in \boldsymbol{R}^n \mid |x - x_0| < \sigma\}$ of x_0, where σ will be chosen later. Let φ be a cut-off function in $C_0^\infty(B_\sigma(x_0))$ such that $\varphi(x) = 1$ on $B_{\sigma/2}(x_0)$. Let $\mathscr{O}$ be an orthogonal matrix such that $\mathscr{O}\nu(x_0) = {}^t(0, \dots, 0, -1)$. Consider the change of variable: $x = x_0 + \mathscr{O}y$ and set $(\varphi u)(x) = v(y), (\varphi p)(x) = \theta(y)$ and $w(y) = {}^t\mathscr{O}v(y)$. Then, we have

$$
\begin{aligned}
&\lambda w - \operatorname{Div} T(w, \theta) = F, \ \operatorname{div} w = G && \text{in } \tilde{\Omega} \\
&\tilde{\nu} \cdot w = 0, \ \alpha w + \beta(D(w)\tilde{\nu} - \langle D(w)\tilde{\nu}, \tilde{\nu}\rangle\tilde{\nu}) = F_\partial && \text{on } \tilde{\Gamma}
\end{aligned}
$$

where $\tilde{\Omega} = \{{}^t\mathscr{O}(x - x_0) \mid x \in \Omega\}$, $\tilde{\nu}(y) = {}^t\mathscr{O}\nu(y)$, $F(y) = {}^t\mathscr{O}f_\varphi(x)$, $G(y) = g_\varphi(x)$ and $F_\partial(y) = {}^t\mathscr{O}h_\varphi(x)$. Let ϵ_0 and ϵ_1 be two small positive numbers such that $0 < 2\epsilon_0 < \epsilon_1$ and

$$
\begin{aligned}
B_{\epsilon_0} \cap \tilde{\Omega} &\subset \big\{y = (y_1, \dots, y_n) \mid y_n > \psi(y'), \ \ y' \in B'_{\epsilon_1}\big\} \\
B_{\epsilon_0} \cap \tilde{\Gamma} &\subset \big\{y = (y_1, \dots, y_n) \mid y_n = \psi(y'), \ \ y' \in B'_{\epsilon_1}\big\}
\end{aligned}
$$

for some $\psi \in C^{2,1}(B'_{\epsilon_1})$ that satisfies the conditions:

$$
\psi(0) = 0, \quad \nabla'\psi(0) = 0, \quad \tilde{\nu} = (\nabla'\psi, -1)/\sqrt{1 + |\nabla'\psi|^2}
$$

where $\nabla'\psi = (\partial_1\psi, \dots, \partial_{n-1}\psi)$, $B_{\epsilon_0} = \{y \in \boldsymbol{R}^n \mid |y| < \epsilon_0\}$, and $B'_{\epsilon_1} = \{y' \in \boldsymbol{R}^{n-1} \mid |y'| < \epsilon_1\}$. Let $\rho(y')$ be a function in $C_0^\infty(\boldsymbol{R}^{n-1})$ such that $\rho(y') = 1$ for $|y'| \leqq 1$ and $\rho(y') = 0$ for $|y'| \geqq 2$ and set $\omega(y') = \rho(y'/\epsilon_0)\psi(y')$,

$$
\begin{aligned}
H &= \big\{y = (y_1, \dots, y_n) \in \boldsymbol{R}^n \mid y_n > \omega(y'), \ \ y' \in \boldsymbol{R}^{n-1}\big\}, \\
\partial H &= \big\{y = (y_1, \dots, y_n) \in \boldsymbol{R}^n \mid y_n = \omega(y'), \ \ y' \in \boldsymbol{R}^{n-1}\big\}, \\
\nu_\omega &= (\nabla'\omega, -1)/\sqrt{1 + |\nabla'\omega|^2}.
\end{aligned}
$$

Let us choose $\sigma > 0$ so small that $\operatorname{supp} w$ and $\operatorname{supp} \theta \subset B_{\epsilon_0}$, and then we finally arrive at the equation:

$$
\begin{aligned}
&\lambda w - \operatorname{Div} T(w, \theta) = F, \ \operatorname{div} w = G && \text{in } H \\
&\nu_\omega \cdot w = 0, \ \alpha w + \beta(D(w)\nu_\omega - \langle D(w)\nu_\omega, \nu_\omega\rangle\nu_\omega) = F_\partial && \text{on } \partial H.
\end{aligned} \tag{5.3}
$$

Moreover, we have $\|\nabla'\omega\|_{L_\infty(\boldsymbol{R}^{n-1})} \leqq C(\epsilon_1)\epsilon_0$ with some constant $C(\epsilon_1)$ that depends on ϵ_1 but does not depend on ϵ_0. Let $K_0 = K_0(q, \epsilon)$ be the positive number given in Theorem 4.1. Let us choose $\epsilon_0 > 0$ so small that $C(\epsilon_1)\epsilon_0 \leqq K_0$. Since $\tilde{\nu} = \nu_\omega$ on $\operatorname{supp} F_\partial$, we see that $\nu_\omega \cdot F_\partial = 0$ on ∂H. Then, applying Theorem 4.1 to (5.3), we see that there exist constants $\Lambda \geqq 1$ and $C > 0$ depending on ϵ, q, ϵ_0 and ϵ_1 such that

$$\mathscr{I}_\lambda(w,\theta,H) \leqq C\mathscr{D}_\lambda(F,G,F_\partial,H) \tag{5.4}$$

provided that $\lambda \in \Sigma_\epsilon$ and $|\lambda| \geqq \Lambda$. We may assume that $\operatorname{supp}\varphi \subset B_R$, and then noting that $|\lambda| \geqq \Lambda \geqq 1$, we see easily that

$$\begin{aligned}
&\|F\|_{L_q(H)} \leqq C(x_0)\Big\{\|u\|_{W^1_q(\Omega_R)} + \|p\|_{L_q(\Omega_R)} + \|f\|_{L_q(\Omega_R)}\Big\};\\
&\|\nabla(G,F_\partial)\|_{L_q(H)} + |\lambda|^{1/2}\|(G,F_\partial)\|_{L_q(H)}\\
&\quad \leqq C(x_0)\big\{\|\nabla(g,h)\|_{L_q(\Omega_R)} + |\lambda|^{1/2}\|(g,h)\|_{L_q(\Omega_R)} + \|\nabla u\|_{L_q(\Omega_R)} + |\lambda|^{1/2}\|u\|_{L_q(\Omega_R)}\big\}.
\end{aligned} \tag{5.5}$$

Now, we shall estimate $\|G\|_{\hat W^{-1}_q(H)}$. Recall that

$$\|G\|_{\hat W^{-1}_q(H)} = \sup\Big\{|(G,v)_H| \mid v \in \hat W^1_{q'}(H),\ \|\nabla v\|_{L_{q'}(H)} = 1\Big\} \tag{5.6}$$

Given $v \in \hat W^1_{q'}(H)$, we set $\Phi(x) = v(z)$ with $z = {}^t\mathscr{O}(x - x_0)$. Defining the constant c by the formulas:

$$c = \begin{cases} \int_\Omega \varphi\Phi\,dx/\int_\Omega \varphi\,dx & \text{when } \Omega \text{ is a bounded domain}\\ \frac{1}{\mu((\operatorname{supp}\varphi)\cap\Omega)}\int_{\operatorname{supp}\varphi\cap\Omega}\Phi\,dx & \text{when } \Omega \text{ is an exterior domain}\end{cases}$$

where μ denotes the Lebesgue measure on $\boldsymbol{R}^n$, we set $\Phi_0 = \Phi - c$. When Ω is a bounded domain, the fact that $\int_\Omega \varphi\Phi_0\,dx = 0$ implies that $\varphi\Phi_0 \in \hat W^1_{q'}(\Omega)$; and when Ω is an exterior domain, the fact that $(\nabla\varphi)\Phi_0$ vanishes on S_R implies that $(\nabla\varphi)\Phi_0 \in \dot W^1_{q'}(\Omega_R)$. Recalling that $G(z) = (\operatorname{div}(\varphi u))(x)$, that $\nu\cdot(\varphi u) = 0$ on Γ, and that $\operatorname{supp}\varphi\cap\Omega \subset \Omega_R$, we have

$$\begin{aligned}
|(G,v)_H| &= |(\operatorname{div}(\varphi u),\Phi)_\Omega| = |(\varphi u,\nabla\Phi_0)_\Omega| \leqq |(g,\varphi\Phi_0)_\Omega| + |(u,(\nabla\varphi)\Phi_0)_\Omega|\\
&\leqq \|g\|_{\hat W^{-1}_q(\Omega)}\|\nabla(\varphi\Phi_0)\|_{L_{q'}(\Omega)} + \|u\|_{\dot W^{-1}_q(\Omega_R)}\|(\nabla\varphi)\Phi_0\|_{\dot W^1_{q'}(\Omega_R)}.
\end{aligned} \tag{5.7}$$

When Ω is an exterior domain, we have $\int_{\operatorname{supp}\varphi\cap\Omega}\Phi_0\,dx = 0$, and therefore by Poincaré's inequality we have

$$\|\nabla(\varphi\Phi_0)\|_{L_{q'}(\Omega)},\quad \|(\nabla\varphi)\Phi_0\|_{\dot W^1_{q'}(\Omega_R)} \leqq C\|\nabla\Phi\|_{L_{q'}(\Omega)}. \tag{5.8}$$

When Ω is a bounded domain, we also have (5.8), because we know the estimate:

$$\|\Phi - c\|_{L_{q'}(\Omega)} \leqq C\|\nabla\Phi\|_{L_{q'}(\Omega)} \tag{5.9}$$

for any $\Phi \in W^1_{q'}(\Omega)$ with $c = \int_\Omega \varphi\Phi\,dx/\int_\Omega\varphi\,dx$. In fact, we can show (5.9) by contradiction as follows. Suppose that for any natural number m there exists a $\Phi_m \in W^1_{q'}(\Omega)$

such that $\|\Phi_m - c_m\|_{L_{q'}(\Omega)} > m\|\nabla\Phi_m\|_{L_{q'}(\Omega)}$ where $c_m = \int_\Omega \varphi\Phi_m\,dx / \int_\Omega \varphi\,dx$. Set $\Psi_m = (\Phi_m - c_m)/\|\Phi_m - c_m\|_{L_{q'}(\Omega)}$, and then $\|\Psi_m\|_{L_q(\Omega)} = 1$ and $\|\nabla\Psi_m\|_{L_{q'}(\Omega)} < 1/m$. Since $\|\Psi_m\|_{W^1_{q'}(\Omega)} \leqq 1 + 1/m \leqq 2$ for any natural number m, passing to a subsequence if necessary, we may assume that there exists a $\Psi \in W^1_{q'}(\Omega)$ such that $\Psi_m \to \Psi$ weakly in $W^1_{q'}(\Omega)$ and strongly in $L_{q'}(\Omega)$ as $m \to \infty$. In particular, we have $\nabla\Psi = 0$, that is $\Psi = c$ (constant), and $\|\Psi\|_{L_{q'}(\Omega)} = 1$.

On the other hand, we see that $\int_\Omega \varphi\Psi\,dx = 0$, because

$$\int_\Omega \varphi\Psi_m\,dx = \frac{1}{\|\Phi_m - c_m\|_{L_{q'}(\Omega)}}\left(\int_\Omega \varphi\Phi_m\,dx - \int_\Omega \varphi\,dx c_m\right) = 0.$$

Therefore, $0 = \int_\Omega \varphi\Psi\,dx = c\int_\Omega \varphi\,dx$, which implies that $0 = c = \Psi$. This contradicts to the fact that $\|\Psi\|_{L_{q'}(\Omega)} = 1$, which shows that (5.9) holds.

By (5.6), (5.7) and (5.8), we have $\|G\|_{\hat W^{-1}_q(H)} \leqq C_\varphi(\|g\|_{\hat W^{-1}_q(\Omega)} + \|u\|_{\dot W^{-1}_q(\Omega_R)})$, which combined with (5.4) and (5.5) implies that

$$\begin{aligned}\mathscr{I}_\lambda(u,p,\Omega_{x_0}) \leqq C_{x_0}\Big[&\|f\|_{L_q(\Omega_R)} + \|\nabla(g,h)\|_{L_q(\Omega_R)} + |\lambda|^{1/2}\|(g,h)\|_{L_q(\Omega_R)}\\ &+ \|\nabla u\|_{L_q(\Omega_R)} + |\lambda|^{1/2}\|u\|_{L_q(\Omega_R)} + |\lambda|\|u\|_{\dot W^{-1}_q(\Omega_R)} + \|p\|_{L_q(\Omega_R)}\Big]\end{aligned}$$

provided that $\lambda \in \Sigma_\epsilon$ and $|\lambda| \geqq \Lambda$, where C_{x_0} is a constant depending on x_0 and $\Omega_{x_0} = \{x \in \boldsymbol{R}^n \mid |x - x_0| < \sigma/2\}$. Since Γ is compact, covering Γ by a finite number of small neighborhoods like Ω_{x_0}, we see that there exist a subdomain Ω' of Ω and two numbers $\lambda_1 \geqq 1$ and C such that $\overline{\Omega'} \supset \Gamma$ and

$$\begin{aligned}\mathscr{I}_\lambda(u,p,\Omega') \leqq C\Big[&\|f\|_{L_q(\Omega_R)} + \|\nabla(g,h)\|_{L_q(\Omega_R)} + |\lambda|^{1/2}\|(g,h)\|_{L_q(\Omega_R)} + |\lambda|\|g\|_{\hat W^{-1}_q(\Omega)}\\ &+ \|\nabla u\|_{L_q(\Omega_R)} + |\lambda|^{1/2}\|u\|_{L_q(\Omega_R)} + |\lambda|\|u\|_{\dot W^{-1}_q(\Omega_R)} + \|p\|_{L_q(\Omega_R)}\Big] \quad (5.10)\end{aligned}$$

provided that $\lambda \in \Sigma_\epsilon$ and $|\lambda| \geqq \lambda_1$.

Now, let λ be any complex number such as $|\lambda| \leqq \lambda_1$. If $(u,p) \in W^2_q(\Omega)^n \times \hat W^1_q(\Omega)$ satisfy (1.1), then it also satisfies the equation:

$$\begin{aligned}&\lambda_1 u - \operatorname{Div} T(u,p) = f + (\lambda_1 - \lambda)u,\ \operatorname{div} u = g \quad &&\text{in } \Omega\\ &\nu \cdot u = 0,\ \ B_{\alpha,\beta}(u) = h &&\text{on } \Gamma.\end{aligned} \quad (5.11)$$

Therefore, applying (5.10) to (5.11), we have

$$\begin{aligned}\|u\|_{W^2_q(\Omega')} + \|\nabla p\|_{L_q(\Omega')} \leqq C\Big[&\|f\|_{L_q(\Omega_R)} + \|(g,h)\|_{W^1_q(\Omega_R)} + \|g\|_{\hat W^{-1}_q(\Omega)}\\ &+ \|u\|_{W^1_q(\Omega_R)} + \|u\|_{\dot W^{-1}_q(\Omega_R)} + \|p\|_{L_q(\Omega_R)}\Big]\end{aligned}$$

which combined with (5.10) implies that

$$\mathscr{I}_\lambda(u,p,\Omega') \leqq C\Big[\mathscr{D}_\lambda(f,g,h,\Omega) + \|\nabla u\|_{L_q(\Omega_R)} + |\lambda|^{1/2}\|u\|_{L_q(\Omega_R)} \\ + |\lambda|\|u\|_{\dot{W}_q^{-1}(\Omega_R)} + \|p\|_{L_q(\Omega_R)}\Big] \tag{5.12}$$

provided that $\lambda \in \Sigma_\epsilon$ and $|\lambda| \geqq \lambda_0$, where C is a constant that depends on ϵ, q, Ω, and λ_0.

Now, we shall estimate u and p on $\Omega \setminus \Omega'$. Let δ be a positive number such that $\Omega' \supset \{x \in \Omega \mid \mathrm{dist}\,(x,\Gamma) < 4\delta\}$ and let φ be a function in $C^\infty(\boldsymbol{R}^n)$ such that $\varphi(x) = 1$ for $x \in D_{3\delta} = \{x \in \Omega \mid \mathrm{dist}\,(x,\Gamma) > 3\delta\}$ and $\varphi(x) = 0$ for $x \in E_\delta = \Omega^c \cup \{x \in \Omega \mid \mathrm{dist}\,(x,\Gamma) < \delta\}$. From (5.2) it follows that

$$\lambda(\varphi u) - \mathrm{Div}\, T(\varphi u, \varphi p) = f_\varphi, \quad \mathrm{div}\,(\varphi u) = g_\varphi \text{ in } \boldsymbol{R}^n$$

and therefore by Theorem 2.1 we have

$$\mathscr{I}_\lambda(\varphi u, \varphi p, \boldsymbol{R}^n) \leqq C_{\epsilon,q}\Big[\|f_\varphi\|_{L_q(\boldsymbol{R}^n)} + \|\nabla g_\varphi\|_{L_q(\boldsymbol{R}^n)} + |\lambda|\|g_\varphi\|_{\hat{W}_q^{-1}(\boldsymbol{R}^n)}\Big] \tag{5.13}$$

provided that $\lambda \in \Sigma_\epsilon$. We see easily that

$$\|f_\varphi\|_{L_q(\boldsymbol{R}^n)} + \|\nabla g_\varphi\|_{L_q(\boldsymbol{R}^n)} \leqq C\Big[\|f\|_{L_q(\Omega)} + \|g\|_{W_q^1(\Omega)} + \|u\|_{W_q^1(\Omega_R)} + \|p\|_{L_q(\Omega_R)}\Big]. \tag{5.14}$$

To estimate $\|g_\varphi\|_{\hat{W}_q^{-1}(\boldsymbol{R}^n)}$, for any $\psi \in C_0^\infty(\boldsymbol{R}^n)$ we observe that

$$\begin{aligned} \big|(g_\varphi,\psi)_{\boldsymbol{R}^n}\big| &= \big|(\mathrm{div}\,(\varphi u),\psi)_{\boldsymbol{R}^n}\big| = \big|(\varphi u, \nabla(\psi - c_\psi))_{\boldsymbol{R}^n}\big| \\ &\leqq |(g,\varphi(\psi - c_\psi))_\Omega| + |(u,(\nabla\varphi)(\psi - c_\psi))_\Omega| \\ &\leqq \|g\|_{\hat{W}_q^{-1}(\Omega)}\|\nabla(\varphi(\psi - c_\psi))\|_{L_{q'}(\Omega)} + \|u\|_{\dot{W}_q^{-1}(\Omega_R)}\|(\nabla\varphi)(\psi - c_\psi)\|_{W_{q'}^1(\Omega_R)} \end{aligned}$$

where we have set

$$c_\psi = \begin{cases} \int_\Omega \varphi\psi\,dx / \int_\Omega \varphi\,dx & \text{when } \Omega \text{ is a bounded domain} \\ \frac{1}{\mu(\mathrm{supp}\,(\nabla\varphi))}\int_{\mathrm{supp}\,(\nabla\varphi)} \psi\,dx & \text{when } \Omega \text{ is an exterior domain.} \end{cases}$$

By Poincaré's inequality and (5.9),

$$\|\nabla(\varphi(\psi - c_\psi))\|_{L_{q'}(\Omega)}, \ \|(\nabla\varphi)(\psi - c_\psi)\|_{W_{q'}^1(\Omega_R)} \leqq C\|\nabla\psi\|_{L_{q'}(\boldsymbol{R}^n)}$$

and therefore we have $\|g_\varphi\|_{\hat{W}_q^{-1}(\boldsymbol{R}^n)} \leqq C[\|g\|_{\hat{W}_q^{-1}(\Omega)} + \|u\|_{\dot{W}_q^{-1}(\Omega_R)}]$, which combined with (5.13) and (5.14) implies that

$$\mathscr{I}_\lambda(u,p,D_{3\delta}) \leqq C\Big[\|f\|_{L_q(\Omega)} + \|g\|_{W^1_q(\Omega)} + |\lambda|\|g\|_{\hat{W}^{-1}_q(\Omega)} + \|u\|_{W^1_q(\Omega_R)} + |\lambda|\|u\|_{\dot{W}^{-1}_q(\Omega_R)} + \|p\|_{L_q(\Omega_R)}\Big]. \tag{5.15}$$

Combining (5.10) and (5.15) completes the proof of Theorem 5.3, because $\Omega' \cup D_{3\delta} = \Omega$. □

A proof of Theorem 5.1. In view of Theorem 5.4, to prove Theorem 5.1 we have to estimate the terms: $\|p\|_{L_q(\Omega_R)}$ and $|\lambda|\|u\|_{\dot{W}^{-1}_q(\Omega_R)}$. First of all, we shall estimate $\|p\|_{L_q(\Omega_R)}$. For this purpose, we shall use the following two propositions.

Proposition 5.5. *Let $1 < q < \infty$ and D be a bounded domain whose boundary ∂D is a $C^{1,1}$ compact hypersurface. Let $f \in L_q(D)$ and $g \in W^1_q(D)$ and assume that $\int_D f\,dx = \int_{\partial D} g\,d\sigma$. Then, there exists a unique $u \in W^2_q(D)$ which solves the equation: $\Delta u = f$ in D and $\partial_\nu u = g$ on ∂D with side condition: $\int_D u\,dx = 0$, and satisfies the estimate: $\|u\|_{W^2_q(D)} \leqq C_q\{\|f\|_{L_q(D)} + \|g\|_{W^1_q(D)}\}$. Here, ν denotes the unit outer normal to ∂D and $\partial_\nu = \nu\cdot\nabla$.*

Proposition 5.6. *Let $1 < q < \infty$ and assume that Ω is an exterior domain whose boundary is a $C^{1,1}$ compact hypersurface. Set*

$$\hat{W}^2_q(\Omega) = \{u \in L_{q,\mathrm{loc}}(\overline{\Omega}) \mid \nabla u \in W^1_q(\Omega)\}$$
$$\hat{L}_{q,R-1}(\Omega) = \Big\{f \in L_q(\Omega) \mid f(x) = 0 \ \ \textit{for } |x| \geqq R-1 \textit{ and } \int_\Omega f\,dx = 0\Big\}.$$

Then, for every $f \in \hat{L}_{q,R-1}(\Omega)$ there exists a $u \in \hat{W}^2_q(\Omega)$ which uniquely solves the equation: $\Delta u = f$ in Ω and $\partial_\nu u = 0$ on Γ and satisfies the estimate:

$$\|u\|_{L_q(\Omega\cap B_R)} + \sup_{|x|\geqq R}|x|^{n-1}|u(x)| + \|\nabla u\|_{W^1_q(\Omega)} \leqq C\|f\|_{L_q(\Omega)}. \tag{5.16}$$

Here, ν denotes the unit outer normal to Γ and $\partial_\nu = \nu\cdot\nabla$.

Propositions 5.5 and 5.6 seems to be well-known. But, to make the paper self-contained as much as possible, we shall prove both propositions in the appendix, because we could not find any proofs in the literatures.

First, we consider the case where Ω is a bounded domain. From the definition of $\hat{W}^1_q(\Omega)$ we may assume that $\int_\Omega p\,dx = 0$. Given $\varphi \in C^\infty_0(\Omega)$, we set $\varphi_0 = \varphi - \mu(\Omega)^{-1}\int_\Omega \varphi\,dx$ and observe that $(p,\varphi)_\Omega = (p,\varphi_0)_\Omega$. Since $\int_\Omega \varphi_0\,dx = 0$, by Proposition 5.5 there exists a $\psi \in W^2_{q'}(\Omega)$ which satisfies the equation: $\Delta\psi = \varphi_0$ in Ω and $\partial_\nu\psi = 0$ on Γ, the side condition: $\int_\Omega \psi\,dx = 0$ and the estimate: $\|\psi\|_{W^2_{q'}(\Omega)} \leqq C\|\varphi\|_{L_{q'}(\Omega)}$. Using such ψ, we have $(p,\varphi_0)_\Omega = (p,\Delta\psi)_\Omega = -(\nabla p,\nabla\psi)_\Omega$. Now, we use the relations: $\nabla p = f - \lambda u + \mathrm{Div}\,D(u)$ in Ω and $\nu\cdot u = 0$ on Γ, and then we proceed the observation as follows:

$$(p, \varphi)_\Omega = -\lambda(\operatorname{div} u, \psi)_\Omega - (D(u)\nu, \nabla\psi)_\Gamma + \sum_{j,k=1}^n (D_{jk}(u), \partial_j\partial_k\psi)_\Omega - (f, \nabla\psi)_\Omega.$$

That $\int_\Omega \psi\, dx = 0$ implies that $\psi \in \hat W^1_{q'}(\Omega)$, and therefore using the relation: $\operatorname{div} u = g$ in Ω, we have

$$|\lambda(\operatorname{div} u, \psi)_\Omega| \leqq |\lambda| \|g\|_{\hat W^{-1}_q(\Omega)} \|\nabla\psi\|_{L_q(\Omega)}.$$

Using the trace theorem, we have

$$|(D(u)\nu, \nabla\psi)_\Gamma| \leqq C\|D(u)\|_{L_q(\Gamma)} \|\psi\|_{W^2_q(\Omega)}.$$

Since $(\mathscr{S}_q f, \nabla\psi)_\Omega = 0$, we arrive at the estimate:

$$|(p, \varphi)_\Omega| \leqq C\Big(|\lambda| \|g\|_{\hat W^{-1}_q(\Omega)} + \|\nabla u\|_{L_q(\Gamma)} + \|\nabla u\|_{L_q(\Omega)} + \|\mathscr{G}_q f\|_{L_q(\Omega)}\Big)\|\varphi\|_{L_{q'}(\Omega)}$$

which implies that

$$\|p\|_{L_q(\Omega)} \leqq C\Big(\|\mathscr{G}_q f\|_{L_q(\Omega)} + |\lambda| \|g\|_{\hat W^{-1}_q(\Omega)} + \|\nabla u\|_{L_q(\Gamma)} + \|\nabla u\|_{L_q(\Omega)}\Big). \tag{5.17}$$

To estimate $\|u\|_{\dot W^{-1}_q(\Omega_R)}$, we take $\varphi \in W^1_{q'}(\Omega)^n$ arbitrarily and observe that

$$\begin{aligned}(\lambda u, \varphi)_\Omega &= (\operatorname{Div} T(u, p), \varphi)_\Omega + (f, \varphi)_\Omega \\ &= (T(u, p)\nu, \varphi)_\Gamma - (1/2)(D(u), D(\varphi))_\Omega + (p, \operatorname{div}\varphi)_\Omega + (f, \varphi)_\Omega\end{aligned}$$

and therefore we have

$$|(\lambda u, \varphi)_\Omega| \leqq C\big[\|(D(u), p)\|_{L_q(\Gamma)} + \|(\nabla u, p)\|_{L_q(\Omega)} + \|f\|_{L_q(\Omega)}\big]\|\varphi\|_{W^1_{q'}(\Omega)}$$

which implies that

$$|\lambda| \|u\|_{\dot W^{-1}_q(\Omega_R)} \leqq C\big(\|(\nabla u, p)\|_{L_q(\Gamma)} + \|(\nabla u, p, f)\|_{L_q(\Omega)}\big). \tag{5.18}$$

When Ω is a bounded domain or an exterior domain, we know that

$$\|v\|_{L_q(\Gamma)} \leqq C_{q,R}\Big(\|\nabla v\|^{1/q}_{L_q(\Omega_R)} \|v\|^{1-1/q}_{L_q(\Omega_R)} + \|v\|_{L_q(\Omega_R)}\Big) \tag{5.19}$$

which combined with Young's inequality implies that

$$\|v\|_{L_q(\Gamma)} \leqq \sigma\|\nabla v\|_{L_q(\Omega_R)} + C_{\sigma,q}\|v\|_{L_q(\Omega_R)} \tag{5.20}$$

for any $\sigma > 0$ with some constant $C_{\sigma,q}$ that depends on σ and q. Combining (5.17), (5.18) and (5.20), for any σ_1, $\sigma_2 > 0$ we have

$$\begin{aligned}|\lambda|\|u\|_{\dot{W}_q^{-1}(\Omega_R)} + \|p\|_{L_q(\Omega)} \leqq C\sigma_1\|\nabla p\|_{L_q(\Omega)} + \sigma_2 C_{\sigma_1,q}\|\nabla^2 u\|_{L_q(\Omega)} \\ + C_{\sigma_1,\sigma_2,q}\Big[\|\nabla u\|_{L_q(\Omega)} + \|f\|_{L_q(\Omega)} + |\lambda|\|g\|_{\hat{W}_q^{-1}(\Omega)}\Big]\end{aligned}$$

which combined with Theorem 5.4 implies that

$$\begin{aligned}&|\lambda|\|u\|_{L_q(\Omega)} + |\lambda|^{1/2}\|\nabla u\|_{L_q(\Omega)} + \|\nabla^2 u\|_{L_q(\Omega)} + \|\nabla p\|_{L_q(\Omega)} \\ &\quad\leqq C_{\sigma_1,\sigma_2,\epsilon,q}\mathscr{D}_\lambda(f,g,h,\Omega) + C\sigma_1\|\nabla p\|_{L_q(\Omega)} + \sigma_2 C_{\sigma_1,q}\|\nabla^2 u\|_{L_q(\Omega)} \\ &\qquad + C_{\sigma_1,\sigma_2,\epsilon,q}\big(\|\nabla u\|_{L_q(\Omega)} + |\lambda|^{1/2}\|u\|_{L_q(\Omega)}\big) \qquad (5.21)\end{aligned}$$

provided that $\lambda \in \Sigma_\epsilon$ and $|\lambda| \geqq 1$. Choosing σ_1, $\sigma_2 > 0$ and $\lambda_1 \geqq 1$ in such a way that $C\sigma_1 \leqq 1/2$, $\sigma_2 C_{\sigma_1,q} \leqq 1/2$ and $C_{\sigma_1,\sigma_2,\epsilon,q}\lambda_1^{-1/2} \leqq 1/2$ in (5.21), we have

$$\mathscr{I}_\lambda(u,p,\Omega) \leqq 2C_{\sigma_1,\sigma_2,\epsilon,q}\mathscr{D}_\lambda(f,g,h,\Omega) \qquad (5.22)$$

provided that $\lambda \in \Sigma_\epsilon$ and $|\lambda| \geqq \lambda_1$, which completes the proof of Theorem 5.1 when Ω is bounded.

When Ω is an exterior domain, to estimate $\|p\|_{L_q(\Omega_R)}$ we shall use Proposition 5.6 instead of Proposition 5.5. Let φ be a function in $C_0^\infty(\Omega_R)$ such that $\int_\Omega \varphi\, dx = 1$. Subtracting a suitable constant from p if necessary, we may assume that

$$\int_\Omega \varphi p\, dx = 0. \qquad (5.23)$$

To estimate $\|p\|_{L_q(\Omega_R)}$ for such p, we choose $\psi \in C_0^\infty(\Omega_R)$ arbitrarily. Set $c = \int_\Omega \psi\, dx$, and then $\int_\Omega(\psi - c\varphi)dx = 0$. By Proposition 5.6 there exists a $\Psi \in \hat{W}_{q'}^2(\Omega)$ such that

$$\Delta\Psi = \psi - c\varphi \text{ in } \Omega \quad \frac{\partial\Psi}{\partial\nu} = 0 \text{ on } \Gamma; \qquad (5.24)$$

$$\|\Psi\|_{L_{q'}(\Omega_{R+1})} + \sup_{|x|\geqq R+1}|x|^{n-1}|\Psi(x)| + \|\nabla\Psi\|_{W_{q'}^1(\Omega)} \leqq C\|\psi\|_{L_{q'}(\Omega)}. \qquad (5.25)$$

Using (5.23) and (5.24), we observe that $(p,\psi)_{\Omega_R} = (p,\psi - c\varphi)_\Omega = (p,\Delta\Psi)_\Omega$. Since $p \in \hat{W}_q^1(\Omega)$, there exists a constant c_1 such that

$$\|(p - c_1)d^{-1}\|_{L_q(\Omega)} \leqq C\|\nabla p\|_{L_q(\Omega)} \qquad (5.26)$$

where d denotes a weight function defined by the formula: $d(x) = (1 + |x|)$ when $q \neq n$ and $d(x) = (1 + |x|)\log(2 + |x|)$ when $q = n$. This assertion is well-known as Hardy type inequality (cf. [**7**], [**17**]). Let $\rho(t)$ be a function in $C^\infty(\boldsymbol{R})$ such that $\rho(t)=1$ for $t \leqq 1/2$ and $\rho(t)=0$ for $t \geqq 1$ and set $\rho_{{}_L}(x)=\rho(\ln\ln|x|/\ln\ln L)$ which is called Sobolev's cut-off function. Since

$$|\nabla\rho_{{}_L}(x)| \leqq C((\ln\ln L)|x|\ln|x|)^{-1}, \quad |\nabla^2\rho_{{}_L}(x)| \leqq C((\ln\ln L)|x|^2\ln|x|)^{-1} \tag{5.27}$$

for $e^{\sqrt{\log L}} \leqq |x| \leqq L$ with large L, it follows from (5.24), (5.25), (5.26) and (5.27) that

$$\begin{aligned}(p,\psi)_{\Omega_R} &= \lim_{L\to\infty}(\rho_{{}_L}p,\Delta\Psi)_\Omega = -\lim_{L\to\infty}(\rho_{{}_L}\nabla p,\nabla\Psi)_\Omega \\ &= \lim_{L\to\infty}(\rho_{{}_L}(\lambda u - \operatorname{Div}D(u)-f),\nabla\Psi)_\Omega \\ &= -\lambda(g,\Psi)_\Omega - (D(u)\nu,\nabla\Psi)_\Gamma + \sum_{j,k=1}^n (D(u)_{jk},\partial_j\partial_k\Psi)_\Omega - (f,\nabla\Psi)_\Omega.\end{aligned}$$

Since $(\mathscr{S}_q f,\nabla\Psi)_\Omega = 0$, we have

$$|(p,\psi)_{\Omega_R}| \leqq C\Big[|\lambda|\|g\|_{\hat W_q^{-1}(\Omega)} + \|\nabla u\|_{L_q(\Gamma)} + \|\nabla u\|_{L_q(\Omega)} + \|\mathscr{G}_q f\|_{L_q(\Omega)}\Big]\|\nabla\Psi\|_{W^1_{q'}(\Omega)}$$

which combined with (5.25) implies that

$$\|p\|_{L_q(\Omega_R)} \leqq C\Big[|\lambda|\|g\|_{\hat W_q^{-1}(\Omega)} + \|\nabla u\|_{L_q(\Gamma)} + \|\nabla u\|_{L_q(\Omega)} + \|\mathscr{G}_q f\|_{L_q(\Omega)}\Big]. \tag{5.28}$$

To estimate $|\lambda|\|u\|_{\dot W_q^{-1}(\Omega_R)}$, we take $\varphi \in \dot W^1_{q'}(\Omega_R)^n$ arbitrarily. Recalling that $\varphi|_{S_R}=0$ when Ω is an exterior domain, we have

$$\begin{aligned}|(\lambda u,\varphi)_{\Omega_R}| &\leqq |(\operatorname{Div}T(u,p),\varphi)_{\Omega_R}| + |(f,\varphi)_{\Omega_R}| \\ &\leqq |(T(u,p)\nu,\varphi)_\Gamma| + (1/2)|(D(u),D(\varphi))_{\Omega_R}| + |(p,\operatorname{div}\varphi)_{\Omega_R}| + |(f,\varphi)_{\Omega_R}| \\ &\leqq C\big[\|(\nabla u,p)\|_{L_q(\Gamma)} + \|(\nabla u,p,f)\|_{L_q(\Omega_R)}\big]\|\varphi\|_{W^1_{q'}(\Omega_R)}\end{aligned}$$

which implies that

$$|\lambda|\|u\|_{\dot W_q^{-1}(\Omega_R)} \leqq C\big[\|(\nabla u,p)\|_{L_q(\Gamma)} + \|(\nabla u,p,f)\|_{L_q(\Omega_R)}\big]. \tag{5.29}$$

Employing the same argument as in the case that Ω is bounded, by (5.28), (5.29), (5.20) and Theorem 5.4 we see that there exists a $\lambda_1 \geqq 1$ depending on ϵ and q such that $\mathscr{I}_\lambda(u,p,\Omega) \leqq C\mathscr{D}_\lambda(f,g,h,\Omega)$ provided that $\lambda \in \Sigma_\epsilon$ and $|\lambda| \geqq \lambda_1$ in the case where Ω is an exterior domain. This completes the proof of Theorem 5.1. $\square$

A PROOF OF THEOREM 5.2. Since K is a compact set in Σ_ϵ and since Σ_ϵ does not contain $\{0\}$, there exists a $\sigma_K>0$ such that $\sigma_K \leqq |\lambda| \leqq (\sigma_K)^{-1}$ for any $\lambda \in K$. Obviously, $\|u\|_{\dot W_q^{-1}(\Omega_R)} \leqq \|u\|_{L_q(\Omega_R)}$. Therefore, by Theorem 5.4 we have

$$\begin{aligned}&\|u\|_{W_q^2(\Omega)}+\|\nabla p\|_{L_q(\Omega)}\\&\quad\leqq C_K\Big[\|f\|_{L_q(\Omega)}+\|(g,h)\|_{W_q^1(\Omega)}+\|g\|_{\hat W_q^{-1}(\Omega_R)}+\|u\|_{W_q^1(\Omega_R)}+\|p\|_{L_q(\Omega_R)}\Big].\end{aligned}\tag{5.30}$$

By contradiction, we shall show that there exists a constant C such that

$$\|u\|_{W_q^1(\Omega_R)}+\|p\|_{L_q(\Omega_R)}\leqq C\Big(\|f\|_{L_q(\Omega)}+\|(g,h)\|_{W_q^1(\Omega)}+\|g\|_{\hat W_q^{-1}(\Omega)}\Big)\tag{5.31}$$

Suppose that for any natural number m there exist $\lambda_m\in K$, $f_m\in L_q(\Omega)^n$, $g_m\in W_q^1(\Omega)\cap\hat W_q^{-1}(\Omega)$, $h_m\in W_{q,\partial}^1(\Omega)$, $u_m\in W_q^2(\Omega)^n$ and $p_m\in\hat W_q^1(\Omega)$ such that

$$\begin{aligned}&\lambda_m u_m-\operatorname{Div}T(u_m,p_m)=f_m,\quad \operatorname{div}u_m=g_m &&\text{in }\Omega\\&\nu\cdot u_m=0,\ \ B_{\alpha,\beta}(u_m)=g_m &&\text{on }\Gamma\end{aligned}\tag{5.32}$$

$$\|u_m\|_{W_q^1(\Omega_R)}+\|p_m\|_{L_q(\Omega_R)}=1\tag{5.33}$$

$$\|f_m\|_{L_q(\Omega)}+\|(g_m,h_m)\|_{W_q^1(\Omega)}+\|g_m\|_{\hat W_q^{-1}(\Omega)}\leqq 1/m.\tag{5.34}$$

Combining (5.30), (5.33) and (5.34), we have $\|u_m\|_{W_q^2(\Omega)}+\|\nabla p_m\|_{L_q(\Omega)}\leqq 2C_K$ for any $m\in\boldsymbol{N}$. Therefore, passing to the subsequence if necessary, we may assume that there exist $\lambda\in K$, $u\in W_q^2(\Omega)^n$ and $p\in\hat W_q^1(\Omega)$ such that $\lambda_m\to\lambda$, $u_m\to u$ weakly in $W_q^2(\Omega)^n$, $\nabla p_m\to\nabla p$ weakly in $L_q(\Omega)$, $u_m\to u$ strongly in $W_q^1(\Omega_R)^n$ and $p_m\to p$ strongly in $L_q(\Omega_R)$ as $m\to\infty$. Letting $m\to\infty$ in (5.32) and (5.33), we see that $(u,p)\in W_q^2(\Omega)^n\times\hat W_q^1(\Omega)$ satisfies the homogeneous equation:

$$\begin{aligned}\lambda u-\operatorname{Div}T(u,p)&=0,\qquad \operatorname{div}u=0\quad\text{in }\Omega\\ \nu\cdot u&=0,\quad B_{\alpha,\beta}(u)=0\quad\text{on }\Gamma\end{aligned}$$

and therefore by the uniqueness assumption we have $u=0$ and $p=c$ (c being a constant). Since $\int_{\Omega_R}p_m\,dx=0$, $\int_{\Omega_R}p\,dx=0$, which implies that $p=c=0$. On the other hand, by (5.33) we have $\|u\|_{W_q^1(\Omega_R)}+\|p\|_{L_q(\Omega_R)}=1$, which contradicts to the fact that $u=0$ and $p=0$. This completes the proof of Theorem 5.2. □

A proof of Theorem 5.3. When Ω is bounded, by (1.3) we know that $\|g\|_{\hat W_q^{-1}(\Omega)}\leqq C_q\|g\|_{L_q(\Omega)}$, and therefore by Theorem 5.4 we have

$$\|u\|_{W_q^2(\Omega)}+\|p\|_{W_q^1(\Omega)}\leqq C_K\Big[\|f\|_{L_q(\Omega)}+\|(g,h)\|_{W_q^1(\Omega)}+\|u\|_{W_q^1(\Omega)}+\|p\|_{L_q(\Omega)}\Big]\tag{5.35}$$

because $\Omega_R=\Omega$ in the bounded domain case. Employing the same argument as in the proof of Theorem 5.2 by contradiction we can show that there exists a constant C such that

$$\|u\|_{W_q^1(\Omega)} + \|p\|_{L_q(\Omega)} \leqq C\Big[\|f\|_{L_q(\Omega)} + \|(g,h)\|_{W_q^1(\Omega)}\Big]$$

which combined with (5.35) implies the theorem. □

Finally, assuming that (1.4) holds we shall give

A PROOF OF THEOREM 1.4. Assume that (1.4) holds. If $f \in J_q(\Omega)$, $g = 0$ and $h = 0$ in (1.1), then noting that $\mathscr{G}_q f = 0$, by (5.17) and (5.28) we have

$$\|p\|_{L_q(\Omega_R)} \leqq C\big[\|\nabla u\|_{L_q(\Gamma)} + \|\nabla u\|_{L_q(\Omega)}\big]$$

which combined with (5.19) and (1.4) implies that

$$\begin{aligned}\|p\|_{L_q(\Omega_R)} &\leqq C_{\epsilon,q}(|\lambda|^{-(1/2)(1-(1/q))} + |\lambda|^{-1/2})\|f\|_{L_q(\Omega)} \\ &\leqq C_{\epsilon,q}|\lambda|^{-(1/2)(1-(1/q))}\|f\|_{L_q(\Omega)}\end{aligned}$$

provided that $\lambda \in \Sigma_\epsilon$ and $|\lambda| \geqq 1$. This completes the proof of Theorem 1.4. □

6. A proof of Theorem 1.3 in the bounded domain case.

Throughout this section, we always assume that Ω is a bounded domain and we shall show Theorem 1.3. In order to prove Theorem 1.3, in view of Theorems 5.1 and 5.3 it suffices to show the unique existence of solutions to (1.1). First of all, we consider the problem in the L_2 framework. Set

$$H_\partial^1(\Omega) = \big\{u \in W_2^1(\Omega)^n \mid \nu \cdot u|_\Gamma = 0\big\}, \quad H_\sigma^1(\Omega) = \big\{u \in H_\partial^1(\Omega) \mid \operatorname{div} u = 0 \text{ in } \Omega\big\}$$

$$\hat{L}_2(\Omega) = \Big\{p \in L_2(\Omega) \mid \int_\Omega p\,dx = 0\Big\}, \qquad \|u\| = \Big\{\sum_{|\alpha|\leqq 1} \|\partial_x^\alpha u\|_{L_2(\Omega)}^2\Big\}^{1/2}.$$

If $(u,p) \in W_2^2(\Omega)^n \times \hat{W}_2^1(\Omega)$ solves (1.1), then we have

$$\lambda(u,v)_\Omega + (1/2)(D(u),D(v))_\Omega + \alpha\beta^{-1}(u,v)_\Gamma - (p,\operatorname{div} v)_\Omega = (f,v)_\Omega + \beta^{-1}(h,v)_\Gamma \quad (6.1)$$

for every $v \in H_\partial^1(\Omega)$. In view of (6.1), we set

$$B_\lambda[u,v] = \lambda(u,v)_\Omega + (1/2)(D(u),D(v))_\Omega + \alpha\beta^{-1}(u,v)_\Gamma$$

and we consider the variational equation:

$$B_\lambda[u,v] = (f,v)_\Omega + \beta^{-1}(h,v)_\Gamma \qquad (6.2)$$

on $H_\sigma^1(\Omega)$.

LEMMA 6.1.
(1) *Let* $0 < \epsilon < \pi/2$ *and* $\sigma > 0$*. Then, there exists a positive constant* $c_{\epsilon,\sigma}$ *such that*

$$B_\lambda[u, u] \geqq c_{\epsilon,\sigma}\|u\|^2$$

for every $\lambda \in \Sigma_\epsilon$ *with* $|\lambda| \geqq \sigma$ *and* $u \in H^1_\partial(\Omega)$*.*
(2) *Assume that* Ω *is not rotationally symmetric when* $\alpha = 0$*. Then, there exist positive constants* λ_0 *and* c *such that*

$$|B_\lambda[u, u]| \geqq c\|u\|^2$$

for any $\lambda \in \boldsymbol{C}$ *with* $|\lambda| \leqq \lambda_0$ *and* $u \in H^1_\sigma(\Omega)$*.*

PROOF.
(1) We know Korn's second inequality:

$$\|D(u)\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\Omega)} \geqq c_0\|u\|^2 \tag{6.3}$$

for every $u \in W^1_2(\Omega)$ with some positive constant c_0 (cf. [**4**], [**19**]). On the other hand, to show that

$$|B_\lambda[u, u]| \geqq \sin(\epsilon/2)\big(|\lambda|\|u\|^2_{L_2(\Omega)} + (1/2)\|D(u)\|^2_{L_2(\Omega)} + \alpha\beta^{-1}\|u\|^2_{L_2(\Gamma)}\big)$$

for any $\lambda \in \Sigma_\epsilon$ and $u \in W^1_2(\Omega)$, we use an elementary calculus:

$$|A\lambda + B| \geqq \sin(\epsilon/2)(A|\lambda| + B)$$

for every nonnegative numbers A, B and $\lambda \in \Sigma_\epsilon$. Therefore, setting

$$c_{\epsilon,\sigma} = c_0 \sin(\epsilon/2) \min(\sigma, 1/2)$$

we have the first assertion.
(2) In view of Korn's second inequality (6.3), by contradiction we shall show that there exists a constant $C > 0$ such that

$$\|u\|^2_{L_2(\Omega)} \leqq C\big(\|D(u)\|^2_{L_2(\Omega)} + \alpha\beta^{-1}\|u\|^2_{L_2(\Gamma)}\big) \tag{6.4}$$

for any $u \in H^1_\sigma(\Omega)$. Therefore, we assume that (6.4) does not hold, that is, there exists a sequence $\{u_n\}_{n=1,2,3,\ldots} \subset H^1_\sigma(\Omega)$ such that

$$\|u_n\|^2_{L_2(\Omega)} = 1 \tag{6.5}$$

$$\|D(u_n)\|^2_{L_2(\Omega)} + \alpha\beta^{-1}\|u_n\|^2_{L_2(\Gamma)} < 1/n \tag{6.6}$$

for all n. By (6.3), (6.5) and (6.6) we have $\|u_n\| \leqq C((1/n)+1) \leqq 2C$ for all n. Therefore, passing to a subsequence if necessary, we see that there exists a $u \in H^1_\sigma(\Omega)$ such that $u_n \to u$ weakly in $W^1_2(\Omega)$ and strongly in $L_2(\Omega)$ as $n \to \infty$. By (6.6) we have $D(u) = 0$. Furthermore, when $\alpha > 0$ we have $\|u\|_{L_2(\Gamma)} = 0$, that is, $u = 0$ on Γ, and therefore we have $u = 0$ (cf. [**4**]). On the other hand, when $\alpha = 0$, the assumption that Ω is not rotationally symmetric implies that $u = 0$ (cf. [**19**] and [**11**]). Therefore, we see that $u = 0$. However, from (6.5) it follows that $\|u\|_{L_2(\Omega)} = 1$, which contradicts the fact that $u = 0$. Therefore, we have that (6.4) does hold with some constant $C > 0$.

Combining (6.3) and (6.4) implies that

$$|B_0[u,u]| \geqq d\|u\|^2$$

for any $u \in H^1_\sigma(\Omega)$ with some positive constant d, and therefore

$$B_\lambda[u,u] \geqq |B_0[u,u]| - |\lambda|\|u\|^2_{L_2(\Omega)} \geqq d\|\nabla u\|^2_{L_2(\Omega)} + (d - |\lambda|)\|u\|^2_{L_2(\Omega)}.$$

Taking $\lambda_0 = d/2$ and $c = d/2$, we have the second assertion, which completes the proof of the lemma. □

In view of Lemma 6.1, by the Lax-Milgram theorem (cf. [**4**]) we have the following theorem.

Lemma 6.2.
(1) *Let $0 < \epsilon < \pi/2$ and $\sigma > 0$. Then, for every $\lambda \in \Sigma_\epsilon$ with $|\lambda| \geqq \sigma$, $f \in H^1_\sigma(\Omega)^*$ and $h \in L_2(\Gamma)$ the variational equation* (6.2) *admits a unique solution $u \in H^1_\sigma(\Omega)$.*
(2) *Assume that Ω is not rotationally symmetric when $\alpha = 0$. Let λ_0 be the same positive number as in Lemma* 6.1 (2). *Then, for every $\lambda \in \boldsymbol{C}$ with $|\lambda| \leqq \lambda_0$, $f \in H^1_\sigma(\Omega)^*$ and $h \in L_2(\Gamma)$ the variational equation* (6.2) *admits a unique solution $u \in H^1_\sigma(\Omega)$.*

Following the argument due to Solonnikov and Ščadilov [**19**] (cf. also [**7**, Theorem 5.2]), we have the following theorem.

Lemma 6.3.
(1) *Let $0 < \epsilon < \pi/2$ and $\sigma > 0$. Then, for every $\lambda \in \Sigma_\epsilon$ with $|\lambda| \geqq \sigma$, $f \in H^1_\sigma(\Omega)^*$ and $h \in L_2(\Gamma)$ there exists a unique $(u,p) \in H^1_\sigma(\Omega) \times \hat{L}_2(\Omega)$ which solves the variational equation*:

$$B_\lambda[u,v] - (p, \operatorname{div} v)_\Omega = (f,v)_\Omega + \beta^{-1}(h,v)_\Gamma$$

for every $v \in H^1_\partial(\Omega)$.
(2) *Assume that Ω is not rotationally symmetric when $\alpha = 0$. Let λ_0 be the same positive number as in Lemma* 6.1 (2). *Then, for every $\lambda \in \boldsymbol{C}$ with $|\lambda| \leqq \lambda_0$, $f \in H^1_\sigma(\Omega)^*$ and $h \in L_2(\Gamma)$ there exists a unique $(u,p) \in H^1_\sigma(\Omega) \times \hat{L}_2(\Omega)$ which solves the variational equation*:

$$B_\lambda[u,v] - (p, \operatorname{div} v)_\Omega = (f,v)_\Omega + \beta^{-1}(h,v)_\Gamma$$

for every $v \in H^1_\partial(\Omega)$.

Theorem 6.4.

(1) *Let* $0 < \epsilon < \pi/2$ *and* $\sigma > 0$. *Then, for every* $\lambda \in \Sigma_\epsilon \cup \{\lambda \in \boldsymbol{C} \mid |\lambda| \geqq \sigma\}$, $f \in H^1_\sigma(\Omega)^*$, $g \in \hat{L}_2(\Omega)$ *and* $h \in L_2(\Gamma)$ *there exists a unique* $(u,p) \in H^1_\sigma(\Omega) \times \hat{L}_2(\Omega)$ *such that* $\operatorname{div} u = g$ *and*

$$B_\lambda[u,v] - (p, \operatorname{div} v)_\Omega = (f,v)_\Omega + \beta^{-1}(h,v)_\Gamma \tag{6.7}$$

for every $v \in H^1_\partial(\Omega)$.

(2) *Assume that* Ω *is not rotationally symmetric when* $\alpha = 0$. *Let* λ_0 *be the same positive number as in Lemma* 6.1 (2). *Then, for every* $\lambda \in \boldsymbol{C}$ *with* $|\lambda| \leqq \lambda_0$, $f \in H^1_\sigma(\Omega)^*$, $g \in \hat{L}_2(\Omega)$ *and* $h \in L_2(\Gamma)$ *there exists a unique* $(u,p) \in H^1_\sigma(\Omega) \times \hat{L}_2(\Omega)$ *such that* $\operatorname{div} u = g$ *and*

$$B_\lambda[u,v] - (p, \operatorname{div} v)_\Omega = (f,v)_\Omega + \beta^{-1}(h,v)_\Gamma \tag{6.8}$$

for every $v \in H^1_\partial(\Omega)$.

Proof. By Proposition 5.5 there exists a unique $w \in \hat{L}_2(\Omega) \cap W^2_2(\Omega)$ such that $\Delta w = g$ in Ω and $\partial w/\partial \nu = 0$ on Γ. Set $u = \nabla w + z$, and then from (6.8) we have

$$B_\lambda[z,v] - (p, \operatorname{div} v)_\Omega = (f,v)_\Omega + \beta^{-1}(h,v)_\Gamma - B_\lambda[\nabla w, v] \tag{6.9}$$

for every $v \in H^1_\partial(\Omega)$. Since $\nabla w \in W^1_2(\Omega)^n$, the map $v \mapsto B_\lambda[\nabla w, v]$ belongs to $H^1_\sigma(\Omega)^*$, and therefore by Lemma 6.3 we see the unique existence of solution $(z,p) \in H^1_\sigma(\Omega) \times \hat{L}_2(\Omega)$ of the variational equation (6.9), which completes the proof of the theorem. □

Now, we shall show the regularity of u and p obtained in Theorem 6.4.

Theorem 6.5.

(1) *Let* $0 < \epsilon < \pi/2$ *and* $\sigma > 0$. *Then, for every* $\lambda \in \Sigma_\epsilon$ *with* $|\lambda| \geqq \sigma$, $f \in C^\infty(\overline{\Omega})^n$, $g \in C^\infty(\overline{\Omega}) \cap \hat{L}_2(\Omega)$ *and* $h \in C^\infty(\overline{\Omega})^n \cap W^1_{2,\partial}(\Omega)$ *there exists a unique solution* $(u,p) \in \bigcap_{1<q<\infty}(W^2_q(\Omega)^n \times \hat{W}^1_q(\Omega))$ *of the equation* (1.1).

(2) *Assume that* Ω *is not rotationally symmetric when* $\alpha = 0$. *Let* λ_0 *be the same positive number as in Lemma* 6.1 (2). *Then, for every* $\lambda \in \boldsymbol{C}$ *with* $|\lambda| \leqq \lambda_0$, $f \in C^\infty(\overline{\Omega})^n$, $g \in C^\infty(\overline{\Omega}) \cap \hat{L}_2(\Omega)$ *and* $h \in C^\infty(\overline{\Omega})^n \cap W^1_{2,\partial}(\Omega)$ *there exists a unique solution* $(u,p) \in \bigcap_{1<q<\infty}(W^2_q(\Omega)^n \times \hat{W}^1_q(\Omega))$ *of the equation* (1.1).

Proof. Since we can show the assertions (1) and (2) by employing the same argument, in the course of the proof we assume that λ satisfies one of the following conditions: $\lambda \in \Sigma_\epsilon$ with $|\lambda| \geqq \sigma$ or $\lambda \in \boldsymbol{C}$ with $|\lambda| \leqq \lambda_0$. To show the theorem, we shall use the same localization procedure as in the proof of Theorem 5.4. Since $f \in H^1_\sigma(\Omega)^*$,

$g \in \hat{L}_2(\Omega)$ and $h \in L_2(\Gamma)$ by Theorem 6.4 we know the existence of $(u, p) \in H^1_\sigma(\Omega)\times\hat{L}_2(\Omega)$ that satisfies the equation: $\operatorname{div} u = g$ and solves (6.8) uniquely. First of all, we shall show that $(u, p) \in W^2_2(\Omega)^n \times W^1_2(\Omega)$. Since Ω is bounded, it is enough to show that $(u, p) \in W^2_{2,\mathrm{loc}}(\overline{\Omega})^n \times W^1_{2,\mathrm{loc}}(\overline{\Omega})$. Pick $x_0 \in \Gamma$ up and let w, θ, F, G, F_∂, H, ∂H and ν_ω be the same as in (5.3). Then, from (6.8) we have

$$\begin{aligned}&\lambda_1(w, v)_H + (1/2)(D(w), D(v))_H + \alpha\beta^{-1}(w, v)_{\partial H} - (\theta, \operatorname{div} v)_H \\ &\quad = ((\lambda_1 - \lambda)w, v)_H + (F, v)_H + \beta^{-1}(F_\partial, v)_{\partial H} \qquad (6.10)\end{aligned}$$

for any $v \in W^1_2(H)^n$ with $\nu_\omega \cdot v|_{\partial H} = 0$. From the assumptions on f, g and h and the fact that $u \in W^1_2(\Omega)^n$ and $p \in L_2(\Omega)$ it follows that $(\lambda_1-\lambda)w+F \in L_2(H)^n$, $G \in W^1_2(H)$ and $F_\partial \in W^1_2(H)^n$ with $\nu_\omega \cdot F_\partial = 0$ on ∂H. Moreover, from the discussion in Section 5 (cf. (5.7)), we have $G \in \hat{W}^{-1}_2(H)$, because $\operatorname{div} u = g \in \hat{W}^1_2(\Omega)$ and $u \in W^1_2(\Omega) \subset \dot{W}^{-1}_2(\Omega_R)$. Therefore, in view of Theorem 4.1 we choose λ_1 so large that there exists a solution $(U, \Phi) \in W^2_2(H)^n \times \hat{W}^1_2(H)$ of the equation:

$$\begin{aligned}&\lambda_1 U - \operatorname{Div} T(U, \Phi) = F + (\lambda_1 - \lambda)w, \ \operatorname{div} U = G && \text{in } H \\ &\nu_\omega \cdot U = 0, \quad \alpha U + \beta(D(U)\nu_\omega - \langle D(U)\nu_\omega, \nu_\omega\rangle\nu_\omega) = F_\partial && \text{on } \partial H. \qquad (6.11)\end{aligned}$$

Since the strong solutions are also the weak ones, by (6.11) we see that U and Φ satisfy (6.11) too. Since $\operatorname{div} w = G$ in H and $\nu_\omega\cdot w = 0$ on ∂H, setting $V = w-U$ and $\Psi = \theta-\Phi$, we have

$$\lambda_1(V, v)_H + (1/2)(D(V), D(v))_H + \alpha\beta^{-1}(V, v)_{\partial H} - (\Psi, \operatorname{div} v)_H = 0 \qquad (6.12)$$

for any $v \in W^1_2(H)^n$ with $\nu_\omega \cdot v|_{\partial H} = 0$, and $\operatorname{div} V = 0$ in H and $\nu_\omega \cdot V = 0$ on ∂H. Therefore, setting $V = v$ in (6.12), we have

$$\lambda_1\|V\|^2_{L_2(H)} + (1/2)\|D(V)\|^2_{L_2(H)} + \alpha\beta^{-1}\|V\|^2_{L_2(\partial H)} = 0$$

which implies that $V = 0$, and therefore (6.12) implies that $(\Psi, \operatorname{div} v)_H = -(\nabla\Psi, v)_H = 0$ for any $v \in W^1_2(H)^n$ with $\nu_\omega \cdot v|_{\partial H} = 0$, which shows that $\nabla\Psi = 0$ on H. Therefore, we have $w \in W^2_2(H)^n$ and $\nabla\theta \in L_2(H)^n$. In this way, we can show that $u \in W^2_2(\Omega)^n$ and $\nabla p \in L_2(\Omega)^n$.

Now, by Sobolev's imbedding theorem we see that $u \in W^1_q(\Omega)^n$ and $p \in L_q(\Omega)$ for every q with $n(1/2 - 1/q) \leqq 1$. Repeating the same argument as above, we can see that $u \in W^2_q(\Omega)^n$ and $\nabla p \in L_q(\Omega)^n$, and therefore repeated use of this argument implies that

$$(u, p) \in \bigcap_{1<q<\infty} \left(W^2_q(\Omega)^n \times W^1_q(\Omega)\right)$$

(cf. for more detailed proof we can refer to [**5**, pp. 629–630], [**17**, Proof of Lemma 7.1] and [**1**, Proof of Lemma 6.2]).

Finally, we shall show that (u,p) actually satisfies (1.1) in the strong sense. Let v be an arbitrary function in $C_0^\infty(\Omega)^n$, then by (6.8) and the divergence theorem of Gauss we have $(\lambda u - \operatorname{Div} T(u,p) - f, v)_\Omega = 0$, which combined with the fact that $(u,p) \in W_2^2(\Omega)^n \times W_2^1(\Omega)$ implies that

$$(\lambda u - \operatorname{Div} T(u,p) - f, \varphi)_\Omega = 0 \tag{6.13}$$

for every $\varphi \in L_2(\Omega)^n$. Let ψ be any vector in $C^1(\Gamma)^n$ such that $\nu \cdot \psi = 0$ on Γ and φ be a vector in $H^1_\partial(\Omega)^n$ such that $\varphi|_\Gamma = \psi$. By (6.13) and (6.8) we have

$$\begin{aligned} 0 &= \lambda(u,\varphi)_\Omega + (1/2)(D(u), D(\varphi))_\Omega + \alpha\beta^{-1}(u,\varphi)_\Gamma - \beta^{-1}(B_{\alpha,\beta}(u), \psi)_\Gamma - (f,\varphi)_\Omega \\ &= \beta^{-1}(h - B_{\alpha,\beta}(u), \psi)_\Gamma. \end{aligned} \tag{6.14}$$

But, we see that (6.14) holds for any $\psi \in C^1(\Gamma)^n$. In fact, let ψ be any vector of $C_0^\infty(\Gamma)^n$ functions and set $\tilde\psi = \psi - (\nu\cdot\psi)\nu$. Since $(D(u)\nu - \langle D(u)\nu, \nu\rangle\nu)\cdot\nu = 0$ identically on Γ and since $\nu\cdot\tilde\psi = 0$ on Γ, by (6.14) we have

$$(h - B_{\alpha,\beta}(u), \psi)_\Gamma = (h - B_{\alpha,\beta}(u), (\nu\cdot\psi)\nu)_\Gamma = (h\cdot\nu - \alpha u\cdot\nu, \nu\cdot\psi)_\Gamma = 0 \tag{6.15}$$

where we have used the facts that $h\cdot\nu = u\cdot\nu = 0$ on Γ. Therefore, the arbitrariness of choice of ψ in (6.15) implies that $h - B_{\alpha,\beta}(u) = 0$ on Γ, which completes the proof of the theorem. □

A proof of Theorem 1.3 in the bounded domain case. Since we can show the assertions (1) and (2) by employing the same argument, in the course of the proof we assume that λ satisfies one of the following conditions: $\lambda \in \Sigma_\epsilon$ with $|\lambda| \geqq \sigma$ or $\lambda \in \boldsymbol{C}$ with $|\lambda| \leqq \lambda_0$. First of all, we shall prove the uniqueness of solutions to (1.1). Let $(u,p) \in W_q^2(\Omega)^n \times \hat W_q^1(\Omega)$ satisfy the homogeneous equation:

$$\begin{aligned} \lambda u - \operatorname{Div} T(u,p) = 0, \qquad \operatorname{div} u = 0 \quad \text{in } \Omega \\ \nu\cdot u = 0, \quad B_{\alpha,\beta}(u) = 0 \quad \text{on } \Gamma. \end{aligned} \tag{6.16}$$

Given any $\varphi \in C_0^\infty(\Omega)^n$, by Theorem 6.5 we know the existence of solution $(v,\theta) \in W_{q'}^2(\Omega)^n \times \hat W_{q'}^1(\Omega)$ to the equation:

$$\begin{aligned} \bar\lambda v - \operatorname{Div} T(v,\theta) = \varphi, \qquad \operatorname{div} v = 0 \quad \text{in } \Omega \\ \nu\cdot v = 0, \quad B_{\alpha,\beta}(v) = 0 \quad \text{on } \Gamma. \end{aligned} \tag{6.17}$$

Therefore, by (6.16) and (6.17) we have

$$(u,\varphi)_\Omega = \big(u, \bar\lambda v - \operatorname{Div} T(v,\theta)\big)_\Omega = (\lambda u - \operatorname{Div} T(u,p), v)_\Omega = 0$$

which shows that $u = 0$. By (6.16) $\nabla p = 0$, which combined with the fact that $\int_\Omega p\,dx = 0$ implies that $p = 0$. Hence, we have shown the uniqueness, which combined with Theorems 5.1 and 5.3 implies the *a priori* estimates (1.4) and (1.5) of solutions to (1.1). To show the existence of solutions, we define the map $\mathscr{A} : W^2_{q,\partial}(\Omega) \times \hat{W}^1_q(\Omega)$ to $L_q(\Omega)^n \times W_{q,\mathrm{div}}(\Omega) \times W^1_{q,\partial}(\Omega)$ by the relation: $\mathscr{A}(u,p) = (\lambda u - \mathrm{Div}\, T(u,p), \mathrm{div}\, u, B_{\alpha,\beta}(u))$. The a priori estimates (1.4) and (1.5) implies that the range of $\mathscr{A}$ is closed. On the other hand, by Theorem 6.5 implies that $C^\infty(\overline{\Omega})^n \times (C^\infty(\overline{\Omega})^n \cap \hat{L}_2(\Omega)) \times (C^\infty(\overline{\Omega})^n \cap W^1_{2,\partial}(\Omega))$ is contained in the range of $\mathscr{A}$. Since $C^\infty(\overline{\Omega})^n \times (C^\infty(\overline{\Omega})^n \cap \hat{L}_2(\Omega)) \times (C^\infty(\overline{\Omega})^n \cap W^1_{2,\partial}(\Omega))$ is dense in $L_q(\Omega)^n \times W_{q,\mathrm{div}}(\Omega) \times W^1_{q,\partial}(\Omega)$, taking its closure implies that the range of $\mathscr{A}$ coincides with $L_q(\Omega)^n \times W_{q,\mathrm{div}}(\Omega) \times W^1_{q,\partial}(\Omega)$, which means the existence of solutions. This completes the proof of Theorem 1.3 when Ω is a bounded domain. □

7. A proof of Theorem 1.3 in the exterior domain case.

To show Theorem 1.3, in view of Theorems 5.1 and 5.2 it suffices to show the unique existence of solutions to (1.1). First of all, we reduce (1.1) to the case where f has a compact support, $g = 0$ and $h = 0$. For this purpose, we shall use the following well-known theorem (cf. [**1**], [**5**], [**7**], [**9**], [**10**] and [**18**]).

Proposition 7.1. *Let $1 < q < \infty$ and Ω be an exterior domain whose boundary Γ is a $C^{1,1}$ compact hypersurface. Set*

$$X_q(\Omega) = \begin{cases} \{u \in W^1_{q,\mathrm{loc}}(\overline{\Omega}) \mid \|u\|_{X_q(\Omega)} < \infty\} & 1 < q < n \\ \{u \in W^1_{q,\mathrm{loc}}(\overline{\Omega}) \mid \int_{\Omega_{R+3}} u\,dx = 0,\ \|u\|_{X_q(\Omega)} < \infty\} & n \leqq q < \infty \end{cases}$$

$$\|u\|_{X_q(\Omega)} = \|\nabla u\|_{L_q(\Omega)} + \|u/d\|_{L_q(\Omega)}, \quad d = d(x) = \begin{cases} 1 + |x| & (q \neq n) \\ (1 + |x|)\log(2 + |x|) & (q = n). \end{cases}$$

Given $F \in L_q(\Omega)^n$ with $\mathrm{div}\, F \in L_q(\Omega)$ *we consider the Laplace equation*:

$$\Delta U = \mathrm{div}\, F \quad \text{in } \Omega \quad \partial U/\partial\nu = \nu \cdot F \quad \text{on } \Gamma. \tag{7.1}$$

Then, the equation (7.1) *admits a unique solution $U \in X_q(\Omega)$ such that $\nabla^2 U \in L_q(\Omega)$ and*

$$\|U\|_{X_q(\Omega)} \leqq C\|F\|_{L_q(\Omega)}, \quad \|\nabla^2 U\|_{L_q(\Omega)} \leqq C\big(\|F\|_{L_q(\Omega)} + \|\mathrm{div}\, F\|_{L_q(\Omega)}\big).$$

To show the existence theorem for (1.1), let $\lambda \in \boldsymbol{C} \setminus (-\infty, 0]$, $f \in L_q(\Omega)^n$, $g = \mathrm{div}\,\tilde{g} \in W_{q,\mathrm{div}}(\Omega)$ and $h \in W^1_{q,\partial}(\Omega)$. Let U be a solution to the Laplace equation: $\Delta U = g = \mathrm{div}\,\tilde{g}$ in Ω and $\partial U/\partial\nu = \nu \cdot \tilde{g} = 0$ on Γ. Since we assume that Γ is a $C^{2,1}$ compact hypersurface, the function u given in Proposition 7.1 actually satisfies the regularity condition: $\nabla^3 u \in L_q(\Omega)$ provided that $\mathrm{div}\, F \in W^1_q(\Omega)$ in addition. And therefore, if we set $u = \nabla U + v$ in (1.1), then we arrive at the zero divergence case. However, we would like to consider the case that Γ is only assumed to be $C^{1,1}$ a compact hypersurface, because we would like to mention that our method can be applied to the non-slip boundary condition case that has been studied by Farwig and Sohr [**5**] under the assumption that $\Gamma \in C^{1,1}$. Therefore, we will use a different method based on a cut-off technique.

Let ψ, φ_0 and φ_∞ be functions in $C^\infty(\boldsymbol{R}^n)$ such that

$$\begin{aligned} \psi(x) &= 1 \text{ for } |x| \leqq R-3 \quad \text{and} \quad \psi(x) = 0 \text{ for } |x| \geqq R-2 \\ \varphi_0(x) &= 1 \text{ for } |x| \leqq R-2 \quad \text{and} \quad \varphi_0(x) = 0 \text{ for } |x| \geqq R-1 \\ \varphi_\infty(x) &= 1 \text{ for } |x| \geqq R-3 \quad \text{and} \quad \varphi_\infty(x) = 0 \text{ for } |x| \leqq R-4. \end{aligned} \tag{7.2}$$

Note that

$$\varphi_0 = 1 \text{ on } \operatorname{supp}\psi; \quad \varphi_\infty = 1 \text{ on } \operatorname{supp}(1-\psi); \quad \varphi_0 = \varphi_\infty = 1 \text{ on } \operatorname{supp}\nabla\psi. \tag{7.3}$$

Let $(u_0, p_0) \in W_q^2(\Omega_R)^n \times \hat{W}_q^1(\Omega_R)$ and $(u_\infty, p_\infty) \in W_q^2(\boldsymbol{R}^n)^n \times \hat{W}_q^1(\boldsymbol{R}^n)$ be solutions to the equations:

$$\begin{cases} \lambda u_0 - \operatorname{Div} T(u_0, p_0) = \varphi_0 f, \quad \operatorname{div} u_0 = \operatorname{div}(\varphi_0 \nabla U) \quad \text{in } \Omega_R \\ \nu \cdot u_0 = 0, \quad \alpha u + \beta(D(u_0)\nu - \langle D(u_0)\nu, \nu\rangle\nu) = h \quad \text{on } \partial\Omega_R \end{cases}$$
$$\lambda u_\infty - \operatorname{Div} T(u_\infty, p_\infty) = \varphi_\infty f, \quad \operatorname{div} u_\infty = \operatorname{div}(\varphi_\infty \nabla U) \quad \text{in } \boldsymbol{R}^n$$

respectively. Here and hereafter, ν denotes not only the unit outer normal to Γ but also that to $\partial\Omega_R = \Gamma \cup S_R$. Since

$$\operatorname{div}(\varphi_N \nabla U) = \varphi_N \Delta U + (\nabla\varphi_N)\cdot(\nabla U) = \varphi_N g + (\nabla\varphi_N)\cdot(\nabla U) \quad (N = 0, \infty)$$

noting that $\nu \cdot \nabla U = 0$ on Γ, we see that $\operatorname{div}(\varphi_0 \nabla U) \in \hat{W}_q^1(\Omega_R)$ and also that $\operatorname{div}(\varphi_\infty \nabla U) \in \hat{W}_q^1(\boldsymbol{R}^n)$. Therefore, by Theorem 1.3 in the bounded domain case and Theorem 2.1 we know the existence of such (u_0, p_0) and (u_∞, p_∞), respectively. From (7.3) it follows that

$$\begin{aligned} \operatorname{div}(\psi u_0 + (1-\psi)u_\infty) &= \psi \operatorname{div}(\varphi_0 \nabla U) + (1-\psi)\operatorname{div}(\varphi_\infty \nabla U) + (\nabla\psi)\cdot(u_0 - u_\infty) \\ &= \operatorname{div}\tilde{g} + (\nabla\psi)\cdot(u_0 - u_\infty). \end{aligned}$$

By Bogovskiĭ's theorem (cf. [**2**], [**3**] and [**7**]) there exists a linear map $\boldsymbol{B}$ from $W_{q,a}^2(D)$ into $W_q^3(\boldsymbol{R}^n)^n$ such that $\operatorname{supp}\boldsymbol{B}[w] \subset D$, $\operatorname{div}\boldsymbol{B}[w] = w$ in $\boldsymbol{R}^n$ and $\|\boldsymbol{B}[w]\|_{W_q^3(\boldsymbol{R}^n)} \leqq C\|w\|_{W_q^2(D)}$, where D is any bounded domain with smooth boundary and we have set

$$W_{q,a}^2(D) = \left\{ w \in W_{q,0}^2(D) \mid \int_D w\,dx = 0 \right\}.$$

To apply the Bogovskiĭ operator $\boldsymbol{B}$ to the term: $(\nabla\psi)\cdot(u_0 - u_\infty)$, we have to observe that

$$
\begin{aligned}
&\int_{R-3\leqq|x|\leqq R-2}(\nabla\psi)\cdot(u_0-u_\infty)dx\\
&\quad=\int_{\Omega_R}(\nabla\psi)\cdot u_0\,dx-\int_{B_R}(\nabla\psi)\cdot u_\infty\,dx\\
&\quad=\int_{\Omega_R}\operatorname{div}(\psi u_0)dx-\int_{\Omega_R}\psi\operatorname{div}u_0\,dx-\int_{B_R}\operatorname{div}(\psi u_\infty)dx+\int_{B_R}\psi\operatorname{div}u_\infty\,dx\\
&\quad=\int_{\Omega_R}\psi\operatorname{div}[(\varphi_\infty-\varphi_0)\nabla U]dx=-\int_{\Omega_R}(\nabla\psi)\cdot((\varphi_\infty-\varphi_0)\nabla U)dx=0
\end{aligned}
$$

where we have used the facts: $\nu\cdot u_0=0$ on $\partial\Omega_R$, $\psi=0$ on S_R, $\nu\cdot\nabla U=0$ on Γ and $(\varphi_\infty-\varphi_0)=0$ on $\operatorname{supp}\nabla\psi$. If we set $u=\psi u_0+(1-\psi)u_\infty-\boldsymbol{B}[(\nabla\psi)\cdot(u_0-u_\infty)]+v$ and $p=\psi p_0+(1-\psi)p_\infty+\theta$ in (1.1), then as the equation of (v,θ) we have

$$
\begin{aligned}
\lambda v-\operatorname{Div}T(v,\theta)=F,\qquad \operatorname{div}v=0\quad\text{in }\Omega\\
\nu\cdot v=0,\quad B_{\alpha,\beta}(v)=0\quad\text{on }\Gamma
\end{aligned}
$$

where the i-th component F_i of n-th vector F is given by the following formula:

$$
\begin{aligned}
F_i=\;&-\frac{\partial\psi}{\partial x_i}(p_0-p_\infty)+\lambda\boldsymbol{B}[(\nabla\psi)\cdot(u_0-u_\infty)]_i-\sum_{j=1}^n\frac{\partial}{\partial x_j}D(\boldsymbol{B}[(\nabla\psi)\cdot(u_0-u_\infty)])_{ij}\\
&+\sum_{j=1}^n\frac{\partial}{\partial x_j}\left(\frac{\partial\psi}{\partial x_i}(u_{0j}-u_{\infty j})+\frac{\partial\psi}{\partial x_j}(u_{0i}-u_{\infty i})\right)\\
&+\sum_{j=1}^n\frac{\partial\psi}{\partial x_j}\left(\frac{\partial}{\partial x_i}(u_{0j}-u_{\infty j})+\frac{\partial}{\partial x_j}(u_{0i}-u_{\infty i})\right).
\end{aligned}
$$

Since $\operatorname{supp}F_i\subset D_{R-3,R-2}$, to complete the proof of Theorem 1.3 in the exterior domain case, it suffices to prove the following theorem.

THEOREM 7.2. *Let $1<q<\infty$. Set*

$$L_{q,R-1}=\{f\in L_q(\Omega)^n\mid f(x)=0\ \textit{for}\ |x|\geqq R-1\}$$

Then, for every $\lambda\in\boldsymbol{C}\setminus(-\infty,0]$ and $f\in L_{q,R-1}(\Omega)$ there exists a unique $(u,p)\in W_q^2(\Omega)^n\times\hat{W}_q^1(\Omega)$ that solves the equation:

$$
\begin{aligned}
\lambda u-\operatorname{Div}(u,p)=f,\qquad \operatorname{div}u=0\quad\textit{in }\Omega\\
\nu\cdot u=0,\quad B_{\alpha,\beta}(u)=0\quad\textit{on }\Gamma.
\end{aligned}
\tag{7.4}
$$

PROOF. Given $f\in L_{q,R-1}(\Omega)$, we set $f_0(x)=f(x)$ for $x\in\Omega$ and $f_0(x)=0$ for $x\notin\Omega$ and γf denotes the restriction of f to Ω_R. Let $(v_0,p_0)\in W_q^2(\Omega_R)^n\times\hat{W}_q^1(\Omega_R)$ and $(v_\infty,p_\infty)\in W_q^2(\boldsymbol{R}^n)^n\times\hat{W}_q^1(\boldsymbol{R}^n)$ be solutions to the equations:

$$\begin{cases} \lambda v_0 - \operatorname{Div} T(v_0, p_0) = \gamma f, \ \operatorname{div} v_0 = 0 & \text{in } \Omega_R, \\ \nu \cdot v_0 = 0, \ B_{\alpha,\beta}(v_0) = 0 & \text{on } \partial\Omega_R, \end{cases}$$
$$\lambda v_\infty - \operatorname{Div} T(v_\infty, p_\infty) = f_0, \ \operatorname{div} v_\infty = 0 \quad \text{in } \boldsymbol{R}^n,$$

respectively. Let us define the operator $A_0 : L_{q,R-1}(\Omega) \to W_q^2(\Omega_R)^n$, $A_\infty : L_{q,R-1}(\Omega) \to W_q^2(\boldsymbol{R}^n)^n$, $B_0 : L_{q,R-1}(\Omega) \to \hat{W}_q^1(\Omega_R)$ and $B_\infty : L_{q,R-1}(\Omega) \to \hat{W}_q^1(\boldsymbol{R}^n)$ by the formulas: $A_0 f = v_0$, $A_\infty f = v_\infty$, $B_0 f = p_0$ and $B_\infty f = p_\infty$, respectively. Let φ be a function in $C_0^\infty(\boldsymbol{R}^n)$ such that $\varphi(x) = 1$ for $|x| \leqq R-3$ and $\varphi(x) = 0$ for $|x| \geqq R-2$, and set

$$\begin{aligned} \Phi f &= (1-\varphi) A_\infty f + \varphi A_0 f + \boldsymbol{B}[(\nabla\varphi) \cdot (A_\infty f - A_0 f)] \\ \Psi f &= (1-\varphi) B_\infty f + \varphi B_0 f \end{aligned}$$

where $\boldsymbol{B} : W_{q,a}^2(D_{R-3,R-2}) \to W_q^3(\boldsymbol{R}^n)^n$ is the Bogovskiĭ operator. To apply $\boldsymbol{B}$ to $(\nabla\varphi) \cdot (A_\infty f - A_0 f)$, we need the following observation:

$$\begin{aligned} \int_{D_{R-3,R-2}} (\nabla\varphi) \cdot (A_\infty f - A_0 f) dx &= \int_{B_R} \operatorname{div}(\varphi A_\infty f) dx - \int_{\Omega_R} \operatorname{div}(\varphi A_0 f) dx \\ &= \int_{S_R} \nu \cdot (\varphi A_\infty f) d\sigma - \int_{\partial\Omega_R} \nu \cdot (\varphi A_0 f) d\sigma = 0. \end{aligned}$$

Subtracting suitable constants if necessary, we may assume that

$$\int_{\Omega_{R-3}} B_0 f \, dx = 0, \quad \int_{\Omega_R} (B_\infty f - B_0 f) dx = 0. \tag{7.5}$$

We have

$$\begin{aligned} \lambda \Phi f - \operatorname{Div} T(\Phi f, \Psi f) = f + Sf, \quad & \operatorname{div}(\Phi f) = 0 \quad \text{in } \Omega, \\ \nu \cdot (\Phi f) = 0, \quad & B_{\alpha,\beta}(\Phi f) = 0 \quad \text{on } \Gamma \end{aligned} \tag{7.6}$$

where we have set

$$\begin{aligned} (Sf)_i = &\sum_{j=1}^n \frac{\partial}{\partial x_j}\Big(\frac{\partial\varphi}{\partial x_i}((A_\infty f)_j - (A_0 f)_j) + \frac{\partial\varphi}{\partial x_j}((A_\infty f)_i - (A_0 f)_i)\Big) \\ &+ \sum_{j=1}^n \frac{\partial\varphi}{\partial x_j}\Big(\frac{\partial}{\partial x_i}((A_\infty f)_j - (A_0 f)_j) + \frac{\partial}{\partial x_j}((A_\infty f)_i - (A_0 f)_i)\Big) \\ &- \frac{\partial\varphi}{\partial x_i}(B_\infty f - B_0 f) + \lambda \boldsymbol{B}[(\nabla\varphi) \cdot (A_\infty f - A_0 f)]_i \\ &- \sum_{j=1}^n \frac{\partial}{\partial x_j} D(\boldsymbol{B}[(\nabla\varphi) \cdot (A_\infty f - A_0 f)])_{ij}. \end{aligned}$$

Since the solution of (7.4) is given by the formula: $u = \Phi(I+S)^{-1}f$ and $p = \Psi(I+S)^{-1}f$ provided that $(I+S)^{-1}$ exists, to prove Theorem 7.2 (and therefore to complete the proof of Theorem 1.3 in the exterior domain case) we start with the following lemma.

LEMMA 7.3. *If the uniqueness of solutions to* (7.4) *holds, then* $(I+S)^{-1}$ *exists.*

PROOF. Since $Sf \in W^1_q(\Omega)$ and $\operatorname{supp} Sf \subset D_{R-3,R-2}$, S is a compact operator on $L_{q,R-1}(\Omega)$. Therefore, to show the existence of $(I+S)^{-1}$ it suffices to show the injectivity of $I+S$. Let $f \in L_{q,R}(\Omega)$ satisfy the equation: $(I+S)f = 0$. We shall show that $f = 0$, in what follows. Set $u = \Phi f$ and $p = \Psi f$, and then $(u,p) \in W^2_q(\Omega)^n \times \hat{W}^1_q(\Omega)$. Moreover, from (7.6) it follows that (u,p) satisfies the homogeneous equation:

$$\begin{aligned} \lambda u - \operatorname{Div} T(u,p) = 0, &\qquad \operatorname{div} u = 0 \quad \text{in } \Omega, \\ \nu \cdot u = 0, &\quad B_{\alpha,\beta}(u) = 0 \quad \text{on } \Gamma \end{aligned} \tag{7.7}$$

because $(I+S)f = 0$. By the assumption we have $u = 0$ and $p = c$ (c being a constant). Since

$$\int_{\Omega_{R-3}} p\,dx = \int_{\Omega_{R-3}} \Psi f\,dx = \int_{\Omega_{R-3}} B_0 f\,dx = 0$$

as follows from (7.5) and the fact that $\varphi = 1$ on B_{R-3}, we have $c = 0$. Therefore, we have

$$\begin{aligned} 0 &= (1-\varphi)A_\infty f + \varphi A_0 f + \boldsymbol{B}[(\nabla\varphi)\cdot(A_\infty f - A_0 f)] \\ 0 &= (1-\varphi)B_\infty f + \varphi B_0 f \end{aligned} \tag{7.8}$$

in Ω, which in particular implies that

$$\begin{aligned} A_\infty f = 0, &\quad B_\infty f = 0 \quad \text{for } |x| \geqq R-2 \\ A_0 f = 0, &\quad B_0 f = 0 \quad \text{for } |x| \leqq R-3. \end{aligned} \tag{7.9}$$

If we set $w = A_0 f$ in Ω_R and $w = 0$ on the outside of Ω and $\theta = B_0 f$ in Ω_R and $\theta = 0$ on the outside of Ω, then $(w,\theta) \in W^2_q(B_R)^n \times \hat{W}^1_q(B_R)$ satisfies the equation:

$$\begin{aligned} \lambda w - \operatorname{Div} T(w,\theta) = f_0, &\qquad \operatorname{div} w = 0 \quad \text{in } B_R, \\ \nu \cdot w = 0, &\quad B_{\alpha,\beta}(w) = 0 \quad \text{on } S_R. \end{aligned} \tag{7.10}$$

On the other hand, from (7.9) it follows that $A_\infty f$ and $B_\infty f$ also satisfy (7.10), and therefore $w = A_\infty f$ and $\theta - B_\infty f = c$ in B_R where c is some constant. But, by (7.5) we have

$$\int_{\Omega_R} (\theta - B_\infty f)dx = \int_{\Omega_R} (B_0 f - B_\infty f)dx = 0$$

which implies that $c = 0$. Namely, $A_\infty f = A_0 f$ and $B_\infty f = B_0 f$ in Ω_R, which combined with (7.8) implies that

$$A_\infty f = \varphi(A_\infty f - A_0 f) - \boldsymbol{B}[(\nabla\varphi)\cdot(A_\infty f - A_0 f)] = 0, \quad B_\infty f = \varphi(B_\infty f - B_0 f) = 0$$

in Ω. Therefore, $f = \lambda A_\infty f - \mathrm{Div}\, T(A_\infty f, B_\infty f) = 0$ in Ω, which completes the proof of the lemma. $\square$

In view of Lemma 7.3, to complete the proof of Theorem 7.2 our task is to show the uniqueness of solutions to (7.4). Let $\lambda \in \boldsymbol{C} \setminus (-\infty, 0]$ and $(u, p) \in W_q^2(\Omega)^n \times \hat W_q^1(\Omega)$ satisfies the homogeneous equation (7.7). By a boot-strap argument we see that $u \in W_{2,\mathrm{loc}}^2(\overline{\Omega})^n$ and $p \in W_{2,\mathrm{loc}}^1(\overline{\Omega})$, because we already know that Theorem 1.4 holds for any bounded domains. Let ψ be a function in $C^\infty(\boldsymbol{R}^n)$ such that $\psi = 1$ for $|x| \geqq R-2$ and $\psi = 0$ for $|x| \leqq R-3$, and set $w = \psi u - \boldsymbol{B}[(\nabla\psi)\cdot u]$ and $\theta = \psi p$. To apply the Bogovskiĭ operator $\boldsymbol{B}$ to $(\nabla\psi)\cdot u$, we need the following observation:

$$\int_{D_{R-3,R-2}} (\nabla\psi)\cdot u\, dx = -\int_{\Omega_R} \mathrm{div}\,((1-\psi)u)dx = \int_\Gamma \nu\cdot u\, d\sigma = 0.$$

Therefore, $(w, \theta) \in W_q^2(\boldsymbol{R}^n)^n \times \hat W_q^1(\boldsymbol{R}^n)$ and (w, θ) satisfies the equation:

$$\lambda w - \mathrm{Div}\, T(w, \theta) = F, \quad \mathrm{div}\, w = 0 \quad \text{in } \boldsymbol{R}^n \tag{7.11}$$

where we have set

$$\begin{aligned} F_i = & -\sum_{j=1}^n \frac{\partial}{\partial x_j}\left(\frac{\partial\psi}{\partial x_i}u_j + \frac{\partial\psi}{\partial x_j}u_i\right) - \sum_{j=1}^n \frac{\partial\psi}{\partial x_j}D_{ij}(u) + \frac{\partial\psi}{\partial x_i}p \\ & - \lambda\boldsymbol{B}[(\nabla\psi)\cdot u]_i + \sum_{j=1}^n \frac{\partial}{\partial x_j}D(\boldsymbol{B}[(\nabla\psi)\cdot u])_{ij}. \end{aligned}$$

Since $F \in L_q(\boldsymbol{R}^n)^n \cap L_2(\boldsymbol{R}^n)^n$, by Theorem 2.1 there exists $(\tilde w, \tilde\theta) \in (W_2^2(\boldsymbol{R}^n)^n \cap W_q^2(\boldsymbol{R}^n)^n) \times (\hat W_2^1(\boldsymbol{R}^n) \cap \hat W_q^1(\boldsymbol{R}^n))$ that also satisfies (7.11). Since the uniqueness holds in the whole space (cf. Theorem 2.1), we have $w = \tilde w$ and $\nabla\theta = \nabla\tilde\theta$, which implies that $u \in W_2^2(\Omega)^n$ and $p \in \hat W_2^1(\Omega)$. Therefore, noting that $C_0^\infty(\boldsymbol{R}^n)$ is dense in $\hat W_2^1(\Omega)$ (cf. [**5**, Lemma 5.1]) and using the divergence theorem of Gauss, by (7.7) we have

$$0 = (\lambda u - \mathrm{Div}\, T(u,p), u)_\Omega = \lambda\|u\|^2_{L_2(\Omega)} + (1/2)\|D(u)\|^2_{L_2(\Omega)} + \alpha\beta^{-1}\|u\|^2_{L_2(\Gamma)} \tag{7.12}$$

which combined with the fact: $\lambda \in \boldsymbol{C} \setminus (-\infty, 0]$ implies immediately that $u = 0$. And therefore, by (7.7) we have also $\nabla p = 0$, which shows the uniqueness. This completes the proof of Theorem 1.3. $\square$

A. Proofs of Propositions 5.5 and 5.6.

A PROOF OF PROPOSITION 5.5. To prove Proposition 5.5, we shall use the following well-known lemma (cf. [**9**], [**10**]).

LEMMA A.1. *Let $1 < q < \infty$ and D be a bounded domain whose boundary ∂D is a $C^{1,1}$ hypersurface. Then, for every $f \in L_q(D)$ and $g \in W_q^1(D)$ there exists a $v \in W_q^2(D)$ that uniquely solves the equation*:

$$-v + \Delta v = f \quad \textit{in } D, \quad \partial_\nu v = g \quad \textit{on } \partial D \tag{A.1}$$

and satisfies the estimate:

$$\|v\|_{W_q^2(D)} \leqq C\Big(\|f\|_{L_q(D)} + \|g\|_{W_q^1(D)}\Big). \tag{A.2}$$

Given $f \in L_q(D)$ and $g \in W_q^1(D)$, let $v \in W_q^2(D)$ be a solution to (A.1). If there holds the relation: $\int_D f\, dx = \int_{\partial D} g\, d\sigma$, then by (A.1) we have

$$\int_D v\, dx = \int_D \Delta v\, dx - \int_D f\, dx = \int_{\partial D} g\, d\sigma - \int_D f\, dx = 0. \tag{A.3}$$

If we set $u = v + w$, then it suffices to solve the equation: $\Delta w = -v$ in D and $\partial_\nu w = 0$ on ∂D with side condition: $\int_D w\, dx = 0$. Therefore, the following lemma implies Proposition 5.5 immediately.

LEMMA A.2. *Let $1 < q < \infty$ and D be a bounded domain whose boundary ∂D is a $C^{1,1}$ hypersurface. Set $\hat{L}_q(D) = \{f \in L_q(D) \mid \int_D f\, dx = 0\}$. Then, for every $f \in \hat{L}_q(D)$ there exists a $u \in W_q^2(D) \cap \hat{L}_q(D)$ that uniquely solves the equation*:

$$\Delta u = f \quad \textit{in } D \quad \partial_\nu u = 0 \quad \textit{on } \partial D \tag{A.4}$$

and satisfies the estimate:

$$\|u\|_{W_q^2(D)} \leqq C\|f\|_{L_q(D)}. \tag{A.5}$$

PROOF. Set $\hat{W}_q^2(D) = \{u \in W_q^2(D) \cap \hat{L}_q(D) \mid \partial_\nu u = 0 \text{ on } \partial D\}$. Given $g \in \hat{L}_q(D)$, let $v \in \hat{W}_q^2(D)$ be a solution to the equation: $-v + \Delta v = g$ in D. Let us define the operator $A : \hat{L}_q(D) \to \hat{W}_q^2(D)$ by the formula: $Ag = v$. Since $\Delta Ag = (I + A)g$ (I being the identity map on $\hat{L}_q(D)$), if we show the existence of $(I + A)^{-1}$, then we see that $u = A(I + A)^{-1}f$ is a required solution to (A.4). Since A is a compact operator on $\hat{L}_q(D)$, to show the existence of $(I + A)^{-1}$, it suffices to show the injectivity of the operator $I + A$. Let f be a function in $\hat{L}_q(D)$ such that $(I + A)f = 0$. Set $u = Af$, and then $u \in \hat{W}_q^2(D)$ and u satisfies the equation: $\Delta u = 0$ in D. By a boot-strap argument, we see that $u \in W_2^2(D)$, and therefore by the divergence theorem of Gauss we have

$\|\nabla u\|_{L_2(D)} = 0$, which implies that u is a constant. But then, what $\int_D u\,dx = 0$ implies that $u = 0$. Since $f = -u + \Delta u = 0$, we have the injectivity of the operator $I + A$, which completes the proof of the lemma. □

A proof of Proposition 5.6. As in the proof of Theorem 7.2, we shall construct a parametrix. Let $E(x)$ be a fundamental soluiton of the Laplace operator Δ given in the proof of Lemma 3.2. Given $f \in \hat{L}_{q,R-1}(\Omega)$, we set

$$Ef(x) = \int_{\boldsymbol{R}^n} E(x-y) f_0(y) dy$$

where $f_0(x) = f(x)$ for $x \in \Omega$ and $f_0(x) = 0$ for $x \notin \Omega$. We see easily that

$$\|Ef\|_{L_q(B_R)} + \sup_{|x| \geqq R} |x|^{n-1} |Ef(x)| + \|\nabla Ef\|_{L_q(\boldsymbol{R}^n)} \leqq C\|f\|_{L_q(\Omega)}. \tag{A.6}$$

On the other hand, given $f \in \hat{L}_{q,R-1}(\Omega)$, γf denotes the restriction of f on Ω_R and let g be a function in $W^1_q(\Omega_R)$ such that $g(x) = \partial_\nu(Ef)$ on S_R and $g(x) = 0$ on Γ. Observe that

$$\begin{aligned}\int_{\partial\Omega_R} g\,d\sigma &= \int_{S_R} g\,d\sigma + \int_\Gamma g\,d\sigma = \int_{S_R} \partial_\nu(Ef) d\sigma = \int_{B_R} \Delta(Ef) dx \\ &= \int_{B_R} f_0\,dx = \int_\Omega f\,dx = 0.\end{aligned}$$

Noting that $\int_{\Omega_R} \gamma f\,dx = \int_\Omega f\,dx = 0$, by Proposition 5.5 there exists a $v \in W^2_q(\Omega)$ that solves the equation:

$$\Delta v = \gamma f \text{ in } \Omega_R, \quad \partial_\nu v = \partial_\nu Ef \text{ on } S_R \text{ and } \partial_\nu v = 0 \text{ on } \Gamma \tag{A.7}$$

and satisfies the estimate:

$$\|v\|_{W^2_q(\Omega_R)} \leqq C\|f\|_{L_q(\Omega)}. \tag{A.8}$$

Moreover, subtracting a suitable constant from v if necessary, we may assume that

$$\int_{\Omega_R} (v - Ef) dx = 0. \tag{A.9}$$

Let A be an operator from $\hat{L}_{q,R-1}(\Omega)$ into $W^2_q(\Omega_R)$ defined by the formula: $v = Af$. Let φ be a function in $C^\infty_0(\boldsymbol{R}^n)$ such that $\varphi(x) = 1$ for $|x| \leqq R-3$ and $\varphi(x) = 0$ for $|x| \geqq R-2$, and set

$$\Phi f = (1-\varphi)Ef + \varphi Af \quad f \in \hat{L}_{q,R-1}(\Omega).$$

Then, we have

$$\Delta\Phi f = f + Sf \quad \text{in } \Omega \quad \partial_\nu \Phi f = 0 \quad \text{on } \Gamma \tag{A.10}$$

where we have set

$$Sf = -\mathrm{div}\,[(\nabla\varphi)(Ef - Af)] - (\nabla\varphi)\cdot\nabla(Ef - Af).$$

We observe that $Sf \in W^1_q(\Omega)$, that $\mathrm{supp}\, Sf \subset D_{R-3,R-2}$, and that

$$\begin{aligned}\int_\Omega Sf\,dx &= -\int_\Omega (\nabla\varphi)\cdot\nabla(Ef - Af)dx = \int_{\Omega_R} \nabla(1-\varphi)\cdot\nabla(Ef - Af)dx \\ &= \int_{S_R} \partial_\nu(Ef - Af)d\sigma - \int_{\Omega_R}(1-\varphi)\Delta(Ef - Af)dx = 0\end{aligned}$$

and therefore S is a compact operator on $\hat{L}_{q,R-1}(\Omega)$. If we show the existence of $(I+S)^{-1}$, then $u = \Phi(I+A)^{-1}f \in \hat{W}^2_q(\Omega)$ solves the equation: $\Delta u = f$ in Ω and $\partial_\nu u = 0$ on Γ. Moreover, combining (A.6) and (A.8) we see that this u satisfies the estimate (5.16). Since the uniqueness follows from the existence of solutions to the dual problem, to complete the proof of Proposition 5.6 it suffice to show the existence of $(I+S)^{-1}$.

Since S is a compact operator on $\hat{L}_{q,R-1}(\Omega)$, to show the existence of $(I+S)^{-1}$, it suffices to show the injectivity of the operator $I+S$. Let f be a function in $\hat{L}_{q,R-1}(\Omega)$ such that $(I+S)f = 0$. Set $u = \Phi f \in \hat{W}^2_q(\Omega)$, and then by (A.10), (A.6) and (A.8) we see that $u \in W^2_q(\Omega)$ satisfies the homogeneous equation: $\Delta u = 0$ in Ω and $\partial_\nu u = 0$ on Γ and the radiation condition: $u(x) = O(|x|^{-(n-1)})$ as $|x| \to \infty$. Therefore, $u = 0$ in Ω. In fact, by using a boot-strap argument, $u \in W^2_{2,\mathrm{loc}}(\overline{\Omega})$. Let ρ be a function in $C^\infty_0(\boldsymbol{R}^n)$ such that $\rho(x) = 1$ for $|x| \leqq 1$ and $\rho(x) = 0$ for $|x| \geqq 2$ and set $\rho_L(x) = \rho(x/L)$. By the divergence theorem of Gauss and the radiation condition we have

$$0 = \lim_{L\to\infty}(\Delta u, \rho_L u)_\Omega = \lim_{L\to\infty}\big\{-(\nabla u, \rho_L\nabla u)_\Omega + (1/2)(u, (\Delta\rho_L)u)_\Omega\big\} = \|\nabla u\|_{L_2(\Omega)}$$

which implies that u is a constant. But then, by the radiation condition we have $u = 0$. Therefore, we have

$$0 = (1-\varphi)Ef + \varphi Af \quad \text{in } \Omega \tag{A.11}$$

which in particular implies that

$$Ef = 0 \quad \text{for } |x| \geqq R-2, \quad Af = 0 \quad \text{for } |x| \leqq R-3. \tag{A.12}$$

If we define $w(x) = Af(x)$ for $x \in \Omega_R$ and $w(x) = 0$ for $x \notin \Omega$, then by (A.12) $w \in W^2_q(B_R)$ and w satisfies the equation: $\Delta w = f_0$ in B_R and $\partial_\nu w = \partial_\nu Ef$ on S_R, which is also satisfied by Ef, and therefore $w - Ef = c$ in B_R (c being a constant). But then, by (A.9) we have $\int_{\Omega_R} c\,dx = \int_{\Omega_R}(Af - Ef)dx = 0$, which implies that $c = 0$, that is $Ef = Af$ in Ω_R. Inserting this equality into (A.11) implies that $Ef = \varphi(Ef - Af) = 0$ in Ω, which shows that $f = \Delta Ef = 0$ in Ω. This show the injectivity of the map $I + S$, which completes the proof of Proposition 5.6. □

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