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The spectrum of the Milnor-Gromoll-Meyer sphere

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Abstract. The spectrum of the Milnor-Gromoll-Meyer sphere is given. For this purpose the system of orthogonal functions on the symplectic group $Sp(2)$ (isomorphic to the covering group of $SO(5)$) is studied.

Introduction.

The purpose of the present paper is to describe the spectrum of the Milnor-Gromoll-Meyer sphere Σ^7, a Riemannian manifold which is homeomorphic to the standard 7-sphere S^7 but not diffeomorphic to S^7. Our idea for finding the spectrum of Σ^7 is simple. Namely, since Σ^7 is given as a base space of a Riemannian submersion $Sp(2) \to \Sigma^7$ due to [**GM**], the spectrum of Σ^7 is a "subspectrum" of the compact Lie group $Sp(2)$ with bi-invariant metric. Here we recall that each space of eigenfunctions, belonging to a common eigenvalue, on $Sp(2)$ is the space $H_{\boldsymbol{m}}$ spanned by matrix elements (representation coefficients) of an irreducible representation $\pi_{\boldsymbol{m}}$ of $Sp(2)$. Therefore, our problem is reduced to the problem of finding the dimension of the subspace $H_{\boldsymbol{m}}^{\Gamma}$ consisting of functions in $H_{\boldsymbol{m}}$ which are constant on each fiber of the submersion. Then the latter problem is solved by studying how the vertical vector fields of the submersion act on the matrix elements—a problem of finding the dimension of the kernel of a linear mapping.

This paper is divided into five sections. We start with the definition of Milnor-Gromoll-Meyer sphere Σ^7 and state our main results, which describe, in addition to the spectrum of Σ^7, the spectrums of some other Riemannian manifolds diffeomorphic to S^7. The next section, §2, is devoted to the representation theory of $SpU(4)$ (which is isomorphic to $Sp(2)$ by the isomorphism $\boldsymbol{c}$ in §3). Following [**Zh**] we consider the Lie group $Sp(4, \boldsymbol{C})$ (complexification of $SpU(4)$), and recall the definition of the representation $\pi_{\boldsymbol{m}}$ with the representation space $\mathfrak{R}_{\boldsymbol{m}}$. Then we express the canonical orthogonal basis (Gel'fand-Cetlin basis) of $\mathfrak{R}_{\boldsymbol{m}}$ as derivatives of the generating function $\phi_{\boldsymbol{m}}$, and get the explicit expression for the matrix elements of $\pi_{\boldsymbol{m}}$. From this expression we obtain the

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formulas which tell us how right or left invariant vector fields on $Sp(4, \boldsymbol{C})$ act on the matrix elements (Corollary 2.3.2). In §3 we give the correspondence $\boldsymbol{c}$ between $Sp(2)$ and $SpU(4)$, which allows us to consider, instead of the original submersion $Sp(2) \to \Sigma^7$, the submersion $SpU(4) \to \Sigma^7$ and to apply the results in §2. In §4, using the fact that the vertical space field of the submersion $SpU(4) \to \Sigma^7$ has a basis consisting of vector fields written as sums of right or left invariant vector fields on $SpU(4)$ (Proposition 3.1.1), we derive the condition for a function in $H_{\boldsymbol{m}}$ to be constant on each fiber of the submersion. Studying this condition through the matrix elements, we see that the problem of finding the dimension of the subspace $H_{\boldsymbol{m}}^{\Gamma}$ becomes the problem of counting the number of lattice points in a certain subset in $\boldsymbol{R}^5$ (Proposition 4.1.1). In Appendix, we give the proof of Theorem 2.2.1 (the expression for Gel'fand-Cetlin basis) by direct computation, and prove the other statements in §2 as its consequences.

Although our result (Theorem 1.1.1) describes the spectrum of Σ^7, we can not "hear" the shape of Σ^7 yet. We hope that our datum makes a contribution to such new development in spectral geometry as statistical properties of spectrums. See e.g. [**KMS**], [**UZ**] for the case in which geodesic flows are completely integrable, and [**Ze**], [**La**] for a more general case. (Incidentally, the geodesic flow of our object Σ^7 is completely integrable.)

Throughout this paper, by functions, vector spaces we mean complex-valued functions, complex vector spaces respectively, unless otherwise specified.

1. The spectrums of some 7-spheres lying under $Sp(2)$.

1.1. Statement of the main results.

Let $\boldsymbol{H}$ be the algebra of quaternions, with basis $1, i, j, k$ satisfying the usual multiplication rules. Let $Sp(n)$ denote the symplectic group for dimension n, that is, the group of $n \times n$ quaternion matrices Q such that $QQ^* = Q^*Q = \mathrm{Id}$, where Q^* is the transposed conjugate matrix of Q. We fix on $Sp(2)$ the bi-invariant metric g_0 normalized so that the tangent vector $i_+ = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \in T_eSp(2) = sp(2)$, at the identity element e, has unit length. Let $\Gamma : Sp(1) \times Sp(2) \to Sp(2)$ be the action defined by the formula

$$\Gamma(q, Q) = \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} Q \begin{pmatrix} \bar{q} & 0 \\ 0 & 1 \end{pmatrix},$$

where $\bar{q}$ is the conjugate of q. Since this action is free and isometric, we can consider the quotient Riemannian manifold $\Sigma^7 = \Gamma \backslash Sp(2)$, which turns out to be an exotic sphere, i.e. a manifold homeomorphic to but not diffeomorphic to the standard sphere S^7 ([**GM**]).

Definition. We call Γ the *Gromoll-Meyer action* on $Sp(2)$, and the quotient Riemannian manifold Σ^7 the *Milnor-Gromoll-Meyer 7-sphere*.

Definition. Let $\gamma = \{0 = \gamma_0 < \gamma_1 \leq \gamma_2 \leq \cdots\}$, $\lambda = \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ be two infinite sequences tending to ∞. We say that γ, λ are *uniformly close* if there exists a positive constant c such that $|\gamma_l - \lambda_l| \leq c$ for all $l = 0, 1, 2, \cdots$. If that is the case, we write $Z_\gamma \doteqdot Z_\lambda$ for the series $Z_\gamma = \sum_{l=0}^{\infty} \exp(-\gamma_l t)$, $Z_\lambda = \sum_{l=0}^{\infty} \exp(-\lambda_l t)$.

Now we can state our result.

Theorem 1.1.1. *Let $\{0 = \gamma_0 < \gamma_1 \leq \gamma_2 \leq \cdots\}$ be the spectrum of Σ^7, and let $Z_{\Sigma^7} = \sum_{l=0}^{\infty} \exp(-\gamma_l t)$ be the partition function. Then Z_{Σ^7} satisfies*

$$Z_{\Sigma^7} \doteqdot \sum \mathfrak{m}_{\boldsymbol{n}} \exp(-\lambda_{\boldsymbol{n}} t), \quad \lambda_{\boldsymbol{n}} = n_1^2 + n_2^2 - 5.$$

Here the summation is taken over all pairs $\boldsymbol{n} = (n_1, n_2)$ of integers such that $n_1 > n_2 \geq 1$, and $\mathfrak{m}_{\boldsymbol{n}}$ is defined as follows:

$$\mathfrak{m}_{\boldsymbol{n}} = \begin{cases} \boldsymbol{i} + n_2^2((-1)^{n_1} 2n_1 - (-1)^{n_2}(n_1 - n_2))/16 & \textit{if } 2n_2 \leq n_1, \\ \boldsymbol{j} + (n_1 - n_2)((-1)^{n_1} n_1^2 - (-1)^{n_2} n_2^2)/16 & \textit{if } 2n_2 \geq n_1, \end{cases}$$

where

$$\boldsymbol{i} = n_2^2(n_1 + n_2)(4n_1^2 - 4n_1 n_2 - 2n_2^2 + 5)/48,$$

$$\boldsymbol{j} = (n_1 - n_2)^2(n_1 + n_2)(-2n_1^2 + 8n_1 n_2 - 2n_2^2 + 5)/48.$$

Remark. Needless to say, the values of $\mathfrak{m}_{\boldsymbol{n}}$ coincide for the case $2n_2 = n_1$, and although $\mathfrak{m}_{\boldsymbol{n}}$ are integers, $\boldsymbol{i}, \boldsymbol{j}$ may be ratinal numbers.

Besides the Gromoll-Meyer action Γ, we can consider other free, isometric actions $Sp(1) \times Sp(2) \to Sp(2)$ and get some 7-dimensional Riemannian manifolds as quotient manifolds. For such a Riemannian manifold M^7, since we have a Riemannian submersion $Sp(2) \to M^7$, we say that M^7 *lies under* $Sp(2)$. We shall discuss the following cases:

(1) the action $(q, Q) \mapsto \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} Q$, and its quotient manifold $\tilde{S}^7$,

(2) the action $(q, Q) \mapsto \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} Q$, and its quotient manifold $\tilde{\tilde{S}}^7$,

(3) the action $(q, Q) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} Q \begin{pmatrix} \bar{q} & 0 \\ 0 & 1 \end{pmatrix}$, and its quotient manifold $\check{S}^7$.

It is easy to see that $\tilde{S}^7, \tilde{\tilde{S}}^7$ are homogeneous Riemannian manifolds diffeomorphic to the standard 7-sphere S^7. On the other hand, $\check{S}^7$ is diffeomorphic to S^7, but now the metric is not homogeneous. For these manifolds we can give their respective spectrums. To describe those we denote by Z_{M^7} the partition function of M^7 (cf. [**BGM**]).

THEOREM 1.1.2. *The partition functions satisfy*

$$Z_{\tilde{S}^7} = \sum \tilde{\mathfrak{m}}_{\boldsymbol{n}} \exp(-\lambda_{\boldsymbol{n}} t), \quad \tilde{\mathfrak{m}}_{\boldsymbol{n}} = n_1 n_2 (n_1 + n_2)(n_1 - n_2)^2/6,$$

$$Z_{\tilde{\tilde{S}}^7} = \sum \tilde{\tilde{\mathfrak{m}}}_{\boldsymbol{n}} \exp(-\lambda_{\boldsymbol{n}} t), \quad \tilde{\tilde{\mathfrak{m}}}_{\boldsymbol{n}} = (1 - (-1)^{n_1+n_2}) n_1 n_2^2 (n_1 + n_2)(n_1 - n_2)/12,$$

$$Z_{\check{S}^7} \doteqdot \sum \check{\mathfrak{m}}_{\boldsymbol{n}} \exp(-\lambda_{\boldsymbol{n}} t), \quad \check{\mathfrak{m}}_{\boldsymbol{n}} = n_1 n_2 (n_1 - n_2)(1 + 2n_2(n_1 - n_2))/6.$$

Here the summations are taken over all pairs $\boldsymbol{n} = (n_1, n_2)$ *of integers such that* $n_1 > n_2 \geq 1$, *and* $\lambda_{\boldsymbol{n}} = n_1^2 + n_2^2 - 5$.

We shall prove these theorems in 1.3, assuming the propositions stated in the next subsection.

1.2. The spaces $H_{\boldsymbol{m}}$ of eigenfunctions on $Sp(2)$ and their subspaces of functions fixed under the actions.

By a signature we mean a pair (m_1, m_2) of integers satisfying $m_1 \geq m_2 \geq 0$. It is known that every irreducible unitary representation of $Sp(2)$ is uniquely defined (up to equivalence) by a signature $\boldsymbol{m} = (m_1, m_2)$ (see [**Zh**]). Now fix a signature $\boldsymbol{m} = (m_1, m_2)$, and let $H_{\boldsymbol{m}}$ denote the vector space spanned by matrix elements (or representation-coefficients) of the representation defined by $\boldsymbol{m}$. Then the representation π of $Sp(2) \times Sp(2)$ on $H_{\boldsymbol{m}}$, $\pi(x, y)\varphi(*) = \varphi(x^{-1} * y)$, is irreducible. Let $H_{\boldsymbol{m}}^{\Gamma}$ denote the subspace of $H_{\boldsymbol{m}}$ consisting of functions which are fixed under the Gromoll-Meyer action Γ, that is,

$$H_{\boldsymbol{m}}^{\Gamma} = \left\{ \varphi \in H_{\boldsymbol{m}} \mid \varphi(*) = \varphi\left(\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} * \begin{pmatrix} \bar{q} & 0 \\ 0 & 1 \end{pmatrix} \right) \text{ for all } q \in Sp(1) \right\}.$$

PROPOSITION 1.2.1. *Let* $\boldsymbol{m} = (m_1, m_2)$ *be a signature. Set* $n_1 = m_1 + 2$, $n_2 = m_2 + 1$. *Then the dimension of the vector space* $H_{\boldsymbol{m}}^{\Gamma}$ *is given by the following formulas*:

(i) *If* $2m_2 \leq m_1$, *then*

$$\dim H_{\boldsymbol{m}}^{\Gamma} = \boldsymbol{i} + n_2^2((-1)^{n_1} 2n_1 - (-1)^{n_2}(n_1 - n_2))/16$$

where

$$\boldsymbol{i} = n_2^2(n_1 + n_2)(4n_1^2 - 4n_1 n_2 - 2n_2^2 + 5)/48.$$

(ii) *If* $2m_2 \geq m_1$, *then*

$$\dim H_{\boldsymbol{m}}^{\Gamma} = \boldsymbol{j} + (n_1 - n_2)((-1)^{n_1} n_1^2 - (-1)^{n_2} n_2^2)/16$$

where

$$\boldsymbol{j} = (n_1 - n_2)^2(n_1 + n_2)(-2n_1^2 + 8n_1n_2 - 2n_2^2 + 5)/48.$$

Next, we study the subspaces of $H_{\boldsymbol{m}}$ consisting of functions which are fixed by the other actions introduced in §1.1. Let $H_{\boldsymbol{m}}^{\sim}$, $H_{\boldsymbol{m}}^{\approx}$, $H_{\boldsymbol{m}}^{\vee}$ be the subspaces consisting of functions $\varphi \in H_{\boldsymbol{m}}$ such that

$$\varphi(*) = \varphi\left(\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} *\right), \quad \varphi(*) = \varphi\left(\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} *\right), \quad \varphi(*) = \varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} * \begin{pmatrix} \bar{q} & 0 \\ 0 & 1 \end{pmatrix}\right)$$

on $Sp(2)$ for all $q \in Sp(1)$, respectively. Then $H_{\boldsymbol{m}}^{\sim}$, $H_{\boldsymbol{m}}^{\approx}$, $H_{\boldsymbol{m}}^{\vee}$ are viewed as spaces of functions on $\tilde{S}^7, \tilde{\tilde{S}}^7, \check{S}^7$, respectively.

Proposition 1.2.2. *Let $\boldsymbol{m} = (m_1, m_2)$ be a signature. Set*

$$n_1 = m_1 + 2, \quad n_2 = m_2 + 1, \quad d = n_1 n_2 (n_1 + n_2)(n_1 - n_2)/6.$$

Then

$$\begin{aligned} \dim H_{\boldsymbol{m}}^{\sim} &= (n_1 - n_2)d, \\ \dim H_{\boldsymbol{m}}^{\approx} &= (1 - (-1)^{n_1+n_2}) n_2 d/2, \\ \dim H_{\boldsymbol{m}}^{\vee} &= n_1 n_2 (n_1 - n_2)(1 + 2n_2(n_1 - n_2))/6. \end{aligned}$$

We shall prove these propositions in §4.

1.3. Derivation of the main results from the propositions in §1.2.

We begin with a general statement (in C^∞ category). This gives, for a Riemannian submersion which is not assumed to be harmonic (see the remark below), a relation between the respective spectrums of the Laplacians on the total space and the base space.

Proposition 1.3.1. *Let W be a closed, connected Riemannian manifold, and let $\Phi : K \times W \to W$ be an action of a compact Lie group K. Suppose that Φ is free and isometric. Let M be the quotient manifold equipped with the metric such that the natural mapping $\pi : W \to M$ is a Riemannian submersion. Let Δ be the Laplacian on W, and $\check{\Delta}$ the Laplacian on M. Let $C_{\mathrm{inv}}^\infty(W)$ be the subspace consisting of functions on W which are constant on each fiber $\pi^{-1}(x)$. Let $\Delta|_{C_{\mathrm{inv}}^\infty(W)}$ denote the restriction of Δ to $C_{\mathrm{inv}}^\infty(W)$. For $x \in M$, let $v(x)$ denote the volume of the fiber $\pi^{-1}(x)$. Set $V = v^{1/2}\Delta(v^{-1/2})$. Let γ be a real number, and let ψ be a function on M, which is also viewed as a function on W. Then*

$$(\Delta - V)\psi = -\gamma\psi \quad \textit{if and only if} \quad \check{\Delta}(v^{1/2}\psi) = -\gamma v^{1/2}\psi.$$

Hence, the spectrum $\{0 = \gamma_0 < \gamma_1 \le \gamma_2 \le \cdots\}$ of M is uniformly close to the spectrum $\{0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots\}$ of the operator $\Delta|_{C_{\mathrm{inv}}^\infty(W)}$. More precisely

$$\lambda_l + \min V \leq \gamma_l \leq \lambda_l + \max V \quad \textit{for any } l = 0, 1, 2, \cdots .$$

REMARK. This proposition is applied to the submersions $Sp(2) \to M^7$ introduced in §1.1. Note that the submersion $Sp(2) \to \Sigma^7$ especially is not harmonic, in other words the volumes of the fibers, $v(x)$, are not constant (cf. [**W**]). For this reason we can not give the spectrum exactly, but we know only the "band" in which the spectrum appears.

PROOF. By the usual argument about Riemannian submersions we have

$$\Delta = \check{\Delta} + \operatorname{grad} \log v$$

for functions on M. (Roughly speaking, we use the formula $\Delta = \operatorname{div} \operatorname{grad}$, and observe that the vector field $\operatorname{grad} \psi$ on W is invariant under the action, the log Lie derivative of the horizontal area element with respect to $\operatorname{grad} \psi$ becomes $\check{\Delta}\psi$, and the log Lie derivative of the vertical area element yields the inner product $\boldsymbol{g}(\operatorname{grad} \log v, \operatorname{grad} \psi)$.) Hence we have $(\Delta - v^{1/2}\Delta(v^{-1/2}))(\psi) = v^{-1/2}\check{\Delta}(v^{1/2}\psi)$ for functions ψ on M. This shows the coincidence of the eigenvalues of the operators $(\Delta - V)|_{C^\infty_{\mathrm{inv}}(W)}$ and $\check{\Delta}$. The first part of our proposition is proved. By the maximum-minimum property of the eigenvalues ([**CH**, VI, §2]) we have the inequalities as above for the eigenvalues of the operators $\Delta|_{C^\infty_{\mathrm{inv}}(W)}, (\Delta - V)|_{C^\infty_{\mathrm{inv}}(W)}$, which gives us immediately the desired estimates for the spectrum of M. □

PROOF OF THEOREM 1.1.1. Applying the preceding proposition to the submersion $Sp(2) \to \Sigma^7$, we see that the spectrum of Σ^7 is uniformly close to the spectrum of the operator $\Delta|_{C^\infty_{\mathrm{inv}}(Sp(2))}$. Hence it suffices to show that the partition function of the spectrum of the operator $\Delta|_{C^\infty_{\mathrm{inv}}(Sp(2))}$ is given by $\sum \mathfrak{m}_{\boldsymbol{n}} \exp(-\lambda_{\boldsymbol{n}} t)$. Indeed, by the irreduciblity of the representation of $Sp(2) \times Sp(2)$ on $H_{\boldsymbol{m}}$ mentioned in §1.2, the Laplacian Δ of $Sp(2)$ is a constant multiplication on each subspace $H_{\boldsymbol{m}}$, and this constant is given by $-((m_1+2)^2 + (m_2+1)^2 - 5)$ (Proposition 3.3.1). Note that the Hilbert space $L^2_\Gamma(Sp(2))$ of square integrable functions on $Sp(2)$ which are fixed under the action Γ has the Hilbert space decomposition $L^2_\Gamma(Sp(2)) = \oplus_{\boldsymbol{m}} H^\Gamma_{\boldsymbol{m}}$, where the summation is taken over all signatures $\boldsymbol{m}$. Hence the operator $\Delta|_{C^\infty_{\mathrm{inv}}(Sp(2))}$ has, for each signature $\boldsymbol{m}$, the eigenvalue $-((m_1+2)^2 + (m_2+1)^2 - 5)$ with multiplicity $\dim H^\Gamma_{\boldsymbol{m}}$. Consequently, since we know this multiplicity by Proposition 1.2.1, we obtain the desired formula. □

PROOF OF THEOREM 1.1.2. The proof is the same as above. In fact, for $\tilde{S}^7$ or $\tilde{\tilde{S}}^7$ the spectrum can be found exactly, because the manifold is homogeneous, and hence the volume $v(x)$ of each fiber is constant. As for $\check{S}^7$, the function v is not constant, and hence the partition function is found only through the relation $\doteqdot$. □

In order to know the constants $\min V, \max V$ concretely, we shall need

LEMMA 1.3.2. *Under the same assumption as in the preceding proposition, the volume of the fiber* $\pi^{-1}(x)$ *is given by*

$$v(x) = v_K \sqrt{\det F(x)}.$$

Here $F(x)$ *is the matrix* $(\boldsymbol{g}(e_i^*, e_j^*); i, j = 1, \dots, \dim K)$, *where* $\boldsymbol{g}$ *is the metric on* W, $\{e_i\}$ *is a basis of the Lie algebra of* K, e_i^* *are the vector fields on* W *induced from* e_i, *and* v_K *is the volume of* K *with respect to the left invariant volume element* ω *satisfying* $\omega(e_1, e_2, \dots, e_{\dim K}) = 1$.

PROOF. Elementary. □

2. The system of orthogonal functions on $SpU(4)$.

2.1. The complex symplectic group $Sp(4, \boldsymbol{C})$ and its representations $\{\pi_{\boldsymbol{m}}\}$.

We begin by recalling the definition of $Sp(4, \boldsymbol{C})$ given in [**Zh**]. The symplectic group $Sp(4, \boldsymbol{C})$ consists of matrices $x = (x_{ij})$ satisfying $\sigma = {}^t x \sigma x$, where σ is the matrix

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

and ${}^t x$ is the transpose of x. The symplectic unitary group is defined as $SpU(4) = Sp(4, \boldsymbol{C}) \cap U(4)$. Let $\boldsymbol{m} = (m_1, m_2)$ be a pair of integers satisfying $m_1 \geq m_2 \geq 0$. We fix $\boldsymbol{m}$ throughout this section. Let $\phi_{\boldsymbol{m}} : Sp(4, \boldsymbol{C}) \to \boldsymbol{C}$ be the function defined by

$$\phi_{\boldsymbol{m}}(x) = x_{11}^{m_1 - m_2} x_{(1,2)(1,2)}^{m_2}.$$

Here $x_{(i,j)(k,l)}$ denotes the minor of a matrix x obtained from the intersection of the i, j-th rows and k, l-th columns. Let $C^\infty(Sp(4, \boldsymbol{C}))$ be the vector space of complex-valued C^∞ functions on $Sp(4, \boldsymbol{C})$ and let $\mathfrak{R}_{\boldsymbol{m}}$ be the subspace spanned by the right translations $R_x \phi_{\boldsymbol{m}} = \phi_{\boldsymbol{m}}(*x)$ of $\phi_{\boldsymbol{m}}$, $x \in Sp(4, \boldsymbol{C})$. Then we have an irreducible complex analytic representation $\pi_{\boldsymbol{m}} : Sp(4, \boldsymbol{C}) \to GL(\mathfrak{R}_{\boldsymbol{m}})$, $\pi_{\boldsymbol{m}}(x)(f) = f(*x)$, $f \in \mathfrak{R}_{\boldsymbol{m}}$. The function $\phi_{\boldsymbol{m}}$ is called the *generating function* of $\pi_{\boldsymbol{m}}$. Let $(,)$ be the inner product of $\mathfrak{R}_{\boldsymbol{m}}$ such that the restriction of $\pi_{\boldsymbol{m}}$ to $SpU(4)$ is unitary, normalized so that $\|\phi_{\boldsymbol{m}}\| = 1$. In order to study $\mathfrak{R}_{\boldsymbol{m}}$ more closely, we introduce some notations. As a basis of the Lie algebra $sp(4, \boldsymbol{C})$ we take

$$\mathrm{h}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \mathrm{h}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{e}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{e}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{e}_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{e}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{f}_1 = {}^t\mathrm{e}_1, \quad \mathrm{f}_2 = {}^t\mathrm{e}_2, \quad \mathrm{f}_3 = {}^t\mathrm{e}_3, \quad \mathrm{f}_4 = {}^t\mathrm{e}_4.$$

For a matrix $\mathrm{f} \in sp(4, \boldsymbol{C})$ we denote by f the left invariant vector field on $Sp(4, \boldsymbol{C})$ which is equal to f at the identity element, and by $\hat{f}$ the right invariant vector field which is equal to the transpose ${}^t\mathrm{f}$ at the identity element (Note that $\hat{f}_3$, for instance, coincides with e_3 at the identity element). The correspondence $f \mapsto \hat{f}$ gives an isomorphism between the two Lie algebras of left invariant vector fields and right invariant vector fields.

2.2. The Gel'fand-Cetlin basis and the matrix elements.

In order to give the Gel'fand-Cetlin basis of $\mathfrak{R}_{\boldsymbol{m}}$ we have to introduce some notations. Let $M_{\boldsymbol{m}}$ be the set of 4-tuples (p_1, p_2, i, j) of integers such that

$$m_1 \geq p_1 \geq m_2 \geq p_2 \geq 0, \quad m_1 + m_2 - (p_1 + p_2) \geq i \geq 0, \quad p_1 - p_2 \geq j \geq 0.$$

For a positive integer r we define $C_r(m)$ to be the polynomial in m of degree r

$$C_r(m) = m(m-1) \cdots (m - (r-1)),$$

and we set $C_r(m) = 1$ for nonpositive integers r. Moreover, for a pair of integers s, t we define $C_{s,t}$ to be the polynomial $C_{s,t} = \prod_{s \geq r \geq t} C_r$. We set $C_{s,t} = 1$ for $s < t$. Now, for each $\mu \in M_{\boldsymbol{m}}$ we define a left invariant differential operator $\Omega_\mu : C^\infty(Sp(4, \boldsymbol{C})) \to C^\infty(Sp(4, \boldsymbol{C}))$, which maps $\mathfrak{R}_{\boldsymbol{m}}$ into itself, as follows. First, for each pair of nonnegative integers p, q, we introduce a differential operator

$$\Omega_{p,q} = \sum_{l=0}^{\min(p,q)} (-1)^l \binom{q}{l} C_l(p) \nabla_{42}^{q-l} f_3^l f_1^{p-l} C_{p,q,l}(h_1, h_2),$$

where

$$\nabla_{42} = f_2(h_2 + 1) - f_4 f_1,$$

and

$$C_{p,q,l}(h_1,h_2)$$
$$= C_{q-l}(h_1+1-p)C_l(h_2+1+p)C_{p,p-l+1}(h_1-h_2)C_{p-l-1,p-q}(h_1-h_2).$$

Then we define

$$\Omega_\mu = f_3^i f_4^j \Omega_{p_1-m_2,p_2} \quad \text{for } \mu=(p_1,p_2,i,j)\in M_{\boldsymbol{m}}.$$

We can now give the Gel'fand-Cetlin basis of $\mathfrak{R}_{\boldsymbol{m}}$.

THEOREM 2.2.1. *For* $\mu=(p_1,p_2,i,j)\in M_{\boldsymbol{m}}$, *set* $\varphi_\mu=\Omega_\mu(\phi_{\boldsymbol{m}})$. *Then* $\{\varphi_\mu\}_{\mu\in M_{\boldsymbol{m}}}$ *is the orthogonal basis of* $\mathfrak{R}_{\boldsymbol{m}}$.

The proof (and the norm of φ_μ) will be given in Appendix.

Next, to give the expression for the matrix elements of $\pi_{\boldsymbol{m}}$ we introduce the right invariant differential operators $\hat{\Omega}_\mu$ in a similar way. Namely, we set

$$\hat{\Omega}_{p,q} = \sum_{l=0}^{\min(p,q)} (-1)^l \binom{q}{l} C_l(p)\hat{\nabla}_{42}^{q-l}\hat{f}_3^l\hat{f}_1^{p-l}C_{p,q,l}(\hat{h}_1,\hat{h}_2),$$

where

$$\hat{\nabla}_{42} = \hat{f}_2(\hat{h}_2+1)-\hat{f}_4\hat{f}_1,$$

and we define

$$\hat{\Omega}_\mu = \hat{f}_3^i\hat{f}_4^j\hat{\Omega}_{p_1-m_2,p_2} \quad \text{for } \mu=(p_1,p_2,i,j)\in M_{\boldsymbol{m}}.$$

Now, for $\mu,\nu\in M_{\boldsymbol{m}}$ let $\varphi_{\mu\nu}: Sp(4,\boldsymbol{C})\to\boldsymbol{C}$ be the function defined by

$$\varphi_{\mu\nu}(x) = (\pi_{\boldsymbol{m}}(x)(\varphi_\nu),\varphi_\mu).$$

THEOREM 2.2.2. *The function* $\varphi_{\mu\nu}$ *is expressed as*

$$\varphi_{\mu\nu} = \hat{\Omega}_\mu\Omega_\nu(\phi_{\boldsymbol{m}}).$$

The proof will be given in Appendix.

2.3. The expression for the action of $sp(4,\boldsymbol{C})$ **on the basis** $\{\varphi_\mu\}$**.**

Finally, we can give the explicit formulas for the action of $sp(4,\boldsymbol{C})$, the differential of $\pi_{\boldsymbol{m}}$, on the basis $\{\varphi_\mu\}$.

PROPOSITION 2.3.1. *Let* $\mu=(p_1,p_2,i,j)\in M_{\boldsymbol{m}}$. *The action of the left invariant vector fields* $e_1,e_2,\dots,h_1,h_2\in sp(4,\boldsymbol{C})$ *on the function* $\varphi_\mu\in\mathfrak{R}_{\boldsymbol{m}}$ *is given by*

$$e_1(\varphi_\mu) = -i(A\varphi_{\mu+\varepsilon_2-\varepsilon_3} + C\varphi_{\mu+\varepsilon_1-\varepsilon_3+\varepsilon_4}) + (I-i)(B\varphi_{\mu-\varepsilon_1} + D\varphi_{\mu-\varepsilon_2+\varepsilon_4}),$$

$$e_2(\varphi_\mu) = -ijA\varphi_{\mu+\varepsilon_2-\varepsilon_3-\varepsilon_4} + j(I-i)B\varphi_{\mu-\varepsilon_1-\varepsilon_4} + i(J-j)C\varphi_{\mu+\varepsilon_1-\varepsilon_3} - (I-i)(J-j)D\varphi_{\mu-\varepsilon_2},$$

$$e_3(\varphi_\mu) = i(I-i)\varphi_{\mu-\varepsilon_3},$$

$$e_4(\varphi_\mu) = j(J-j)\varphi_{\mu-\varepsilon_4},$$

$$f_1(\varphi_\mu) = -j(A\varphi_{\mu+\varepsilon_2-\varepsilon_4} + B\varphi_{\mu-\varepsilon_1+\varepsilon_3-\varepsilon_4}) + (J-j)(C\varphi_{\mu+\varepsilon_1} + D\varphi_{\mu-\varepsilon_2+\varepsilon_3}),$$

$$f_2(\varphi_\mu) = A\varphi_{\mu+\varepsilon_2} + B\varphi_{\mu-\varepsilon_1+\varepsilon_3} + C\varphi_{\mu+\varepsilon_1+\varepsilon_4} + D\varphi_{\mu-\varepsilon_2+\varepsilon_3+\varepsilon_4},$$

$$f_3(\varphi_\mu) = \varphi_{\mu+\varepsilon_3},$$

$$f_4(\varphi_\mu) = \varphi_{\mu+\varepsilon_4},$$

$$h_1(\varphi_\mu) = (I-1-2i)\varphi_\mu,$$

$$h_2(\varphi_\mu) = (J-1-2j)\varphi_\mu,$$

where I, J, A, B, C, D *are constants*

$$I = m_1 + m_2 - (p_1 + p_2) + 1, \quad J = p_1 - p_2 + 1,$$

$$A = \frac{1}{IJC_{p_1-m_2-p_2-1}(m_1 - m_2)},$$

$$B = \frac{(p_1+1)(p_1-m_2)C_{p_2+1}(m_1-p_1+p_2+1)}{IJ},$$

$$C = \frac{(m_1-p_1)(m_1+m_2-p_1+1)}{IJC_{p_2+1}(m_1-p_1+p_2)},$$

$$D = -p_2(m_1-p_2+2)(m_2-p_2+1)(m_1+m_2-p_2+3)C_{p_1-m_2-p_2}(m_1-m_2)/IJ,$$

and ε_i *denote the vectors* $\varepsilon_1 = (1,0,0,0), \varepsilon_2 = (0,1,0,0), \ldots, \varepsilon_4 = (0,0,0,1)$, *which act on the indexing set* $M_{\boldsymbol{m}}$ *in the obvious manner. Here it is to be understood that the function* $\varphi_{\mu'}$ *is zero if* $\mu' \notin M_{\boldsymbol{m}}$.

In order to obtain the system of orthogonal functions on the Milnor-Gromoll-Meyer sphere, we shall use the following.

Corollary 2.3.2. *Let* $\mu = (p_1, p_2, i, j)$, $\nu = (q_1, q_2, k, l) \in M_{\boldsymbol{m}}$, *and set*

$$I = m_1 + m_2 - (p_1 + p_2) + 1, \quad J = p_1 - p_2 + 1,$$

$$K = m_1 + m_2 - (q_1 + q_2) + 1, \quad L = q_1 - q_2 + 1.$$

Then for the matrix element $\varphi_{\mu\nu}$, the following identities hold:

$$\hat{h}_1(\varphi_{\mu\nu}) = (I - 1 - 2i)\varphi_{\mu\nu}, \quad \hat{h}_2(\varphi_{\mu\nu}) = (J - 1 - 2j)\varphi_{\mu\nu},$$

$$h_1(\varphi_{\mu\nu}) = (K - 1 - 2k)\varphi_{\mu\nu}, \quad h_2(\varphi_{\mu\nu}) = (L - 1 - 2l)\varphi_{\mu\nu},$$

$$\hat{e}_3(\varphi_{\mu\nu}) = i(I - i)\varphi_{\mu-\varepsilon_3\nu}, \quad \hat{e}_4(\varphi_{\mu\nu}) = j(J - j)\varphi_{\mu-\varepsilon_4\nu},$$

$$e_3(\varphi_{\mu\nu}) = k(K - k)\varphi_{\mu\nu-\varepsilon_3}, \quad e_4(\varphi_{\mu\nu}) = l(L - l)\varphi_{\mu\nu-\varepsilon_4},$$

$$\hat{f}_3(\varphi_{\mu\nu}) = \varphi_{\mu+\varepsilon_3\nu}, \quad \hat{f}_4(\varphi_{\mu\nu}) = \varphi_{\mu+\varepsilon_4\nu},$$

$$f_3(\varphi_{\mu\nu}) = \varphi_{\mu\nu+\varepsilon_3}, \quad f_4(\varphi_{\mu\nu}) = \varphi_{\mu\nu+\varepsilon_4},$$

where $\varepsilon_3 = (0,0,1,0)$, $\varepsilon_4 = (0,0,0,1)$, *and* $\mu - \varepsilon_3$, *for example, means* $(p_1, p_2, i-1, j)$, *with the convention that* $\varphi_{\mu'\nu} = 0$ *if* $\mu' \notin M_{\boldsymbol{m}}$, *and* $\varphi_{\mu\nu'} = 0$ *if* $\nu' \notin M_{\boldsymbol{m}}$.

The proofs will be given in Appendix.

3. The description of Σ^7 in terms of $SpU(4)$.

3.1. The Gromoll-Meyer action in $SpU(4)$.

Now, in order to study the submersion $Sp(2) \to \Sigma^7$ defined in §1, we consider the corresponding objects in the general linear group $GL(4, \boldsymbol{C})$. To be more precise, let $SpU(n) = Sp(n, \boldsymbol{C}) \cap U(n)$ denote the symplectic unitary group in the terminology of [**Zh**], and let $spu(n)$ be the Lie algebra. Let $h_1, h_2, e_1, \dots, f_4$ (resp. $\hat{h}_1, \hat{h}_2, \hat{e}_1, \dots, \hat{f}_4$) be the basis of left (resp. right) invariant vector fields on $Sp(4, \boldsymbol{C})$ as in §2. We introduce two inclusions $\mathfrak{i}$, $\mathfrak{o} : GL(2, \boldsymbol{C}) \to GL(4, \boldsymbol{C})$ defined by

$$\mathfrak{i}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathfrak{o}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 & 0 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ c & 0 & 0 & d \end{pmatrix}.$$

Define

$$\Gamma^c : GL(2, \boldsymbol{C}) \times GL(4, \boldsymbol{C}) \to GL(4, \boldsymbol{C}) \quad \text{by } \Gamma^c(g, x) = \mathfrak{i}(g)\mathfrak{o}(g)x\mathfrak{o}(g^{-1}),$$

and denote its restriction: $SpU(2) \times SpU(4) \to SpU(4)$ by the same symbol. Set

$$c_i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad c_j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad c_k = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \in spu(2).$$

Proposition 3.1.1. *Let g_0 be the bi-invariant metric on $SpU(4)$ normalized so that $i\mathrm{h}_1 \in spu(4)$ has unit length. Then the action*

$$\Gamma^c : SpU(2) \times SpU(4) \to SpU(4)$$

is free and isometric, and hence yields the quotient Riemannian manifold. This Riemannian manifold is isometric to the Milnor-Gromoll-Meyer sphere Σ^7. For each $A = c_i, c_j, c_k \in spu(2)$, the Killing vector field $d\Gamma_A^c$ on $SpU(4)$ induced by the action is expressed as

$$d\Gamma_{c_i}^c = i\hat{h}_1 + i\hat{h}_2 - ih_1,$$
$$d\Gamma_{c_j}^c = \hat{e}_3 - \hat{f}_3 + \hat{e}_4 - \hat{f}_4 + e_3 - f_3,$$
$$d\Gamma_{c_k}^c = -i(\hat{e}_3 + \hat{f}_3) - i(\hat{e}_4 + \hat{f}_4) + i(e_3 + f_3),$$

respectively. Therefore, for a function φ on $SpU(4)$ which is written as the restriction of a holomorphic function on $Sp(4, \boldsymbol{C})$, in order that φ is invariant under the action Γ^c it is necessary and sufficient that φ satisfies

$$(\hat{h}_1 + \hat{h}_2 - h_1)\varphi = 0, \quad (\hat{f}_3 + \hat{f}_4 - e_3)\varphi = 0, \quad (\hat{e}_3 + \hat{e}_4 - f_3)\varphi = 0.$$

For the proof we have to study the correspondence between $Sp(2)$ and $SpU(4)$, which is done in the following.

3.2. The correspondence of $Sp(2)$ to $SpU(4)$.

In order to define a correspondence of $Sp(2)$ to $SpU(4)$ (slightly different from [**Ch**] because of the different choice of "J"), we let $\mathfrak{M}_n(\boldsymbol{C})$ (resp. $\mathfrak{M}_n(\boldsymbol{H})$) denote the algebra of $n \times n$ matrices with coefficients in $\boldsymbol{C}$ (resp. $\boldsymbol{H}$). Let $\boldsymbol{c}$ be an injective $\boldsymbol{R}$-algebra homomorphism $\boldsymbol{c} : \boldsymbol{H} \to \mathfrak{M}_2(\boldsymbol{C})$ defined by

$$\boldsymbol{c}(a + jb) = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \quad a, b \in \boldsymbol{C}.$$

It is easy to verify that its restrictions give the isomorphisms $Sp(1) \cong SpU(2)$, $sp(1) \cong spu(2)$. Moreover, we denote by $\boldsymbol{c}$ again the mapping $\mathfrak{M}_2(\boldsymbol{H}) \to \mathfrak{M}_4(\boldsymbol{C})$ defined by the formula

$$\boldsymbol{c}(Q) = \begin{pmatrix} A & -\bar{B}S \\ SB & S\bar{A}S \end{pmatrix}$$

where Q is written in the form $Q = A + jB$ with $A, B \in \mathfrak{M}_2(\boldsymbol{C})$, and where $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then using the relation $jB = \bar{B}j$, $B \in \mathfrak{M}_2(\boldsymbol{C})$, we see that $\boldsymbol{c}$ is an injective $\boldsymbol{R}$-algebra homomorphism and satisfies $\boldsymbol{c}({}^t\bar{Q}) = {}^t\overline{(\boldsymbol{c}(Q))}$. Hence the restrictions of $\boldsymbol{c}$ yield the isomorphisms $Sp(2) \cong SpU(4)$, $sp(2) \cong spu(4)$.

Lemma 3.2.1. *The Gromoll-Meyer action Γ and the action Γ^c are equivalent through $\boldsymbol{c}$, that is, the following diagram is commutative*:

$$\begin{array}{ccc} Sp(1) \times Sp(2) & \xrightarrow{\Gamma} & Sp(2) \\ {\scriptstyle \boldsymbol{c}\times\boldsymbol{c}}\downarrow \wr\| & & {\scriptstyle \boldsymbol{c}}\downarrow \wr\| \\ SpU(2) \times SpU(4) & \xrightarrow{\Gamma^c} & SpU(4) \end{array}$$

Hence, for $\alpha \in sp(1)$ the Killing vector field $d\Gamma_\alpha$ on $Sp(2)$ induced by the action Γ corresponds to the Killing vector field $d\Gamma^c_{\boldsymbol{c}(\alpha)}$ on $SpU(4)$ under $\boldsymbol{c}$.

Proof. Immediate from the identity $\boldsymbol{c}\left(\left(\begin{smallmatrix} q & 0 \\ 0 & q' \end{smallmatrix}\right)\right) = \mathfrak{o}(\boldsymbol{c}(q))\mathfrak{i}(\boldsymbol{c}(q'))$, $q, q' \in \boldsymbol{H} - \{0\}$. □

3.3. Proof of Proposition 3.1.1.

For simplicity of notation we set

$$\alpha_+ = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}, \quad \alpha_- = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}, \quad 1_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \alpha_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$$

for each $\alpha = i, j, k \in \boldsymbol{H}$. Then these matrices form the orthonormal basis for the Lie algebra $sp(2)$ (with respect to g_0). Moreover, the isomorphism $\boldsymbol{c} : sp(2) \to spu(4)$ gives the following correspondence between the respective basis:

$$\boldsymbol{c}(i_+) = i\mathrm{h}_1, \quad \boldsymbol{c}(j_+) = -\mathrm{e}_3 + \mathrm{f}_3, \quad \boldsymbol{c}(k_+) = -i\mathrm{e}_3 - i\mathrm{f}_3,$$

$$\boldsymbol{c}(i_-) = i\mathrm{h}_2, \quad \boldsymbol{c}(j_-) = -\mathrm{e}_4 + \mathrm{f}_4, \quad \boldsymbol{c}(k_-) = -i\mathrm{e}_4 - i\mathrm{f}_4,$$

$$\boldsymbol{c}(1_0) = \frac{1}{\sqrt{2}}(\mathrm{e}_1 - \mathrm{f}_1), \quad \boldsymbol{c}(i_0) = \frac{i}{\sqrt{2}}(\mathrm{e}_1 + \mathrm{f}_1),$$

$$\boldsymbol{c}(j_0) = \frac{1}{\sqrt{2}}(-\mathrm{e}_2 + \mathrm{f}_2), \quad \boldsymbol{c}(k_0) = -\frac{i}{\sqrt{2}}(\mathrm{e}_2 + \mathrm{f}_2).$$

Hence, in particular, our definition of the metric g_0 on $SpU(4)$ in Proposition 3.1.1 implies that $\boldsymbol{c}$ gives the isometry between $Sp(2)$ and $SpU(4)$. By Lemma 3.2.1 we conclude that the action Γ^c satisfies the same properties as Γ and the quotient manifold $\Gamma^c \backslash SpU(4)$ is isometric to Σ^7. Next, to find the expressions for Killing vector fields induced by Γ^c, let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in spu(2)$. Then the vector field $d\Gamma^c_A$ on $SpU(4)$ induced by Γ^c is expressed as

$$d\Gamma_A^c(x) = R_{x*}\left(\begin{pmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ c & 0 & 0 & d \end{pmatrix}\right) - L_{x*}\left(\begin{pmatrix} a & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ c & 0 & 0 & d \end{pmatrix}\right) \in T_x SpU(4)$$

where R_{x*} (resp. L_{x*}) denotes the map: $T_e SpU(4) \to T_x SpU(4)$ between the tangent spaces, induced from the right (resp. left) translation by x on $SpU(4)$. Taking $A = c_i = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) \in spu(2)$ and recalling the definition of $h_1, h_2, \hat{h}_1, \hat{h}_2$ in §2, we observe that the above formula becomes

$$\begin{aligned} d\Gamma_{c_i}^c(x) &= R_{x*}\left(\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}\right) - L_{x*}\left(\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}\right) \\ &= i\hat{h}_{1x} + i\hat{h}_{2x} - ih_{1x}. \end{aligned}$$

Similarly, using such obvious relations as $R_{x*}e_3 = (\hat{f}_3)_x$ we get the expressions for $d\Gamma_{c_j}^c, d\Gamma_{c_k}^c$. There remains to prove the last part. Clearly, on $Sp(4, \boldsymbol{C})$ the equation $d\Gamma_{c_i}^c(\varphi) = d\Gamma_{c_j}^c(\varphi) = d\Gamma_{c_k}^c(\varphi) = 0$ is equivalent to $(\hat{h}_1 + \hat{h}_2 - h_1)\varphi = (\hat{f}_3 + \hat{f}_4 - e_3)\varphi = (\hat{e}_3 + \hat{e}_4 - f_3)\varphi = 0$. Since $Sp(4, \boldsymbol{C})$ is the complexification of $SpU(4)$, we have also such equivalence on $SpU(4)$. This completes the proof of Proposition 3.1.1. □

The following was mentioned in §1.3.

Proposition 3.3.1. *Let $\boldsymbol{g}_0$ be the bi-invariant metric on the symplectic group $Sp(2)$ as in §1.1. Let $\boldsymbol{m} = (m_1, m_2)$ be a signature and let $H_{\boldsymbol{m}}$ be the space as in §1.2. Then the Laplacian Δ of the Riemannian manifold $Sp(2)$ satisfies*

$$\Delta\varphi = -((m_1 + 2)^2 + (m_2 + 1)^2 - 5)\varphi$$

for any $\varphi \in H_{\boldsymbol{m}}$.

Proof. This is a special case of a well known fact about compact Lie groups (cf. [**Gu**]). We give a direct proof, as a consequence of our argument. By means of the isometry $\boldsymbol{c}$ we identify the space $H_{\boldsymbol{m}}$ with the vector space $\oplus_{\mu\nu} \boldsymbol{C}\varphi_{\mu\nu}$, where μ, ν range over $M_{\boldsymbol{m}}$, and $\varphi_{\mu\nu}$ are as in §2.2. It suffices to prove that the Laplacian Δ of $SpU(4)$ satisfies $\Delta\phi_{\boldsymbol{m}} = -((m_1 + 2)^2 + (m_2 + 1)^2 - 5)\phi_{\boldsymbol{m}}$ for the generating function $\phi_{\boldsymbol{m}}$ (defined in §2.1). Indeed, since the left invariant vector fields $ih_1, ih_2, -e_3 + f_3, \ldots, -(i/\sqrt{2})(e_2 + f_2)$ on $SpU(4)$ mentioned above form the orthonormal basis of $T_e SpU(4)$ at the identity element e, we see that

$$\Delta = -(h_1^2 + h_2^2 + 4h_1 + 2h_2 + 2f_1e_1 + 2f_2e_2 + 4f_3e_3 + 4f_4e_4).$$

Then our contention follows from the identities

$$h_1(\phi_{\boldsymbol{m}}) = m_1\phi_{\boldsymbol{m}}, \quad h_2(\phi_{\boldsymbol{m}}) = m_2\phi_{\boldsymbol{m}},$$

$$e_1(\phi_{\boldsymbol{m}}) = e_2(\phi_{\boldsymbol{m}}) = e_3(\phi_{\boldsymbol{m}}) = e_4(\phi_{\boldsymbol{m}}) = 0$$

which are noted in Appendix. □

4. The dimension of $H_{\boldsymbol{m}}^{\Gamma}$ or $H_{\boldsymbol{m}}^{\Gamma}(SpU(4))$.

4.1. The space $H_{\boldsymbol{m}}^{\Gamma}(SpU(4))$ and the lattice points $L_{\boldsymbol{m}}$.

Fix a pair of integers $\boldsymbol{m} = (m_1, m_2)$, $m_1 \geq m_2 \geq 0$, and let $\pi_{\boldsymbol{m}}$ be the unitary representation of $SpU(4)$ defined in §2. Let $M_{\boldsymbol{m}}$ be defined as in §2.2. Denote by $H_{\boldsymbol{m}}(SpU(4))$ the vector space spanned by matrix elements of $\pi_{\boldsymbol{m}}$. Then we have

$$H_{\boldsymbol{m}}(SpU(4)) = \oplus \boldsymbol{C}\varphi_{\mu\nu}$$

where the direct sum is taken over all $\mu, \nu \in M_{\boldsymbol{m}}$, and $\varphi_{\mu\nu}$ are the functions on $SpU(4)$ defined by $\varphi_{\mu\nu}(*) = (\pi_{\boldsymbol{m}}(*)\varphi_\nu, \varphi_\mu)$ with the Gel'fand-Cetlin basis $\{\varphi_\mu\}$. Let $\Gamma^c : SpU(2) \times SpU(4) \to SpU(4)$ be the Gromoll-Meyer action defined in §3.1, and let $H_{\boldsymbol{m}}^{\Gamma}(SpU(4))$ be the subspace of $H_{\boldsymbol{m}}(SpU(4))$ consisting of functions fixed under Γ^c. In order to find the dimension of $H_{\boldsymbol{m}}^{\Gamma}(SpU(4))$ we introduce a set of lattice points in $\boldsymbol{R}^5$. Namely, we let $L_{\boldsymbol{m}}$ be the set of 5-tuples (p_1, p_2, q_1, q_2, l) of integers such that

(1) $m_1 \geq p_1 \geq m_2 \geq p_2 \geq 0$, $m_1 \geq q_1 \geq m_2 \geq q_2 \geq 0$, $q_1 - q_2 \geq l \geq 0$,
(2) $q_1 + q_2$ is even,
(3) $(q_1 + q_2)/2 \leq m_1 + m_2 - p_1$, $p_2 \leq (q_1 + q_2)/2 \leq p_1$.

Proposition 4.1.1. *The dimension of the vector space $H_{\boldsymbol{m}}^{\Gamma}(SpU(4))$ is equal to the cardinality of the set $L_{\boldsymbol{m}}$.*

To prove our proposition we begin by considering a direct sum decomposition of $H_{\boldsymbol{m}}(SpU(4))$. For integers p_1, p_2, q_1, q_2, l satisfying the condition (1) above, let $H_{\boldsymbol{m},(p,q,l)}$ be the subspace of $H_{\boldsymbol{m}}(SpU(4))$ defined by

$$H_{\boldsymbol{m},(p,q,l)} = \oplus_{i,j,k}\boldsymbol{C}\varphi_{(p_1,p_2,i,j)(q_1,q_2,k,l)}$$

where the direct sum is taken over all integers i, j, k such that 4-tuples (p_1, p_2, i, j), (q_1, q_2, k, l) belong to the set $M_{\boldsymbol{m}}$. Then we have

$$H_{\boldsymbol{m}}(SpU(4)) = \oplus_{p,q,l}H_{\boldsymbol{m},(p,q,l)}$$

where the sum is taken over all integers p_1, p_2, q_1, q_2, l satisfying (1). Thus Proposition 4.1.1 is reduced to the following assertion.

Proposition 4.1.2. *Let p_1, p_2, q_1, q_2, l be integers satisfying* (1) *above. Then the space $H_{\boldsymbol{m},(p,q,l)}$ is invariant under the action Γ^c. Let $H^{\Gamma}_{\boldsymbol{m},(p,q,l)}$ denote the subspace of functions in $H_{\boldsymbol{m},(p,q,l)}$ which are fixed under Γ^c. Then*

$$H^{\Gamma}_{\boldsymbol{m},(p,q,l)} \cong \begin{cases} \boldsymbol{C} & \text{if } p_1, p_2, q_1, q_2, l \text{ satisfy (2) and (3)}, \\ 0 & \text{otherwise}. \end{cases}$$

4.2. Proof of Proposition 4.1.2.

First we note that p_1, p_2, q_1, q_2, l are fixed, and set

$$I = m_1 + m_2 - (p_1 + p_2) + 1, \quad J = p_1 - p_2 + 1,$$
$$K = m_1 + m_2 - (q_1 + q_2) + 1, \quad L = q_1 - q_2 + 1.$$

For any integers i, j, k such that $0 \le i \le I - 1$, $0 \le j \le J - 1$, $0 \le k \le K - 1$, we set

$$\varphi_{(i,j)(k)} = \varphi_{(p_1,p_2,i,j)(q_1,q_2,k,l)}.$$

As a matter of convention we set $\varphi_{(i,j)(k)} = 0$ if integers i, j, k do not satisfy the condition $0 \le i \le I - 1$, $0 \le j \le J - 1$, $0 \le k \le K - 1$. Note that $H_{\boldsymbol{m},(p,q,l)}$ is a vector space with the basis $\varphi_{(i,j)(k)}$, $0 \le i \le I - 1$, $0 \le j \le J - 1$, $0 \le k \le K - 1$. By Proposition 3.1.1 we see that a subspace S of $H_{\boldsymbol{m}}(SpU(4))$ is invariant under Γ^c if and only if S is invariant under the three vector fields

$$d\Gamma^c_{c_i} = i\hat{h}_1 + i\hat{h}_2 - ih_1, \quad d\Gamma^c_{c_j} = \hat{e}_3 - \hat{f}_3 + \hat{e}_4 - \hat{f}_4 + e_3 - f_3,$$
$$d\Gamma^c_{c_k} = -i(\hat{e}_3 + \hat{f}_3) - i(\hat{e}_4 + \hat{f}_4) + i(e_3 + f_3).$$

By Corollary 2.3.2 we observe that the space $H_{\boldsymbol{m},(p,q,l)}$ is invariant under the operators $\hat{h}_1, \hat{h}_2, h_1, \hat{e}_3, \hat{f}_3, \hat{e}_4, \hat{f}_4, e_3, f_3$, and hence the space $H_{\boldsymbol{m},(p,q,l)}$ is Γ^c invariant. The former part of Proposition 4.1.2 is proved. For the proof of the latter part we have to look at the action Γ^c on $H_{\boldsymbol{m},(p,q,l)}$ more closely. For any integer η we introduce the subspace $H(\eta)$ of $H_{\boldsymbol{m},(p,q,l)}$ defined by

$$H(\eta) = \bigoplus_{\substack{k=i+j+\eta \\ 0 \le i \le I-1 \\ 0 \le j \le J-1}} \boldsymbol{C}\varphi_{(i,j)(k)}.$$

By our definition, $H(\eta) \ne 0$ if and only if $-(I + J - 2) \le \eta \le K - 1$. Geometrically, $H(\eta)$ is the vector space with the basis $\varphi_{(i,j)(k)}$ whose index (i, j, k) ranges over lattice points in $[0, I-1] \times [0, J-1] \times [0, K-1]$ of the plane $\{(x, y, z) \mid z = x + y + \eta\}$ in $\boldsymbol{R}^3$. Obviously we have the direct sum decomposition

$$H_{\boldsymbol{m},(p,q,l)} = \bigoplus_{-(I+J-2) \le \eta \le K-1} H(\eta).$$

On the other hand, by Proposition 3.1.1 we know that $H^{\Gamma}_{\boldsymbol{m},(p,q,l)}$ consists of all $\varphi \in H_{\boldsymbol{m},(p,q,l)}$ such that

$$(\hat{h}_1 + \hat{h}_2 - h_1)\varphi = 0, \quad (\hat{f}_3 + \hat{f}_4 - e_3)\varphi = 0, \quad (\hat{e}_3 + \hat{e}_4 - f_3)\varphi = 0.$$

Therefore the latter part of Proposition 4.1.2 is an immediate consequence of the following two lemmas.

Lemma 4.2.1. *Let η be an integer.*
(i) *If $\eta \neq p_2 - (q_1 + q_2)/2$, then $(\hat{h}_1 + \hat{h}_2 - h_1)\varphi \neq 0$ for any $\varphi \neq 0 \in H(\eta)$.*
(ii) *If $\eta = p_2 - (q_1 + q_2)/2$, then $(\hat{h}_1 + \hat{h}_2 - h_1)\varphi = 0$ for any $\varphi \in H(\eta)$.*

Proof. Using the obvious relation $I + J - K - 1 = q_1 + q_2 - 2p_2$, by Corollary 2.3.2 we have

$$(\hat{h}_1 + \hat{h}_2 - h_1)\varphi_{(i,j)(k)} = (-2p_2 - 2(i+j) + q_1 + q_2 + 2k)\varphi_{(i,j)(k)}.$$

Hence $(\hat{h}_1 + \hat{h}_2 - h_1)\varphi = (q_1 + q_2 - 2p_2 + 2\eta)\varphi$ for $\varphi \in H(\eta)$. This proves our lemma. □

Lemma 4.2.2. *Assume that $q_1 + q_2$ is even, and set $\eta_0 = p_2 - (q_1 + q_2)/2$. Set $D = \hat{f}_3 + \hat{f}_4 - e_3$, $U = \hat{e}_3 + \hat{e}_4 - f_3$, and let $\varphi \in H(\eta_0)$.*
(i) *Suppose that $\eta_0 \geq 1$. If $D(\varphi) = 0$, then $\varphi = 0$.*
(ii) *Suppose that $\eta_0 \leq -I$ or $\eta_0 \leq -J$. If $D(\varphi) = U(\varphi) = 0$, then $\varphi = 0$.*
(iii) *Suppose that $-I + 1 \leq \eta_0 \leq 0$ and $-J + 1 \leq \eta_0 \leq 0$. Then $H^{\Gamma}_{\boldsymbol{m},(p,q,l)} \cong \boldsymbol{C}$.*

In order to prove Lemma 4.2.2 we have to consider, moreover, a direct sum decomposition of $H(\eta)$. For any integers η, k, we set

$$H(\eta; k) = \bigoplus_{\substack{i+j=k-\eta \\ 0 \leq i \leq I-1 \\ 0 \leq j \leq J-1}} \boldsymbol{C}\varphi_{(i,j)(k)}.$$

Obviously, $H(\eta; k) \neq 0$ if and only if

$$-(I + J - 2) \leq \eta \leq K - 1, \quad \max(\eta, 0) \leq k \leq \min(K - 1, I + J - 2 + \eta).$$

Now, for each η the subspace $H(\eta)$ can be written as

$$H(\eta) = \bigoplus_{0 \leq k \leq K-1} H(\eta; k).$$

For any fixed η and for each k satisfying $0 \leq k \leq K - 1$, we denote by proj_k the projection of $H(\eta)$ onto the factor $H(\eta; k)$ (which may be trivial).

Lemma 4.2.3. *Let* η *be an integer.*

(i) *The operator* $D = \hat{f}_3 + \hat{f}_4 - e_3$ *maps the space* $H(\eta)$ *into* $H(\eta - 1)$. *Moreover, for each integer* k, $0 \le k \le K - 1$, *the mapping* D *satisfies*

$$D : H(\eta; k) \to H(\eta - 1; k - 1) \oplus H(\eta - 1; k),$$

$$D(\varphi_{(i,j)(k)}) = (-k(K-k)\varphi_{(i,j)(k-1)}, \varphi_{(i+1,j)(k)} + \varphi_{(i,j+1)(k)}).$$

(ii) *The operator* $U = \hat{e}_3 + \hat{e}_4 - f_3$ *maps the space* $H(\eta)$ *into* $H(\eta + 1)$. *Moreover, for each* k, $0 \le k \le K - 1$, *the mapping* U *satisfies*

$$U : H(\eta; k) \to H(\eta + 1; k) \oplus H(\eta + 1; k + 1),$$

$$U(\varphi_{(i,j)(k)}) = (i(I-i)\varphi_{(i-1,j)(k)} + j(J-j)\varphi_{(i,j-1)(k)}, -\varphi_{(i,j)(k+1)}).$$

Proof. By Corollary 2.3.2 we have

$$(\hat{f}_3 + \hat{f}_4 - e_3)\varphi_{\mu\nu} = \varphi_{\mu+\varepsilon_3 \nu} + \varphi_{\mu+\varepsilon_4 \nu} - k(K-k)\varphi_{\mu\nu-\varepsilon_3},$$

$$(\hat{e}_3 + \hat{e}_4 - f_3)\varphi_{\mu\nu} = i(I-i)\varphi_{\mu-\varepsilon_3 \nu} + j(J-j)\varphi_{\mu-\varepsilon_4 \nu} - \varphi_{\mu\nu+\varepsilon_3}$$

where $\mu = (p_1, p_2, i, j)$, $\nu = (q_1, q_2, k, l)$. Recalling our definition of $\varphi_{(i,j)(k)}$, we get immediately the expressions for $D(\varphi_{(i,j)(k)})$, $U(\varphi_{(i,j)(k)})$. Lemma 4.2.3 is proved. □

Lemma D. *Let* η *be an integer, and set* $D = \hat{f}_3 + \hat{f}_4 - e_3$.

(i) *The following identities hold*:

$$D \circ \mathrm{proj}_0 = \mathrm{proj}_0 \circ D \circ \mathrm{proj}_0 : H(\eta) \to H(\eta - 1; 0),$$

$$\mathrm{proj}_k \circ D = \mathrm{proj}_k \circ D \circ (\mathrm{proj}_k + \mathrm{proj}_{k+1}) : H(\eta) \to H(\eta - 1; k), 0 \le k \le K - 2,$$

$$\mathrm{proj}_{K-1} \circ D = \mathrm{proj}_{K-1} \circ D \circ \mathrm{proj}_{K-1} : H(\eta) \to H(\eta - 1; K - 1).$$

(ii) *For any* k *satisfying* $1 \le k \le K - 1$, *the composite mapping* $\mathrm{proj}_{k-1} \circ D : H(\eta; k) \to H(\eta - 1; k - 1)$ *is an isomorphism.*

(iii) *If* $\eta \ge 1$, *then the mapping* $D : H(\eta) \to H(\eta - 1)$ *is injective.*

(iv) *Suppose that* $\eta \le 0$, *and let* $\varphi \in H(\eta)$. *If* $D(\varphi) = 0$ *and* $\mathrm{proj}_0(\varphi) = 0$, *then* $\varphi = 0$.

Proof. Assertion (i) is an immediate consequence of (i) of the preceding lemma. To prove (ii), suppose that $1 \le k \le K - 1$. Then note that $H(\eta; k)$ is isomorphic to $H(\eta - 1; k - 1)$ under the correspondence $\varphi_{(i,j)(k)} \mapsto \varphi_{(i,j)(k-1)}$ between their respective basis. By (i) of Lemma 4.2.3 we see that the mapping $\mathrm{proj}_{k-1} \circ D \,|\, H(\eta; k)$ is expressed, with respect these basis, as a diagonal matrix with nonzero diagonals. This proves (ii). To prove (iii), suppose that $\eta \geq 1$, and $\varphi \in H(\eta)$ satisfies $D(\varphi) = 0$. To prove $\varphi = 0$ it will suffice to prove that $\mathrm{proj}_k(\varphi) = 0$, $k = 0, \dots, K-1$. Clearly, $\mathrm{proj}_0 \varphi = 0$, because $H(\eta; 0) = 0$ for $\eta \geq 1$. Using this fact and (i) we observe that $\mathrm{proj}_0 D \, \mathrm{proj}_1 \varphi = \mathrm{proj}_0 D(\mathrm{proj}_0 + \mathrm{proj}_1)(\varphi) = \mathrm{proj}_0 D(\varphi) = 0$. Since $\mathrm{proj}_0 D : H(\eta; 1) \to H(\eta - 1; 0)$ is injective by (ii), we get $\mathrm{proj}_1 \varphi = 0$. In this way, using the injectivity of $\mathrm{proj}_{k-1} \circ D \,|\, H(\eta; k)$, we get $\mathrm{proj}_k \varphi = 0$ for $k = 2, \dots, K-1$, as desired. A similar argument proves (iv). □

Lemma U. *Let η be an integer, and set $U = \hat{e}_3 + \hat{e}_4 - f_3$.*

(i) *The following identities hold*:

$$\mathrm{proj}_0 \circ U = \mathrm{proj}_0 \circ U \circ \mathrm{proj}_0 : H(\eta) \to H(\eta + 1; 0),$$

$$\mathrm{proj}_k \circ U = \mathrm{proj}_k \circ U \circ (\mathrm{proj}_{k-1} + \mathrm{proj}_k) : H(\eta) \to H(\eta + 1; k), 1 \leq k \leq K - 1,$$

$$U \circ \mathrm{proj}_{K-1} = \mathrm{proj}_{K-1} \circ U \circ \mathrm{proj}_{K-1} : H(\eta) \to H(\eta + 1; K - 1).$$

(ii) *Suppose that $\eta \leq -I$ or $\eta \leq -J$, and let $\varphi \in H(\eta)$. Then*, $\mathrm{proj}_0(\varphi) = 0$ *if and only if* $\mathrm{proj}_0(U(\varphi)) = 0$.

(iii) *Suppose that $-\min(I - 1, J - 1) \leq \eta \leq 0$. Then the kernel of the operator* $\mathrm{proj}_0 \circ U : H(\eta; 0) \to H(\eta + 1; 0)$ *is of dimension one.*

(iv) *If $\eta < -(I - K)$ or $\eta < -(J - K)$, then*

$$U : H(\eta; K - 1) \to H(\eta + 1; K - 1)$$

is injective.

Proof. Assertion (i) follows from Lemma 4.2.3, (ii). As for (ii), since $\mathrm{proj}_0 \circ U = \mathrm{proj}_0 \circ U \circ \mathrm{proj}_0$, the condition $\mathrm{proj}_0(\varphi) = 0$ implies $\mathrm{proj}_0 U(\varphi) = 0$. Conversely, suppose that, say $\eta \leq -I$. Then, note that there exists a number $j_0 \geq 1$ such that

$$\varphi_{(I-1, j_0)(0)}, \varphi_{(I-2, j_0+1)(0)}, \varphi_{(I-3, j_0+2)(0)}, \cdots$$

form a basis of $H(\eta; 0)$. Write $\mathrm{proj}_0 \varphi = c_0 \varphi_{(I-1, j_0)(0)} + c_1 \varphi_{(I-2, j_0+1)(0)} + \cdots$. By Lemma 4.2.3, (ii) we observe that the coefficient of $\mathrm{proj}_0 U(\varphi) = \mathrm{proj}_0 U \, \mathrm{proj}_0(\varphi)$ with respect to $\varphi_{(I-1, j_0-1)(0)}$ is $c_0 j_0 (J - j_0)$. Hence the assumption $\mathrm{proj}_0 U(\varphi) = 0$ implies $c_0 = 0$, and inductively $c_1 = 0, \cdots$. In this way we see that $\mathrm{proj}_0(U(\varphi)) = 0$ implies $\mathrm{proj}_0 \varphi = 0$. To prove (iii), suppose that $-\min(I - 1, J - 1) \leq \eta \leq 0$. Then note that

$$\varphi_{(-\eta, 0)(0)}, \varphi_{(-\eta-1, 1)(0)}, \cdots, \varphi_{(0, -\eta)(0)}$$

form a basis of $H(\eta;0)$, and $\varphi_{(-\eta-1,0)(0)}, \varphi_{(-\eta-2,1)(0)}, \cdots, \varphi_{(0,-\eta-1)(0)}$ form a basis of $H(\eta+1;0)$. Now, using Lemma 4.2.3, (ii), we observe that the matrix of $\mathrm{proj}_0 \circ U$ with respect to these basis is of full rank. This shows (iii), because $\dim H(\eta;0) = \dim H(\eta+1;0)+1$. A similar argument proves (iv). Indeed, suppose that, say $\eta < -(I-K)$. Then we can take as the basis of $H(\eta;K-1)$ (resp. $H(\eta+1;K-1)$)

$$\varphi_{(I-1,j_0)(K-1)}, \varphi_{(I-2,j_0+1)(K-1)}, \cdots \text{(resp. } \varphi_{(I-1,j_0-1)(K-1)}, \varphi_{(I-2,j_0)(K-1)}, \cdots)$$

for some $j_0 \geq 1$. It is easy to see that the matrix of U with respect to these basis is of full rank. Therefore, since $\dim H(\eta;K-1) \leq \dim H(\eta+1;K-1)$ we get the injectivity of $U \mid H(\eta;K-1)$. This completes the proof of Lemma U. □

PROOF OF LEMMA 4.2.2, (i), (ii). Assertion (i) follows from Lemma D, (iii). Assertion (ii) follows from Lemma U, (ii) and Lemma D, (iv). □

PROOF OF LEMMA 4.2.2, (iii). Suppose that q_1+q_2 is even, and $-I+1 \leq \eta_0 \leq 0$, $-J+1 \leq \eta_0 \leq 0$. By Lemma 4.2.1 we know that

$$H^{\Gamma}_{\boldsymbol{m},(p,q,l)} = \{\varphi \in H(\eta_0) \mid D(\varphi) = U(\varphi) = 0\}.$$

First, we contend that $\dim H^{\Gamma}_{\boldsymbol{m},(p,q,l)} \leq 1$. Indeed, let $\varphi, \psi \in H^{\Gamma}_{\boldsymbol{m},(p,q,l)}$. Then $\mathrm{proj}_0\, U(\varphi) = \mathrm{proj}_0\, U(\psi) = 0$, and thus $\mathrm{proj}_0\, U(\mathrm{proj}_0(\varphi)) = \mathrm{proj}_0\, U(\mathrm{proj}_0(\psi)) = 0$. By Lemma U, (iii) we have $c\,\mathrm{proj}_0(\varphi) + d\,\mathrm{proj}_0(\psi) = 0$ for some nontrivial $c, d \in \boldsymbol{C}$. Hence, by Lemma D, (iv) we get $c\varphi + d\psi = 0$, as desired.

To complete the proof of Lemma 4.2.2, (iii), we have to give a nonzero $\phi \in H(\eta_0)$ such that $D(\phi) = U(\phi) = 0$. For this purpose, let $\phi_0 \in H(\eta_0;0)$ be a nonzero element such that $\mathrm{proj}_0\, U(\phi_0) = 0$ (it exists by Lemma U, (iii)).

ASSERTION 1. *There exists* $\phi \in H(\eta_0)$ *such that* $\mathrm{proj}_0(\phi) = \phi_0$, $\mathrm{proj}_k(D(\phi)) = 0$ *for each* $0 \leq k \leq K-2$.

PROOF. To construct ϕ we shall use, for every $k = 1, \cdots, K-1$, the surjectivity of $\mathrm{proj}_{k-1} \circ D : H(\eta_0;k) \to H(\eta_0-1;k-1)$ ((ii) of Lemma D). First, we find $\phi_1 \in H(\eta_0;1)$ such that $\mathrm{proj}_0\, D(\phi_1) = -D(\phi_0)$. Second, we find $\phi_2 \in H(\eta_0;2)$ such that $\mathrm{proj}_1\, D(\phi_2) = -\mathrm{proj}_1\, D(\phi_1)$. In this way we find finally $\phi_{K-1} \in H(\eta_0;K-1)$ such that $\mathrm{proj}_{K-2}\, D(\phi_{K-1}) = -\mathrm{proj}_{K-2}\, D(\phi_{K-2})$. Thus, setting $\phi = \phi_0 + \phi_1 + \cdots + \phi_{K-1}$ proves Assertion 1. □

We contend that this ϕ is what we wanted. Indeed, it suffices to prove the following two assertions.

ASSERTION 2. $U(\phi) = 0$.

Proof. First, note that $UD\varphi = DU\varphi$ for any $\varphi \in H(\eta_0)$, because $[U, D] = \hat{h}_1 + \hat{h}_2 - h_1$. Then, recalling Lemma U, (i) and the identities $\mathrm{proj}_k(D(\phi)) = 0$, $0 \le k \le K - 2$, we see that $\mathrm{proj}_k DU(\phi) = 0$ for $0 \le k \le K - 2$. Now, using this fact and the injectivity of $\mathrm{proj}_k D \,|\, H(\eta; k+1)$ (Lemma D, (ii)), we shall prove $\mathrm{proj}_k U(\phi) = 0$ for $k = 0, \ldots, K-1$, and hence $U(\phi) = 0$. Indeed, first, $\mathrm{proj}_0 U(\phi) = 0$, because $\mathrm{proj}_0 U(\phi) = \mathrm{proj}_0 U \mathrm{proj}_0(\phi) = \mathrm{proj}_0 U(\phi_0) = 0$ (by the choice of ϕ_0). Next, $\mathrm{proj}_1 U(\phi) = 0$, because $\mathrm{proj}_0 D \mathrm{proj}_1 U(\phi) = \mathrm{proj}_0 D(\mathrm{proj}_0 + \mathrm{proj}_1) U(\phi) = \mathrm{proj}_0 DU(\phi) = 0$, and $\mathrm{proj}_0 D \,|\, H(\eta_0 + 1; 1)$ is injective. Inductively, $\mathrm{proj}_k U(\phi) = 0$ for any positive $k \le K - 1$. This proves Assertion 2. □

Assertion 3. $\mathrm{proj}_{K-1} D(\phi) = 0$, *and hence* $D(\phi) = 0$.

Proof. Note that the mapping $U : H(\eta_0 - 1; K-1) \to H(\eta_0; K-1)$ is injective by Lemma U (iv), because η_0 satisfies $\eta_0 - 1 < -(I - K), -(J - K)$ (because of the obvious relation $I + J - K = 1 - 2\eta_0$). Hence to prove $\mathrm{proj}_{K-1} D(\phi) = 0$ it suffices to prove that $U \mathrm{proj}_{K-1} D(\phi) = 0$. But, this is obvious, because we know that $UD(\phi) = 0$ and $D(\phi) = \mathrm{proj}_{K-1} D(\phi)$. Assertion 3 is proved. □

This completes the proof of Lemma 4.2.2. □

4.3. Counting the cardinality of L_m and the proof of Proposition 1.2.1.

Proposition 4.3.1. *Let* $L_{\boldsymbol{m}}$ *be the set defined in* §4.1, *and let* $\sharp L_{\boldsymbol{m}}$ *be its cardinality. Set* $n_1 = m_1 + 2$, $n_2 = m_2 + 1$.

(i) *If* $2m_2 \le m_1$, *then*

$$\sharp L_{\boldsymbol{m}} = \boldsymbol{i} + n_2^2((-1)^{n_1} 2n_1 - (-1)^{n_2}(n_1 - n_2))/16$$

where

$$\boldsymbol{i} = n_2^2(n_1 + n_2)(4n_1^2 - 4n_1 n_2 - 2n_2^2 + 5)/48.$$

(ii) *If* $2m_2 \ge m_1$, *then*

$$\sharp L_{\boldsymbol{m}} = \boldsymbol{j} + (n_1 - n_2)((-1)^{n_1} n_1^2 - (-1)^{n_2} n_2^2)/16$$

where

$$\boldsymbol{j} = (n_1 - n_2)^2(n_1 + n_2)(-2n_1^2 + 8n_1 n_2 - 2n_2^2 + 5)/48.$$

We need two lemmas.

Lemma 4.3.2. *The cardinality of* $L_{\boldsymbol{m}}$ *is given by*

$$\sharp L_{\boldsymbol{m}} = \sum_{\substack{m_2/2 \le s \le (m_1 + m_2)/2 \\ s \in \boldsymbol{Z}}} \sharp P_{\boldsymbol{m}}(s) \sharp Q_{\boldsymbol{m}}(s),$$

where $P_{\boldsymbol{m}}(s), Q_{\boldsymbol{m}}(s)$ *are the sets defined by*

$$P_{\boldsymbol{m}}(s) = \{(p_1, p_2) \in \boldsymbol{Z}^2 \quad \text{such that } m_1 \geq p_1 \geq m_2 \geq p_2 \geq 0, m_1 + m_2 - s \geq p_1 \geq s \geq p_2\},$$

$$Q_{\boldsymbol{m}}(s) = \{(q_1, q_2, l) \in \boldsymbol{Z}^3 \quad \text{such that } m_1 \geq q_1 \geq m_2 \geq q_2 \geq 0, q_1 - q_2 \geq l \geq 0, q_1 + q_2 = 2s\}.$$

PROOF. It is easy to verify that the set $L_{\boldsymbol{m}}$ is written as a disjoint union $L_{\boldsymbol{m}} = \bigcup_s P_{\boldsymbol{m}}(s) \times Q_{\boldsymbol{m}}(s)$, which yields at once our desired formula. □

It is elementary to find the cardinalities of $P_{\boldsymbol{m}}(s)$, $Q_{\boldsymbol{m}}(s)$. Indeed, we have

LEMMA 4.3.3. *Let* s *be an integer such that* $m_2 \leq 2s \leq m_1 + m_2$. *Then*

$$\sharp P_{\boldsymbol{m}}(s) = \begin{cases} (m_1 - m_2 + 1)(s + 1) & \text{if } s \leq m_2, \\ (m_2 + 1)(m_1 + m_2 + 1 - 2s) & \text{if } s \geq m_2, \end{cases}$$

$$\sharp Q_{\boldsymbol{m}}(s) = \begin{cases} (m_2 + 1)(2s - m_2 + 1) & \text{if } m_2 \leq 2s \leq m_1, \\ (m_1 - m_2 + 1)(m_1 + m_2 + 1 - 2s) & \text{if } m_1 \leq 2s \leq m_1 + m_2. \end{cases}$$

PROOF OF PROPOSITION 4.3.1. Suppose that $2m_2 \leq m_1$. Then the righthand side of the formula in Lemma 4.3.2 is written as

$$\sum_{m_2/2 \leq s \leq m_2} * + \sum_{m_2 < s \leq m_1/2} * + \sum_{m_1/2 < s \leq (m_1+m_2)/2} *$$

with $* = \sharp P_{\boldsymbol{m}}(s) \sharp Q_{\boldsymbol{m}}(s)$. Here $s \in \boldsymbol{Z}$. Applying Lemma 4.3.3, we have

$$\begin{aligned} \sharp L_{\boldsymbol{m}} = {} & (m_1 - m_2 + 1)(m_2 + 1) \\ & \cdot \left\{ (m_2/2 + 1)(1 + (-1)^{m_2})/2 + \sum_{m_2/2 < s \leq m_2} (s + 1)(2s - m_2 + 1) \right\} \\ & + (m_2 + 1)^2 \sum_{m_2 < s \leq m_1/2} (m_1 + m_2 + 1 - 2s)(2s - m_2 + 1) \\ & + (m_1 - m_2 + 1)(m_2 + 1) \sum_{m_1/2 < s \leq (m_1+m_2)/2} (m_1 + m_2 + 1 - 2s)^2. \end{aligned}$$

Using the elementary formulas

$$\sum_{a < s \leq b} 1 = [b] - [a], \quad \sum_{a < s \leq b} s = (1/2)([b] - [a])([b] + [a] + 1),$$

$$\sum_{a < s \leq b} s^2 = ([b] - [a])(2([b] + [a])^2 - 2[b][a] + 3([b] + [a]) + 1)/6,$$

$$[l/2] = l/2 - 1/4 + (-1)^l/4$$

for real numbers a, b satisfying $0 \leq a < b$ and an integer $l \geq 0$, where [ ] denotes Gauss's symbol, and substituting $m_1 = n_1 - 2$, $m_2 = n_2 - 1$, we see that $\sharp L_{\boldsymbol{m}}$ becomes the desired formula. Similarly, we obtain the formula for the case $2m_2 \geq m_1$. Proposition 4.3.1 is proved. □

Proof of Proposition 1.2.1. By Proposition 4.1.1 we see that $\dim H^{\Gamma}_{\boldsymbol{m}}(SpU(4))$ is given by the formula in Proposition 4.3.1. Since $H^{\Gamma}_{\boldsymbol{m}} \cong H^{\Gamma}_{\boldsymbol{m}}(SpU(4))$ by Lemma 3.2.1, we obtain the desired expression for $\dim H^{\Gamma}_{\boldsymbol{m}}$. □

4.4. Proof of Proposition 1.2.2.

Let $H_{\boldsymbol{m}}(SpU(4))$ be as in §4.1. Let $\boldsymbol{c}$ be the correspondence $Sp(1) \cong SpU(2)$, $Sp(2) \cong SpU(4)$ in §3.2. By means of $\boldsymbol{c}$, instead of the action $Sp(1) \times Sp(2) \to Sp(2)$ giving the quotient manifold $\tilde{S}^7$ (resp. $\tilde{\tilde{S}}^7$, resp. $\check{S}^7$), we consider the corresponding action $\Gamma^{\sim}$ (resp. $\Gamma^{\approx}$, resp. $\Gamma^{\vee}$): $SpU(2) \times SpU(4) \to SpU(4)$. In order to describe these actions, let $A = c_i, c_j, c_k \in spu(2)$ be as in §3.1. Then, as in the proof of Proposition 3.1.1 we see that the Killing vector fields $d\Gamma^{\sim}_A$ on $SpU(4)$ induced by $\Gamma^{\sim}$ are expressed as

$$(1) \qquad d\Gamma^{\sim}_{c_i} = i\hat{h}_1, \quad d\Gamma^{\sim}_{c_j} = \hat{e}_3 - \hat{f}_3, \quad d\Gamma^{\sim}_{c_k} = -i\hat{e}_3 - i\hat{f}_3.$$

Similarly we have

$$(2) \qquad d\Gamma^{\approx}_{c_i} = i\hat{h}_1 + i\hat{h}_2, \quad d\Gamma^{\approx}_{c_j} = \hat{e}_3 - \hat{f}_3 + \hat{e}_4 - \hat{f}_4,$$

$$d\Gamma^{\approx}_{c_k} = -i(\hat{e}_3 + \hat{f}_3) - i(\hat{e}_4 + \hat{f}_4),$$

$$(3) \qquad d\Gamma^{\vee}_{c_i} = i\hat{h}_2 - ih_1, \quad d\Gamma^{\vee}_{c_j} = \hat{e}_4 - \hat{f}_4 + e_3 - f_3,$$

$$d\Gamma^{\vee}_{c_k} = -i(\hat{e}_4 + \hat{f}_4) + i(e_3 + f_3).$$

Now, let $H^{\sim}_{\boldsymbol{m}}(SpU(4))$ (resp. $H^{\approx}_{\boldsymbol{m}}(SpU(4))$, resp. $H^{\vee}_{\boldsymbol{m}}(SpU(4))$) be the subspace of $H_{\boldsymbol{m}}(SpU(4))$ consisting of functions which are fixed by the action $\Gamma^{\sim}$ (resp. $\Gamma^{\approx}$, resp. $\Gamma^{\vee}$). Then, the description of the actions above yields immediately

Lemma 4.4.1.

$$H^{\sim}_{\boldsymbol{m}}(SpU(4)) = \{\varphi \mid \hat{h}_1(\varphi) = \hat{e}_3(\varphi) = \hat{f}_3(\varphi) = 0\},$$

$$H^{\approx}_{\boldsymbol{m}}(SpU(4)) = \{\varphi \mid (\hat{h}_1 + \hat{h}_2)\varphi = (\hat{e}_3 + \hat{e}_4)\varphi = (\hat{f}_3 + \hat{f}_4)\varphi = 0\},$$

$$H^{\vee}_{\boldsymbol{m}}(SpU(4)) = \{\varphi \mid (\hat{h}_2 - h_1)\varphi = (\hat{e}_4 - f_3)\varphi = (\hat{f}_4 - e_3)\varphi = 0\}$$

where $\varphi \in H_{\boldsymbol{m}}(SpU(4))$.

In order to find the dimension of $H_{\boldsymbol{m}}^{\sim}(SpU(4))$, we proceed as in §4.2. Namely, for any integers p_1, p_2, j such that $m_1 \geq p_1 \geq m_2 \geq p_2 \geq 0$, $p_1 - p_2 \geq j \geq 0$ and for any $v \in M_{\boldsymbol{m}}$ we set

$$H_{\boldsymbol{m},(p,j,v)} = \oplus_{0 \leq i \leq I-1} \boldsymbol{C}\varphi_{(p_1,p_2,i,j)v},$$

where $I = m_1 + m_2 - (p_1 + p_2) + 1$ as in §4.2. Then $H_{\boldsymbol{m}}(SpU(4)) = \oplus_{p,j,v} H_{\boldsymbol{m},(p,j,v)}$. To see how $\Gamma^{\sim}$ acts on $H_{\boldsymbol{m},(p,j,v)}$, we set

$$\varphi_{(i)} = \varphi_{(p_1,p_2,i,j)v}$$

with the convention that $\varphi_{(i)} = 0$ if i does not satisfy $0 \leq i \leq I-1$. Then by Corollary 2.3.2 we have

$$\hat{h}_1(\varphi_{(i)}) = (I - 1 - 2i)\varphi_{(i)}, \quad \hat{e}_3(\varphi_{(i)}) = i(I-i)\varphi_{(i-1)}, \quad \hat{f}_3(\varphi_{(i)}) = \varphi_{(i+1)}.$$

These formulas show that each subspace $H_{\boldsymbol{m},(p,j,v)}$ is invariant under $\hat{h}_1, \hat{e}_3, \hat{f}_3$, and moreover

$$\{\varphi \in H_{\boldsymbol{m},(p,j,v)} \mid \hat{h}_1(\varphi) = \hat{e}_3(\varphi) = \hat{f}_3(\varphi) = 0\} = \begin{cases} \boldsymbol{C}\varphi_{(0)} & \text{if } I = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, we conclude that $\dim H_{\boldsymbol{m}}^{\sim}(SpU(4))$ is equal to the cardinality of the set $\{j \in \boldsymbol{Z} \mid 0 \leq j \leq m_1 - m_2\} \times M_{\boldsymbol{m}}$, that is, $(m_1 - m_2 + 1)d$, where $d = (m_1+2)(m_2+1)(m_1-m_2+1)(m_1+m_2+3)/6$. This proves the first formula of Proposition 1.2.2.

To find the dimension of $H_{\boldsymbol{m}}^{\approx}(SpU(4))$, for any integers p_1, p_2 such that $m_1 \geq p_1 \geq m_2 \geq p_2 \geq 0$ and any $v \in M_{\boldsymbol{m}}$, we set

$$H_{\boldsymbol{m},(p,v)} = \oplus_{0 \leq i \leq I-1, 0 \leq j \leq J-1} \boldsymbol{C}\varphi_{(p_1,p_2,i,j)v},$$

where I, J are as in §4.2. Then $H_{\boldsymbol{m}}(SpU(4)) = \oplus_{p,v} H_{\boldsymbol{m},(p,v)}$. We set $\varphi_{(i,j)} = \varphi_{(p_1,p_2,i,j)v}$, and $\varphi_{(i,j)} = 0$ if i, j do not satisfy $0 \leq i \leq I-1$, $0 \leq j \leq J-1$. By Corollary 2.3.2 we have

$$(\hat{h}_1 + \hat{h}_2)\varphi_{(i,j)} = (I + J - 2 - 2i - 2j)\varphi_{(i,j)},$$

$$(\hat{e}_3 + \hat{e}_4)\varphi_{(i,j)} = i(I-i)\varphi_{(i-1,j)} + j(J-j)\varphi_{(i,j-1)},$$

$$(\hat{f}_3 + \hat{f}_4)\varphi_{(i,j)} = \varphi_{(i+1,j)} + \varphi_{(i,j+1)}.$$

Using these formulas, we see that $H_{\boldsymbol{m},(p,v)}$ is invariant under $\hat{h}_1 + \hat{h}_2$, $\hat{e}_3 + \hat{e}_4$, $\hat{f}_3 + \hat{f}_4$, and

$$\{\varphi \in H_{\boldsymbol{m},(p,v)} \mid (\hat{h}_1 + \hat{h}_2)\varphi = (\hat{e}_3 + \hat{e}_4)\varphi = (\hat{f}_3 + \hat{f}_4)\varphi = 0\} = \begin{cases} \boldsymbol{C}\phi & \text{if } I = J, \\ 0 & \text{otherwise,} \end{cases}$$

where $\phi = \varphi_{(I-1,0)} - \varphi_{(I-2,1)} + \cdots + (-1)^{I-1}\varphi_{(0,I-1)}$. Hence, $H_{\boldsymbol{m}}^{\approx}(SpU(4))$ is non-trivial if and only if $m_1 + m_2$ is even, and if that is the case, $\dim H_{\boldsymbol{m}}^{\approx}(SpU(4))$ is equal to the cardinality of the set $\{0 \le p_2 \le m_2\} \times M_{\boldsymbol{m}}$, that is $(m_2+1)d$.

There remains to find the dimension of $H_{\boldsymbol{m}}^{\vee}(SpU(4))$. For any integers p_1, p_2, i, q_1, q_2, l such that

$$(*) \qquad m_1 \ge p_1 \ge m_2 \ge p_2 \ge 0, \quad I-1 \ge i \ge 0,$$

$$(**) \qquad m_1 \ge q_1 \ge m_2 \ge q_2 \ge 0, \quad L-1 \ge l \ge 0,$$

we set

$$H_{\boldsymbol{m},(p,i,q,l)} = \oplus_{0 \le j \le J-1, 0 \le k \le K-1} \boldsymbol{C}\varphi_{(p_1,p_2,i,j)(q_1,q_2,k,l)}.$$

Here I, J, K, L are as in §4.2. Then $H_{\boldsymbol{m}}(SpU(4)) = \oplus_{p,i,q,l} H_{\boldsymbol{m},(p,i,q,l)}$. We set $\varphi_{(j)(k)} = \varphi_{(p_1,p_2,i,j)(q_1,q_2,k,l)}$, and set $\varphi_{(j)(k)} = 0$ if j, k do not satisfy $0 \le j \le J-1$, $0 \le k \le K-1$. Then, by Corollary 2.3.2 we have

$$(\hat{h}_2 - h_1)\varphi_{(j)(k)} = (J - K + 2(k-j))\varphi_{(j)(k)},$$

$$(\hat{e}_4 - f_3)\varphi_{(j)(k)} = j(J-j)\varphi_{(j-1)(k)} - \varphi_{(j)(k+1)},$$

$$(\hat{f}_4 - e_3)\varphi_{(j)(k)} = \varphi_{(j+1)(k)} - k(K-k)\varphi_{(j)(k-1)}.$$

Using these formulas we observe that $H_{\boldsymbol{m},(p,i,q,l)}$ is invariant under $\hat{h}_2 - h_1$, $\hat{e}_4 - f_3$, $\hat{f}_4 - e_3$, and moreover

$$\{\varphi \in H_{\boldsymbol{m},(p,i,q,l)} \mid (\hat{h}_2 - h_1)\varphi = (\hat{e}_4 - f_3)\varphi = (\hat{f}_4 - e_3)\varphi = 0\} = \begin{cases} \boldsymbol{C}\phi & \text{if } J = K, \\ 0 & \text{otherwise,} \end{cases}$$

where $\phi = \sum_{0 \le j \le J-1}((J-j-1)!/j!)\varphi_{(j)(j)}$. Hence $\dim H_{\boldsymbol{m}}^{\vee}(SpU(4))$ is equal to $\sharp C_{\boldsymbol{m}}$, where $C_{\boldsymbol{m}}$ is the set of 6-tuples of integers $(p_1, p_2, i, q_1, q_2, l)$ satisfying $(*), (**)$ above and the condition $p_1 - p_2 + q_1 + q_2 = m_1 + m_2$. To find $\sharp C_{\boldsymbol{m}}$, for each integer s such that $0 \le s \le m_1$, let $P(s)$ be the set of points $(p_1, p_2, i) \in \boldsymbol{Z}^3$ satisfying $(*)$ and the condition $p_1 - p_2 = s$, and let $Q(s)$ be the set of points $(q_1, q_2, l) \in \boldsymbol{Z}^3$ satisfying $(**)$ and the condition $q_1 + q_2 = m_1 + m_2 - s$. Then $C_{\boldsymbol{m}} = \bigcup_{0 \le s \le m_1} P(s) \times Q(s)$ (disjoint union). It is directly verified that

$$\sharp P(s) = (m_1+2)(m_1+1-\max(m_2,s)) + \sum_{m_2 \le p_1 \le m_1} (m_2 + s - 1 - 2p_1),$$

$$\sharp Q(s) = (m_1+2)(m_1+1-\max(m_2,s)) + \sum_{m_2 \le q_1 \le m_1} (2q_1 + s - 1 - 2m_1 - m_2).$$

In this way we obtain

$$\sharp C_{\boldsymbol{m}} = (m_1+2)(m_2+1)(m_1-m_2+1)(3+2m_1+2m_1m_2-2m_2^2)/6.$$

Substituting $m_1 = n_1 - 2$, $m_2 = n_2 - 1$ yields the desired expression for $\sharp C_{\boldsymbol{m}}$. This completes the proof of Proposition 1.2.2. □

Appendix.

A.1. PROOF OF THEOREM 2.2.1.

We fix a pair $\boldsymbol{m} = (m_1, m_2)$ of integers satisfying $m_1 \geq m_2 \geq 0$, and use the notation in §2. We begin by recalling the Gauss decomposition in $Sp(4, \boldsymbol{C})$ (see [**Zh**]). Almost every $x = (x_{ij}) \in Sp(4, \boldsymbol{C})$ is decomposed uniquely into the product

$$x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ \vdots & \ddots & 1 & 0 \\ & \cdots & * & 1 \end{pmatrix} \begin{pmatrix} \delta_1 & 0 & 0 & 0 \\ 0 & \delta_2 & 0 & 0 \\ 0 & 0 & \delta_2^{-1} & 0 \\ 0 & 0 & 0 & \delta_1^{-1} \end{pmatrix} \begin{pmatrix} 1 & z_{12} & z_{13} & z_{14} \\ 0 & 1 & z_{23} & z_{24} \\ 0 & 0 & 1 & -z_{12} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

of matrices in $Sp(4, \boldsymbol{C})$, and we know that

$$\delta_1 = x_{11}, \quad \delta_2 = \frac{x_{(1,2)(1,2)}}{x_{11}}, \quad z_{12} = \frac{x_{12}}{x_{11}}, \quad z_{13} = \frac{x_{13}}{x_{11}}, \quad z_{14} = \frac{x_{14}}{x_{11}},$$

$$z_{23} = \frac{x_{(1,2)(1,3)}}{x_{(1,2)(1,2)}}, \quad z_{24} = z_{13} - z_{12}z_{23}.$$

From the formula $z_{24} = x_{(1,2)(1,4)}/x_{(1,2)(1,2)}$, we see that if $0 \leq p \leq m_1 - m_2$, $0 \leq q \leq m_2$, then $z_{12}^p z_{24}^q \phi_{\boldsymbol{m}} \in C^\infty(Sp(4, \boldsymbol{C}))$, and moreover $z_{12}^p z_{24}^q \phi_{\boldsymbol{m}} \in \mathfrak{R}_{\boldsymbol{m}}$ by [**Zh**, §113, Theorem 6], or by Proposition A.1.3 below. Now consider the subgroup

$$G_0 = \left\{ \begin{pmatrix} a & 0 & 0 & b \\ 0 & a' & b' & 0 \\ 0 & c' & d' & 0 \\ c & 0 & 0 & d \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in Sp(2, \boldsymbol{C}) \right\}$$

of $Sp(4, \boldsymbol{C})$, and denote by $\pi_{\boldsymbol{m}}|_{G_0}$ the restriction of the representation $\pi_{\boldsymbol{m}}$ to the subgroup G_0.

LEMMA A.1.1. *If $0 \leq p \leq m_1 - m_2$, $0 \leq q \leq m_2$, then $z_{12}^p z_{24}^q \phi_{\boldsymbol{m}} \in \mathfrak{R}_{\boldsymbol{m}}$ is a highest weight vector of $\pi_{\boldsymbol{m}}|_{G_0}$, whose weight is given by*

$$\begin{pmatrix} \delta_1 & 0 & 0 & 0 \\ 0 & \delta_2 & 0 & 0 \\ 0 & 0 & \delta_2^{-1} & 0 \\ 0 & 0 & 0 & \delta_1^{-1} \end{pmatrix} \mapsto \delta_1^{m_1-(p+q)} \delta_2^{m_2+p-q}.$$

Conversely, each highest weight vector of $\pi_{\boldsymbol{m}}|_{G_0}$ *has the form* $z_{12}^p z_{24}^q \phi_{\boldsymbol{m}}$ (*up to a constant factor*) *with* $0 \leq p \leq m_1 - m_2$, $0 \leq q \leq m_2$.

Proof. By definition, a function $f \in \mathfrak{R}_{\boldsymbol{m}}$ is a highest weight vector, with weight α, of $\pi_{\boldsymbol{m}}|_{G_0}$ if and only if f satisfies the functional equations

$$f(*z_0) = f(*) \text{ for any } z_0 \in Z_0, \quad f(*\delta) = \alpha(\delta) f(*) \text{ for any } \delta \in D_0,$$

where Z_0 is the subgroup of G_0 consisting of upper triangular matrices with 1's along the diagonal, and D_0 is the subgroup of G_0 consisting of diagonal matrices. From the identities

$$z \begin{pmatrix} 1 & 0 & 0 & b \\ 0 & 1 & b' & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z_{12} & z_{13} + b' z_{12} & z_{14} + b \\ 0 & 1 & z_{23} + b' & z_{24} \\ 0 & 0 & 1 & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\delta^{-1} z \delta = \begin{pmatrix} 1 & \delta_1^{-1}\delta_2 z_{12} & \delta_1^{-1}\delta_2^{-1} z_{13} & \delta_1^{-2} z_{14} \\ 0 & 1 & \delta_2^{-2} z_{23} & \delta_1^{-1}\delta_2^{-1} z_{24} \\ 0 & 0 & 1 & \delta_1^{-1}\delta_2 z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for matrices

$$z = \begin{pmatrix} 1 & z_{12} & z_{13} & z_{14} \\ 0 & 1 & z_{23} & z_{24} \\ 0 & 0 & 1 & z_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \delta = \begin{pmatrix} \delta_1 & 0 & 0 & 0 \\ 0 & \delta_2 & 0 & 0 \\ 0 & 0 & \delta_2^{-1} & 0 \\ 0 & 0 & 0 & \delta_1^{-1} \end{pmatrix},$$

we see that the function $z_{12}^p z_{24}^q \phi_{\boldsymbol{m}}$ satisfies the functional equations above. To prove the converse we have to use the indicator system in [**Zh**]. By [**Zh**, §113, Theorem 6] we know that if $z_{12}^p z_{24}^q \phi_{\boldsymbol{m}} \in \mathfrak{R}_{\boldsymbol{m}}$ for some $p, q \geq 0$, then we have $0 \leq p \leq m_1 - m_2$, $0 \leq q \leq m_2$. Now it is easy to complete the proof of the converse. □

Corollary A.1.2. *The representation space* $\mathfrak{R}_{\boldsymbol{m}}$ *has the orthogonal direct sum decomposition*

$$\mathfrak{R}_{\boldsymbol{m}} = \oplus_{p,q} \mathfrak{R}_{\boldsymbol{m}}(p,q),$$

where p, q *range over integers such that* $0 \leq p \leq m_1 - m_2$, $0 \leq q \leq m_2$, *and* $\mathfrak{R}_{\boldsymbol{m}}(p,q)$ *denotes the subspace of* $\mathfrak{R}_{\boldsymbol{m}}$ *spanned by the right translations* $(z_{12}^p z_{24}^q \phi_{\boldsymbol{m}})(*g)$, $g \in G_0$. *The dimension of* $\mathfrak{R}_{\boldsymbol{m}}(p,q)$ *is equal to* $(m_1 - (p+q) + 1)(m_2 + p - q + 1)$.

Proof. The method of Z-invariants ([**Zh**]) gives us the orthogonal direct sum decomposition. To find the dimension of $\mathfrak{R}_{\boldsymbol{m}}(p,q)$, note that the irreducible representation $\pi_{\boldsymbol{m}}|_{G_0}$ in $\mathfrak{R}_{\boldsymbol{m}}(p,q)$ can be written as a tensor product of two irreducible representations π_α, π_β of $Sp(2,\boldsymbol{C})$ $(=SL(2,\boldsymbol{C}))$, whose highest weights α, β are given by

$$\alpha\left(\begin{pmatrix}\delta_1 & 0\\ 0 & \delta_1^{-1}\end{pmatrix}\right) = \delta_1^{m_1-(p+q)}, \quad \beta\left(\begin{pmatrix}\delta_2 & 0\\ 0 & \delta_2^{-1}\end{pmatrix}\right) = \delta_2^{m_2+p-q}$$

respectively. Since the dimensions of the representation spaces of π_α, π_β are $m_1-(p+q)+1$, $m_2+p-q+1$ respectively, we get the desired formula. $\square$

Proposition A.1.3. *For any integers $p,q\geq 0$, the operator $\Omega_{p,q}$ satisfies*

$$\Omega_{p,q}(\phi_{\boldsymbol{m}}) = N_{p,q} z_{12}^p z_{24}^q \phi_{\boldsymbol{m}},$$

where $N_{p,q}$ is defined to be

$$N_{p,q} = C_q(m_1+1)C_q(m_2)C_q(m_1+m_2+2)C_{p,p-q}(m_1-m_2).$$

The proof will be given in the end of this appendix.

Remark. Since for a nonnegative integer p, the number $C_r(p)$ does not vanish if and only if $r\leq p$, we see that for nonnegative integers p,q, the number $N_{p,q}$ does not vanish if and only if $p\leq m_1-m_2$ and $q\leq m_2$.

For the proof of Theorem 2.2.1 we have to study the operators f_3, f_4, which map $\mathfrak{R}_{\boldsymbol{m}}(p,q)$ into itself, more closely.

Lemma A.1.4. *The commutation relations*

$$[h_1, \nabla_{42}] = -\nabla_{42}, \quad [h_2, \nabla_{42}] = -\nabla_{42},$$

hold, and moreover for nonnegative integers p,q,i,j,

$$[h_1, f_3^i f_4^j \Omega_{p,q}] = -(p+q+2i) f_3^i f_4^j \Omega_{p,q},$$

$$[h_2, f_3^i f_4^j \Omega_{p,q}] = (p-q-2j) f_3^i f_4^j \Omega_{p,q}.$$

Hence, the generating function $\phi_{\boldsymbol{m}}$ satisfies

$$h_1(f_3^i f_4^j \Omega_{p,q}(\phi_{\boldsymbol{m}})) = (m_1-p-q-2i) f_3^i f_4^j \Omega_{p,q}(\phi_{\boldsymbol{m}}),$$

$$h_2(f_3^i f_4^j \Omega_{p,q}(\phi_{\boldsymbol{m}})) = (m_2+p-q-2j) f_3^i f_4^j \Omega_{p,q}(\phi_{\boldsymbol{m}}).$$

Proof. The former parts are direct consequences of the relations:

$$[h_1, f_1] = -f_1, \quad [h_2, f_1] = f_1, \quad [h_1, f_2] = -f_2, \quad [h_2, f_2] = -f_2,$$

$$[h_1, f_3] = -2f_3, \quad [h_2, f_3] = 0, \quad [h_1, f_4] = 0, \quad [h_2, f_4] = -2f_4.$$

The fact that $h_1(\phi_{\boldsymbol{m}}) = m_1\phi_{\boldsymbol{m}}$, $h_2(\phi_{\boldsymbol{m}}) = m_2\phi_{\boldsymbol{m}}$ yields the latter part. $\square$

PROOF OF THEOREM 2.2.1. First, note that each φ_μ is nonzero. This follows from Lemma A.1.5 below. Next, we contend that for fixed p_1, p_2 satisfying $0 \le p_2 \le m_2 \le p_1 \le m_1$, the functions

$$\varphi_{\mu_p},\ \mu_p = (p_1, p_2, i, j), \quad 0 \le i \le m_1 + m_2 - (p_1 + p_2), \quad 0 \le j \le p_1 - p_2,$$

constitute an orthogonal basis for $\mathfrak{R}_{\boldsymbol{m}}(p_1 - m_2, p_2)$. In fact, by Proposition A.1.3 we observe that

$$\varphi_{\mu_p} = \Omega_{\mu_p}(\phi_{\boldsymbol{m}}) = f_3^i f_4^j \Omega_{p_1 - m_2, p_2}(\phi_{\boldsymbol{m}}) \in \mathfrak{R}_{\boldsymbol{m}}(p_1 - m_2, p_2).$$

Comparing the weights of φ_{μ_p} by Lemma A.1.4, we see that φ_{μ_p}'s are mutually orthogonal. Thus, counting the dimension of $\mathfrak{R}_{\boldsymbol{m}}(p_1 - m_2, p_2)$ we conclude that $\{\varphi_{\mu_p}\}$ is a basis for $\mathfrak{R}_{\boldsymbol{m}}(p_1 - m_2, p_2)$. Now by Corollary A.1.2 it is clear that $\{\varphi_\mu\}_{\mu \in M_{\boldsymbol{m}}}$ is an orthogonal basis of $\mathfrak{R}_{\boldsymbol{m}}$. Theorem 2.2.1 is proved. □

Assuming Proposition A.1.3, we can find the norm of φ_μ.

PROPOSITION A.1.5. *The norm $N_\mu = \|\varphi_\mu\|$ of φ_μ is given by the formula*

$$N_\mu^2 = i!j!C_i(m_1 + m_2 - (p_1 + p_2))C_j(p_1 - p_2)N_{p_1 - m_2, p_2}N^*_{p_1 - m_2, p_2}$$

where $N_{p,q}$ is as in Proposition A.1.3, *and $N^*_{p,q}$ is defined to be*

$$N^*_{p,q} = p!q!C_q(m_1 + 1 - p)C_q(m_2 + 1 + p)C_{p-1, p-q}(m_1 - m_2).$$

PROOF. The same argument as in [**Zh**, §69] proves our proposition. Namely, we consider the adjoint operator Ω^*_μ of Ω_μ. It is given by

$$\Omega^*_{p,q} = \sum_{l=0}^{\min(p,q)} (-1)^l \binom{q}{l} C_l(p) C_{p,q,l}(h_1, h_2) e_1^{p-l} e_3^l \nabla_{42}^{*q-l},$$

$$\nabla^*_{42} = (h_2 + 1)e_2 - e_1 e_4 = e_2(h_2 + 1) - e_4 e_1.$$

Direct computation yields

$$\Omega^*_{p,q}(z_{12}^p z_{24}^q \phi_{\boldsymbol{m}}) = N^*_{p,q} \phi_{\boldsymbol{m}} \quad \text{for nonnegative integers } p, q.$$

On the other hand

$$e_3^i e_4^j f_3^i f_4^j (z_{12}^p z_{24}^q \phi_{\boldsymbol{m}}) = i!j!C_i(m_1 - p - q)C_j(m_2 + p - q) z_{12}^p z_{24}^q \phi_{\boldsymbol{m}}.$$

Using these formulas and Proposition A.1.3 we observe that

$$\Omega^*_\mu \Omega_\mu(\phi_{\boldsymbol{m}}) = N_{p_1 - m_2, p_2} N^*_{p_1 - m_2, p_2} i!j!C_i(m_1 + m_2 - (p_1 + p_2))C_j(p_1 - p_2)\phi_{\boldsymbol{m}}.$$

Recalling our assumption $(\phi_{\boldsymbol{m}}, \phi_{\boldsymbol{m}}) = 1$, we see that this formula shows Proposition A.1.5. □

There remains to prove Proposition A.1.3. We need some lemmas.

Lemma A.1.6. *The operators* f_1, ∇_{42} *satisfy*

$$f_1(\phi_{\boldsymbol{m}}) = (m_1 - m_2) z_{12} \phi_{\boldsymbol{m}}, \quad \nabla_{42}(\phi_{\boldsymbol{m}}) = m_2(m_1 + m_2 + 2) z_{24} \phi_{\boldsymbol{m}},$$

and moreover, for any nonnegative integers r, k,

$$f_1^r(\phi_{\boldsymbol{m}}) = C_r(m_1 - m_2) z_{12}^r \phi_{\boldsymbol{m}},$$
$$\nabla_{42}^k(\phi_{\boldsymbol{m}}) = C_k(m_2) C_k(m_1 + m_2 + 2) z_{24}^k \phi_{\boldsymbol{m}}.$$

Proof. Recall that the left invariant vector field f_1 is expressed as $f_1 = E_{21} - E_{43}$, where $E_{ij} = \sum_{s=1}^4 x_{si}(\partial/\partial x_{sj})$, and that E_{ij} is the operator replacing the j-th column by the i-th column [**Zh**, §68]. Then we observe that

$$f_1(\phi_{\boldsymbol{m}}) = E_{21}(\phi_{\boldsymbol{m}}) = (m_1 - m_2) \frac{x_{12}}{x_{11}} \phi_{\boldsymbol{m}} = (m_1 - m_2) z_{12} \phi_{\boldsymbol{m}}.$$

Since $f_1(z_{12}) = -z_{12}^2$, we get immediately the expression for $f_1^r(\phi_{\boldsymbol{m}})$. Similarly, we have

$$f_2(\phi_{\boldsymbol{m}}) = ((m_1 + m_2) z_{13} - 2m_2 z_{12} z_{23}) \phi_{\boldsymbol{m}},$$
$$f_4(\phi_{\boldsymbol{m}}) = m_2 z_{23} \phi_{\boldsymbol{m}}, \quad f_4(z_{12}) = z_{13}.$$

(To get the expression for $f_2(\phi_{\boldsymbol{m}})$ we have to use the identity $x_{(1,2)(1,4)} + x_{(1,2)(2,3)} = 0$.) Using these identities, we get the expression for $\nabla_{42}(\phi_{\boldsymbol{m}})$. Moreover, using the identities

$$h_2(z_{24}) = -z_{24}, \quad f_1(z_{24}) = z_{12} z_{24} - z_{14},$$
$$f_2(z_{24}) = -z_{24}^2 + (z_{12} z_{24} - z_{14}) z_{23}, \quad f_4(z_{14}) = 0, \quad f_4(z_{24}) = -z_{23} z_{24},$$

by direct computation we obtain

$$\nabla_{42}(z_{24}^r) = r(r-2) z_{24}^{r+1},$$
$$\nabla_{42}(z_{24}^r \phi_{\boldsymbol{m}}) = (m_2 - r)(m_1 + m_2 + 2 - r) z_{24}^{r+1} \phi_{\boldsymbol{m}},$$

and then the desired expression for $\nabla_{42}^k(\phi_{\boldsymbol{m}})$. Lemma A.1.6 is proved. □

Lemma A.1.7. *For nonnegative integers* p, q *the following identities hold*:

$$f_3(z_{12}^{p-1} z_{24}^q \phi_{\boldsymbol{m}}) = (q - m_2) z_{12}^p z_{24}^{q+1} \phi_{\boldsymbol{m}}$$
$$+ (m_1 - p - q + 1) z_{12}^{p-1} z_{24}^q z_{14} \phi_{\boldsymbol{m}},$$

$$\nabla_{42}(z_{12}^p z_{24}^q \phi_{\boldsymbol{m}}) = (m_2 - q)(m_1 + m_2 + 2 + p - q) z_{12}^p z_{24}^{q+1} \phi_{\boldsymbol{m}}$$
$$+ p(m_2 + p + 1) z_{12}^{p-1} z_{24}^q z_{14} \phi_{\boldsymbol{m}}.$$

Hence, eliminating the terms containing z_{14} *yields*

$$(m_1 + 1 - q)(m_2 - q)(m_1 + m_2 + 2 - q) z_{12}^p z_{24}^{q+1} \phi_{\boldsymbol{m}}$$
$$= (m_1 - (p+q) + 1)\nabla_{42}(z_{12}^p z_{24}^q \phi_{\boldsymbol{m}}) - p(m_2 + p + 1) f_3(z_{12}^{p-1} z_{24}^q \phi_{\boldsymbol{m}}).$$

Proof. From the identities

$$f_3(z_{12}) = -z_{12}z_{14}, \quad f_3(z_{24}) = (-z_{14} + z_{12}z_{24})z_{24},$$
$$f_3(\phi_{\boldsymbol{m}}) = (m_1 z_{14} - m_2 z_{12} z_{24})\phi_{\boldsymbol{m}}$$

we get at once the expression for $f_3(z_{12}^{p-1} z_{24}^q \phi_{\boldsymbol{m}})$. As for the second formula we note that

$$f_2(z_{12}) = z_{14} - z_{12}z_{13}, \quad h_2(z_{12}) = z_{12}, \quad \text{and hence } \nabla_{42}(z_{12}) = 2z_{14}.$$

Then we get

$$\nabla_{42}(z_{12}^p) = p(p+1) z_{12}^{p-1} z_{14},$$

and hence the expression for $\nabla_{42}(z_{12}^p z_{24}^q \phi_{\boldsymbol{m}})$. Lemma A.1.7 is proved. □

Lemma A.1.8. *For any nonnegative integers* p, q,

$$\Omega_{p,q+1} = \nabla_{42}\Omega_{p,q}(h_1 - (p+q) + 1) C_{p-q-1}(h_1 - h_2)$$
$$- p f_3 \Omega_{p-1,q}(h_2 + p + 1) C_p(h_1 - h_2).$$

Proof. Recalling the definition of $\Omega_{p,q}$, and using the fact that $C_l(p) = 0$ if $l \geq p+1$, we can write the first half of the right-hand side of our formula as

$$\nabla_{42}\Omega_{p,q}(h_1 - (p+q) + 1) C_{p-q-1}(h_1 - h_2) = \sum_{l=0}^{q} (-1)^l C_l(p) \nabla_{42}^{q-l+1} f_3^l f_1^{p-l} D_l$$

with $D_l = \binom{q}{l} C_{p,q,l}(h_1, h_2)(h_1 - (p+q) + 1) C_{p-q-1}(h_1 - h_2)$. Similarly, using $[f_3, \nabla_{42}] = 0$, $[h_2, f_3] = 0$, $[h_2, \Omega_{p-1,q}] = (p - q - 1)\Omega_{p-1,q}$ and replacing $l+1$ by l, we can write the second half of the right-hand side as

$$-p f_3 \Omega_{p-1,q}(h_2 + p + 1) C_p(h_1 - h_2) = \sum_{l=1}^{q+1} (-1)^l C_l(p) \nabla_{42}^{q-l+1} f_3^l f_1^{p-l} E_l$$

with $E_l = \binom{q}{l-1}(h_2+p+1)C_{p-1,q,l-1}(h_1,h_2)C_p(h_1-h_2)$. On the other hand, by definition we have

$$\Omega_{p,q+1} = \sum_{l=0}^{q+1}(-1)^l C_l(p)\nabla_{42}^{q+1-l} f_3^l f_1^{p-l} F_l$$

with $F_l = \binom{q+1}{l} C_{p,q+1,l}(h_1,h_2)$. Now, note the following obvious relations:

$$C_{s,t} = C_{s,t+1}C_t = C_s C_{s-1,t} \quad \text{for any integers } s \geq t,$$

$$C_r(m) = mC_{r-1}(m-1) = (m-r+1)C_{r-1}(m) \quad \text{for any integer } r \geq 1.$$

Then, using the identities

$$\binom{q+1}{l}(h_1+1+l-(p+q)) = \binom{q}{l}(h_1+1-(p+q)) + \binom{q}{l-1}(h_1+2-p),$$

$1 \leq l \leq q$, we obtain $D_l + E_l = F_l$ for $1 \leq l \leq q$. Clearly $D_0 = F_0$, $E_{q+1} = F_{q+1}$. These relations yield the desired formula. Lemma A.1.8 is proved. □

Proof of Proposition A.1.3. We shall prove Proposition A.1.3 by double induction on p, q. It suffices to prove the following two assertions.

Assertion 1. *If p or q is zero, then Proposition* A.1.3 *holds, that is*

$$\Omega_{0,q}(\phi_{\boldsymbol{m}}) = N_{0,q} z_{24}^q \phi_{\boldsymbol{m}}, \quad \Omega_{p,0}(\phi_{\boldsymbol{m}}) = N_{p,0} z_{12}^p \phi_{\boldsymbol{m}}.$$

Indeed, we have $\Omega_{0,q} = \nabla_{42}^q C_q(h_1+1)$, $N_{0,q} = C_q(m_1+1)C_q(m_2)C_q(m_1+m_2+2)$, and $\Omega_{p,0} = f_1^p$, $N_{p,0} = C_p(m_1-m_2)$. Then Lemma A.1.6 shows our formulas.

Assertion 2. *Fix any integers $p \geq 1$, $q \geq 0$, and assume that*

$$\Omega_{p-1,q}(\phi_{\boldsymbol{m}}) = N_{p-1,q} z_{12}^{p-1} z_{24}^q \phi_{\boldsymbol{m}}, \quad \Omega_{p,q}(\phi_{\boldsymbol{m}}) = N_{p,q} z_{12}^p z_{24}^q \phi_{\boldsymbol{m}}.$$

Then

$$\Omega_{p,q+1}(\phi_{\boldsymbol{m}}) = N_{p,q+1} z_{12}^p z_{24}^{q+1} \phi_{\boldsymbol{m}}.$$

Indeed, this is a consequence of Lemmas A.1.7, A.1.8 and the relation

$$C_{p-q-1}(m_1-m_2)N_{p,q} = C_p(m_1-m_2)N_{p-1,q}.$$

This completes the proof of Proposition A.1.3. □

A.2. Proof of Theorem 2.2.2.

For representations of the general linear groups, the explicit formulas of matrix elements are given in [**Zh**]. For the purpose of proving our theorem, the argument in [**Zh**, §71] may be summarized as in the following lemma. For a function $\psi : Sp(4, \boldsymbol{C}) \to \boldsymbol{C}$ let $\hat{\psi}$ denote the function defined by $\hat{\psi}(x) = \psi({}^t x)$. For a left invariant vector field X on $Sp(4, \boldsymbol{C})$ we denote by $\hat{X}$ the right invariant vector field such that the tangent vector $(\hat{X})_e$ at the identity element e is equal to the transpose ${}^t(X_e)$ of the tangent vector $X_e \in sp(4, \boldsymbol{C})$. Fix $\boldsymbol{m} = (m_1, m_2)$ as in §2, and recall that $\varphi_{\mu\nu}(x) = (\pi_{\boldsymbol{m}}(x)(\varphi_\nu), \varphi_\mu)$.

Lemma A.2.1.

(i) *For any* $x \in Sp(4, \boldsymbol{C})$, *the adjoint operator* $\pi_{\boldsymbol{m}}(x)^*$ *of* $\pi_{\boldsymbol{m}}(x)$ *satisfies*

$$\pi_{\boldsymbol{m}}(x)^* = \pi_{\boldsymbol{m}}({}^t\bar{x}).$$

(ii) *The identities*

$$\widehat{X(\psi)} = \hat{X}(\hat{\psi}), \quad \widehat{X(Y(\psi))} = \hat{X}(\hat{Y}(\hat{\psi}))$$

hold for any left invariant vector fields X, Y *and any function* ψ *on* $Sp(4, \boldsymbol{C})$.

(iii) *The complex conjugate of the function* $\varphi_{\mu\nu}$ *satisfies*

$$\overline{\varphi_{\mu\nu}(x)} = \varphi_{\mu\nu}(\bar{x}) \quad \text{for } x \in Sp(4, \boldsymbol{C}),$$

and hence $\widehat{\varphi_{\mu\nu}} = \varphi_{\nu\mu}$.

Proof. As noted in [**Zh**, p. 198], using the assumption that the restriction $\pi_{\boldsymbol{m}}|_{SpU(4)}$ is unitary, we observe that the differential representation $d\pi_{\boldsymbol{m}} : sp(4, \boldsymbol{C}) \to gl(\mathfrak{R}_{\boldsymbol{m}})$ satisfies $(d\pi_{\boldsymbol{m}}(X))^* = d\pi_{\boldsymbol{m}}({}^t\overline{X})$. Hence we see that $\pi_{\boldsymbol{m}}(x)^* = \pi_{\boldsymbol{m}}({}^t\bar{x})$ for $x \in Sp(4, \boldsymbol{C})$ which is close to e, and thus for all x. To prove (ii) it suffices to verify the first half. We see that $\hat{X}(\hat{\psi})(x) = (d/ds)|_{s=0}\hat{\psi}(\exp s\,{}^t(X_e)x) = (d/ds)|_{s=0}\psi({}^t x \exp s(X_e)) = (X(\psi))({}^t x) = \widehat{X(\psi)}(x)$. This proves (ii). In order to prove (iii), since the function $\varphi_{\mu\nu} : Sp(4, \boldsymbol{C}) \to \boldsymbol{C}$ is holomorphic, it suffices to verify that the restriction of $\varphi_{\mu\nu}$ to $Sp(4, \boldsymbol{R})$ is real-valued. For this purpose consider the real vector space $\mathfrak{R}_{\boldsymbol{m},0} = \{\psi \in \mathfrak{R}_{\boldsymbol{m}} \mid \psi(x) \in \boldsymbol{R}$ for any $x \in Sp(4, \boldsymbol{R})\}$. It is clear that $\mathfrak{R}_{\boldsymbol{m},0}$ is invariant under $\pi_{\boldsymbol{m}}(x)$ $(x \in Sp(4, \boldsymbol{R}))$, and that $\phi_{\boldsymbol{m}} \in \mathfrak{R}_{\boldsymbol{m},0}$. Moreover $\varphi_\mu \in \mathfrak{R}_{\boldsymbol{m},0}$ for any $\mu \in M_{\boldsymbol{m}}$ (because the operator Ω_μ leaves $\mathfrak{R}_{\boldsymbol{m},0}$ invariant). Since $\{\varphi_\mu\}_{\mu \in M_{\boldsymbol{m}}}$ constitutes a basis of $\mathfrak{R}_{\boldsymbol{m},0}$, we conclude that the value $\varphi_{\mu\nu}(x)$ is real for $x \in Sp(4, \boldsymbol{R})$. This completes the proof of Lemma A.2.1. □

Proof of Theorem 2.2.2. Set $o = (m_2, 0, 0, 0) \in M_{\boldsymbol{m}}$. Then clearly Ω_o is the identity operator, and $\varphi_o = \phi_{\boldsymbol{m}}$.

Assertion 1. $\varphi_{oo} = \phi_{\boldsymbol{m}}$, *that is*, $(\pi_{\boldsymbol{m}}(x)\phi_{\boldsymbol{m}}, \phi_{\boldsymbol{m}}) = \phi_{\boldsymbol{m}}(x)$ *for* $x \in Sp(4, \boldsymbol{C})$.

This is an immediate consequence of the Gauss decomposition of x, (i) of Lemma A.2.1, and the fact that $\pi_{\boldsymbol{m}}(\delta)\phi_{\boldsymbol{m}} = \delta_1^{m_1}\delta_2^{m_2}\phi_{\boldsymbol{m}}$ for any δ as in Appendix A.1 (see [**Zh**, p. 205]).

Assertion 2. $\varphi_{ov} = \varphi_v$.

For $\lambda \in M_{\boldsymbol{m}}$ let N_λ denote the norm of $\varphi_\lambda \in \mathfrak{R}_{\boldsymbol{m}}$, and let $\phi_\lambda = (1/N_\lambda)\varphi_\lambda$ the normalization. Set $\phi_{\mu v}(x) = (\pi_{\boldsymbol{m}}(x)(\phi_v), \phi_\mu)$. Then we have $\phi_{\mu v}(xy) = \sum_\lambda \phi_{\mu\lambda}(x) \cdot \phi_{\lambda v}(y)$, $x, y \in Sp(4, \boldsymbol{C})$. Taking $\mu = v = o$ we get $\phi_{\boldsymbol{m}}(xy) = \sum_\lambda \phi_{o\lambda}(x)\phi_{\lambda o}(y)$. On the other hand we have $\pi_{\boldsymbol{m}}(y)(\phi_{\boldsymbol{m}}) = \sum_\lambda \phi_{\lambda o}(y)\phi_\lambda$, i.e. $\phi_{\boldsymbol{m}}(xy) = \sum_\lambda \phi_{\lambda o}(y) \cdot \phi_\lambda(x)$. Thus the irreducibility of $\pi_{\boldsymbol{m}}$ yields $\phi_{o\lambda}(x) = \phi_\lambda(x)$. This proves Assertion 2.

Assertion 3. $\varphi_{\mu o} = \hat{\Omega}_\mu(\phi_{\boldsymbol{m}})$.

By Assertion 2 we know that $\varphi_{ov}(x) = \varphi_v(x) = \Omega_v(\phi_{\boldsymbol{m}})(x)$. Hence $\widehat{\varphi_{ov}} = \widehat{\Omega_v(\phi_{\boldsymbol{m}})}$. Applying (ii), (iii) of Lemma A.2.1 and recalling the definition of $\hat{\Omega}_v$, we see that $\varphi_{vo} = \hat{\Omega}_v(\phi_{\boldsymbol{m}})$.

Now we can prove Theorem 2.2.2. From the proof of Assertion 2 we have $\phi_{\boldsymbol{m}}(xy) = \sum_\lambda \phi_{o\lambda}(x)\phi_{\lambda o}(y)$, $\phi_{\mu o}(xy) = \sum_\lambda \phi_{\mu\lambda}(x)\phi_{\lambda o}(y)$. Fixing y, and applying the right invariant operator $\hat{\Omega}_\mu$ to the first formula, by Assertions 2, 3 we get $\varphi_{\mu o}(xy) = (1/N_\lambda)\sum_\lambda \hat{\Omega}_\mu\Omega_\lambda(\phi_{\boldsymbol{m}}(x))\phi_{\lambda o}(y)$. Comparing this formula and the second one above, by the irreducibility of $\pi_{\boldsymbol{m}}$ we obtain $(1/N_\mu N_\lambda)\hat{\Omega}_\mu\Omega_\lambda(\phi_{\boldsymbol{m}}(x)) = \phi_{\mu\lambda}(x)$. Returning to the unnormalized functions, we get $\varphi_{\mu\lambda} = \hat{\Omega}_\mu\Omega_\lambda(\phi_{\boldsymbol{m}})$, which is what we wanted. Theorem 2.2.2 is proved. □

A.3. Proofs of Proposition 2.3.1 and Corollary 2.3.2.

Let $\mu = (p_1, p_2, i, j)$ and set $\mu_0 = (p_1, p_2, 0, 0)$. Recalling the definition $\varphi_{\mu_0} = \Omega_{p_1-m_2,p_2}(\phi_{\boldsymbol{m}})$, by Proposition A.1.3 we have

$$\varphi_{\mu_0} = N_{p_1-m_2,p_2} z_{12}^{p_1-m_2} z_{24}^{p_2}\phi_{\boldsymbol{m}}.$$

From this expression we get directly the following formulas for the action of $sp(4, \boldsymbol{C})$ on φ_{μ_0} (special case $i = j = 0$ of Proposition 2.3.1).

Lemma A.3.1. *Let* I, J, A, B, C, D *be as in Proposition* 2.3.1. *Then*

$$e_1(\varphi_{\mu_0}) = I(B\varphi_{\mu_0-\varepsilon_1} + D\varphi_{\mu_0-\varepsilon_2+\varepsilon_4}),$$

$$e_2(\varphi_{\mu_0}) = -IJD\varphi_{\mu_0-\varepsilon_2}, \quad e_3(\varphi_{\mu_0}) = 0, \quad e_4(\varphi_{\mu_0}) = 0,$$

$$f_1(\varphi_{\mu_0}) = J(C\varphi_{\mu_0+\varepsilon_1} + D\varphi_{\mu_0-\varepsilon_2+\varepsilon_3}),$$

$$f_2(\varphi_{\mu_0}) = A\varphi_{\mu_0+\varepsilon_2} + B\varphi_{\mu_0-\varepsilon_1+\varepsilon_3} + C\varphi_{\mu_0+\varepsilon_1+\varepsilon_4} + D\varphi_{\mu_0-\varepsilon_2+\varepsilon_3+\varepsilon_4}.$$

PROOF. The identities

$$e_1(z_{12}) = 1, \quad e_1(z_{24}) = -z_{23}, \quad e_1(\phi_{\boldsymbol{m}}) = 0$$

yield immediately

$$e_1(z_{12}^p z_{24}^q \phi_{\boldsymbol{m}}) = p z_{12}^{p-1} z_{24}^q \phi_{\boldsymbol{m}} - q z_{12}^p z_{24}^{q-1} z_{23} \phi_{\boldsymbol{m}}$$

for nonnegative integers p, q. On the other hand, using the relation $z_{13} = z_{24} + z_{12} z_{23}$ and the identities in the proof of Lemma A.1.6 we get

$$f_4(z_{12}^p z_{24}^{q-1} \phi_{\boldsymbol{m}}) = (m_2 + p - q + 1) z_{12}^p z_{24}^{q-1} z_{23} \phi_{\boldsymbol{m}} + p z_{12}^{p-1} z_{24}^q \phi_{\boldsymbol{m}}.$$

Eliminating the term $z_{12}^p z_{24}^{q-1} z_{23} \phi_{\boldsymbol{m}}$ from these two formulas, we get

$$(m_2 + p - q + 1) e_1(z_{12}^p z_{24}^q \phi_{\boldsymbol{m}}) = p(m_2 + p + 1) z_{12}^{p-1} z_{24}^q \phi_{\boldsymbol{m}} - q f_4(z_{12}^p z_{24}^{q-1} \phi_{\boldsymbol{m}}).$$

Then the identity $N_{p,q} = C_{q+1}(m_1 - m_2 - p + q + 1) N_{p-1,q}$ yields the desired expression for $e_1(\varphi_{\mu_0})$. Next, the expression for $e_2(\varphi_{\mu_0})$ follows immediately from

$$e_2(z_{12}) = 0, \quad e_2(z_{24}) = 1, \quad e_2(\phi_{\boldsymbol{m}}) = 0,$$

$$N_{p,q} = (m_1 - q + 2)(m_2 - q + 1)(m_1 + m_2 - q + 3) C_{p-q}(m_1 - m_2) N_{p,q-1}.$$

The identities $e_3(\varphi_{\mu_0}) = e_4(\varphi_{\mu_0}) = 0$ are obvious by the infinitesimal version of Lemma A.1.1. As for $f_1(\varphi_{\mu_0})$, we use the formula

$$f_1(z_{12}^p z_{24}^q \phi_{\boldsymbol{m}}) = (m_1 - m_2 - p + q) z_{12}^{p+1} z_{24}^q \phi_{\boldsymbol{m}} - q z_{12}^p z_{24}^{q-1} z_{14} \phi_{\boldsymbol{m}}$$

obtained from the identities in the proof of Lemma A.1.6. Moreover, we need the formula for $f_3(z_{12}^p z_{24}^{q-1} \phi_{\boldsymbol{m}})$ obtained from the first formula in Lemma A.1.7. Eliminating the term $z_{12}^p z_{24}^{q-1} z_{14} \phi_{\boldsymbol{m}}$ from these two formulas, we obtain the desired expression for $f_1(\varphi_{\mu_0})$. Finally, to find the expression for $f_2(\varphi_{\mu_0})$ we proceed similarly. We use the formula

$$\begin{aligned} f_2(z_{12}^p z_{24}^q \phi_{\boldsymbol{m}}) = {} & (m_1 + m_2 - (p+q)) z_{12}^p z_{24}^{q+1} \phi_{\boldsymbol{m}} + p z_{12}^{p-1} z_{24}^q z_{14} \phi_{\boldsymbol{m}} \\ & + (m_1 - m_2 + q - p) z_{12}^{p+1} z_{24}^q z_{23} \phi_{\boldsymbol{m}} - q z_{12}^p z_{24}^{q-1} z_{14} z_{23} \phi_{\boldsymbol{m}}. \end{aligned}$$

Moreover, we need

$$\begin{aligned} f_3 f_4(z_{12}^p z_{24}^{q-1} \phi_{\boldsymbol{m}}) = {} & p f_3(z_{12}^{p-1} z_{24}^q \phi_{\boldsymbol{m}}) - p(q - m_2) z_{12}^p z_{24}^{q+1} \phi_{\boldsymbol{m}} \\ & + (q - 1 - m_2) f_4(z_{12}^{p+1} z_{24}^q \phi_{\boldsymbol{m}}) \\ & + (m_1 - p - q + 1)(m_2 + p - q + 1) z_{12}^p z_{24}^{q-1} z_{14} z_{23} \phi_{\boldsymbol{m}} \end{aligned}$$

which is a direct consequence of the expression for $f_4(z_{12}^p z_{24}^{q-1}\phi_{\boldsymbol{m}})$ given above, $f_3(z_{23}) = -z_{24}^2$ and the first formulas in the proof of Lemma A.1.7. Eliminating the term $z_{12}^p z_{24}^{q-1} z_{14} z_{23} \phi_{\boldsymbol{m}}$ from these two formulas, we obtain the desired expression for $f_2(\varphi_{\mu_0})$. Lemma A.3.1 is proved. □

To complete the proof of Proposition 2.3.1, we need the following commutation relations.

LEMMA A.3.2. *Let i, j be nonnegative integers. Then*

$$[e_1, f_3^i f_4^j] = -i f_3^{i-1} f_4^j f_2,$$
$$[e_2, f_3^i f_4^j] = i f_3^{i-1} f_4^j f_1 - ij f_3^{i-1} f_4^{j-1} f_2 + j f_3^i f_4^{j-1} e_1,$$
$$[e_3, f_3^i f_4^j] = i(h_1 + i - 1) f_3^{i-1} f_4^j,$$
$$[e_4, f_3^i f_4^j] = j(h_2 + j - 1) f_3^i f_4^{j-1},$$
$$[f_1, f_3^i f_4^j] = -j f_3^i f_4^{j-1} f_2,$$
$$[f_2, f_3^i f_4^j] = 0.$$

PROOF. Direct consequences of the identities used in the proof of Lemma A.1.4 and

$$[e_1, f_3] = -f_2, \quad [e_1, f_4] = 0, \quad [e_2, f_3] = f_1, \quad [e_2, f_4] = e_1,$$
$$[e_3, f_3] = h_1, \quad [e_3, f_4] = 0, \quad [e_4, f_3] = 0, \quad [e_4, f_4] = h_2,$$
$$[f_1, f_3] = 0, \quad [f_1, f_4] = -f_2, \quad [f_2, f_3] = [f_2, f_4] = [f_3, f_4] = 0.$$ □

PROOF OF PROPOSITION 2.3.1. The expressions for $f_3(\varphi_\mu)$, $f_4(\varphi_\mu)$ are obvious by our definition of φ_μ. As for $h_1(\varphi_\mu), h_2(\varphi_\mu)$, recalling that

$$\varphi_\mu = N_{p_1-m_2, p_2} f_3^i f_4^j (z_{12}^{p_1-m_2} z_{24}^{p_2} \phi_{\boldsymbol{m}}),$$

we see that the desired formulas are already given in Lemma A.1.4. Note that $\varphi_\mu = f_3^i f_4^j(\varphi_{\mu_0})$. Then the others are direct consequences of Lemmas A.3.1 and A.3.2. □

PROOF OF COROLLARY 2.3.2. Immediate consequence of Theorem 2.2.2, Lemma A.2.1 and the fact that the operator $\hat{\Omega}_\mu$ (resp. Ω_ν) commutes with every left (resp. right) invariant vector field. □

References

[BGM] M. Berger, P. Gauduchon and E. Mazet, Le Spectre d'une Variété Riemannienne, Lecture Notes in Math., **194**, Springer-Verlag, 1971.

[CH] R. Courant and D. Hilbert, Methods of mathematical physics, Interscience, 1953.

[Ch] C. Chevalley, Theory of Lie Groups, Princeton Univ. Press, New Jersey, 1946.

[Gu] D. Gurarie, Symmetries and Laplacians, North-Holland Math. Stud., **174**, 1992.

[GM] D. Gromoll and W. Meyer, An exotic sphere with nonnegative sectional curvature, Ann. Math., **100(1)** (1974), 401–406.

[KMS] D. V. Kosygin, A. A. Minasov and Ya. G. Sinai, Statistical properties of the spectra of Laplace-Beltrami operators on Liouville surfaces, Russian Math. Surveys, **48** (1993), 1–142.

[La] V. Lazutkin, KAM Theory and Semiclassical Approximations to Eigenfunctions, Springer-Verlag, 1993.

[UZ] A. Uribe and S. Zelditch, Spectral Statistics on Zoll Surfaces, Comm. Math. Phys., **154** (1993), 313–346.

[W] B. Watson, Manifold maps commuting with the Laplacian, J. Differential Geom., **8** (1973), 85–94.

[Ze] S. Zelditch, Spectrum and Geodesic Flow, In Zeta Functions in Geometry, Adv. Stud. Pure Math., **21** (1992), 1–19.

[Zh] D. P. Zhelobenko, Compact Lie Groups and their representations, Trans. Math. Monogr., **40**, Amer. Math. Soc., 1973.

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