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Diophantine approximations for a constant related to elliptic functions

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Abstract. This paper is devoted to the study of rational approximations of the ratio $\eta(\lambda)/\omega(\lambda)$, where $\omega(\lambda)$ and $\eta(\lambda)$ are the real period and real quasi-period, respectively, of the elliptic curve $y^2 = x(x-1)(x-\lambda)$. Using monodromy principle for hypergeometric function in the logarithm case we obtain rational approximations of $(\eta/\omega)(\lambda)$ with $\lambda \in \boldsymbol{Q}$ and we shall find new measures of irrationality, both in the archimedean and non archimedean case.

1. Results and notations.

The Gauss hypergeometric function is defined by the power series

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| x\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n, \quad (c \neq 0, -1, -2, \dots), \tag{1} $$

where $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$ for $n \geq 1$.

This series is an analytic solution in $|x| < 1$ to the differential equation.

$$ x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0 \tag{2} $$

Let E_λ be the family of elliptic curves

$$ y^2 = x(x-1)(x-\lambda) \tag{3} $$

where $\lambda \in \boldsymbol{C}^*$, $0 < |\lambda| < 1$. Legendre notations for the periods of these curves (i.e. complete integrals of the first kind) and quasi-period (i.e. complete integrals of the second kind) are, respectively

$$ \omega(\lambda) \stackrel{\text{def}}{=} \frac{\pi}{2} {}_2F_1\left(\begin{matrix} 1/2, 1/2 \\ 1 \end{matrix} \middle| \lambda\right) \tag{4} $$

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$$\eta(\lambda) \stackrel{\text{def}}{=} \frac{\pi}{2} {}_2F_1\left(\begin{matrix} -1/2, 1/2 \\ 1 \end{matrix} \middle| \lambda\right) \tag{5}$$

$$G(\lambda) = \eta(\lambda)/\omega(\lambda). \tag{6}$$

There is no loss of generality in studying the diophantine approximations of

$${}_2F_1\left(\begin{matrix} 3/2, 3/2 \\ 2 \end{matrix} \middle| \lambda\right) \Big/ {}_2F_1\left(\begin{matrix} 1/2, 1/2 \\ 1 \end{matrix} \middle| \lambda\right). \tag{7}$$

Indeed, from the contiguous relation, we have

$${}_2F_1\left(\begin{matrix} 3/2, 3/2 \\ 2 \end{matrix} \middle| \lambda\right) \Big/ {}_2F_1\left(\begin{matrix} 1/2, 1/2 \\ 1 \end{matrix} \middle| \lambda\right) = \frac{1}{\lambda(1-\lambda)}(G(\lambda) + \lambda - 1), \tag{8}$$

so we can study equivalently the fraction (7).

We shall denote by $\boldsymbol{Q}_p$ the p-adic completion of $\boldsymbol{Q}$, where $p \in \{\infty, \text{primes } p\}$, in particular $\boldsymbol{Q}_\infty = \boldsymbol{R}$. For an irrational number $\theta \in \boldsymbol{Q}_p$, we shall call the irrationality measure $m_p(\theta)$ of θ the infimum of the m's satisfying the following condition: for any $\varepsilon > 0$ there exists an $H_0 = H_0(\varepsilon)$ such that

$$|\alpha - P/Q|_p > H^{-m-\varepsilon}$$

for any rational P/Q satisfying $H = \max\{|P|, |Q|\} \geq H_0$. From now on we denote $m_\infty(\theta) = m(\theta)$. All our measures are effective in the sense that H_0 can be effectively determined. Throughout this paper, we also suppose that λ is a rational number, p a prime number and $v_p(\lambda) = v$ the p-adic valuation of $\lambda = p^v \cdot \alpha/\beta$, $\alpha \in \boldsymbol{Z}$; $\beta \in \boldsymbol{Z}^+$, $p \nmid \alpha\beta$ and we set $|\lambda|_p = p^{-v}$. We also use the notation

$$u(\alpha) = \begin{cases} 2^{v_2(\alpha)}, & \text{if } v_2(\alpha) < 4 \\ 2^4, & \text{if } v_2(\alpha) \geq 4 \end{cases}$$

In the following theorems let $\alpha/\beta \in \boldsymbol{Q}^*$ be such that $(\alpha, \beta) = 1$ and $\beta \in \boldsymbol{Z}^+$. In the archimedean case we have

THEOREM 1. *Let* $\lambda = \alpha/\beta$, $\lambda \in [-1, 1]$ *satisfy*

$$\frac{\beta}{u(\alpha)}[e(1 - \sqrt{1-\lambda})]^2 < 1.$$

Then $G(\lambda)$ *is irrational and*

$$m_\infty(G(\lambda)) \leq 1 - \frac{\operatorname{Log}[\beta(2e(1 + |\lambda|/4))^2/u(\alpha)]}{\operatorname{Log}[\beta(e(1 - \sqrt{1-\lambda}))^2/u(\alpha)]}.$$

As numerical examples we give the following list

$$m_\infty G(1/3) \le 16{,}884\cdots$$

$$m_\infty G(-1/2) \le 15{,}755\cdots$$

$$m_\infty G(-1/4) \le 8{,}720\cdots$$

$$m_\infty G(-2/3) \le 65{,}631\cdots$$

$$m_\infty G(-2/5) \le 10{,}402\cdots$$

$$m_\infty G(-4/11) \le 9{,}179\cdots$$

Let $K(k)$ and $E(k)$ be the complete integrals of the first and second kind respectively.

Then

$$K(k) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{1}{2}\omega(k^2)$$

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\varphi}\, d\varphi = \frac{1}{2}\eta(k^2).$$

Corollary 1. *Let $k = 1/q$, then $G(k^2) = E(k)/K(k)$ is irrational for every integer $q \ge 2$.*

In the non archimedean case

Theorem 2. *Let p be a prime such that $p \nmid \beta$ and $\lambda = \alpha/\beta$ a rational number satisfying $|\alpha|_p^2 < |16^2 u(\alpha)|_p$ and ${}_2F_1\left(\begin{matrix}1/2, 1/2\\ 1\end{matrix}\middle|\frac{\alpha}{\beta}\right) \neq 0$.*

Then $G(\lambda)$ is irrational and

$$m_p(G(\lambda)) \le 1 - \frac{\operatorname{Log} Q}{\operatorname{Log} QR},$$

if $QR < 1$, where $R = |\alpha^2/(16^2 u(\alpha))|_p$ and

$$Q = \begin{cases} \dfrac{4e^2\beta}{u(\alpha)}\left(1 + \dfrac{|\alpha|}{4\beta}\right)^2; & \text{if } |\alpha/\beta| < 1,\\[2ex] \dfrac{16e^2}{u(\alpha)}|\alpha|; & \text{if } |\alpha/\beta| > 1.\end{cases}$$

In particular

$$m_p(G_p(p^h)) \le \frac{2h\operatorname{Log} p}{h\operatorname{Log} p - 2 - 4\operatorname{Log} 2}$$

for all $p \neq 2$ satisfying $p^h > 16e^2$.

2. Derivation of the approximation form.

Let $n \geq 1$ be an integer. We want to find a remainder function $R_n(x)$ such that $\operatorname{ord}_0 R_n \geq 2n$ and

$$R_n(x) = P_n(x)_2F_1\left(\begin{matrix} a+1, b+1 \\ c+1 \end{matrix} \middle| x\right) - Q_{n-1}(x)_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| x\right) \tag{9}$$

for some polynomials $P_n(x)$ and $Q_{n-1}(x)$ of degree n and $n-1$, respectively.

LEMMA 1. *With the above notations*

$$R_n(x) = C_n x^{2n}{}_2F_1\left(\begin{matrix} a+n+1, b+n+1 \\ c+2n+1 \end{matrix} \middle| x\right)$$

where $C_n \neq 0$ *depends only on* a, b, c *and* n.

PROOF. Let us recall the main ideas of the Riemann's proof. Linear algebra ensures us of the existence of two non zero polynomials $P_n(x)$ and $Q_{n-1}(x)$ of degree at most n and $n-1$ respectively such that the linear form (11) satisfies $\operatorname{ord}_0 R(x) \geq 2n$. The remainder function $R_n(x)$ is an holomorphic solution in a neighborhood of zero of a family of Fuchsian second order linear differential equation (2).

The non apparent singularities of (2) in $P_1(\boldsymbol{C})$ are $0, 1, \infty$. We have to find the roots of indicial equations of (2) (Exponents of (2)). There is no loss of generality in assuming that a, b and c are real.

In "0", lower bounds $\operatorname{mod} \boldsymbol{N}$, for exponents are $(2n, -c)$.

In "1": $(0, c-a-b-1)$.

In "∞": (Using local parameter $t = 1/x$), $(-n+a+1, -n+b+1)$.

The problem's datas and Fuchs's relation show that (2) doesn't have apparent singularities and one can conclude that we obtain the following Riemann's scheme

$$P\left(\begin{matrix} \underline{0} & \underline{\infty} & \underline{1} \\ 2n & -n+a+1 & 0 \\ -c & -n+b+1 & c-a-b-1 \end{matrix} \middle| x\right). \tag{H}$$

The proof above works also when two of the exponents differ by an integer (Logarithm singularities or reducible case). In this case (2) has no accessory parameter [**Hu3**]. We can transform this Riemann's scheme into

$$x^{2n}P\left(\begin{matrix} \underline{0} & \underline{\infty} & \underline{1} \\ 0 & n+a+1 & 0 \\ -c-2n & n+b+1 & c-a-b-1 \end{matrix} \middle| x\right),$$

which shows us without any calculation that

$$R_n(x) = C_n x^{2n} {}_2F_1\left(\begin{matrix} n+a+1, n+b+1 \\ 2n+1+c \end{matrix} \middle| x\right)$$

In the following lemma, we shall compute the monodromy representation of the hypergeometric series in the logarithmic case (i.e. $a+b=c=1$).

If $g(x) = \sum_{k=0}^{\infty} g_k x^k$, then the notation

$$[g(x)]_n = \sum_{k=0}^{n} g_k x^k$$

is used for the truncated part of $g(x)$.

LEMMA 2. *Suppose that $a+b=c=1$ and let us denote*

$$P(x) = \sum_{k=0}^{2n} \frac{(a-n)_k (b-n)_k}{(-2n)_k k!} x^k.$$

Then the polynomials in lemma 1 are given by

$$P_n(x) = \left[(1-x)P(x) {}_2F_1\left(\begin{matrix} a, b \\ 1 \end{matrix} \middle| \right)\right]_{(n)}$$

$$Q_{n-1}(x) = \left[(1-x)P(x) {}_2F_1\left(\begin{matrix} a+1, b+1 \\ 2 \end{matrix} \middle| x\right)\right]_{(n-1)}$$

$$C_n = \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(a+n+1)\Gamma(b+n+1)}{\Gamma(2n+1)\Gamma(2n+2)\Gamma^2(a)\Gamma^2(b)ab} = \frac{(a)_n (b)_n (a)_{n+1} (b)_{n+1}}{(2n)!(2n+1)!ab}$$

PROOF. The proof of this lemma is an interesting application of monodromy method for Fuchsian differential equations and we shall give it in details. A sketch can be found in [**Hu2**]. We use a monodromy's argument to obtain an analytic continuation of the relation (9) in lemma 1 along the following curve: Assume that $x \in \mathbf{C}$, in a neighborhood of zero, then γ is a loop with base point x which encircles the point "1" once counterclockwise, cuts the segment $]0,1[$ and returns to its starting point. Now, we must compute the action of the monodromy matrix M_γ for the following hypergeometric series

$${}_2F_1\left(\begin{matrix} a, b \\ 1 \end{matrix} \middle| x\right); \quad {}_2F_1\left(\begin{matrix} a+1, b+1 \\ 2 \end{matrix} \middle| x\right) \quad \text{and} \quad {}_2F_1\left(\begin{matrix} a+n+1, b+n+1 \\ 2n+2 \end{matrix} \middle| x\right)$$

when the parameters a, b satisfy $a+b=1$ First, we shall solve in this case, the connection problem, i.e, the analytic continuation of a fundamental system of solutions of the hypergeometric equation (1) in a simply connected neighborhood at the origin to another fundamental system at the point "1".

Writing the following connection formula [**A**,**S**, 559–560] (known since Gauss in the case $a = b = 1/2$)

If $x \in \boldsymbol{C}$ is such that $|1 - x| < 1$ and $|\arg(1 - x)| < \pi$, then

$$(10)\qquad {}_2F_1\left(\begin{matrix} a, b \\ 1 \end{matrix}\,\middle|\, x\right) = \frac{1}{\Gamma(a)\Gamma(b)}\left\{\psi_0(1-x) - \mathrm{Log}(1-x)\,{}_2F_1\left(\begin{matrix} a, b \\ 1 \end{matrix}\,\middle|\, 1-x\right)\right\}.$$

Using analytic continuation along γ, we obtain

$${}_2F_1\left(\begin{matrix} a, b \\ 1 \end{matrix}\,\middle|\, x\right) \xrightarrow{\gamma} {}_2F_1\left(\begin{matrix} a, b \\ 1 \end{matrix}\,\middle|\, x\right)^{\gamma} = {}_2F_1\left(\begin{matrix} a, b \\ 1 \end{matrix}\,\middle|\, x\right) - 2i\pi\frac{1}{\Gamma(a)\Gamma(b)}\,{}_2F_1\left(\begin{matrix} a, b \\ 1 \end{matrix}\,\middle|\, 1-x\right)$$

which yields

$$(11)\qquad {}_2F_1\left(\begin{matrix} a, b \\ 1 \end{matrix}\,\middle|\, x\right) \xrightarrow{\gamma} {}_2F_1\left(\begin{matrix} a, b \\ 1 \end{matrix}\,\middle|\, x\right) + 2i\pi/\Gamma(a)^2\Gamma(b)^2\left\{{}_2F_1\left(\begin{matrix} a, b \\ 1 \end{matrix}\,\middle|\, x\right)\mathrm{Log}\,x - \psi_0(x)\right\},$$

where $\psi_0(x)$ denotes an analytic function at zero.

To deal with the analytic continuation along γ for ${}_2F_1\left(\begin{matrix} a+1, b+1 \\ 2 \end{matrix}\,\middle|\, x\right)$ and ${}_2F_1\left(\begin{matrix} a+n+1, b+n+1 \\ 2n+2 \end{matrix}\,\middle|\, x\right)$, we use another connection formula [**A**,**S**, 559–560]. Namely, assuming $m \in N$, $m \geq 1$,

$$(12)\qquad {}_2F_1\left(\begin{matrix} \alpha, \beta \\ \alpha+\beta-m \end{matrix}\,\middle|\, x\right) = \frac{\Gamma(m)\Gamma(\alpha+\beta-m)}{\Gamma(\alpha)\Gamma(\beta)}(1-x)^{-m}A_{m-1}(1-x)$$
$$- (-1)^m\frac{\Gamma(\alpha+\beta-m)}{\Gamma(\alpha-m)\Gamma(\beta-m)m!}$$
$$\times\left\{{}_2F_1\left(\begin{matrix} \alpha, \beta \\ 1+m \end{matrix}\,\middle|\, 1-x\right)\mathrm{Log}(1-x) - \psi_m(1-x)\right\},$$

where

$$A_{m-1}(x) = \sum_{k=0}^{m-1}\frac{(\alpha-m)_k(\beta-m)_k}{(1-m)_k k!}x^k$$

and $\psi_m(x)$ denotes analytic function at zero. Using "monodromy" to transform ${}_2F_1\left(\begin{matrix} a, b \\ 2 \end{matrix}\,\middle|\, x\right)$, we obtain by (12),

$$(13)\qquad {}_2F_1\left(\begin{matrix} a+1, b+1 \\ 2 \end{matrix}\,\middle|\, x\right) \xrightarrow{\gamma} {}_2F_1\left(\begin{matrix} a+1, b+1 \\ 2 \end{matrix}\,\middle|\, x\right) + 2i\pi/\Gamma(a)\Gamma(b)\cdot{}_2F_1\left(\begin{matrix} a+1, b+1 \\ 2 \end{matrix}\,\middle|\, 1-x\right)$$

which yields

$$(14)\quad {}_2F_1\left(\begin{matrix}a+1,b+1\\2\end{matrix}\,\middle|\,x\right) \to {}_2F_1\left(\begin{matrix}a+1,b+1\\2\end{matrix}\,\middle|\,x\right)$$
$$+\,2i\pi/\Gamma(a)^2\Gamma(b)^2\left\{1/abx-\psi_1(x)+{}_2F_1\left(\begin{matrix}a+1,b+1\\2\end{matrix}\,\middle|\,x\right)\operatorname{Log}x\right\}.$$

In the same manner, using (12) we can see that

$$\begin{aligned}&{}_2F_1\left(\begin{matrix}a+n+1,b+n+1\\2n+2\end{matrix}\,\middle|\,x\right)\\&\quad={}_2F_1\left(\begin{matrix}a+n+1,b+n+1\\(a+n+1)+(b+n+1)-1\end{matrix}\,\middle|\,x\right)\xrightarrow{\gamma}{}_2F_1\left(\begin{matrix}a+n+1,b+n+1\\2n+2\end{matrix}\,\middle|\,x\right)\\&\qquad+\frac{2i\pi\Gamma(2n+2)}{\Gamma(a+n)\Gamma(b+n)}\,{}_2F_1\left(\begin{matrix}a+n+1,b+n+1\\2\end{matrix}\,\middle|\,1-x\right).\end{aligned}$$

According to (12), we find

$$\begin{aligned}&{}_2F_1\left(\begin{matrix}a+n+1,b+n+1\\2\end{matrix}\,\middle|\,1-x\right)\\&\quad={}_2F_1\left(\begin{matrix}a+n+1,b+n+1\\(2n+3)-(2n+1)\end{matrix}\,\middle|\,1-x\right)\\&\quad=\frac{\Gamma(2n+1)}{\Gamma(n+1+a)\Gamma(n+1+b)}x^{-(2n+1)}A_{2n}(x)+\left(\frac{1}{\Gamma(a-n)\Gamma(b-n)(2n+1)!}\right)\\&\qquad\times\left\{{}_2F_1\left(\begin{matrix}a+n+1,b+n+1\\2n+2\end{matrix}\,\middle|\,x\right)\operatorname{Log}x-\psi_{2n+1}(x)\right\}\end{aligned}$$

hence

$$\begin{aligned}{}_2F_1&\left(\begin{matrix}a+n+1,b+n+1\\2n+2\end{matrix}\,\middle|\,x\right)\xrightarrow{\gamma}{}_2F_1\left(\begin{matrix}a+n+1,b+n+1\\2n+2\end{matrix}\,\middle|\,x\right)\\&+\frac{2i\pi(2n)!(2n+1)!}{\Gamma(a+n)\Gamma(a+n+1)\Gamma(b+n)\Gamma(b+n+1)}\cdot\frac{A_{2n}(x)}{x^{2n+1}}\\&+\frac{2i\pi}{\Gamma(a+n)\Gamma(a-n)\Gamma(b+n)\Gamma(b-n)}\\&\times\left\{{}_2F_1\left(\begin{matrix}a+n+1,b+n+1\\2n+2\end{matrix}\,\middle|\,x\right)\operatorname{Log}x-\psi_{2n+1}(x)\right\}.\end{aligned}$$

Since the monodromy action leaves the polynomials P_n and Q_{n-1} invariant, we obtain the following system

$$\begin{cases} P_n(x){}_2F_1\left(\begin{matrix} a+1, b+1 \\ 2 \end{matrix}\middle| x\right) - Q_{n-1}(x){}_2F_1\left(\begin{matrix} a, b \\ 1 \end{matrix}\middle| x\right) \\ \qquad = C_n x^{2n}{}_2F_1\left(\begin{matrix} a+n+1, b+n+1 \\ 2n+2 \end{matrix}\middle| x\right) \\ P_n(x)\left\{\psi_1(x) - \dfrac{1}{abx}\right\} + Q_{n-1}(x)\psi_0(x) = C_n'\left\{\dfrac{A_{2n}(x)}{x} + D_n x^{2n}\psi_{2n+1}(x)\right\}, \end{cases}$$

where

$$C_n' = \frac{(2n)!(2n+1)!\Gamma(a)^2\Gamma(b)^2 \cdot C_n}{\Gamma(a+n)\Gamma(b+n)\Gamma(a+n+1)\Gamma(b+n+1)}.$$

The determinant $\Delta(x)$ of this system

$$\Delta(x) = \begin{vmatrix} {}_2F_1\left(\begin{matrix} a+1, b+1 \\ 2 \end{matrix}\middle| x\right) & -{}_2F_1\left(\begin{matrix} a, b \\ 1 \end{matrix}\middle| x\right) \\ \psi_1(x) - \dfrac{1}{abx} & \psi_0(x) \end{vmatrix}$$

can be transformed into

$$\Delta(x) = \frac{1}{ab}\begin{vmatrix} \dfrac{d}{dx}{}_2F_1\left(\begin{matrix} a, b \\ 2 \end{matrix}\middle| x\right) & {}_2F_1\left(\begin{matrix} a, b \\ 1 \end{matrix}\middle| x\right) \\ \dfrac{d}{dx}\phi(x) & \phi(x) \end{vmatrix},$$

where $\phi(x)$ is the logarithmic solution of the differential equation (2). Hence $\Delta(x)$ is a multiple of the Wronskian and satisfies the following differential equation

$$\Delta'(x) = \frac{2x-1}{x(1-x)}\Delta(x).$$

It follows that

$$\Delta(x) = K/x(1-x), \quad \text{where } K \in \boldsymbol{C}^*.$$

Since $\lim_{x\to 0} x(1-x)\Delta(x) = K$, we can deduce that $K = -1/ab$.

Now the system (14) gives

$$(15)\qquad \Delta(x)P_n(x) = C_n'\left\{\frac{A_{2n}(x)}{x} + x^{2n}\psi_{2n+1}(x)\right\}{}_2F_1\left(\begin{matrix} a,b \\ 1 \end{matrix}\middle| x\right) - C_n{}_2F_1\left(\begin{matrix} a+n+1, b+n+1 \\ 2n+2 \end{matrix}\middle| x\right)\psi_0(x),$$

$$(16)\qquad \Delta(x)Q_{n-1}(x) = C_n'\left\{\frac{A_{2n}(x)}{x} + x^{2n}\psi_{2n+1}(x)\right\}{}_2F_1\left(\begin{matrix} a+1,b+1 \\ 2 \end{matrix}\middle| x\right) - C_n x^{2n}{}_2F_1\left(\begin{matrix} a+n+1, b+n+1 \\ 2n+2 \end{matrix}\middle| x\right)\left\{\psi_1(x) - \frac{1}{abx}\right\}.$$

Setting $abC_n' = 1$, we get our claim.

We note that

$$P_n(x){}_2F_1\left(\begin{matrix} a+1,b+1 \\ 2 \end{matrix}\middle| x\right) - Q_{n-1}{}_2F_1\left(\begin{matrix} a,b \\ 1 \end{matrix}\middle| x\right) = C_n x^{2n}{}_2F_1\left(\begin{matrix} a+n+1, b+n+1 \\ 2n+2 \end{matrix}\middle| x\right)$$

is a formal series identity and may be used also in the p-adic fields $\boldsymbol{Q}_p$, when the series converge.

In the following, we shall need the non-vanishing of

$$\Delta_n(x) = \begin{vmatrix} P_n(x) & Q_{n-1}(x) \\ P_{n+1}(x) & Q_n(x) \end{vmatrix}.$$

Lemma 3. *With the above notations, we obtain*

$$\Delta_n(x) = C_n x^{2n}.$$

Proof. Clearly $\Delta_n(x)$ is a polynomial in x of $\deg \Delta_n(x) \le 2n$. We have $\Delta_n(x) = P_{n+1}(x)R_n(x) - P_n(x)R_{n+1}(x)$, thus $\mathrm{ord}_{x=0}\,\Delta_n(x) \ge 2n$ and our lemma follows, by comparing the coefficients.

From now on we set $a = b = 1/2$. □

Lemma 4. *If $|x| < 1$, then*

a) $|R_n(x)| \le u|(1 - \sqrt{1-x})/2|^{2n}$,

b) $|P_n(x)| \le v(1 + |x|/4)^{2n}$,

where u and v denote constants depending on x.

Proof. a) Using the integral representation

$$R_n(x) = C_n \frac{\Gamma(2n+2)}{\Gamma(n+3/2)\Gamma(n+1/2)} x^{2n} \int_0^1 \left(\frac{t(1-t)}{1-tx}\right)^n \frac{t^{1/2}(1-t)^{-1/2}}{(1-tx)^{3/2}}\, dt$$

we find that

$$|R_n(x)| \leq C_n \frac{\Gamma(2n+2)}{\Gamma(n+3/2)\Gamma(n+1/2)} |x|^{2n} \sup\left|\frac{t(1-t)}{1-tx}\right|^n \left|\int_0^1 \frac{t^{1/2}(1-t)^{-1/2}\,dt}{(1-tx)^{3/2}}\right|$$

which yields

$$|R_n(x)| \leq 2\frac{\Gamma(n+1/2)\Gamma(n+3/2)}{\Gamma(2n+1)} |1-\sqrt{1-x}|^{2n} \left|{}_2F_1\left(\begin{matrix}3/2, 3/2\\ 2\end{matrix}\middle| x\right)\right|$$

and by Stirling formula

$$|R_n(x)| \leq C_1(x)\left|\frac{1-\sqrt{1-x}}{2}\right|^{2n}.$$

Using the truncated series of lemma 2, namely

$$P_n(x) = \left[(1-x)P(x)\,{}_2F_1\left(\begin{matrix}1/2, 1/2\\ 1\end{matrix}\middle| x\right)\right]_{(n)}.$$

It is possible to have elementary bounds for $P_n(x)$ and $Q_{n-1}(x)$.

Let us consider

$$P(x) = \sum_{k=0}^{2n} \binom{n-1/2}{k}^2 \Big/ \binom{2n}{k} (-x)^k,$$

we see that we only need the first n terms of this polynomial.

But with the notation

$$|g|(x) = \sum_{k=0}^{\infty} |g_k| x^k.$$

We obtain

$$|P_n(x)| \leq (1+|x|)[|P|(|x|)]_{(n)} \left[{}_2F_1\left(\begin{matrix}1/2, 1/2\\ 1\end{matrix}\middle| |x|\right)\right]_{(n)}.$$

An easy computation shows that for $0 \leq k \leq n$,

$$\left|\binom{n-1/2}{k}\right| \leq \binom{2n}{k}/2^k.$$

We thus get

$$[|P|(|x|)]_{(n)} \leq \sum_{k=0}^{n} \binom{2n}{k} |x/4|^k \leq (1+|x/4|)^{2n}.$$

It follows that

$$|P_n(x)| \leq C(x) \cdot (1 + |x/4|)^{2n} \quad \text{and} \quad |Q_{n-1}(x)| \leq C(x) \cdot (1 + |x/4|)^{2n}.$$

This ends the proof. □

Remark. Finding the asymptotic formula $|P_n(x)| \sim |(1 + \sqrt{1-x})/2|^{2n}$ requires a more delicate analysis. In the following we shall give a sketch of its poof. Recall the formula in the proof of lemma 2

$$ {}_2F_1\left(\begin{matrix} n+3/2, n+3/2 \\ 2n+2 \end{matrix} \middle| x \right) \xrightarrow{\gamma} {}_2F_1\left(\begin{matrix} n+3/2, n+3/2 \\ 2n+2 \end{matrix} \middle| x \right) + 2i\pi \frac{\Gamma(2n+2)}{\Gamma(n+1/2)^2} \cdot {}_2F_1\left(\begin{matrix} n+3/2, n+3/2 \\ 2 \end{matrix} \middle| 1-x \right).$$

Using the proof of lemma 2, we see that the asymptotics for $|P_n(x)|$ comes from the behaviour of

$$\frac{\Gamma(2n+2)}{\Gamma(n+1/2)^2} {}_2F_1\left(\begin{matrix} n+3/2, n+3/2 \\ 2 \end{matrix} \middle| 1-x \right).$$

For this, we need another integral formula for the hypergeometric function [**B**,**E**, 60, formula 2]

$$ {}_2F_1\left(\begin{matrix} n+3/2, n+3/2 \\ 2 \end{matrix} \middle| 1-x \right) = \frac{\Gamma(2) \exp i\pi(n-1/2)}{2\Gamma(1/2+n)\Gamma(1/2-n) \sin \pi(1/2-n)} \int_\gamma \frac{t^{1/2+n}(1-t)^{-n-1/2}}{(1-t(1-x))^{n+3/2}}\, dt,$$

where γ denotes a closed counter-clockwise contour enclosing the $t = 1$ of the integrand, which do not enclose the pole $1/1-x$. (Suitable determinations for $t^{1/2}, (1-t)^{1/2}, (1-t(1-x))^{1/2}$ are choosen).

Now we may apply the saddle-point method as described for example by Dieudonné [**Dieu**, Chapter IX] to estimate the main part of

$$\int_\gamma \frac{t^{1/2+n}\, dt}{(1-t)^{n+1/2}(1-t(1-x))^{n+3/2}}.$$

If we use asymptotic formula, $|P_n(x)| \sim |(1+\sqrt{1+x})/2|^{2n}$, (see also [**Rie**]) then we must do a slight modification of the irrationality measures in the Theorems I and II, but in this case measures are not completely effective.

3. Estimation of the denominator D_n.

Let D_n be the denominator of the polynomials $P_n(x)$ and $Q_{n-1}(x)$ i.e. $D_nP_n(x),\ D_nQ_{n-1}(x) \in Z[x]$.

First we shall determine the denominator of the polynomial

$$P(x) = \sum_{k=0}^{2n} \binom{n-1/2}{k}^2 \Big/ \binom{2n}{k} (-x)^k,$$

using "the divisibility criterion" Lemma 8 of [**He**,**Ma**,**Vä1**].

Therefore we use for $r \in \boldsymbol{Q}$ the notation $p|r$ or $r \equiv 0 \pmod p$, if p is a prime such that $v_p(r) \geq 1$. Furthermore, if $v_p(r) \geq 0$, then there exists a unique $\bar{r} \in \{0, 1, \dots, p-1\}$ satisfying $\bar{r} \equiv r \pmod p$. Let now $r = R/S \in Q$, $(R, S) = 1$, $v_p(r) \geq 0$, $\max_{1 \leq j \leq k}\{|R + (j-1)S|\} < p^2$ and $k = Ap + \bar{k}$, then

(17) $$p \,\Big|\, \binom{r}{k} \quad \text{iff} \quad \bar{r} + 1 \leq \bar{k}.$$

LEMMA 5. *Let $n \geq 2$, $D_n = 4D_0D_1 16^n$, where*

$$D_0 = \mathrm{lcm}(1, 2, \dots, [\sqrt{2n}]) \quad \text{and} \quad D_1 = \prod_{\substack{\bar{n} \leq (p-1)/2 \\ \sqrt{2n} < p \leq 2n}} p,$$

then

$$D_n \binom{n-1/2}{k}^2 \Big/ \binom{2n}{k} \in \boldsymbol{Z}$$

for all $0 \leq k \leq 2n$ and (asymptotically)

$$\log D_n \sim n \log(4e^2).$$

PROOF. If p is a prime such that $\sqrt{2n} < p \leq 2n$ and

$$p \,\Big|\, \binom{2n}{k} \quad \text{then by (17) we get}$$

(18) $$\overline{2n} + 1 \leq \bar{k}.$$

Let us also suppose

(19) $$\bar{n} \geq \frac{p+1}{2}.$$

Then (18) and (19) give

$$2\bar{n} - p + 1 \leq \bar{k},$$

which is equivalent to

$$\bar{n} + \frac{p-1}{2} - p + 1 \leq \frac{\bar{k}}{2}. \tag{20}$$

Using again (19) we get from (20)

$$\overline{n - 1/2} + 1 \leq \bar{k}. \tag{21}$$

So by (17)

$$p \mid \binom{n-1/2}{k}.$$

Thus the primes $p \in (\sqrt{2n}, 2n]$ in the denominator of

$$\binom{n-1/2}{k}^2 \bigg/ \binom{2n}{k}$$

which do not divide the numerator satisfy

$$\bar{n} \leq \frac{p-1}{2}, \tag{22}$$

and therefore we need the factor D_1.

If we set $\bar{n} = n - Np$, then the primes

$$p \in I_N = \left(\frac{2n+1}{2N+1}, \frac{n}{N}\right)$$

satisfy (22) for all $N \in \boldsymbol{Z}^+$. For D_1 these intervals imply the asymptotic

$$\begin{aligned}
\exp\left(\sum_{\substack{\bar{n} \leq (p-1)/2 \\ \sqrt{2n} < p \leq 2n}} \log p\right) &\sim \exp \sum_{N=1}^{+\infty} \left(\sum_{p \in I_N} \log p\right) \\
&= \exp n \left\{\sum_{N=0}^{\infty} \frac{1}{N+1} - \frac{1}{N+3/2}\right\} \\
&= \exp n\{\psi(3/2) - \psi(1)\} = \exp(2 - 2\log 2)n,
\end{aligned}$$

where ψ denotes the digamma function [**A**,**S**].

It is known [**Beu**,**Ma**,**Vä**] that

$$\operatorname{lcm}\left[\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}\right] \quad \text{divides} \quad \operatorname{lcm}[1, 2, \ldots, n]$$

and so we need the factor D_0. Analogously to the works [**A**,**R**] and [**He**,**Ma**,**Vä1**], we get an asymptotic relation

$$\log(D_0D_1) \sim n\log(e^2/4).$$

Hence,

$$\log D_n \sim n\log(4e^2).$$

Set now

$$p_n(x) = D_nP_n(x) = \sum_{i=0}^{n} p_{n,i}x^i,$$

$$q_n(x) = D_nQ_{n-1}(x) = \sum_{i=0}^{n-1} q_{n,i}x^i,$$

$$r_n(x) = D_nR_n(x) = x^{2n}\sum_{i=0}^{\infty} r_{n,2n+i}x^i.$$

We shall show that $p_n(x), q_n(x) \in \boldsymbol{Z}[x]$.
Let us set moreover

$$F(x) = {}_2F_1\left(\begin{matrix}1/2, 1/2\\ 1\end{matrix}\,\middle|\, x\right) = \sum_{i=0}^{\infty} f_ix^i,$$

$$\tilde{F}(x) = {}_2F_1\left(\begin{matrix}3/2, 3/2\\ 2\end{matrix}\,\middle|\, x\right) = \sum_{i=0}^{\infty} \tilde{f}_ix^i,$$

where

$$f_i = \frac{1}{16^i}\binom{2i}{i}^2, \quad \tilde{f}_i = \frac{4(i+1)}{16^{i+1}}\binom{2i+2}{i+1}^2,$$

and

$$P(x) = \sum_{k=0}^{2n} P_{n,k}x^k, \quad P_{n,k} = (-1)^k\frac{\binom{n-1/2}{k}^2}{\binom{2n}{k}}.$$

Consequently

$$P_n(x) = \left[(1-x)\sum_{k=0}^{n}\left(\sum_{i+j=k} f_iP_{n,j}\right)x^k\right]_{(n)},$$

$$Q_{n-1}(x) = \left[(1-x)\sum_{k=0}^{n-1}\left(\sum_{i+j=k} \tilde{f}_iP_{n,j}\right)x^k\right]_{(n-1)}.$$

The coefficients of these polynomials have a common denominator D_n and more precisely we have

$$|D_n f_i P_{n,j}| = 4 \cdot 16^{n-i} D_0 D_1 \binom{2i}{i}^2 \binom{n-1/2}{j}^2 \Big/ \binom{2n}{j} \in 4 \cdot 16^{n-k} \boldsymbol{Z}$$

for all $i + j = k \le n$ and

$$|D_n \tilde{f}_i P_{n,j}| = 16^{n-i}(i+1) D_0 D_1 \binom{2i+2}{i+1}^2 \binom{n-1/2}{j}^2 \Big/ \binom{2n}{j} \in 16^{n-k} \boldsymbol{Z}$$

for all $i + j = k \le n - 1$. Thus

$$p_n(x),\ q_n(x) \in \boldsymbol{Z}[x].$$

and when D_n is replaced by

$$D_n^* = 4 \cdot 16^n D_0 D_1 / u(\alpha)^n$$

we get

$$p_n = D_n^* \beta^n P_n(\alpha/\beta) \in \boldsymbol{Z}, \quad q_n = D_n^* \beta^n Q(\alpha/\beta) \in \boldsymbol{Z}.$$

4. Proofs.

1) Proof of the archimedean case (Theorem 1).
Lemma 3 gives the nonvanishing property of the determinant

$$p_n q_{n+1} - p_{n+1} q_{n-1} \neq 0$$

and from Lemmas 4 and 5 we get bounds for q_n, D_n and the remainder term r_n. We apply Lemma 3 of [**A**,**R**] to deduce the irrationality measure given in Theorem 1.

2) Proof of the non archimedean case (Theorem 2).
By Lemma 1 we have the following formula for the remainder

$$r_n(x) = p_n(x)\tilde{F}(x) - q_n(x)F(x).$$

Using the fact that $p_{n,i},\ q_{n,i} \in \boldsymbol{Z}$, it is possible to estimate the remainder term when $|x|_p < 1$. In the p-adic case we need polynomial bounds also when $|x| \ge 1$.

Lemma 6. *Let* $R(p) = |x/16|_p^2 < 1$, *then*

$$|r_n(x)|_p < |4|_p^{-1} R(p)^n.$$

Let $|x| \ge 1$, *then*

$$\max\{|p_n(x)|, |q_n(x)|\} \le (16e^2)^{n+\varepsilon} |x|^n.$$

PROOF. For the coefficients of $r_n(x)$ we have the upper bound

$$|r_{n,2n+k}|_p \le \max_{i+j=2n+k} \left\{ \left| p_{n,i} \frac{\binom{2j}{j}^2}{16^j} - q_{n,i} \frac{4(j+1)}{16^{j+1}} \binom{2j+2}{j+1}^2 \right|_p \right\}$$

$$\le |4|_p^{-1} |16|_p^{-(2n+k)}$$

and using the assumption $|x|_p < |16|_p$ we get

$$|r_n(x)|_p \le |x|_p^{2n} \max_{k \ge 0} \{ |r_{n,2n+k}|_p |x|_p^k \}$$

$$\le |x|_p^{2n} |4|_p^{-1} |16|_p^{-2n} = |4|_p^{-1} R(p)^n.$$

The coefficients of the polynomials can be estimated elementary as follows

$$|D_n f_i P_{n,j}| \le 16^{n-i} D_0 D_1 \binom{2i}{i}^2 \binom{n-1/2}{j}^2 \Big/ \binom{2n}{j}$$

$$\le 16^{n-i} e^{(\sqrt{2n}+\varepsilon_1)} (e^2/4)^{n+\varepsilon_2} 2^{4i} 2^{2n} \le (16e^{2+\varepsilon_3})^n, \quad (i+j=k \le n),$$

which gives

$$|p_n(x)| \le 2n^2 (16e^{2+\varepsilon_3})^n |x|^n.$$

Also $|q_n(x)|$ has a similar bound and we achieve

$$\max\{|p_n(x)|, |q_n(x)|\} \le (16e^2)^{n+\varepsilon} |x|^n$$

where $\varepsilon = \varepsilon(n)$, $\lim_{n\to\infty} \varepsilon(n) = 0$.

To prove Theorem 2 we shall apply Lemma 7 of [**He**,**Ma**,**Vä1**], which gives for the p-adic number an irrationality measure

$$m_p(\theta) \le 1 - \frac{\log Q}{\log QR}, \quad QR < 1,$$

when $\max\{|p_n|, |q_n|\} \le Q^{n+\varepsilon}$, $|r_n|_p \le R^{n-\varepsilon}$ and $\Delta_n = p_n q_{n+1} - q_n p_{n+1} \ne 0$ in the approximation formulae $q_n\theta - p_n = r_n$, $p_n, q_n \in \mathbf{Z}$ $(n = 1, 2, \ldots)$. Let now $x = \alpha/\beta \in Q^*$, $p \nmid \beta \in \mathbf{Z}^+$, $|\alpha|_p^2 < |16^2 u(\alpha)|_p$ and in this case from the lemmas 4, 5 and 6 we shall get

$$|p_n| = |D_n\beta^n P_n(\alpha/\beta)/u(\alpha)^n|$$
$$\leq (4e^2)^{n+\varepsilon}(\beta/u(\alpha))^n\left(1+\frac{|\alpha|}{4\beta}\right)^{2n},$$

if $|\alpha/\beta| \leq 1$ and if $|\alpha/\beta| > 1$ $|p_n| \leq (4e^2)^{n+\varepsilon}(\beta/u(\alpha))^n(4|\alpha|/\beta)^n$. These bounds are valid also for $|q_n|$.

Further

$$|r_n|_p = |D_n\beta^n R_n(\alpha/\beta)/u(\alpha)^n|_p$$
$$\leq \frac{1}{|4|_p}|\alpha^2/16^2u(\alpha)|_p^n.$$

Hence we may take Q and R given in theorem 2.

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