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On J-orders of elements of $KO(\boldsymbol{C}P^m)$

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Abstract. Let $KO(\boldsymbol{C}P^m)$ be the KO-ring of the complex projective space $\boldsymbol{C}P^m$. By means of methods of rational D-series [**4**], a formula for the J-orders of elements of $KO(\boldsymbol{C}P^m)$ is given. Explicit formulas are given for computing the J-orders of the canonical generators of $KO(\boldsymbol{C}P^m)$ and the J-order of any complex line bundle over $\boldsymbol{C}P^m$.

1. Introduction.

Let X be a connected finite-dimensional CW complex. Let $KO(X)$ be the KO-ring of X and $\tilde{K}O(X)$ (resp. $\tilde{K}SO(X)$) be the subgroup of $KO(X)$ of elements (resp. orientable elements) of virtual dimension zero. For a real vector bundle E over X, let $S(E)$ be the sphere bundle associated to E with respect to some inner product on E. Let $JO(X) = KO(X)/TO(X)$ be the J-group of X, where $TO(X) = \{E - F \in \tilde{K}O(X) : S(E \oplus n)$ is fibre homotopy equivalent to $S(F \oplus n)$ for some $n \in \boldsymbol{N}\}$. Then by Adams [**1**] and Quillen [**9**], it is shown that

$$TO(X) = \Big\{x \in \tilde{K}SO(X) : \text{there exists } u \in \tilde{K}SO(X) \text{ such that } \theta_k(x) = \frac{\psi^k(1+u)}{1+u} \text{ in } 1 + \tilde{K}SO(X) \otimes \boldsymbol{Q}_k \text{ for all } k \in \boldsymbol{N}\Big\} \tag{1}$$

where $\theta_k : \tilde{K}SO(X) \to 1 + \tilde{K}SO(X) \otimes \boldsymbol{Q}_k$ is the Bott exponential map, ψ^k is the Adams operation, $\boldsymbol{Q}_k = \{n/k^m : n, m \in \boldsymbol{Z}\}$, and $1 + \tilde{K}SO(X) \otimes \boldsymbol{Q}_k$ is the multiplicative group of elements $1 + w$ with $w \in \tilde{K}SO(X) \otimes \boldsymbol{Q}_k$.

Now, X is connected implies that $KO(X) = \tilde{K}O(X) \oplus \boldsymbol{Z}$. So, $JO(X) = \tilde{J}O(X) \oplus \boldsymbol{Z}$ where $\tilde{J}O(X) = \tilde{K}O(X)/TO(X)$. For $x \in KO(X)$, the J-order of x is the order of $x + TO(X)$ in $JO(X)$. By Atiyah [**3**], $\tilde{J}O(X)$ is a finite group. Hence, $x \in KO(X)$ has a finite J-order if and only if $x \in \tilde{K}O(X)$. Let $y_m = r\xi_m(\boldsymbol{C}) - 2$ where $\xi_m(\boldsymbol{C})$ is the complex Hopf line bundle over the complex projective space $\boldsymbol{C}P^m$. Every element of $\tilde{K}O(\boldsymbol{C}P^m)$ has the form

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$P_m(y_m; m_1, \dots, m_s) = m_1 y_m + \cdots + m_s y_m^s$ for some $m_i \in \boldsymbol{Z}$ and $s \in \boldsymbol{N}$. The purpose of this paper is to compute the J-order of $P_m(y_m; m_1, \dots, m_s)$ which we denote by $b_m(P_m(y_m; m_1, \dots, m_s))$. In addition to their self importance, these orders are of great importance in geometric topology, for instance, it is well-known that the Stiefel fibration $U(n)/U(n-m-1) \to S^{2n-1}$ has a cross-section if and only if n is a multiple of $b_m(P_m(y_m; 1, 0, \dots, 0))$, computed by Adams-Walker [**2**]. Dibağ [**5**] has used (1) to give another proof of Lam's results [**6**], we rediscover his proof as a special case of Example 2 below.

In section 2, we first obtain a useful formula for $\theta_p(y_m^n)$ for $n = 1, \dots, s$. Then we use (1) and some facts of rational D-series [**4**] to give, in Theorem 2.5, a formula for $b_m(P_m(y_m; m_1, \dots, m_s))$. The formula given in Theorem 2.5 involes many difficulties and one can obtain a little information about the range of the J-orders of elements of $\tilde{K}O(\boldsymbol{C}P^m)$. So instead, we use a well-known computations of the J-order of y_m to obtain, in Theorem 2.6, upper and lower bounds for $b_m(P_m(y_m; m_1, \dots, m_s))$.

There are two important examples in mind, namely the J-orders of the canonical generators of $\tilde{K}O(\boldsymbol{C}P^m)$ and the J-orders of complex line bundles over $\boldsymbol{C}P^m$. In section 3 we first obtain an explicit formula for the p-component of the J-order of y_m^k for $k = 2, 3$, or 4. For $k \geq 5$, we show that the formula is true if $p = 2, 3$ or $p > k$. Then, we give an explicit formula for the J-order of any complex line bundle over $\boldsymbol{C}P^m$.

2. The J-order of $P_m(y_m; m_1, \dots, m_s) \in \tilde{K}O(\boldsymbol{C}P^m)$.

Let L be a non-trivial complex line bundle over $\boldsymbol{C}P^m$. Then $L \cong \xi_m(\boldsymbol{C})^n$, or $L \cong \overline{\xi_m(\boldsymbol{C})^n}$ for some $n \in \boldsymbol{N}$, where $\overline{\xi_m(\boldsymbol{C})}$ denotes the conjugate bundle to $\xi_m(\boldsymbol{C})$. Let $KU(\boldsymbol{C}P^m)$ be the KU-ring of $\boldsymbol{C}P^m$. In Lemma 2.1, we find the image of $\xi_m(\boldsymbol{C})^n$ under the realification homomorphism

$$r : KU(\boldsymbol{C}P^m) \to KO(\boldsymbol{C}P^m).$$

Note that $r(\overline{\xi_m(\boldsymbol{C})^n}) = r(\xi_m(\boldsymbol{C})^n)$.

Recall that from [**2**], $KO(\boldsymbol{C}P^m)$ is a truncated polynomial ring over the integers generated by y_m with the following relations:

$$y_m^{t+1} = 0 \quad \text{if } m = 2t;$$

$$2y_m^{2t+1} = 0, \quad y_m^{2t+2} = 0 \quad \text{if } m = 4t+1;$$

$$y_m^{2t+2} = 0 \quad \text{if } m = 4t+3.$$

For each $m \in \boldsymbol{N}$, let

$$d_m = \begin{cases} t & \text{if } m = 2t \\ 2t+1 & \text{if } m = 4t+1 \text{ or } m = 4t+3. \end{cases}$$

For each $r, s \in \boldsymbol{N}$ with $r \geq s$, let $b_{r,s}$ be the coefficient of $(\xi_m(\boldsymbol{C})^s + \overline{\xi_m(\boldsymbol{C})^s})$ in $(\xi_m(\boldsymbol{C}) + \overline{\xi_m(\boldsymbol{C})} - 2)^r$. Using the fact that $\xi_m(\boldsymbol{C})\overline{\xi_m(\boldsymbol{C})} = 1$, we easily see that:

$$b_{r,s} = (-1)^{r-s}\binom{2r}{r-s}. \tag{2}$$

Further, $b_{r,0}$ is the constant term of $(\xi_m(\boldsymbol{C}) + \overline{\xi_m(\boldsymbol{C})} - 2)^r$.

Let $n \in \boldsymbol{N}$. For each $s = 1, \dots, n$, define $d_{n,s}$ by the recurrence relation $d_{n,n} = 1$ and for $s = n-1, n-2, \dots, 1$

$$d_{n,s} = -(d_{n,s+1}b_{s+1,s} + d_{n,s+2}b_{s+2,s} + \cdots + d_{n,n}b_{n,s}). \tag{3}$$

Convention. Let $f \in \boldsymbol{Z}[\![y_m]\!]$, where $\boldsymbol{Z}[\![y_m]\!]$ is the ring of formal power series with coefficients in $\boldsymbol{Z}$. If we consider f as an element of $KO(\boldsymbol{C}P^m)$, then we implicitly mean that $f (\bmod\, y_m^{d_m+1})$.

Lemma 2.1. *Let* $n, m \in \boldsymbol{N}$. *Then*

(i) $r(\xi_m(\boldsymbol{C})^n) = d_{n,1}y_m + d_{n,2}y_m^2 + \cdots + d_{n,n}y_m^n + 2.$

(ii) $\psi^n(y_m) = d_{n,1}y_m + d_{n,2}y_m^2 + \cdots + d_{n,n}y_m^n$, *and for* $k = 2, \dots, d_m$,
$\psi^n(y_m^k) = d_{kn,1}y_m + \cdots + d_{kn,kn}y_m^{kn} - d_{k,1}\psi^n(y_m) - \cdots - d_{k,k-1}\psi^n(y_m^{k-1})$.

(iii) $d_{n,1} = n^2$.

Proof. (i) Let $c : KO(\boldsymbol{C}P^m) \to KU(\boldsymbol{C}P^m)$ be the complexification homomorphism. $cr(\xi_m(\boldsymbol{C})^n) = \xi_m(\boldsymbol{C})^n + \overline{\xi_m(\boldsymbol{C})}^n$. On the other hand, by (2) and (3), we have

$$c(d_{n,1}y_m + \cdots + d_{n,n}y_m^n + 2) = \xi_m(\boldsymbol{C})^n + \overline{\xi_m(\boldsymbol{C})}^n.$$

Using the fact that c is a monomorphism for $m = 2t$ and $m = 4t+3$, we get

$$r(\xi_m(\boldsymbol{C})^n) = d_{n,1}y_m + \cdots + d_{n,n}y_m^n + 2.$$

To prove the case $m = 4t+1$, let $i : \boldsymbol{C}P^{4t+1} \to \boldsymbol{C}P^{4t+2}$ be the inclusion map. Then $i^* : KO(\boldsymbol{C}P^{4t+2}) \to KO(\boldsymbol{C}P^{4t+1})$ is an epimorphism and maps $r(\xi_{4t+2}(\boldsymbol{C})^n)$ to $r(\xi_{4t+1}(\boldsymbol{C})^n)$. Hence,

$$\begin{aligned} r(\xi_{4t+1}(\boldsymbol{C})^n) &= i^*(r(\xi_{4t+2}(\boldsymbol{C})^n)) = i^*(d_{n,1}y_{4t+2} + \cdots + d_{n,n}y_{4t+2}^n + 2) \\ &= d_{n,1}y_{4t+1} + d_{n,2}y_{4t+1}^2 + \cdots + d_{n,n}y_{4t+1}^n + 2. \end{aligned}$$

(ii) Let $l \in \{1, \dots, d_m\}$. By (i),

$$\begin{aligned} d_{ln,1}y_m + \cdots + d_{ln,ln}y_m^{ln} &= r(\xi_m(\boldsymbol{C})^{ln}) - 2 = r\psi^n(\xi_m(\boldsymbol{C})^l) - 2 \\ &= \psi^n(r\xi_m(\boldsymbol{C})^l) - 2 = \psi^n(d_{l,1}y_m + \cdots + d_{l,l}y_m^l). \end{aligned}$$

The result follows.

(iii) $d_{n,1}$ is the constant term of

$$\frac{cr(\xi_m(\boldsymbol{C})^n) - 2}{\xi_m(\boldsymbol{C}) + \overline{\xi_m(\boldsymbol{C})} - 2} = \frac{\xi_m(\boldsymbol{C})^n + \overline{\xi_m(\boldsymbol{C})}^n - 2}{\xi_m(\boldsymbol{C}) + \overline{\xi_m(\boldsymbol{C})} - 2}$$
$$= (\xi_m(\boldsymbol{C})^{n-1} + \xi_m(\boldsymbol{C})^{n-2} + \cdots + \xi_m(\boldsymbol{C}) + 1)$$
$$\times (\overline{\xi_m(\boldsymbol{C})}^{n-1} + \overline{\xi_m(\boldsymbol{C})}^{n-2} + \cdots + \overline{\xi_m(\boldsymbol{C})} + 1).$$

Hence,

$$d_{n,1} = n + 2(n-1) + 2(n-2) + \cdots + 2$$
$$= n + 2((n-1) + (n-2) + \cdots + 1) = n + 2\left(\frac{n(n-1)}{2}\right) = n^2.$$

This completes the proof of Lemma 2.1. □

Now, we use the above lemma to find $\theta_p(y_m^n) \in 1 + \tilde{K}O(\boldsymbol{C}P^m) \otimes \boldsymbol{Q}_p$. For each $n, m, p \in \boldsymbol{N}$ with $n \leq d_m$, let

$$A(p;n,m) = \left(\frac{\psi^p(d_{n,1} + \cdots + d_{n,n}y_m^{n-1})}{d_{n,1} + \cdots + d_{n,n}y_m^{n-1}}\right)^{1/2} \in 1 + \tilde{K}O(\boldsymbol{C}P^m) \otimes \boldsymbol{Q}_p,$$

and for $n \geq 2$, let

$$B(p;n,m) = (\theta_p(y_m)^{d_{n,1}-1}\theta_p(y_m^2)^{d_{n,2}} \cdots \theta_p(y_m^{n-1})^{d_{n,n-1}})^{-1} \in 1 + \tilde{K}O(\boldsymbol{C}P^m) \otimes \boldsymbol{Q}_p.$$

Theorem 2.2. *Let $p \geq 2$ and $m = 2t$ for some $t \geq 1$. Then*

(i) $\theta_p(y_m) = \left(\dfrac{\psi^p(y_m)}{p^2 y_m}\right)^{1/2}$.

(ii) $\theta_p(y_m^n) = A(p;n,m)B(p;n,m)$, *for each* $2 \leq n \leq t$.

Proof. (i) This is Lemma 5.4 of [7].

(ii) If η is a complex $4n$-dimensional vector bundle over a finite CW complex X such that $\bigwedge^{4n} \eta = 1$, then $c\theta_p(r\eta) = \theta_p(\eta)$. Let $\eta = 2\xi_m(\boldsymbol{C})^n + 2\overline{\xi_m(\boldsymbol{C})}^n$. Then η is a 4-dimensional complex vector bundle over $\boldsymbol{C}P^m$ with

$$\bigwedge^4 (\eta) = \bigwedge^2 (2\xi_m(\boldsymbol{C})^n) \bigwedge^2 (2\overline{\xi_m(\boldsymbol{C})}^n) = 1.$$

Hence, $c\theta_p(r\eta) = \theta_p(\eta)$. By Lemma 2.1,

$$\xi_m(\boldsymbol{C})^n + \overline{\xi_m(\boldsymbol{C})}^n - 2 = cr(\xi_m(\boldsymbol{C})^n) - 2 = c(d_{n,1}y_m + \cdots + d_{n,n}y_m^n).$$

Also,

$$r\eta = 2r(cr(\xi_m(\boldsymbol{C})^n)) = 2rc(r(\xi_m(\boldsymbol{C})^n))$$
$$= 4r(\xi_m(\boldsymbol{C})^n) = 4d_{n,1}y_m + \cdots + 4d_{n,n}y_m^n + 8.$$

Thus,

$$c\theta_p(r\eta) = c((p\theta_p(y_m)^{d_{n,1}}\theta_p(y_m^2)^{d_{n,2}}\cdots\theta_p(y_m^{n-1})^{d_{n,n-1}}\theta_p(y_m^n))^4).$$

On the other hand,

$$\theta_p(\eta) = \left(\frac{\xi_m(\boldsymbol{C})^{np} + \overline{\xi_m(\boldsymbol{C})}^{np} - 2}{\xi_m(\boldsymbol{C})^n + \overline{\xi_m(\boldsymbol{C})}^n - 2}\right)^2 = \left(\frac{\psi^p(\xi_m(\boldsymbol{C})^n + \overline{\xi_m(\boldsymbol{C})}^n - 2)}{\xi_m(\boldsymbol{C})^n + \overline{\xi_m(\boldsymbol{C})}^n - 2}\right)^2$$

$$= \left(\frac{c(\psi^p(y_m))c(\psi^p(d_{n,1} + d_{n,2}y_m + \cdots + d_{n,n}y_m^{n-1}))}{c(y_m)c(d_{n,1} + d_{n,2}y_m + \cdots + d_{n,n}y_m^{n-1})}\right)^2.$$

By (i),

$$c\left(\frac{\psi^p(y_m)}{y_m}\right) = c(p^2\theta_p(y_m)^2).$$

Hence,

$$\theta_p(\eta) = c\left((p\theta_p(y_m))^4\left(\frac{\psi^p(d_{n,1} + d_{n,2}y_m + \cdots + d_{n,n}y_m^{n-1})}{d_{n,1} + d_{n,2}y_m + \cdots + d_{n,n}y_m^{n-1}}\right)^2\right).$$

Now, c is a monomorphism implies that

$$p\theta_p(y_m)^{d_{n,1}}\theta_p(y_m^2)^{d_{n,2}}\cdots\theta_p(y_m^{n-1})^{d_{n,n-1}}\theta_p(y_m^n)$$

$$= p\theta_p(y_m)\left(\frac{\psi^p(d_{n,1} + d_{n,2}y_m + \cdots + d_{n,n}y_m^{n-1})}{d_{n,1} + d_{n,2}y_m + \cdots + d_{n,n}y_m^{n-1}}\right)^{1/2}.$$

Hence,

$$\theta_p(y_m^n)B(p;n,m)^{-1} = A(p;n,m).$$

This completes the proof of Theorem 2.2. □

Remark. By Using a method similar to that used in proving Lemma 2.1 when $m = 4t+1$, we easily obtain a similar formula for $\theta_p(y_m^n)$ when m is an odd integer.

Corollary 2.3. *$\theta_p \circ \psi^n = \psi^n \circ \theta_p$ on $\tilde{K}O(\boldsymbol{C}P^m)$ for all $m, n, p \in \boldsymbol{N}$.*

Proof. Clearly, we only need to show that $\theta_p \circ \psi^n(y_m^k) = \psi^n \circ \theta_p(y_m^k)$ for each $k = 1, \ldots, d_m$. By induction on k, if $k = 1$ then by Theorem 2.2 and Lemma 2.1 (ii),

$$\theta_p(\psi^n(y_m)) = \theta_p(d_{n,1}y_m + \cdots + d_{n,n}y_m^n)$$

$$= \left(\frac{\psi^p(d_{n,1} + d_{n,2}y_m + \cdots + d_{n,n}y_m^{n-1})}{d_{n,1} + d_{n,2}y_m + \cdots + d_{n,n}y_m^{n-1}}\right)^{1/2}\theta_p(y_m)$$

$$= \left(\frac{\psi^p(\psi^n(y_m))}{p^2\psi^n(y_m)}\right)^{1/2} = \psi^n\left(\frac{\psi^p(y_m)}{p^2y_m}\right)^{1/2} = \psi^n(\theta_p(y_m)).$$

Now, suppose $\theta_p(\psi^n(y_m^l)) = \psi^n(\theta_p(y_m^l))$ for $l = 1, \ldots, k-1$. Then

$$\begin{aligned}\theta_p(\psi^n(y_m^k)) &= \theta_p(d_{kn,1}y_m + \cdots + d_{kn,kn}y_m^{kn} - d_{k,1}\psi^n(y_m) - \cdots - d_{k,k-1}\psi^n(y_m^{k-1}))\\ &= \theta_p(y_m)^{d_{kn,1}} \cdots \theta_p(y_m^{kn})^{d_{kn,kn}}\theta_p(\psi^n(y_m))^{-d_{k,1}} \cdots \theta_p(\psi^n(y_m^{k-1}))^{-d_{k,k-1}}.\end{aligned}$$

On the other hand, $\psi^n(\theta_p(y_m^k)) = \psi^n(A(p;k,m)B(p;k,m))$. So, we only need to show that

$$\theta_p(y_m)^{d_{kn,1}} \cdots \theta_p(y_m^{kn})^{d_{kn,kn}} = \psi^n(A(p;k,m)\theta_p(y_m)).$$

$$\begin{aligned}&\theta_p(y_m)^{d_{kn,1}} \cdots \theta_p(y_m^{kn})^{d_{kn,kn}}\\ &\quad = \left(\frac{\psi^p(d_{kn,1} + d_{kn,2}y_m + \cdots + d_{kn,kn}y_m^{kn-1})}{d_{kn,1} + d_{kn,2}y_m + \cdots + d_{kn,kn}y_m^{kn-1}}\right)^{1/2}\theta_p(y_m)\\ &\quad = \left(\frac{\psi^p(d_{kn,1}y_m + d_{kn,2}y_m^2 + \cdots + d_{kn,kn}y_m^{kn})}{p^2(d_{kn,1}y_m + d_{kn,2}y_m^2 + \cdots + d_{kn,kn}y_m^{kn})}\right)^{1/2} = \left(\frac{\psi^p(\psi^{kn}(y_m))}{p^2\psi^{kn}(y_m)}\right)^{1/2}\\ &\quad = \left(\psi^n\left(\frac{\psi^p(\psi^k(y_m))}{p^2\psi^k(y_m)}\right)\right)^{1/2} = \psi^n(A(p;k,m)\theta_p(y_m)).\end{aligned}$$

The result follows. □

Let $s = d_m$ and $P_m(y_m; m_1, \ldots, m_s) = m_1 y_m + \cdots + m_s y_m^s \in \tilde{K}O(\boldsymbol{CP}^m)$. Let $b_m(P_m(y_m; m_1, \ldots, m_s))$ be the J-order of $P_m(y_m; m_1, \ldots, m_s)$, that is the order of $P_m(y_m; m_1, \ldots, m_s) + TO(\boldsymbol{CP}^m)$ in $\tilde{J}O(\boldsymbol{CP}^m)$. By using the same method of Proposition 5.7 of [**7**], we easily see that

$$b_{4t+3}(P_{4t+3}(y_{4t+3}; m_1, \ldots, m_{2t+1})) = b_{4t+2}(P_{4t+2}(y_{4t+2}; m_1, \ldots, m_{2t+1})),$$

and $b_{4t+1}(P_{4t+1}(y_{4t+1}; m_1, \ldots, m_{2t+1}))$.

$$= \begin{cases} b_{4t}(P_{4t}(y_{4t}; m_1, \ldots, m_{2t})) & \text{if } m_{2t+1} = 0\\ lcm\{b_{4t}(P_{4t}(y_{4t}; m_1, \ldots, m_{2t})), 2\} & \text{if } m_{2t+1} = 1.\end{cases}$$

Therefore, in the remainder of this paper we shall assume that $m = 2t$ for some $t \geq 1$, unless othewise indicated.

To compute $b_m(P_m(y_m; m_1, \ldots, m_t))$, we use the notion of rational D-series introduced and developed in [4]. Let $\boldsymbol{Q}[\![x]\!]$ be the ring of formal power series with coefficients in $\boldsymbol{Q}$ and $\boldsymbol{Q}^*[\![x]\!] = \{f(x) \in \boldsymbol{Q}[\![x]\!] : f(0) = \pm 1\}$. Let

$$f(x) = \pm 1 + \sum_{i \geq 1} a_i x^i \in \boldsymbol{Q}^*[\![x]\!].$$

For each $k \geq 1$, let $e_k(f)$ denote the smallest positive integer e_k such that

$(f(x))^{e_k} \in \boldsymbol{Z}[\![x]\!](\operatorname{mod} x^{k+1})$. Let $S_k(f)$ be the set of all primes dividing the denominators of the coefficients a_i for $i = 1, \dots, k$. For a rational number q, let $v_p(q)$ be the exponent of p in the prime factorization of q and let $D(q)$ be the denominator of q in its lowest term. For convenience, we assume that $v_p(0) = -\infty$ and $D(0) = 1$. It follows from Lemma 1.3 [**4**] that $p \in S_k(f)$ if and only if $v_p(e_k(f)) > 0$.

For each prime p, let $\alpha_p, \beta_p \in \boldsymbol{Z}^+$, and let $\alpha = (\alpha_2, \alpha_3, \alpha_5, \dots)$, and $\beta = (\beta_2, \beta_3, \beta_5, \dots)$. A series $f = \pm 1 + \sum_{i \geq 1} a_i x^i \in \boldsymbol{Q}^*[\![x]\!]$ is called a rational D-series of type (α, β) if $v_p(a_{\alpha_p}) = -\beta_p$ and $v_p(a_k) \geq -\beta_p[k/\alpha_p]$ for each k with $a_k \neq 0$. A rational D-series f is called strict at a prime p if $v_p(a_k) > -\beta_p(k/\alpha_p)$ for each $k \neq \alpha_p$ with $a_k \neq 0$. If f is strict at p then, by Theorem 3.5 of [**4**],

$$v_p(e_k(f)) = \max\left\{0, \beta_p r + v_p(r) : 0 \leq r \leq \left[\frac{k}{\alpha_p}\right]\right\}.$$

With these facts on hand, we return to our problem.

Let $\boldsymbol{Z}_{(p)} = \{r/s : r, s \in \boldsymbol{Z} \text{ with } v_p(s) = 0\}$ be the localization of $\boldsymbol{Z}$ at p. The following lemma is (5.2) and Lemma 5.5 of [**7**] with minor changes.

Lemma 2.4. (i) *Let $1 + u$ be an element of $\boldsymbol{Q}^*[\![y_m]\!](mod\ y_m^{t+1})$. Then*

$$\frac{\psi^p(1+u)}{1+u} \in \boldsymbol{Z}^*_{(p)}[\![y_m]\!](mod\ y_m^{t+1})$$

if and only if $u \in \boldsymbol{Z}_{(p)}[\![y_m]\!](mod\ y_m^{t+1})$ with $u(0) = 0$.

(ii) $(\theta_p(y_m))^h \in \boldsymbol{Z}^*_{(p)}[\![y_m]\!](mod\ y_m^{t+1})$ *if and only if*

$$v_p(h) \geq v_p(b_m(y_m)) = \max\left\{0, s + v_p(s) : 0 \leq s \leq \left[\frac{m}{p-1}\right]\right\}.$$

Now, we compute $b_m(P_m(y_m; m_1, \dots, m_t))$. For each $n = 1, \dots, t$, let $\theta_p(y_m^n) = 1 + \alpha_{n,1}(p) y_m + \cdots + \alpha_{n,t}(p) y_m^t$ where $\alpha_{n,i}(p)$ is the coefficient of y_m^i given by Theorem 2.2. According to (1) $b_m(P_m(y_m; m_1, \dots, m_t))$ is the smallest positive integer h such that

$$\theta_p(hP_m(y_m; m_1, \dots, m_t)) = \frac{\psi^p(1+u)}{1+u} \quad \text{in } \tilde{K}O(\boldsymbol{C}P^m) \otimes \boldsymbol{Q}$$

for some $u \in \tilde{K}O(\boldsymbol{C}P^m)$ and all primes p. Let

$$\begin{aligned}\beta_m(p; m_1, \dots, m_t) &= \theta_p(P_m(y_m; m_1, \dots, m_t)) = \theta_p(y_m)^{m_1} \cdots \theta_p(y_m^t)^{m_t} \\ &= 1 + \alpha_1(p; m_1, \dots, m_t) y_m + \cdots + \alpha_t(p; m_1, \dots, m_t) y_m^t,\end{aligned}$$

for some $\alpha_i(p; m_1, \dots, m_t) \in \boldsymbol{Q}$. Then

$$\theta_p(hP_m(y_m; m_1, \dots, m_t)) = \beta_m(p; m_1, \dots, m_t)^h.$$

Let

$$\beta_m(p; m_1, \dots, m_t)^h = \frac{\psi^p(1+u)}{1+u}$$

for some $u \in \tilde{K}O(\boldsymbol{C}P^m)$. Then $\beta_m(p; m_1, \dots, m_t)^h$ has integer coefficients. Hence

$$v_p(h) \geq v_p(e_t(\beta_m(p; m_1, \dots, m_t))).$$

On the other hand, if $b = e_t(\beta_m(p; m_1, \dots, m_t))$ then

$$\beta_m(p; m_1, \dots, m_t)^b = \left(\frac{\psi^p(1+u)}{1+u}\right)^{b/h} = \frac{\psi^p(1+w)}{1+w}$$

for some $w \in \boldsymbol{Q}^*[y_m](\mathrm{mod}\, y_m^{t+1})$. $\beta_m(p; m_1, \dots, m_t)^b \in \boldsymbol{Z}^*[y_m](\mathrm{mod}\, y_m^{t+1})$ implies that

$$\frac{\psi^p(1+w)}{1+w} \in \boldsymbol{Z}^*[y_m](\mathrm{mod}\, y_m^{t+1}).$$

So, by Lemma 2.4 $w \in \tilde{K}O(\boldsymbol{C}P^m)$ and hence $v_p(h) \leq v_p(b)$. By Corollary 1.3 of [**4**], $e_t(\beta_m(p; m_1, \dots, m_t)) = D(b_{p,1}) \cdots D(b_{p,t})$ where $b_{p,1} = \alpha_1(p; m_1, \dots, m_t)$ and for $k = 2, \dots, t$, $b_{p,k}$ is the coefficient of y_m^k in

$$\beta_m(p; m_1, \dots, m_t)^{D(b_{p,1}) \cdots D(b_{p,k-1})}.$$

So far, we have proved:

Theorem 2.5. $v_p(b_m(P_m(y_m; m_1, \dots, m_t))) = v_p(D(b_{p,1})) + \cdots + v_p(D(b_{p,t}))$.

Although Theorem 2.5 gives $b_m(P_m(y_m; m_1, \dots, m_t))$ by a formula, it is difficult to use this formula to find $b_m(P_m(y_m; m_1, \dots, m_t))$ for specific values of $m_1, \dots, m_t$, because one needs first to find the coefficients of y_m^k in $\theta_p(y_m)^{m_1} \cdots \theta_p(y_m^t)^{m_t}$ and then to find $e_t(\beta_m(p; m_1, \dots, m_t))$ which involves tedious calculations. So, alternatively, we next try to obtain information about $b_m(P_m(y_m; m_1, \dots, m_t))$ by using what we know about $b_m(y_m)$.

By Theorem 2.2, we directly obtain

$$\theta_p(y_m^k) = \frac{\alpha_k}{\theta_p(y_m)^{N_k}} \quad \text{where } \alpha_2 = \left(\frac{\psi^p(d_{2,1} + y_m)}{d_{2,1} + y_m}\right)^{1/2}, \quad N_2 = d_{2,1} - 1$$

and for $k = 3, \dots, t$,

$$\begin{aligned} \alpha_k &= \left(\frac{\psi^p(d_{k,1} + d_{k,2} y_m + \cdots + d_{k,k} y_m^{k-1})}{d_{k,1} + d_{k,2} y_m + \cdots + d_{k,k} y_m^{k-1}}\right)^{1/2} \alpha_2^{-d_{k,2}} \alpha_3^{-d_{k,3}} \cdots \alpha_{k-1}^{-d_{k,k-1}}, \\ N_k &= (d_{k,1} - 1) - N_2 d_{k,2} - \cdots - N_{k-1} d_{k,k-1}. \end{aligned} \tag{4}$$

For each $m_1, \dots, m_t \in \boldsymbol{Z}$, let

$$E(m_1, \dots, m_t) = \mathrm{lcm}\{e_t(\alpha_2^{m_2}), \dots, e_t(\alpha_t^{m_t})\},$$

$$N(m_1, \dots, m_t) = m_1 - m_2 N_2 - \cdots - m_t N_t,$$

$$L(p; m_1, \dots, m_t) = v_p(b_m(y_m)) - v_p(N(m_1, \dots, m_t)) - v_p(E(m_1, \dots, m_t)),$$

and

$$U(p; m_1, \dots, m_t) = \max\{v_p(b_m(y_m)) - v_p(N(m_1, \dots, m_t)), v_p(E(m_1, \dots, m_t))\}.$$

THEOREM 2.6. *Let* $P_m(y_m; m_1, \dots, m_t) = m_1 y_m + \cdots + m_t y_m^t \in \tilde{K}O(\boldsymbol{CP}^m)$. *Then*

$$L(p; m_1, \dots, m_t) \le v_p(b_m(P_m(y_m; m_1, \dots, m_t))) \le U(p; m_1, \dots, m_t).$$

PROOF. Let $h = b_m(P_m(y_m; m_1, \dots, m_t))$. Then

$$\theta_p(m_1 y_m + \cdots + m_t y_m^t)^h = \frac{\psi^p(1+u)}{1+u}$$

for some $u \in \tilde{K}O(\boldsymbol{CP}^m)$ and all primes p. So,

$$\theta_p(y_m)^{N(m_1,\dots,m_t)h} = \frac{\psi^p(1+u)}{1+u} \alpha_2^{-m_2 h} \cdots \alpha_t^{-m_t h}.$$

Thus,

$$\theta_p(y_m)^{N(m_1,\dots,m_t)E(m_1,\dots,m_t)h}$$

has integer coefficients. Hence, by Lemma 2.4 (ii)

$$v_p(N(m_1, \dots, m_t)) + v_p(E(m_1, \dots, m_t)) + v_p(h) \ge v_p(b_m(y_m)),$$

namely $v_p(h) \ge L(p; m_1, \dots, m_t)$. On the other hand, let $b \in \boldsymbol{N}$ such that $v_p(b) = U(p; m_1, \dots, m_t)$. Then

$$\theta_p(P_m(y_m; m_1, \dots, m_t))^b = \left(\frac{\psi^p(1+u)}{1+u}\right)^{b/h}.$$

So,

$$\theta_p(y_m)^{N(m_1,\dots,m_t)b} = \left(\frac{\psi^p(1+u)}{1+u}\right)^{b/h} \alpha_2^{-m_2 b} \cdots \alpha_t^{-m_t b}.$$

Let

$$\left(\frac{\psi^p(1+u)}{1+u}\right)^{b/h} = \frac{\psi^p(1+w)}{1+w}$$

for some $w \in \boldsymbol{Q}^*[\![y_m]\!] (\mathrm{mod}\, y_m^{t+1})$ with $w(0) = 0$. Now,

$$\alpha_2^{-m_2 b} \cdots \alpha_t^{-m_t b} \quad \text{and} \quad \theta_p(y_m)^{N(m_1,\ldots,m_t)b}$$

have integer coefficients. Hence,

$$\frac{\psi^p(1+w)}{1+w}$$

has integer coefficients, which implies that $w \in \boldsymbol{Z}^*[\![y_m]\!] (\mathrm{mod}\, y_m^{t+1})$. Hence,

$$v_p(h) \le v_p(b).$$

This completes the proof of Theorem 2.6. □

Corollary 2.7. *Let $P_m(y_m; m_1, \ldots, m_t) = m_1 y_m + \cdots + m_t y_m^t \in \tilde{K}O(\boldsymbol{C}P^m)$. Let s be the smallest positive integer such that $m_i = 0$ for all $i > s$, if $m_t \neq 0$, let $s = t$. Then*

$$v_p(b_m(P_m(y_m; m_1, \ldots, m_t))) = \max\{0, v_p(b_m(y_m)) - v_p(N(m_1, \ldots, m_t))\}$$

for all $p > s$.

Proof. Using Lemma 2.1 (iii), we easily see that $S_t(\alpha_k^{m_k}) \subseteq \{2, 3, \ldots, k\}$ for each $k = 2, \ldots, t$. So, if $p > s$ then $v_p(E(m_1, \ldots, m_t)) = 0$. Now, the result follows from Theorem 2.6. □

3. Two important examples.

Let $m = 2t$. Then

$$\tilde{J}O(\boldsymbol{C}P^m) = \langle \alpha_1 = y_m + TO(\boldsymbol{C}P^m), \ldots, \alpha_t = y_m^t + TO(\boldsymbol{C}P^m) \rangle.$$

In Example 1, we give a simple formula for the J-orders of α_2, α_3, and α_4.

Let $L \cong \xi_m(\boldsymbol{C})^n$ for some $n \in \boldsymbol{N}$. By the J-order of L we mean the order of $r\xi_m(\boldsymbol{C})^n - 2 + TO(\boldsymbol{C}P^m)$ in $JO(\boldsymbol{C}P^m)$. Lam [**6**] has used complex K-theory to find the J-order of L when n is a prime power. Also, Dibağ [**5**] has used (1) to give another proof of Lam's results. In Example 2, we give a simple formula for the J-order of L for each $n \in \boldsymbol{N}$.

Example 1. For each $k = 2, \ldots, t$, the J-order of $\alpha_k = y_m^k + TO(\boldsymbol{C}P^m)$ is

$$b_m(P_m(y_m; 0, \ldots, 0, m_k = 1, 0, \ldots, 0)).$$

So, by Corollary 2.7, if $p > k$ then

$$v_p(b_m(y_m^k)) = v_p(b_m(y_m)) - v_p(N_k).$$

According to (2) and (4),

$$\begin{aligned}
N_k &= -\sum_{i=1}^{k} b_{k,i} \\
&= -\sum_{i=1}^{k}(-1)^{k-i}\binom{2k}{k-i} \\
&= -\sum_{i=1}^{k-1}(-1)^{k-i}\binom{2k}{k-i} - 1 \\
&= -\sum_{i=1}^{k-1}(-1)^{k-i}\left(\binom{2k-1}{k-i} + \binom{2k-1}{k-i-1}\right) - 1 \\
&= -\sum_{i=1}^{k-1}(-1)^{k-i}\binom{2k-1}{k-i} - \sum_{i=1}^{k-i}(-1)^{k-i}\binom{2k-1}{k-i-1} - 1 \\
&= -\sum_{i=1}^{k-1}(-1)^{k-i}\binom{2k-1}{k-i} + \sum_{i=2}^{k-i}(-1)^{k-i}\binom{2k-1}{k-i-1} \\
&= -(-1)^{k-1}\binom{2k-1}{k-1} = (-1)^{k}\binom{2k-1}{k-1}.
\end{aligned}$$

Hence, if $p > k$ then $v_p(N_k) = [2(k-1)/(p-1)]$.

In [**8**], we proved that if $p = 2$, or 3, then

$$v_p(b_m(y_m^k)) = \max\left\{0, r - \left[\frac{2(k-1)}{p-1}\right] + v_p(r) : \left[\frac{2k}{p-1}\right] \le r \le \left[\frac{m}{p-1}\right]\right\}$$

for each $k = 1, \dots, t$. So, we have:

Theorem 3.1. *If $k = 2, 3$, or 4, then*

$$v_p(b_m(y_m^k)) = \max\left\{0, r - \left[\frac{2(k-1)}{p-1}\right] + v_p(r) : \left[\frac{2k}{p-1}\right] \le r \le \left[\frac{m}{p-1}\right]\right\}$$

for each $p \ge 2$. Further, this formula is true if $p = 2, 3$ and $k = 5, \dots, t$ or if $p > k$.

Remark. If $p \neq 2, 3$ and $p \le k$, then Theorem 3.1 is not necessarily true for $k \ge 5$. For instance, by the method of [**8**],

$$v_5(b_{20}(y_{20}^5)) = 3 \quad \text{while } \max\{0, r - 2 - v_5(r) : 2 \le r \le 5\} = 4.$$

Now, we compute the J-order of any complex line bundle over $\boldsymbol{CP}^m$.

Example 2. Let $n \in \boldsymbol{N}$. Then the J-order of $\xi_m(\boldsymbol{C})^n$ is $b_m(r\xi_m(\boldsymbol{C})^n - 2)$. By Theorem 2.5, $v_p(b_m(r\xi_m(\boldsymbol{C}) - 2)) = v_p(e_t(\theta_p(r\xi_m(\boldsymbol{C})^n - 2)))$.

$$r\xi_m(\boldsymbol{C})^n - 2 = r\psi^n(\xi_m(\boldsymbol{C}) - 1) = \psi^n(r\xi_m(\boldsymbol{C}) - 2) = \psi^n(y_m).$$

So, by Corollary 2.3,

$$\theta_p(r\xi_m(\boldsymbol{C})^n - 2) = \theta_p(\psi^n(y_m)) = \psi^n(\theta_p(y_m)).$$

Let $n = p_1^{r_1} \cdots p_s^{r_s} p^d$ where $r_i > 0$ for $i = 1, \dots, s$, and $d \geq 0$. Let $p = 2q + 1$ be any odd prime number. By, Lemma 5.4 [**7**],

$$\theta_p(y_m) = 1 + \sum_{j=1}^{q-1} m_j y_m^j + \frac{1}{p} y_m^q$$

where $m_j \in \boldsymbol{Z}$ with $v_p(m_j) = 0$ for $j = 1, \dots, q-1$. Hence,

$$\psi^n(\theta_p(y_m)) = 1 + \sum_{j=1}^{q-1} m_j \psi^n(y_m)^j + \frac{1}{p} \psi^n(y_m)^q.$$

Using Lemma 3.6 of [**7**], we easily obtain that

$$\psi^{p^d}(y_m) = \sum_{j=1}^{p^d - 1} n_j y_m^j + y_m^{p^d}$$

with $v_p(n_j) > 0$ for $j = 1, \dots, p^d - 1$. Now, by using Lemma 2.1 (iii) and the fact that Adams operations are ring homomorphisms with $\psi^{l_1} \circ \psi^{l_2} = \psi^{l_1 l_2}$ for each $l_1, l_2 \in \boldsymbol{Z}$, we get

$$\psi^n(y_m) = \sum_{j=1}^{n} a_j y_m^j$$

with $v_p(a_j) > 0$ for $j < p^d$ and $v_p(a_{p^d}) = 0$. Hence,

$$\psi^n(\theta_p(y_m)) = 1 + \sum_{j=1}^{nq} b_j y_m^j$$

with $v_p(b_{p^d q}) = -1$, $v_p(b_j) \geq 0$ for $j < p^d q$, and $v_p(b_j) \geq -1$ for $j > p^d q$. Hence, $\theta_p(\psi^n(y_m))$ is a strict D-series at p of type (α, β) where

$$\alpha_{p'} = \begin{cases} p^d q & p' = p \\ \infty & p' \neq p, \end{cases} \qquad \beta_{p'} = \begin{cases} 1 & p' = p \\ \infty & p' \neq p. \end{cases}$$

Hence,

$$v_p(b_m(r\xi_m(\boldsymbol{C})^n - 2)) = \max\left\{0, r + v_p(r) : 0 \le r \le \left[\frac{m}{p^{v_p(n)}(p-1)}\right]\right\}.$$

If $p = 2$, then

$$\theta_p(y_m) = \left(1 + \frac{1}{4} y_m\right)^{1/2}.$$

So,

$$\psi^n(\theta_p(y_m)) = \left(1 + \frac{1}{4}\psi^n(y_m)\right)^{1/2}.$$

Let $n = p_1^{r_1} \cdots p_s^{r_s} 2^d$. Then $1 + (1/4)\psi^n(y_m) = 1 + \sum_{j=1}^{n} c_j y_m^j$ with $v_p(c_{2^d}) = -2$, $v_p(c_j) \ge 0$ for $j < 2^d$, and $v_p(c_j) \ge -2$ for $j > 2^d$. Hence, $1 + (1/4)\psi^n(y_m)$ is a strict D-series at 2 of type (α, β) where

$$\alpha_{p'} = \begin{cases} 2^d & p' = 2 \\ \infty & p' \ne 2, \end{cases} \qquad \beta_{p'} = \begin{cases} 2 & p' = 2 \\ \infty & p' \ne 2. \end{cases}$$

So,

$$v_2\left(e_t\left(1 + \frac{1}{4}\psi^n(y_m)\right)\right) = \max\left\{0, 2r + v_2(r) : 0 \le r \le \left[\frac{t}{2^d}\right]\right\}.$$

Hence,

$$\begin{aligned} &v_2(b_m(r\xi_m(\boldsymbol{C})^n - 2)) \\ &\quad = v_2\left(e_t\left(\left(1 + \frac{1}{4}\psi^n(y_m)\right)^{1/2}\right)\right) \\ &\quad = \max\left\{0, 2r + v_2(2r) : 0 \le r \le \left[\frac{t}{2^d}\right]\right\} = \max\left\{0, r + v_2(r) : 0 \le r \le \left[\frac{m}{2^d}\right]\right\}. \end{aligned}$$

So, we have:

THEOREM 3.2. *Let* $n \in \boldsymbol{N}$ *and* p *be any prime number then*

$$v_p(b_m(r\xi_m(\boldsymbol{C})^n - 2)) = \max\left\{0, r + v_p(r) : 0 \le r \le \left[\frac{m}{p^{v_p(n)}(p-1)}\right]\right\}.$$

REMARK. A similar proof of Theorem 3.2 when n is a power of a prime p has been obtained independently by Dibağ [**5**].

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