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Invariance of Hochschild cohomology algebras under stable equivalences of Morita type

By Zygmunt POGORZAŁY*

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Abstract. There is proved that the Hochschild cohomology algebras of finite-dimensional self-injective K-algebras over a field K are invariants of stable equivalences of Morita type.

1. Introduction.

Let K be a fixed field. In representation theory of finite-dimensional associative K-algebras with units, stable equivalences of Morita type seem to be of particular relevance. They arose in representation theory of finite groups (see [**2**], [**5**]). It was proved by Rickard that if two self-injective K-algebras are derived equivalent, then they are stably equivalent of Morita type [**6**].

In [**6**] Rickard generalized a result of Happel [**3**] and proved that the Hochschild cohomology algebras of finite-dimensional K-algebras are invariant under derived equivalences. Our objective is to prove a similar result for stable equivalences of Morita type between self-injective K-algebras. Unfortunately we are not able to lift a stable equivalence of Morita type between two self-injective K-algebras to a stable equivalence of their enveloping algebras (see [**7**]). Thus the idea of the proof of our result is quite different. We are able to show that any stable equivalence of Morita type between two self-injective K-algebras induces a stable equivalence between certain subcategories of the module categories over the enveloping algebras of the original algebras, which are enough to compute the Hochschild cohomology algebras.

The main result of this note is the following

THEOREM 1.1. *Let A, B be two self-injective finite-dimensional K-algebras. If A and B are stably equivalent of Morita type then their Hochschild cohomology algebras are isomorphic.*

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2. Preliminaries.

Let C be a finite-dimensional associative K-algebra with a unit element 1. Let C^o denote the opposite algebra. Then the enveloping algebra of C is the algebra $C^e = C^o \otimes_K C$. It is well-known that every C-bimodule M is a right C^e-module and conversely, every right C^e-module N is a C-bimodule.

For any K-algebra C, we denote by $\mathrm{mod}(C)$ the category of all finite-dimensional right C-modules. Let $\mathscr{P}$ be the two-sided ideal in $\mathrm{mod}(C)$ consisting of the morphisms which factor through projective C-modules. Then the factor category $\mathrm{mod}(C)/\mathscr{P}$ is said to be the *stable category* of $\mathrm{mod}(C)$ (or shortly of C) and is denoted by $\underline{\mathrm{mod}}(C)$. For every two objects $M, N \in \underline{\mathrm{mod}}(C)$, we shall denote by $\underline{\mathrm{Hom}}_C(M,N)$ the K-vector space of the morphisms from M to N in $\underline{\mathrm{mod}}(C)$. This is the factor space $\mathrm{Hom}_C(M,N)/\mathscr{P}(M,N)$ and for any $f \in \mathrm{Hom}_C(M,N)$ its coset $\underline{f}$ modulo $\mathscr{P}(M,N)$ is an element of $\underline{\mathrm{Hom}}_C(M,N)$.

If C is a self-injective K-algebra then there is an equivalence $\Omega_C : \underline{\mathrm{mod}}(C) \to \underline{\mathrm{mod}}(C)$ which is the Heller's loop-space functor [**4**]. This equivalence can be applied for computing of the extension groups $\mathrm{Ext}^n_C(M,N)$. In fact the following lemma holds.

LEMMA 2.1. *If C is a self-injective K-algebra then for every positive integer n and every $M, N \in \mathrm{mod}(C)$ there is an isomorphism* $\mathrm{Ext}^n_C(M,N) \cong \underline{\mathrm{Hom}}_C(\Omega^n_C(M), N)$ *of K-vector spaces.*

PROOF. See 2.6 in [**1**].

For every K-algebra C its *Hochschild cohomology algebra* is the algebra $HH(C) = \bigoplus_{i=0}^{\infty} \mathrm{Ext}^i_{C^e}(C,C)$, where the multiplication is given by the Yoneda product.

We can define on $\bigoplus_{i=0}^{\infty} \underline{\mathrm{Hom}}_{C^e}(\Omega^i_{C^e}(C), C)$ the following multiplication $*$. For any non-negative integers n, m and any two morphisms $\underline{f} \in \underline{\mathrm{Hom}}_{C^e}(\Omega^m_{C^e}(C), C)$, $\underline{g} \in \underline{\mathrm{Hom}}_{C^e}(\Omega^n_{C^e}(C), C)$ we put $\underline{g} * \underline{f} = \underline{g}\Omega^n_{C^e}(\underline{f})$. Then we extend the operation $*$ bilinearly to a multiplication $*$ on $\bigoplus_{i=0}^{\infty} \underline{\mathrm{Hom}}_{C^e}(\Omega^i_{C^e}(C), C)$. A routine verification shows that the K-vector space $\bigoplus_{i=0}^{\infty} \underline{\mathrm{Hom}}_{C^e}(\Omega^i_{C^e}(C), C)$ with the multiplication $*$ forms an associative K-algebra $\overline{HH}(C)$ with a unit element. Moreover, the following fact is true. □

LEMMA 2.2. *The K-algebras $HH(C)$ and $\overline{HH}(C)$ are isomorphic.*

PROOF. See 2.6 in [**1**]. □

3. Stable equivalences of Morita type.

Two K-algebras A and B are said to be *stably equivalent* if there is an equivalence of categories $\Phi : \underline{\mathrm{mod}}(A) \to \underline{\mathrm{mod}}(B)$. The following stable equivalences are of particular interest especially in representation theory of blocks of group algebras (see [**2**], [**5**], [**7**]).

Two K-algebras A and B are said to be *stably equivalent of Morita type* provided that there is an A-B-bimodule N and a B-A-bimodule M such that the following conditions are satisfied:

(i) M, N are projective as left modules and as right modules,

(ii) $M \otimes_A N \cong B \oplus \Pi$ as B-bimodules for some projective B-bimodule Π,

(iii) $N \otimes_B M \cong A \oplus \Pi'$ as A-bimodules for some projective A-bimodule Π'.

For a K algebra C a C-bimodule X is said to be *left-right projective* if it is projective as a left C-module and as a right C-module. $\square$

Lemma 3.1. *Let C be a self-injective finite-dimensional K-algebra. For every non-negative integer n the C-bimodule $\Omega^n_{C^e}(C)$ is left-right projective.*

Proof. We shall show our lemma inductively on n. For $n = 0$ we have $\Omega^0_{C^e}(C) = C$ and the required condition is obvious.

Assume that for some non-negative integer n the C-bimodule $\Omega^n_{C^e}(C)$ is left-right projective. Then there is the following short exact sequence $0 \to \Omega^{n+1}_{C^e}(C) \to P \to \Omega^n_{C^e}(C) \to 0$ in $\mathrm{mod}(C^e)$, where P is a right projective C^e-module. But if we consider the above sequence as a sequence of right C-modules then it splits, because $\Omega^n_{C^e}(C)$ is a right projective C-module by the inductive assumption. Thus $\Omega^{n+1}_{C^e}(C)$ is a right projective C-module since P is a right projective C-module. Similarly one obtains that $\Omega^{n+1}_{C^e}(C)$ is a left projective C-module. Consequently, the lemma follows. $\square$

Lemma 3.2. *Let A and B be two finite-dimensional self-injective K-algebras which are stably equivalent of Morita type. Let ${}_BM_A$ and ${}_AN_B$ be bimodules which establish this equivalence between A and B. Then for any non-negative integer n there is an isomorphism $M \otimes_A \Omega^n_{A^e}(A) \otimes_A N \cong \Omega^n_{B^e}(B)$ in $\underline{\mathrm{mod}}(B^e)$.*

Proof. We shall show our lemma inductively on n. First observe that for $n = 0$ we have $M \otimes_A A \otimes_A N \cong M \otimes_A N \cong B \oplus \Pi$, where Π is a right projective B^e-module. Thus $M \otimes_A A \otimes_A N \cong B$ in $\underline{\mathrm{mod}}(B^e)$.

Now we assume that for some non-negative integer n there is an isomorphism $M \otimes_A \Omega^n_{A^e}(A) \otimes_A N \cong \Omega^n_{B^e}(B)$ in $\underline{\mathrm{mod}}(B^e)$. Consider the following short exact sequence $0 \to \Omega^{n+1}_{A^e}(A) \to P \to \Omega^n_{A^e}(A) \to 0$ in $\mathrm{mod}(A^e)$, where P is a right projective A^e-module. Since N is a left projective A-module, we obtain the following short exact sequence $0 \to \Omega^{n+1}_{A^e}(A) \otimes_A N \to P \otimes_A N \to \Omega^n_{A^e}(A) \otimes_A N \to 0$ in $\mathrm{mod}(B^o \otimes_K A)$. Since M is a right projective A-module, we get the following short exact sequence $0 \to M \otimes_A \Omega^{n+1}_{A^e}(A) \otimes_A N \to M \otimes_A P \otimes_A N \to M \otimes_A \Omega^n_{A^e}(A) \otimes_A N \to 0$ in $\mathrm{mod}(B^e)$. But M is a left projective B-module and P is a projective A-bimodule, so $M \otimes_A P$ is a projective B-A-bimodule. Since N is a right projective

B-module, $M \otimes_A P \otimes_A N$ is a projective B-bimodule. Moreover, we infer by the inductive assumption that $M \otimes_A \Omega^n_{A^e}(A) \otimes_A N \cong \Omega^n_{B^e}(B) \oplus Q$ for some projective B-bimodule Q. Hence we get by the exactness of the last sequence that $M \otimes_A \Omega^{n+1}_{A^e}(A) \otimes_A N \cong \Omega^{n+1}_{B^e}(B)$ in $\underline{\mathrm{mod}}(B^e)$. Consequently, the lemma follows. □

COROLLARY 3.3. *Let A, B be finite-dimensional self-injective K-algebras which are stably equivalent of Morita type. Let ${}_BM_A$ and ${}_AN_B$ be bimodules which establish this equivalence between A and B. Then for every non-negative integer n there is an isomorphism $N \otimes_B M \otimes_A \Omega^n_{A^e}(A) \otimes_A N \otimes_B M \cong \Omega^n_{A^e}(A)$ in $\underline{\mathrm{mod}}(A^e)$.*

PROOF. Apply Lemma 3.2 twice. □

4. Proof of the main result.

Let C be a self-injective finite-dimensional K-algebra. It is well-known that its enveloping algebra C^e is also self-injective. Consider the full subcategory in $\underline{\mathrm{mod}}(C^e)$ which is formed by the finite direct sums of objects isomorphic to $\Omega^n_{C^e}(C)$ for non-negative integers n. We shall denote this subcategory by $\underline{\mathrm{mod}}_C(C^e)$. It plays the crucial role in our proof of the main result.

Let A, B be self-injective finite-dimensional K-algebras which are stably equivalent of Morita type. Suppose that the bimodules ${}_BM_A$ and ${}_AN_B$ yield their stable equivalence of Morita type. Now our goal is to show that the functor $M \otimes_A - \otimes_A N : \mathrm{mod}(A^e) \to \mathrm{mod}(B^e)$ induces an equivalence of the categories $\underline{\mathrm{mod}}_A(A^e)$ and $\underline{\mathrm{mod}}_B(B^e)$.

PROPOSITION 4.1. *There exists an equivalence $F : \underline{\mathrm{mod}}_A(A^e) \to \underline{\mathrm{mod}}_B(B^e)$ such that for every non-negative integer n it holds that $F(\Omega^n_{A^e}(A)) \cong \Omega^n_{B^e}(B)$ in $\underline{\mathrm{mod}}_B(B^e)$.*

PROOF. In order to prove the proposition we have to define a functor $F : \underline{\mathrm{mod}}_A(A^e) \to \underline{\mathrm{mod}}_B(B^e)$. For every object X in $\underline{\mathrm{mod}}_A(A^e)$ we put $F(X) = M \otimes_A X \otimes_A N$. For every morphism $\underline{f} : X \to Y$ in $\underline{\mathrm{mod}}_A(A^e)$ we put $F(\underline{f}) = \underline{1_M \otimes f \otimes 1_N}$. A direct verification shows that a morphism $f : X \to Y$ in $\mathrm{mod}(A^e)$ between objects from $\underline{\mathrm{mod}}_A(A^e)$ factors through a right projective A^e-module if and only if the morphism $1_M \otimes f \otimes 1_N$ factors through a right projective B^e-module. Thus F is well-defined.

Now we can define a quasi-inverse G of the functor F similarly. We put $G(U) = N \otimes_B U \otimes_B M$ for every object U in $\underline{\mathrm{mod}}_B(B^e)$. For every morphism $\underline{g} : U \to V$ in $\underline{\mathrm{mod}}_B(B^e)$ we put $F(\underline{g}) = \underline{1_N \otimes g \otimes 1_M}$. A simple analysis shows that G is a quasi-inverse of F. Therefore F is an equivalence of categories. Moreover, we infer by Lemma 3.2 that $F(\Omega^n_{A^e}(A)) \cong \Omega^n_{B^e}(B)$ for every non-negative integer n which finishes our proof. □

Proof of Theorem 1.1. Let A and B be self-injective finite-dimensional K-algebras which are stably equivalent of Morita type. Then there are bimodules ${}_BM_A$ and ${}_AN_B$ which yield their stable equivalence of Morita type. Then we know from Proposition 4.1 that $\underline{\mathrm{End}}_{A^e}(A) \cong \underline{\mathrm{End}}_{B^e}(B)$. Combining Lemma 2.1 and Proposition 4.1 we obtain that there is an isomorphism of K-vector spaces $HH(A) \cong HH(B)$.

In order to finish our proof we need to show the following fact. For any morphism $\underline{g} : \Omega^n_{A^e}(A) \to \Omega^m_{A^e}(A)$ for some non-negative integers n, m, it holds that $\underline{1_M \otimes \Omega_{A^e}(g) \otimes 1_N} = \Omega_{B^e}(\underline{1_M \otimes g \otimes 1_N})$, where $\Omega_{A^e}(g)$ is a representative of the coset $\Omega_{A^e}(\underline{g})$.

There is the following commutative diagram in $\mathrm{mod}(A^e)$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^{n+1}_{A^e}(A) & \longrightarrow & P & \longrightarrow & \Omega^n_{A^e}(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Omega_{A^e}(g)} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & \Omega^{m+1}_{A^e}(A) & \longrightarrow & Q & \longrightarrow & \Omega^m_{A^e}(A) & \longrightarrow & 0
\end{array}$$

whose rows are exact, where $P \to \Omega^n_{A^e}(A)$, $Q \to \Omega^m_{A^e}(A)$ are minimal projective covers in $\mathrm{mod}(A^e)$. Hence we obtain the following commutative diagram in $\mathrm{mod}(B^e)$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M \otimes_A \Omega^{n+1}_{A^e}(A) \otimes_A N & \longrightarrow & M \otimes_A P \otimes_A N & \longrightarrow & M \otimes_A \Omega^n_{A^e}(A) \otimes_A N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle \tilde{h}} & & \downarrow{\scriptstyle \tilde{g}} & & \\
0 & \longrightarrow & M \otimes_A \Omega^{m+1}_{A^e}(A) \otimes_A N & \longrightarrow & M \otimes_A Q \otimes_A N & \longrightarrow & M \otimes_A \Omega^m_{A^e}(A) \otimes_A N & \longrightarrow & 0
\end{array}$$

whose rows are exact, where $\tilde{f} = 1_M \otimes \Omega_{A^e}(g) \otimes 1_N$, $\tilde{h} = 1_M \otimes h \otimes 1_N$, $\tilde{g} = 1_M \otimes g \otimes 1_N$. Since $M \otimes_A P \otimes_A N$, $M \otimes_A Q \otimes_A N$ are projective B-modules, we have $\Omega_{B^e}(\underline{1_M \otimes g \otimes 1_N}) = \underline{1_M \otimes \Omega_{A^e}(g) \otimes 1_N}$, which shows the above fact.

Using the above fact and Proposition 4.1 we obtain that for any morphisms $\underline{g}$, $\underline{h}$ in $\underline{\mathrm{mod}}_A(A^e)$ it holds $F(\underline{g} * \underline{h}) = F(\underline{g}) * F(\underline{h})$, where $F : \underline{\mathrm{mod}}_A(A^e) \to \underline{\mathrm{mod}}_B(B^e)$ is the equivalence induced by the functor $M \otimes_A - \otimes_A N : \mathrm{mod}(A^e) \to \mathrm{mod}(B^e)$. Then applying Lemma 2.2 we obtain that $HH(A) \cong HH(B)$ as K-algebras and Theorem 1.1 is proved. □

Final remarks. Let bimodules ${}_BM_A$, ${}_AN_B$ establish a stable equivalence of Morita type between finite-dimensional self-injective K-algebras A and B. Then we can consider the Hochschild cohomology algebra $HH({}_AN_B) = \bigoplus_{i=0}^{\infty} \mathrm{Ext}^i_{A^o \otimes_K B}(N, N)$. Repeating all the arguments from the proof of Theorem 1.1 we can obtain the following result.

Theorem. *There is an isomorphism $HH(A) \cong HH({}_AN_B)$ of the Hochschild cohomology algebras.*

References

[1] D. J. Benson, Representations and cohomology I: Basic representation theory of finite groups and associative algebras, Cambridge Studies in Advanced Mathematics, **30**, Cambridge University Press (Cambridge, 1991).

[2] M. Broué, Equivalences of Blocks of Group Algebras, in: V. Dlab and L. L. Scott (eds.) Finite Dimensional Algebras and Related Topics, NATO ASI, Series C, Vol. 424, Kluwer Academic Press (Dodrecht, 1992), 1–26.

[3] D. Happel, Hochschild cohomology of finite-dimensional algebras, in: Seminair d'Algebre P. Dubriel et M-P. Maliavin, Lecture Notes in Math., **1404** (Springer-Verlag, Berlin, 1989), 108–126.

[4] A. Heller, The loop-space functor in homological algebra, Trans. Amer. Math. Soc., **96** (1960), 382–394.

[5] M. Linckelmann, Stable equivalences of Morita type for self-injective algebras and p-groups, Math. Z., **223** (1996), 87–100.

[6] J. Rickard, Derived equivalences as derived functors, J. London Math. Soc., (2) **43** (1991), 37–48.

[7] J. Rickard, Some Recent Advances in Modular Representation Theory, Canadian Math. Soc. Conference Proceedings, Vol. 23 (1998), 157–178.

Zygmunt POGORZAŁY
Faculty of Mathematics and Informatics
Nicholas Copernicus University
ul. Chopina 12/18
87-100 Toruń
Poland
E-mail: zypo@mat.uni.torun.pl