J. Math. Soc. Japan
Vol. 5, No. 4, 2001

Asymptotic behavior of classical solutions to a system of semilinear wave equations in low space dimensions

Dedicated to Professor Kiyoshi Mochizuki on the occasion of his 60th birthday

By Hideo KUBO* and Kôji KUBOTA

(Received Nov. 12, 1999)
(Revised May 15, 2000)

Abstract. We give a new a priori estimate for a classical solution of the inhomogeneous wave equation in $\boldsymbol{R}^n \times \boldsymbol{R}$, where $n = 2, 3$. As an application of the estimate, we study the asymptotic behavior as $t \to \pm\infty$ of solutions $u(x,t)$ and $v(x,t)$ to a system of semilinear wave equations: $\partial_t^2 u - \Delta u = |v|^p$, $\partial_t^2 v - \Delta v = |u|^q$ in $\boldsymbol{R}^n \times \boldsymbol{R}$, where $(n+1)/(n-1) < p \le q$ with $n = 2$ or $n = 3$. More precisely, it is known that there exists a critical curve $\Gamma = \Gamma(p,q,n) = 0$ on the p-q plane such that, when $\Gamma > 0$, the Cauchy problem for the system has a global solution with small initial data and that, when $\Gamma \le 0$, a solution of the problem generically blows up in finite time even if the initial data are small. In this paper, when $\Gamma > 0$, we construct a global solution $(u(x,t), v(x,t))$ of the system which is asymptotic to a pair of solutions to the homogeneous wave equation with small initial data given, as $t \to -\infty$, in the sense of both the energy norm and the pointwise convergence. We also show that the scattering operator exists on a dense set of a neighborhood of 0 in the energy space.

1. Introduction and statement of main results.

The initial value problem for semilinear wave equations with small initial data and the related nonlinear scattering theory have been developed by many authors, since the work of F. John [**12**] was established in 1979. (See for instance [**1**]–[**38**]). In those works, the "basic estimates" for solutions to the following inhomogenious wave equations play an essential role in an explicit or implicit manner:

$$\partial_t^2 u - \Delta u = F \quad \text{in } \boldsymbol{R}^n \times \boldsymbol{R}, \tag{1.1}$$

where $\partial_t = \partial/\partial t$ and $\Delta = \sum_{j=1}^n \partial_j^2$ with $\partial_j = \partial/\partial x_j$ $(j = 1, \dots, n)$. The aim of this

2000 *Mathematics Subject Classification*. Primary 35L70, Secondly 35B40, 35B45.
Key Words and Phrases. inhomogeneous wave equation, semilinear wave equation, scattering operator, asymptotic behavior.
*Partially supported by Grant-in-Aid for Science Research (No. 09304012), Ministry of Education, Culture, Sports, Science and Technology, Japan.

paper is to give a new basic estimate for the solution to (1.1) in the case where $n = 2$ or $n = 3$, which refines previous ones, especially for $n = 2$. (See also [**20**], [**26**], [**36**] and 2) of the remarks following Theorem 1.2). To state this more precisely, we introduce an integral operator $L(F)(x,t)$ as follows:

$$L(F)(x,t) = \frac{1}{2\pi}\int_{-\infty}^{t} ds \int_0^{t-s} \frac{\rho}{\sqrt{(t-s)^2-\rho^2}}\, d\rho \int_{|\omega|=1} F(x+\rho\omega, s)\, dS_\omega \tag{1.2}$$

for $(x,t) \in \boldsymbol{R}^2 \times \boldsymbol{R}$ and

$$L(F)(x,t) = \frac{1}{4\pi}\int_{-\infty}^{t} (t-s)\, ds \int_{|\omega|=1} F(x+(t-s)\omega, s)\, dS_\omega \tag{1.3}$$

for $(x,t) \in \boldsymbol{R}^3 \times \boldsymbol{R}$. Note that $L(F)(x,t)$ satisfies (1.1) under a suitable assumption on $F(x,t)$. The main result of this paper is summarized as follows.

Theorem 1.1. *Let $n = 2$ or $n = 3$. Let $(x,t) \in \boldsymbol{R}^n \times \boldsymbol{R}$ and $r = |x|$. Let $F \in C(\boldsymbol{R}^n \times \boldsymbol{R})$. Then we have*

$$\begin{aligned} &|L(F)(x,t)|(1+r+|t|)^{(n-1)/2}/\Phi_n(r,t;\nu) \\ &\quad \le C \sup_{(y,s)\in \boldsymbol{R}^n\times\boldsymbol{R}} \{|y|^{(n-1)/2}(1+|y|+|s|)^{1+\nu}(1+||y|-|s||)^{1+\mu}|F(y,s)|\} \end{aligned} \tag{1.4}$$

for any $\mu > 0$ and $\nu > 0$, where C is a constant depending only on μ and ν. Here we have set

$$\Phi_3(r,t;\nu) = (1+|r-t|)^{-\nu}, \tag{1.5}$$

and

$$\Phi_2(r,t;\nu) = \begin{cases} (1+|r-t|)^{-\nu} & \text{if } -\infty < t \le r, \\ (1+t-r)^{-1/2}(1+t-r)^{[1/2-\nu]_+} & \text{if } r < t \end{cases} \tag{1.6}$$

with $[a]_+ = \max\{a, 0\}$ and $A^{[0]_+} = 1 + \log A$.

Remark. When $t \le 0$, the assumption $\mu > 0$ may be relaxed so that $\mu > -(n-1)/2$. More precisely, one can replace the $\Phi_n(r,t;\nu)$ in (1.4) by

$$\Phi_n(r,t;\nu)(1+|r-t|)^{[-\mu]_+},$$

provided $\nu > 0$, $\mu > -(n-1)/2$ and $t \le 0$. This would be attained by modifying a little the proofs of Lemmas 4.2 and 4.4 below.

As an application of Theorem 1.1, we consider a system of semilinear wave equations:

$$\partial_t^2 u - \Delta u = |v|^{p-1}v \quad \text{in } \boldsymbol{R}^n \times \boldsymbol{R}, \tag{1.7}$$

$$\partial_t^2 v - \Delta v = |u|^{q-1}u \quad \text{in } \boldsymbol{R}^n \times \boldsymbol{R}, \tag{1.8}$$

where $1 < p \leq q$ and either $n = 2$ or $n = 3$. Concerning the initial value problem for (1.7) and (1.8) with small initial data, Del Santo, Georgiev and Mitidieri [**6**] proved the existence of global solutions to (1.7) and (1.8) in $\boldsymbol{R}^n \times (0, \infty)$, provided

$$\Gamma = \Gamma(p, q, n) > 0, \tag{1.9}$$

$$(n+1)/(n-1) < p \leq q, \quad \text{i.e.,} \quad 0 < p^* \leq q^*, \tag{1.10}$$

where we have set

$$\begin{aligned} &\Gamma = \alpha + p\beta, \quad \alpha = pq^* - 1, \quad \beta = qp^* - 1, \\ &p^* = \frac{n-1}{2}p - \frac{n+1}{2}, \quad q^* = \frac{n-1}{2}q - \frac{n+1}{2}. \end{aligned} \tag{1.11}$$

On the other hand, when $\Gamma \leq 0$, there is a solution which blows up in finite time, even if the initial data are sufficiently small. (See [**6**], [**5**], [**8**] for the case $\Gamma < 0$ and [**7**], [**2**], [**17**], [**18**] for the case $\Gamma = 0$). In this article, we study asymptotic behavior of classical solutions to (1.7) and (1.8), when (1.9) and (1.10) hold. To this end, we introduce the following function space $X_{\nu,\kappa}$ for $\nu > 0$, $\kappa > 0$:

$$X_{\nu,\kappa} = \{(u, v) \in C(\boldsymbol{R}^n \times \boldsymbol{R}) \times C(\boldsymbol{R}^n \times \boldsymbol{R}); \|u\|_\nu + \|v\|_\kappa < +\infty\}, \tag{1.12}$$

where the norm $\|\cdot\|_\nu$ is defined by

$$\|u\|_\nu = \sup_{(x,t) \in \boldsymbol{R}^n \times \boldsymbol{R}} \{|u(x,t)|(1 + r + |t|)^{(n-1)/2} / \Phi_n(r, |t|; \nu)\} \tag{1.13}$$

with $r = |x|$. In addition, we set for $\delta > 0$

$$X_{\nu,\kappa}(\delta) = \{(u, v) \in X_{\nu,\kappa} : \|u\|_\nu + \|v\|_\kappa \leq \delta\}. \tag{1.14}$$

Next, let us denote by $u^-(x,t)$ and $v^-(x,t)$ the solutions to the homogeneous wave equation

$$\partial_t^2 w - \Delta w = 0 \quad \text{in } \boldsymbol{R}^n \times \boldsymbol{R}, \tag{1.15}$$

satisfying the following initial conditions, respectively:

$$u^-(x, 0) = f_1(x), \quad \partial_t u^-(x, 0) = g_1(x) \quad \text{in } \boldsymbol{R}^n, \tag{1.16}$$

$$v^-(x, 0) = f_2(x), \quad \partial_t v^-(x, 0) = g_2(x) \quad \text{in } \boldsymbol{R}^n. \tag{1.17}$$

We assume that $f_j \in C^3(\boldsymbol{R}^n)$ and $g_j \in C^2(\boldsymbol{R}^n)$ $(j = 1, 2)$ satisfy

$$
\begin{aligned}
&|f_1(x)| \le \varepsilon(1+r)^{-(n-1)/2-p^*}, \quad |f_2(x)| \le \varepsilon(1+r)^{-(n-1)/2-q^*}, \\
&\sum_{|\gamma|=1} |\partial_x^\gamma f_1(x)| + |g_1(x)| \le \varepsilon(1+r)^{-(n+1)/2-p^*}, \\
&\sum_{|\gamma|=1} |\partial_x^\gamma f_2(x)| + |g_2(x)| \le \varepsilon(1+r)^{-(n+1)/2-q^*},
\end{aligned} \tag{1.18}
$$

and

$$
\begin{aligned}
&\sup_{x \in \boldsymbol{R}^n} \left\{ (1+r)^{(n+1)/2+p^*} \left(\sum_{2 \le |\gamma| \le 3} |\partial_x^\gamma f_1(x)| + \sum_{1 \le |\gamma| \le 2} |\partial_x^\gamma g_1(x)| \right) \right\} < +\infty, \\
&\sup_{x \in \boldsymbol{R}^n} \left\{ (1+r)^{(n+1)/2+q^*} \left(\sum_{2 \le |\gamma| \le 3} |\partial_x^\gamma f_2(x)| + \sum_{1 \le |\gamma| \le 2} |\partial_x^\gamma g_2(x)| \right) \right\} < +\infty,
\end{aligned} \tag{1.19}
$$

where $\varepsilon > 0$ and $r = |x|$. Then there is a positive constant $C_0 = C_0(p, q, n)$ such that

$$
(u^-, v^-) \in X_{p^*, q^*}(C_0 \varepsilon), \quad (\partial_x^\gamma u^-, \partial_x^\gamma v^-) \in X_{p^*, q^*} \quad \text{for } |\gamma| \le 2, \tag{1.20}
$$

provided (1.10) holds. For the proof, see Lemma 2 in [**26**] for 3-dimensional case and Proposition 1.1 in [**20**] (or Proposition 2.1 in [**19**]) for 2-dimensional case. (See also those proofs).

It follows from the definitions of Γ, α and β that

$$
\beta \le \Gamma/(p+1) \le \alpha \quad \text{for } 1 < p \le q. \tag{1.21}
$$

Here we shall state a part of our results for a special case where

$$
\beta > 0, \quad \textit{i.e.}, \quad qp^* > 1. \tag{1.22}
$$

The general case will be discussed in Section 5 below. Notice that (1.22) implies (1.9) according to (1.21) and that (1.22) is equivalent to (1.9) when $p = q$. Moreover, since $q \ge p$, we have

$$
\beta > 0 \quad \text{if } pp^* > 1, \quad \textit{i.e.}, \quad p > p_0(n) := \frac{n+1+\sqrt{n^2+10n-7}}{2(n-1)}. \tag{1.23}
$$

The number $p_0(n)$ is known as a critical exponent for the initial value problem for a single wave equation (1.30) below with small initial data.

Theorem 1.2. *Let $n = 2$ or $n = 3$. Suppose that* (1.10), (1.18), (1.19) *and* (1.22) *hold. Then for any ν and κ satisfying*

$$
1/q < \nu \le p^*, \quad 1/p < \kappa \le q^*, \tag{1.24}
$$

there is a positive number $\varepsilon_0 = \varepsilon_0(\nu, \kappa, p, q)$ *such that for any* ε *with* $0 < \varepsilon \le \varepsilon_0$, *there exists uniquely a classical solution* $(u, v) \in X_{\nu,\kappa}(2C_0\varepsilon)$ *of* (1.7) *and* (1.8) *verifying the following properties*:

$$\|\!|(u - u^-)(t)\|\!|_e \le C\|v\|_\kappa^p(1 + |t|)^{-p^*} \quad \text{for } t \le 0, \tag{1.25}$$

$$\|\!|(v - v^-)(t)\|\!|_e \le C\|u\|_\nu^q(1 + |t|)^{-q^*} \quad \text{for } t \le 0, \tag{1.26}$$

where

$$\|\!|u(t)\|\!|_e^2 = \frac{1}{2}\{\|\nabla u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2\}.$$

Moreover, *for any* $(x, t) \in \boldsymbol{R}^n \times \boldsymbol{R}$ *we have*

$$|u(x, t) - u^-(x, t)| \le C\|v\|_\kappa^p(1 + r + |t|)^{-(n-1)/2}\Phi_n(r, t; p^*), \tag{1.27}$$

$$|v(x, t) - v^-(x, t)| \le C\|u\|_\nu^q(1 + r + |t|)^{-(n-1)/2}\Phi_n(r, t; q^*). \tag{1.28}$$

Furthermore, *there exists a unique solution* $(u^+, v^+) \in X_{p^*,q^*}$ *of the homogeneous wave equation* (1.15) *satisfying* (5.54) *through* (5.57) *below*. *Here* C *is a constant depending only on* ν, κ, p *and* q.

Remarks. 1) The existnce of ν and κ satisfying (1.24) follows from the assumption that $\alpha \ge \beta > 0$. Moreover, the scattering operator

$$(f_1, f_2, g_1, g_2) \mapsto (u^+(0), v^+(0), \partial_t u^+(0), \partial_t v^+(0)) \tag{1.29}$$

is defined for such (f_1, f_2, g_1, g_2) satisfying (1.18) and (1.19) with $0 < \varepsilon \le \varepsilon_0$.

2) We shall here compare the basic estimate (1.4) with the ones in the previous works [**26**], [**36**] and [**20**]. Let us consider a single semilinear wave equation

$$\partial_t^2 u - \Delta u = F(u) \quad \text{in } \boldsymbol{R}^n \times \boldsymbol{R}, \tag{1.30}$$

where $n = 2$ or $n = 3$, and $F(u) = |u|^p$ or $F(u) = |u|^{p-1}u$ with $p > 1$. Assume that

$$pp^* > 1, \quad \text{i.e.,} \quad p > p_0(n), \tag{1.31}$$

where p^* and $p_0(n)$ are given respectively in (1.11) and (1.23). Note that

$$p^* > 1/p_0(n) \quad \text{for } p > p_0(n). \tag{1.32}$$

Besides, we have $p^* = p - 2$, $p_0(3) = 1 + \sqrt{2}$ for $n = 3$ and $p^* = (p - 3)/2$, $p_0(2) = (3 + \sqrt{17})/2$ for $n = 2$. Then it is shown in Pecher [**26**] for $n = 3$, Tsutaya [**36**] for $n = 2$ and Kubota and Mochizuki [**20**] for $n = 2$ that the scattering operator for (1.30) is defined on a dense set of a neighborhood of 0 in the energy space by establishing the following basic estimates (1.33), (1.35), (1.39) and (1.40).

When $n = 3$, Pecher [**26**] proved that

$$|L(F(u))(x,t)|(1+r+|t|)(1+|r-t|)^{p^*} \le C\|u\|_{p^*}^p \tag{1.33}$$

for $(x,t) \in \boldsymbol{R}^3 \times \boldsymbol{R}$ with $r = |x|$, $u(x,t) \in C(\boldsymbol{R}^3 \times \boldsymbol{R})$ and $p_0(3) < p \le 3$, where the norm $\|u\|_\nu$ is defined by (1.13) with (1.5), and L is the linear operator given by (1.3). We shall compare (1.33) with our basic estimate (1.4) with $F = F(u)$ and $\nu = p^*$. Note that the latter implies that the left hand side of (1.33) is dominated by

$$C \sup_{(y,s) \in \boldsymbol{R}^3 \times \boldsymbol{R}} \{|u(y,s)|^p (1+|y|+|s|)^p (1+||y|-|s||)^{1+\mu}\}$$

for $\mu > 0$, since $2 + p^* = p$ for $n = 3$. Choosing ν such that $\nu > 1/p$ and taking $\mu = p\nu - 1$, we get from (1.13) with (1.5)

$$|u(y,s)|^p (1+|y|+|s|)^p (1+||y|-|s||)^{1+\mu} \le \|u\|_\nu^p$$

for $(y,s) \in \boldsymbol{R}^3 \times \boldsymbol{R}$, hence

$$|L(F(u))(x,t)|(1+r+|t|)(1+|r-t|)^{p^*} \le C\|u\|_\nu^P \tag{1.34}$$

for $(x,t) \in \boldsymbol{R}^3 \times \boldsymbol{R}$. Note that (1.34) refines (1.33) when $1/p < \nu < p^*$.

Next let us consider the case of two space dimensions, *i.e.*, $n = 2$. Tsutaya [**36**] proved that

$$|L(F(u))(x,t)|(1+r+|t|)^{1/2} \le C|u|_m^p \Psi_m(r,t) \tag{1.35}$$

for $(x,t) \in \boldsymbol{R}^2 \times \boldsymbol{R}$ with $r = |x|$ and $u(x,t) \in C(\boldsymbol{R}^2 \times \boldsymbol{R})$, where L is the linear operator defined by (1.2),

$$\Psi_m(r,t) = (1+|r-t|)^{-m}\{1+\log(1+|r-t|)\},$$

$$|u|_m = \sup_{(y,s) \in \boldsymbol{R}^2 \times \boldsymbol{R}} \{|u(y,s)|(1+|y|+|s|)^{1/2}/\Psi_m(|y|,|s|)\}$$

and

$$m = \min\{1/2, p^*\},$$

provided we take the parameter k in the hypothesis (H2) of [**36**] so that $k \ge (p-2)/2$. (See Lemma 4.1 in that paper). On the other hand, our basic estimate (1.4) with $F = F(u)$ and $\nu = p^*$ gives an upper bound of the left hand side of (1.35) by

$$C\Phi_2(r,t;p^*) \times \sup_{(y,s)\in \boldsymbol{R}^2\times\boldsymbol{R}} \{|u(y,s)|^p(1+|y|+|s|)^{p/2}(1+||y|-|s||)^{1+\mu}\}$$

for $\mu > 0$, since $(3/2) + p^* = p/2$ for $n = 2$. Choosing ν such that

$$\frac{1}{p} < \nu < \frac{1}{2} \tag{1.36}$$

and taking $\mu = p\nu - 1$, we get from (1.13) with (1.6)

$$|u(y,s)|^p(1+|y|+|s|)^{p/2}(1+||y|-|s||)^{1+\mu} \le \|u\|_\nu^p$$

for $(y,s) \in \boldsymbol{R}^3 \times \boldsymbol{R}$, hence

$$|L(F(u))(x,t)|(1+r+|t|)^{1/2} \le C\|u\|_\nu^p \Phi_2(r,t;p^*) \tag{1.37}$$

for $(x,t) \in \boldsymbol{R}^2 \times \boldsymbol{R}$.

Notice that (1.6) implies

$$\Phi_2(r,t;p^*) = (1+|r-t|)^{-p^*} \tag{1.38}$$

for either $0 < p^* < 1/2$ or $p^* \ge 1/2$ and $-\infty < t \le r$. Therefore we see that (1.37) refines (1.35), provided

$$\frac{1}{p} < \nu < m = \min\{1/2, p^*\}.$$

Indeed, as for the decay rates of $L(F(u))(x,t)$ we have

$$\Phi_2(r,t;p^*) \le \Psi_m(r,t)/\{1+\log(1+|r-t|)\}$$

if $p^* \ne 1/2$, while $\Phi_2(r,t;p^*) \le \Psi_m(r,t)$ if $p^* = 1/2$, *i.e.*, $p = 4$. Moreover

$$\Psi_m(|y|,|s|)/\Phi_2(|y|,|s|;\nu) = \{1+\log(1+||y|-|s||)\}/(1+||y|-|s||)^{m-\nu}$$

implies $\|u\|_\nu \le C|u|_m$, since $\nu < m$.

Finally we shall compare our (1.4) with the basic estimates (2.4) and (2.21) with (1.27) in [**20**]. For simplicity of description we suppose that the parameter κ in (1.3) of that paper satisfies $\kappa \ge p/2$. Then the second author and Mochizuki [**20**] proved that if

$$\frac{1}{p} < \nu < p^* \le \frac{1}{2},$$

then

$$|L(F(u))(x,t)|(1+r+|t|)^{1/2}(1+|r-t|)^{p^*-\delta} \le C\|u\|_\nu^p \tag{1.39}$$

for $0 < \delta < p^*$, and if

$$\frac{1}{p} < \nu < \frac{1}{2} < p^*,$$

then

$$|L(F(u))(x,t)|(1+r+|t|)^{1/2}(1+|r-t|)^{1/2} \leq C\|u\|_\nu^p \tag{1.40}$$

for $(x,t) \in \boldsymbol{R}^2 \times \boldsymbol{R}$ and $u(x,t) \in C(\boldsymbol{R}^2 \times \boldsymbol{R})$. (See Proposition 2.1 and (1.27) in that paper). It is easy to see that (1.37) with (1.36) refines (1.39) and (1.40) according to (1.38).

It is also shown in [**20**] that

$$|L(F(u))(x,t)|(1+r+|t|)^{(1/2)+p^*+p\nu-1} \leq C\|u\|_\nu^p \tag{1.41}$$

for $t \leq 0$ and $x \in \boldsymbol{R}^2$ with $r = |x|$, provided

$$\frac{1-p^*}{p-1} < \nu < \frac{1}{p} \quad \text{and} \quad 0 < \nu, \tag{1.42}$$

which implies $p^* + p\nu - 1 > \nu$. (See Proposition 1.8 and (1.27) there). Making use of the remark following Theorem 1.1, one can prove also (1.41). In fact, when $1/2 < p\nu < 1$, the proof of (1.37) is still valid, if we replace $\Phi_2(r,t;p^*)$ by

$$\Phi_2(r,t;p^*)(1+|r-t|)^{[-\mu]_+} \leq (1+r+|t|)^{-p^*-(p\nu-1)},$$

since $t \leq 0$ and $-1/2 < \mu = p\nu - 1 < 0$. If $p\nu \leq 1/2$, we replace p^* in $\Phi_2(r,t;p^*)$ by $p^* + p\nu - 1 - \mu$ with some $0 > \mu > -1/2$. Then we obtain (1.41), since $p\nu - 1 - \mu < 0$.

The plan of this paper is as follows: In Section 2, we collect some notations. In Section 3, we prepare a couple of lemmas which are needed to prove Theorem 1.1, and we carry out the proof of the theorem in Section 4. In Section 5, we state our results for a system of semilinear wave equations. In Section 6, we establish a priori estimates by making use of Theorem 1.1, and we prove Theorems 5.1 and 5.2 in Section 7.

2. Notations.

In this section we collect some notations which will be used in the sequel. We set

$$a \vee b = \max\{a,b\} \quad \text{for } a,b \in \boldsymbol{R}. \tag{2.1}$$

In particular, we put

$$[a]_+ = a \vee 0, \quad A^{[0]_+} = 1 + \log A. \tag{2.2}$$

Next we define several norms for a real valued function $u(x,t)$:

$$\|u\|_\nu = \sup_{(y,s)\in \boldsymbol{R}^n\times \boldsymbol{R}} \{|u(y,s)|(1+|y|+|s|)^{(n-1)/2}/\Phi_n(|y|,|s|;\nu)\}, \tag{2.3}$$

$$[u]_\nu = \max_{|\gamma|\le 2} \|\partial_x^\gamma u\|_\nu, \tag{2.4}$$

$$|||u(t)|||_e^2 = \frac{1}{2}\{\|\nabla u(t)\|_{L^2}^2 + \|\partial_t u(t)\|_{L^2}^2\}, \tag{2.5}$$

where $\Phi_n(r,t;\nu)$ is given by (1.5) and (1.6).

We set for $\nu > 0$ and $\mu > 0$

$$z_{\nu,\mu}(\lambda,s) = (1+|s|+\lambda)^{1+\nu}(1+||s|-\lambda|)^{1+\mu}, \tag{2.6}$$

$$M_{\nu,\mu}(F) = \sup_{(y,s)\in \boldsymbol{R}^n\times \boldsymbol{R}} \{|y|^{(n-1)/2} z_{\nu,\mu}(|y|,s)|F(y,s)|\}. \tag{2.7}$$

3. Preliminaries.

In this section we collect a basic identity and elementary inequalities. The first is a fundamental identity concerning the spherical mean. For the proof, see [**27**], also Lemma 2.3 in [**19**].

LEMMA 3.1. *Let $b \in C([0,\infty))$, $x \in \boldsymbol{R}^n\backslash\{0\}$, $n \ge 2$, $r = |x|$ and $\rho > 0$. Then we have*

$$\int_{|\omega|=1} b(|x+\rho\omega|)\,dS_\omega = 2^{3-n}\omega_{n-1}(r\rho)^{2-n}\int_{|\rho-r|}^{\rho+r} \lambda b(\lambda)h(\lambda,\rho,r)\,d\lambda, \tag{3.1}$$

where $\omega_k = 2\pi^{k/2}/\Gamma(k/2)$ and

$$h(\lambda,\rho,r) = (\rho^2-(\lambda-r)^2)^{(n-3)/2}((\lambda+r)^2-\rho^2)^{(n-3)/2}. \tag{3.2}$$

The following inequalities are due to Lemmas 1.2 and 2.2 in [**20**].

LEMMA 3.2. *For $a < b < d$, we set*

$$J(a,b,d) = \int_a^b \frac{d\sigma}{\sqrt{b-\sigma}\sqrt{\sigma-a}\sqrt{d-\sigma}}. \tag{3.3}$$

Then we have

$$J(a,b,d) \le \frac{\pi}{\sqrt{d-b}} \tag{3.4}$$

and for any $\theta > 0$

$$J(a,b,d) \le C\left(\frac{d-a}{d-b}\right)^\theta \frac{1}{\sqrt{b-a}}, \tag{3.5}$$

where C is a constant depending only on θ.

Finally, we prepare the following useful lemma, which is an extended version of Lemmas 1.3 and 1.5 in [**20**]. This lemma will be repeatedly used in Section 4 below.

LEMMA 3.3. *Let* $\kappa > 0$, $0 \le \gamma < 1$ *and* $\kappa + \gamma > 1$. *Then we have*

$$\int_{|b|}^{\infty} (1+\sigma)^{-\kappa}(a+\sigma)^{-\gamma}\, d\sigma \le C(1+|b|)^{-\kappa-\gamma+1} \quad \text{for } a \ge -|b|. \tag{3.6}$$

Moreover, we have

$$\int_{-\infty}^{b} (1+|\sigma|)^{-\kappa}(a-\sigma)^{-\gamma}\, d\sigma \le C(1+|a|)^{-\gamma}(1+|a|)^{[1-\kappa]_+} \quad \text{for } a \ge b, \tag{3.7}$$

or equivalently,

$$\int_{b}^{\infty} (1+|\sigma|)^{-\kappa}(a+\sigma)^{-\gamma}\, d\sigma \le C(1+|a|)^{-\gamma}(1+|a|)^{[1-\kappa]_+} \quad \text{for } a \ge -b. \tag{3.8}$$

Here C *is a constant depending only on* κ *and* γ.

REMARK. If $a \le 0$, one can replace the right hand side of (3.7) by that of (3.6).

PROOF. First we show (3.6). Since $(a+\sigma)^{-\gamma} \le (\sigma - |b|)^{-\gamma}$, by integration by parts, we have

$$\int_{|b|}^{\infty} (1+\sigma)^{-\kappa}(a+\sigma)^{-\gamma}\, d\sigma \le \frac{\kappa}{1-\gamma}\int_{|b|}^{\infty} (1+\sigma)^{-\kappa-\gamma}\, d\sigma,$$

which yields (3.6), because $\kappa + \gamma > 1$.

Next we show (3.8). If we set

$$P_1 = \int_{-a}^{|a|} (1+|\sigma|)^{-\kappa}(a+\sigma)^{-\gamma}\, d\sigma, \quad P_2 = \int_{|a|}^{\infty} (1+|\sigma|)^{-\kappa}(a+\sigma)^{-\gamma}\, d\sigma,$$

we see from the assumption $b \ge -a$ that the left hand side of (3.8) is dominated by $P_1 + P_2$. Then (3.6) with $b = a$ gives

$$P_2 \le C(1+|a|)^{-\kappa-\gamma+1}.$$

While P_1 is estimated as follows: when $0 < a \le 1$, it is enough to show that P_1 is bounded. It follows that

$$P_1 \le \int_{-a}^{a} (a+\sigma)^{-\gamma}\, d\sigma,$$

hence P_1 is bounded, because $\gamma < 1$. On the other hand, when $a > 1$, we split the integral at $\sigma = -a/2$ to get

$$P_1 \le \left(1+\frac{a}{2}\right)^{-\kappa}\int_{-a}^{-a/2}(a+\sigma)^{-\gamma}\,d\sigma + \left(\frac{a}{2}\right)^{-\gamma}\int_{-a/2}^{a}(1+|\sigma|)^{-\kappa}\,d\sigma.$$

Since $\gamma < 1$ and $a > 1$, we get $P_1 \le C(1+a)^{-\gamma}(1+a)^{[1-\kappa]_+}$. The proof is complete. □

4. Basic estimates.

In this section, we shall prove Theorem 1.1. By (2.7), we have

$$\left|\int_{|\omega|=1} F(x+\rho\omega,s)\,dS_\omega\right| \le M_{\nu,\mu}(F)\int_{|\omega|=1}\frac{dS_\omega}{\lambda^{(n-1)/2}z_{\nu,\mu}(\lambda,s)} \tag{4.1}$$

with $\lambda = |x+\rho\omega|$. Applying Lemma 3.1, we get

$$\int_{|\omega|=1}\frac{dS_\omega}{\lambda^{(n-1)/2}z_{\nu,\mu}(\lambda,s)} = 2^{3-n}\omega_{n-1}(r\rho)^{2-n}\int_{|\rho-r|}^{\rho+r}\frac{\lambda^{(3-n)/2}h(\lambda,\rho,r)}{z_{\nu,\mu}(\lambda,s)}\,d\lambda. \tag{4.2}$$

Case 1: $\mathbf{n=2}$. If we set

$$I(r,t) = \frac{2}{\pi}\int_{-\infty}^{t}ds\int_0^{t-s}\frac{\rho}{\sqrt{(t-s)^2-\rho^2}}\,d\rho\int_{|\rho-r|}^{\rho+r}\frac{\lambda^{1/2}h(\lambda,\rho,r)}{z_{\nu,\mu}(\lambda,s)}\,d\lambda, \tag{4.3}$$

it follows from (1.2), (4.1) and (4.2) with $n=2$ that

$$|L(F)(x,t)| \le M_{\nu,\mu}(F)\times I(r,t). \tag{4.4}$$

Changing the order of the integrals, we get

$$I(r,t) = I_1(r,t) + I_2(r,t), \tag{4.5}$$

where we have set

$$I_1(r,t) = \int_{-\infty}^{t}ds\int_{|t-s-r|}^{t-s+r}\frac{1}{z_{\nu,\mu}(\lambda,s)}K_1(\lambda,s,r,t)\,d\lambda, \tag{4.6}$$

$$I_2(r,t) = \int_{-\infty}^{t-r}ds\int_0^{t-s-r}\frac{1}{z_{\nu,\mu}(\lambda,s)}K_2(\lambda,s,r,t)\,d\lambda. \tag{4.7}$$

Here

$$K_1(\lambda,s,r,t) = \frac{2\sqrt{\lambda}}{\pi}\int_{|\lambda-r|}^{t-s}\frac{\rho h(\lambda,\rho,r)}{\sqrt{(t-s)^2-\rho^2}}\,d\rho \quad \text{for } |t-s-r|<\lambda<t-s+r, \tag{4.8}$$

$$K_2(\lambda,s,r,t) = \frac{2\sqrt{\lambda}}{\pi}\int_{|\lambda-r|}^{\lambda+r}\frac{\rho h(\lambda,\rho,r)}{\sqrt{(t-s)^2-\rho^2}}\,d\rho \quad \text{for } 0<\lambda<t-s-r. \tag{4.9}$$

We introduce new variables

$$\alpha = \lambda + s \quad \text{and} \quad \beta = \lambda - s. \tag{4.10}$$

If we denote by $I_1^{\pm}$ and $I_2^{\pm}$ the integrals over $\pm s \geq 0$ of I_1 and I_2, respectively, then we have

$$I_1^+ = \frac{1}{2}\chi(t)\int_{|t-r|}^{t+r}(1+\alpha)^{-1-\nu}\,d\alpha\int_{r-t}^{\alpha}(1+|\beta|)^{-1-\mu}K_1\,d\beta, \tag{4.11}$$

$$I_1^- = \frac{1}{2}\int_{t-r}^{t+r}(1+|\alpha|)^{-1-\mu}\,d\alpha\int_{\alpha\vee|r-t|}^{\infty}(1+\beta)^{-1-\nu}K_1\,d\beta, \tag{4.12}$$

and

$$I_2^+ = \frac{1}{2}\int_{0}^{[t-r]_+}(1+\alpha)^{-1-\nu}\,d\alpha\int_{-\alpha}^{\alpha}(1+|\beta|)^{-1-\mu}K_2\,d\beta, \tag{4.13}$$

$$I_2^- = \frac{1}{2}\int_{-\infty}^{t-r}(1+|\alpha|)^{-1-\mu}\,d\alpha\int_{|\alpha|}^{\infty}(1+\beta)^{-1-\nu}K_2\,d\beta, \tag{4.14}$$

where $\chi(t) = 1$ for $t > 0$ and $\chi(t) = 0$ for $t \leq 0$. In addition, we further divide I_2^- into J_1 and J_2 which are defined by

$$J_1 = \frac{1}{2}\int_{0}^{[t-r]_+}(1+\beta)^{-1-\nu}\,d\beta\int_{-\beta}^{\beta}(1+|\alpha|)^{-1-\mu}K_2\,d\alpha, \tag{4.15}$$

$$J_2 = \frac{1}{2}\int_{|r-t|}^{\infty}(1+\beta)^{-1-\nu}\,d\beta\int_{-\beta}^{t-r}(1+|\alpha|)^{-1-\mu}K_2\,d\alpha. \tag{4.16}$$

Then we have

$$I_1(r,t) = I_1^+ + I_1^-, \quad I_2(r,t) = I_2^+ + J_1 + J_2. \tag{4.17}$$

Next we derive several estimates of K_1 and K_2 in the following lemma.

LEMMA 4.1. *Let* $t - r < \alpha < t + r$ *and* $\beta > r - t$ *with* (4.10). *Then it holds that*

$$K_1(\lambda,s,r,t) \leq \frac{\sqrt{\alpha}}{\sqrt{\beta+r+t}\sqrt{\alpha+r-t}} \quad \text{for } \alpha \geq \beta, \tag{4.18}$$

$$K_1(\lambda,s,r,t) \leq \frac{\sqrt{\beta}}{\sqrt{\beta+r+t}\sqrt{\alpha+r-t}} \quad \text{for } \alpha \leq \beta \tag{4.19}$$

and

$$K_1(\lambda,s,r,t) \le \frac{1}{\sqrt{\alpha+r-t}}. \tag{4.20}$$

Moreover, we have for any $\theta > 0$

$$K_1(\lambda,s,r,t) \le \frac{C\sqrt{\beta}}{\sqrt{t+r-\alpha}\sqrt{\beta+t-r}}\frac{\beta^\theta}{(\alpha+r-t)^\theta} \quad \textit{for } \alpha \le \beta. \tag{4.21}$$

Let $-\beta < \alpha < t-r$. *Then it holds that*

$$K_2(\lambda,s,r,t) \le \frac{\sqrt{\alpha}}{\sqrt{t-r-\alpha}\sqrt{t+r+\beta}} \quad \textit{for } \alpha \ge \beta, \tag{4.22}$$

$$K_2(\lambda,s,r,t) \le \frac{\sqrt{\beta}}{\sqrt{t-r-\alpha}\sqrt{t+r+\beta}} \quad \textit{for } \alpha \le \beta \tag{4.23}$$

and

$$K_2(\lambda,s,r,t) \le \frac{1}{\sqrt{t-r-\alpha}}. \tag{4.24}$$

Moreover, we have for any $\theta > 0$

$$K_2(\lambda,s,r,t) \le \frac{C}{\sqrt{r}}\frac{\beta^\theta}{(t-r-\alpha)^\theta} \quad \textit{for } \beta > t-r. \tag{4.25}$$

Proof. First we consider K_1. It is easy to see that $t-r < \alpha < t+r$ and $\beta > r-t$ imply $|\lambda - r| < t-s < \lambda + r$. It follows from (4.8), (3.2) and (3.3) that

$$K_1(\lambda,s,r,t) = \frac{1}{\pi}\sqrt{\lambda}J((\lambda-r)^2,(t-s)^2,(\lambda+r)^2). \tag{4.26}$$

By (3.4) and (4.10), we have

$$\begin{aligned} K_1(\lambda,s,r,t) &\le \frac{\sqrt{\lambda}}{\sqrt{(\lambda+r)^2-(t-s)^2}} \\ &= \frac{\sqrt{\lambda}}{\sqrt{\beta+r+t}\sqrt{\alpha+r-t}}. \end{aligned} \tag{4.27}$$

Noting that

$$\lambda \le \alpha \quad \text{for } \alpha \ge \beta, \quad \lambda \le \beta \quad \text{for } \alpha \le \beta, \tag{4.28}$$

we get (4.18) and (4.19). Moreover, since $\lambda < \beta + t$ for $s < t$, we get (4.20) from (4.27). Furthermore, it follows from (4.26), (3.5) and (4.10) that

$$(4.29)\qquad K_1(\lambda,s,r,t) \le C\left(\frac{(\lambda+r)^2-(\lambda-r)^2}{(\lambda+r)^2-(t-s)^2}\right)^{\theta}\frac{\sqrt{\lambda}}{\sqrt{(t-s)^2-(\lambda-r)^2}}$$

$$= C\left(\frac{4\lambda r}{(\beta+r+t)(\alpha+r-t)}\right)^{\theta}\frac{\sqrt{\lambda}}{\sqrt{t+r-\alpha}\sqrt{\beta+t-r}},$$

which implies (4.21), by (4.28) and $\beta+r+t>r$ for $s<t$.

Next we consider K_2. It is easy to see that $-\beta<\alpha<t-r$ implies $|\lambda-r|<\lambda+r<t-s$. It follows from (4.9), (3.2) and (3.3) that

$$(4.30)\qquad K_2(\lambda,s,r,t) = \frac{1}{\pi}\sqrt{\lambda}J((\lambda-r)^2,(\lambda+r)^2,(t-s)^2).$$

By (3.4) and (4.10), we have

$$(4.31)\qquad K_2(\lambda,s,r,t) \le \frac{\sqrt{\lambda}}{\sqrt{(t-s)^2-(\lambda+r)^2}}$$

$$= \frac{\sqrt{\lambda}}{\sqrt{t-r-\alpha}\sqrt{t+r+\beta}}.$$

Noting (4.28), we get (4.22) and (4.23). Moreover, since $\lambda \le \beta + t$ for $s<t$, we get (4.24) from (4.31). Furthermore, it follows from (4.30), (3.5) and (4.10) that

$$(4.32)\qquad K_2(\lambda,s,r,t) \le C\left(\frac{(t-s)^2-(\lambda-r)^2}{(t-s)^2-(\lambda+r)^2}\right)^{\theta}\frac{\sqrt{\lambda}}{\sqrt{(\lambda+r)^2-(\lambda-r)^2}}$$

$$= C\left(\frac{(t-r+\beta)(t+r-\alpha)}{(t-r-\alpha)(t+r+\beta)}\right)^{\theta}\frac{1}{\sqrt{4r}}$$

which yields (4.25), because $t-r<\beta$ and $-\alpha<\beta$. This completes the proof. $\square$

Now we shall prove for $\mu>0$, $\nu>0$

$$(4.33)\qquad I(r,t) \le C(1+r+|t|)^{-1/2}\Phi_2(r,t;\nu)\quad \text{for } r>0,\ t\in \boldsymbol{R},$$

by establishing the estimates in Propositions 4.1 and 4.2 below. Once we obtain those estimates, it is easy to see that (4.33) follows from them via (4.5), (4.17) and (1.6).

PROPOSITION 4.1. *Let $t - r > 0$ and let $\mu > 0$ and $\nu > 0$. Then we have*

$$I_2^+ \le C(1 + r + t)^{-1/2}(1 + t - r)^{-1/2}(1 + t - r)^{[1/2-\nu]_+}, \tag{4.34}$$

$$J_1 \le C(1 + r + t)^{-1/2}(1 + t - r)^{-1/2}(1 + t - r)^{[1/2-\nu]_+}. \tag{4.35}$$

PROOF. First we consider I_2^+. It follows from (4.13) and (4.22) that for $t - r > 0$

$$I_2^+ \le \frac{1}{2}\int_0^{t-r} (1+|\alpha|)^{-1/2-\nu}(t-r-\alpha)^{-1/2}\,d\alpha \int_{-\alpha}^{\infty} (1+|\beta|)^{-1-\mu}(t+r+\beta)^{-1/2}\,d\beta. \tag{4.36}$$

Using (3.8) as $a = t + r$, $b = -\alpha$, $\kappa = 1 + \mu$ and $\gamma = 1/2$, we have

$$(1 + r + t)^{1/2} I_2^+ \le C\int_{-\infty}^{t-r} (1 + |\alpha|)^{-1/2-\nu}(t - r - \alpha)^{-1/2}\,d\alpha.$$

Using (3.7) as $a = b = t - r$, $\kappa = (1/2) + \nu$ and $\gamma = 1/2$, we get (4.34).

Next we consider J_1. It follows from (4.15) and (4.23) that for $t - r > 0$

$$\begin{aligned} J_1 &\le \frac{1}{2}\int_0^{t-r} (1 + |\beta|)^{-1/2-\nu}(t+r+\beta)^{-1/2}\,d\beta \int_{-\beta}^{\beta} (1 + |\alpha|)^{-1-\mu}(t - r - \alpha)^{-1/2}\,d\alpha \\ &\le C\int_0^{t-r} (1 + |\beta|)^{-1/2-\nu}(t+r+\beta)^{-1/2}(t-r-\beta)^{-1/2}\,d\beta, \end{aligned} \tag{4.37}$$

because $\mu > 0$. When $0 \le r + t \le 1$, it is enough to show that J_1 is bounded. It follows from (4.37) that

$$J_1 \le C\int_0^{t-r} \beta^{-1/2}(t - r - \beta)^{-1/2}\,d\beta,$$

hence J_1 is bounded. When $r + t \ge 1$, it follows from (4.37) that

$$J_1 \le C(1 + t + r)^{-1/2}\int_{-\infty}^{t-r} (1 + |\beta|)^{-1/2-\nu}(t - r - \beta)^{-1/2}\,d\beta.$$

Using (3.7) as $a = b = t - r$, $\kappa = (1/2) + \nu$ and $\gamma = 1/2$, we get (4.35). The proof is complete. □

PROPOSITION 4.2. *Let $\mu > 0$ and $\nu > 0$. Then we have*

$$I_1^+ \le C(1 + r + |t|)^{-1/2}(1 + |r - t|)^{-\nu}, \tag{4.38}$$

$$I_1^- \le C(1 + r + |t|)^{-1/2}(1 + |r - t|)^{-\nu}, \tag{4.39}$$

$$J_2 \le C(1 + r + |t|)^{-1/2}(1 + |r - t|)^{-\nu}. \tag{4.40}$$

Proof. First we show (4.38). It follows from (4.11) and (4.18) that for $t > 0$

$$(4.41)\quad I_1^+ \le \frac{1}{2}\int_{|t-r|}^{\infty} (1+|\alpha|)^{-1/2-\nu}(\alpha+r-t)^{-1/2}\,d\alpha \int_{r-t}^{\infty} (1+|\beta|)^{-1-\mu}(t+r+\beta)^{-1/2}\,d\beta.$$

To deal with the α-integral, we use (3.6) as $a = b = r - t$, $\kappa = (1/2) + \nu$ and $\gamma = 1/2$. While, to handle the β-integral, we employ (3.8) as $a = t + r$, $b = r - t$, $\kappa = 1 + \mu$ and $\gamma = 1/2$. Then we obtain (4.38), because $t > 0$.

Next we shall show (4.39) and (4.40), by proving the following Lemmas 4.2 and 4.3. Indeed, for such (r,t) that $r + t \ge 4$ and $0 \le t \le 3r$, the desired estimates follow from Lemma 4.3. While for the other case, Lemma 4.2 yields them, because $1 + |r - t|$ is equivalent to $1 + r + |t|$ for the case.

Lemma 4.2. *Let $\mu > 0$ and $\nu > 0$. Then we have*

$$(4.42)\qquad I_1^- \le C(1+|r-t|)^{-(1/2)-\nu}, \quad J_2 \le C(1+|r-t|)^{-(1/2)-\nu}.$$

Proof. First we consider I_1^-. It follows from (4.12) and (4.20) that

$$\begin{aligned} I_1^- &\le \frac{1}{2}\int_{t-r}^{\infty} (1+|\alpha|)^{-1-\mu}(\alpha+r-t)^{-1/2}\,d\alpha \int_{|r-t|}^{\infty} (1+\beta)^{-1-\nu}\,d\beta \\ &\le \frac{1}{2\nu}(1+|r-t|)^{-\nu}\int_{t-r}^{\infty} (1+|\alpha|)^{-1-\mu}(\alpha+r-t)^{-1/2}\,d\alpha. \end{aligned}$$

Using (3.8) as $a = r - t$, $b = t - r$, $\kappa = 1 + \mu$ and $\gamma = 1/2$, we get (4.42) for I_1^-.

Next we consider J_2. It follows from (4.16) and (4.24) that

$$\begin{aligned} J_2 &\le \frac{1}{2}\int_{|r-t|}^{\infty} (1+\beta)^{-1-\nu}\,d\beta \int_{-\infty}^{t-r} (1+|\alpha|)^{-1-\mu}(t-r-\alpha)^{-1/2}\,d\alpha \\ &\le \frac{1}{2\nu}(1+|r-t|)^{-\nu}\int_{-\infty}^{t-r} (1+|\alpha|)^{-1-\mu}(t-r-\alpha)^{-1/2}\,d\alpha. \end{aligned}$$

Using (3.7) as $a = b = t - r$, $\kappa = 1 + \mu$ and $\gamma = 1/2$, we get (4.42) for J_2. The proof is complete. □

Lemma 4.3. *Let $\mu > 0$ and $\nu > 0$. If $r + t \ge 4$ and $0 \le t \le 3r$, then we have*

$$(4.43)\quad I_1^- \le C(1+r+t)^{-1/2}(1+|r-t|)^{-\nu}, \quad J_2 \le C(1+r+t)^{-1/2}(1+|r-t|)^{-\nu}.$$

Proof. First we consider J_2. We choose θ such that $0 < \theta < \min\{\nu, 1/2\}$. Then it follows from (4.16) and (4.25) that

$$J_2 \le \frac{C}{\sqrt{r}} \int_{|r-t|}^{\infty} (1+\beta)^{-1-\nu+\theta}\, d\beta \int_{-\infty}^{t-r} (1+|\alpha|)^{-1-\mu}(t-r-\alpha)^{-\theta}\, d\alpha$$

$$\le C(1+r+t)^{-1/2}(1+|r-t|)^{-\nu+\theta} \int_{-\infty}^{t-r} (1+|\alpha|)^{-1-\mu}(t-r-\alpha)^{-\theta}\, d\alpha,$$

because $4r \ge r+t \ge 4$. Using (3.7) as $a=b=t-r$, $\kappa = 1+\mu$ and $\gamma = \theta$, we get (4.43) for J_2.

Next we consider I_1^-. It follows from (4.12) and (4.21) that

$$I_1^- \le C \int_{t-r}^{t+r} (1+|\alpha|)^{-1-\mu}(t+r-\alpha)^{-1/2}(\alpha+r-t)^{-\theta}\, d\alpha$$

$$\times \int_{|r-t|}^{\infty} (1+\beta)^{-1/2-\nu+\theta}(\beta+t-r)^{-1/2}\, d\beta$$

$$\le C(1+|r-t|)^{-\nu+\theta} \int_{t-r}^{t+r} (1+|\alpha|)^{-1-\mu}(t+r-\alpha)^{-1/2}(\alpha+r-t)^{-\theta}\, d\alpha,$$

where we have taken θ to be $0<\theta<\min\{\nu, 1/2\}$ and we have used (3.6) as $a=b=t-r$, $\kappa = (1/2)+\nu-\theta$ and $\gamma = 1/2$.

Note that $(t+r)/2 \ge t-r$ for $t \le 3r$. Since $t+r-\alpha \ge (t+r)/2$ for $t-r \le \alpha \le (t+r)/2$ and $|\alpha| \ge (t+r)/2$ for $(t+r)/2 \le \alpha \le t+r$, we have

$$(1+|r-t|)^{\nu-\theta} I_1^- \le C(t+r)^{-1/2} \int_{t-r}^{\infty} (1+|\alpha|)^{-1-\mu}(\alpha+r-t)^{-\theta}\, d\alpha \tag{4.44}$$

$$+ (1+r+t)^{-1-\mu} \int_{t-r}^{t+r} (t+r-\alpha)^{-1/2}(\alpha+r-t)^{-\theta}\, d\alpha.$$

Using (3.8) as $a=r-t$, $b=t-r$, $\kappa = 1+\mu$ and $\gamma = \theta$, we get

$$\int_{t-r}^{\infty} (1+|\alpha|)^{-1-\mu}(\alpha+r-t)^{-\theta}\, d\alpha \le C(1+|r-t|)^{-\theta}. \tag{4.45}$$

Moreover, since $\theta < 1/2$, we have

$$\int_{t-r}^{t+r} (t+r-\alpha)^{-1/2}(\alpha+r-t)^{-\theta}\, d\alpha \le (2r)^{1/2-\theta} \int_{t-r}^{t+r} (t+r-\alpha)^{-1/2}(\alpha+r-t)^{-1/2}\, d\alpha$$

$$\le C(1+r+t)^{1/2-\theta}.$$

Therefore, we see that the right hand side of (4.44) is dominated by $C(1+r+t)^{-1/2} \cdot (1+|r-t|)^{-\theta}$, because $r+t \ge 4$. Hence, we get (4.43) for I_1^-. The proof is complete. $\square$

CASE 2: $\mathbf{n = 3}$. If we set

$$I(r,t) = \frac{1}{2r}\int_{-\infty}^{t} ds \int_{|t-s-r|}^{t-s+r} \frac{1}{z_{\nu,\mu}(\lambda,s)}\, d\lambda, \tag{4.46}$$

it follows from (1.3), (4.1) and (4.2) with $n = 3$ and $\rho = t - s$ that

$$|L(F)(x,t)| \le M_{\nu,\mu}(F) \times I(r,t). \tag{4.47}$$

Moreover, if we denote by $I^{\pm}$ the integrals over $\pm s \ge 0$ of I, respectively, then we have

$$I^{+}(r,t) = \frac{1}{4r}\chi(t)\int_{|t-r|}^{t+r} (1+\alpha)^{-1-\nu}\, d\alpha \int_{r-t}^{\alpha} (1+|\beta|)^{-1-\mu}\, d\beta, \tag{4.48}$$

$$I^{-}(r,t) = \frac{1}{4r}\int_{t-r}^{t+r} (1+|\alpha|)^{-1-\mu}\, d\alpha \int_{\alpha \vee |r-t|}^{\infty} (1+\beta)^{-1-\nu}\, d\beta, \tag{4.49}$$

where $\chi(t) = 1$ for $t > 0$, $\chi(t) = 0$ for $t \le 0$. Since

$$I(r,t) = I^{+}(r,t) + I^{-}(r,t), \tag{4.50}$$

it suffices to show

$$I^{\pm}(r,t) \le C(1+r+|t|)^{-1}(1+|r-t|)^{-\nu}. \tag{4.51}$$

To this end, we prepare the following:

LEMMA 4.4. *Let $\kappa > 1$. For $r > 0$ and $t > 0$, we have*

$$\frac{1}{r}\int_{|t-r|}^{t+r} (1+\sigma)^{-\kappa}\, d\sigma \le C(1+r+t)^{-1}(1+|r-t|)^{-\kappa+1}. \tag{4.52}$$

Moreover, for $r > 0$ and $t \in \boldsymbol{R}$, we have

$$\frac{1}{r}\int_{t-r}^{t+r} (1+|\sigma|)^{-\kappa}\, d\sigma \le C(1+r+|t|)^{-1}. \tag{4.53}$$

PROOF. We shall show only (4.53), because (4.52) was handled in a similar fashion. When $r + |t| \ge 1$ and $|t| \le 2r$, (4.53) follows from the fact that $r^{-1} \le C(1+r+|t|)^{-1}$, because $\kappa > 1$. In the other case, we have $(1+|\sigma|)^{-\kappa} \le C(1+r+|t|)^{-\kappa}$ for $t - r \le \sigma \le t + r$, hence we get (4.53). This completes the proof. □

Now we estimate $I^{\pm}$. Since $\mu > 0$, we have

$$I^{+}(r,t) \le \frac{1}{2\mu r}\chi(t)\int_{|t-r|}^{t+r} (1+\alpha)^{-1-\nu}\, d\alpha, \tag{4.54}$$

which yields (4.51) for I^+ by (4.52) with $\kappa = 1 + \nu$. Moreover, since $\nu > 0$, we have

$$(1 + |r - t|)^{\nu} I^-(r, t) \le \frac{C}{r} \int_{t-r}^{t+r} (1 + |\alpha|)^{-1-\mu} \, d\alpha, \tag{4.55}$$

from which we get (4.51) for I^- by (4.53). This completes the proof of Theorem 1.1.

5. An application.

As we have mentioned in Section 1, we shall consider a system of semilinear wave equations as an application of Theorem 1.1:

$$\partial_t^2 u - \Delta u = F(v) \quad \text{in } \boldsymbol{R}^n \times \boldsymbol{R}, \tag{5.1}$$

$$\partial_t^2 v - \Delta v = G(u) \quad \text{in } \boldsymbol{R}^n \times \boldsymbol{R}. \tag{5.2}$$

Suppose that $F \in C^2(\boldsymbol{R})$ and $G \in C^2(\boldsymbol{R})$ satisfy

$$F(0) = F'(0) = F''(0) = 0, \quad G(0) = G'(0) = G''(0) = 0, \tag{5.3}$$

and that there are $p > 2$, $q > 2$ and $A > 0$ such that for $|u_i| \le 1$, $|v_i| \le 1$ $(i = 1, 2)$

$$|F''(v_1) - F''(v_2)| \le \begin{cases} Ap(p-1)|v_1 - v_2|^{p-2} & \text{if } 2 < p \le 3, \\ Ap(p-1)|v_1 - v_2|(|v_1| + |v_2|)^{p-3} & \text{if } p > 3, \end{cases} \tag{5.4}$$

$$|G''(u_1) - G''(u_2)| \le \begin{cases} Aq(q-1)|u_1 - u_2|^{q-2} & \text{if } 2 < q \le 3, \\ Aq(q-1)|u_1 - u_2|(|u_1| + |u_2|)^{q-3} & \text{if } q > 3. \end{cases} \tag{5.5}$$

Remark. Typical examples of F and G are

$$F(v) = |v|^{p-1} v \quad \text{or} \quad F(v) = |v|^p, \tag{5.6}$$

$$G(u) = |u|^{q-1} u \quad \text{or} \quad G(u) = |u|^q. \tag{5.7}$$

Before stating the main result, we prepare the following lemma:

Lemma 5.1. *Assume that* (1.9) *and* (1.10) *hold.*
(i) *If* ν *satisfies*

$$0 < \nu \le p^*, \tag{5.8}$$

$$\nu > p^* - \frac{\Gamma}{pq - 1} = \frac{1 - p^* + p(1 - q^*)}{pq - 1}, \tag{5.9}$$

then there is a number κ verifying

$$0 < \kappa \leq q^*, \tag{5.10}$$

$$\kappa > \frac{1}{p} - \frac{p^* - v}{p} \tag{5.11}$$

and

$$\kappa < q^* - 1 + qv = \frac{1}{p} + q\left(v - p^* + \frac{\Gamma}{pq}\right). \tag{5.12}$$

(ii) *Let v satisfy* (5.8) *and* (5.9). *Furthermore, if $\Gamma > (pq-1)/2$, we assume*

$$v > p^* - \frac{\Gamma}{pq} - \frac{1}{2pq} = \frac{1}{q}\left(1 + \frac{1}{2p} - q^*\right). \tag{5.13}$$

Then there is a number κ verifying (5.10), (5.11), (5.12) *and*

$$\kappa > \frac{1}{2p}. \tag{5.14}$$

(iii) *If v satisfies* (5.8) *and*

$$v > p^* - \frac{\Gamma}{pq} = \frac{1}{q} - \frac{\alpha}{pq} = \frac{1}{q}\left(1 + \frac{1}{p} - q^*\right), \tag{5.15}$$

then there is a number κ verifying (5.10), (5.12) *and*

$$\kappa > \frac{1}{p}. \tag{5.16}$$

Remark. Note that the conditions (5.9), (5.13) and (5.15) are meaningful only when the right hand sides are positive, by (5.8). Moreover, it is easy to see that (5.15) implies (5.9) and (5.13).

Proof of Lemma 5.1. Firstly, we prove the statement (iii). It suffices to check

$$\frac{1}{p} < q^*, \quad \frac{1}{p} < \frac{1}{p} + q\left(v - p^* + \frac{\Gamma}{pq}\right). \tag{5.17}$$

Notice that (1.9) and (1.21) yield $\alpha = pq^* - 1 > 0$, hence the first inequality in (5.17) holds. While the other follows from (5.15) immediately.

Secondly, we prove the statement (i), by dividing the argument into two cases.

Case 1. $1/q < \nu \le p^*$.
In this case, we have

$$q^* < q^* - 1 + q\nu.$$

Since $q^* > 0$, it is enough to show

$$\frac{1}{p} - \frac{p^* - \nu}{p} < q^*. \tag{5.18}$$

Since

$$q^* - \left(\frac{1}{p} - \frac{p^* - \nu}{p}\right) = \frac{1}{p}(\alpha + p^* - \nu),$$

we get (5.18), by $\alpha > 0$ and (5.8).

Case 2. $0 < \nu \le 1/q$.
In this case, we have

$$q^* - 1 + q\nu \le q^*. \tag{5.19}$$

Moreover, it holds that

$$\frac{1}{p} - \frac{p^* - \nu}{p} < \frac{1}{p} + q\left(\nu - p^* + \frac{\Gamma}{pq}\right). \tag{5.20}$$

Indeed, this is equivalent to

$$p^* - \nu < \frac{\Gamma}{pq - 1},$$

which follows from (5.9). Namely, we obtain by the equality of (5.12)

$$\frac{1}{p} - \frac{p^* - \nu}{p} < q^* - 1 + q\nu. \tag{5.21}$$

Note that

$$q^* - 1 + q\nu > 0 \quad \text{if } q^* > 1, \tag{5.22}$$

and that

$$\frac{1}{p} - \frac{p^* - \nu}{p} > 0 \quad \text{if } q^* \le 1, \tag{5.23}$$

because $p^* \le q^*$ and $\nu > 0$. Therefore, the statement (i) follows from (5.19), (5.21), (5.22) and (5.23).

Finally, we prove the statement (ii). Since we have assumed (5.8) and (5.9), we see from the statement (i) that there is a number κ verifying (5.10), (5.11) and (5.12). When either $\Gamma \le (pq-1)/2$ or $\Gamma > (pq-1)/2$ and $\nu \ge p^* - 1/2$, we have (5.14) by (5.11). Next we consider the case where $\Gamma > (pq-1)/2$ and $\nu < p^* - 1/2$. Then we have

$$\frac{1}{p} - \frac{p^* - \nu}{p} < \frac{1}{2p},$$

hence it suffices to show

$$\frac{1}{2p} < q^*, \quad \frac{1}{2p} < \frac{1}{p} + q\left(\nu - p^* + \frac{\Gamma}{pq}\right). \tag{5.24}$$

Since $\alpha > 0$, we have $q^* > 1/p$. Moreover, the latter inequality follows from (5.13). This completes the proof of Lemma 5.1. $\square$

THEOREM 5.1. *Let $n = 2$ or $n = 3$. Suppose that* (1.9), (1.10), (1.18), (1.19) *and* (5.8) *hold.*

(A) *Let ν and κ satisfy* (5.9) *through* (5.12). *Then there is a positive number $\varepsilon_0 = \varepsilon_0(\nu, \kappa, p, q, A)$ such that for any ε with $0 < \varepsilon \le \varepsilon_0$, there exists uniquely a classical solution $(u, v) \in X_{\nu,\kappa}(2C_0\varepsilon)$ of* (5.1) *and* (5.2) *verifying*

$$|\partial_x^\alpha(u(x,t) - u^-(x,t))| \le C[v]_\kappa^p (1 + r + |t|)^{-(n-1)/2} \Phi_n(r, t; \nu), \tag{5.25}$$

$$|\partial_x^\alpha(v(x,t) - v^-(x,t))| \le C[u]_\nu^q (1 + r + |t|)^{-(n-1)/2} \Phi_n(r, t; \kappa) \tag{5.26}$$

for any $(x,t) \in \boldsymbol{R}^n \times \boldsymbol{R}$ and $|\alpha| \le 2$, and

$$|||(u - u^-)(t)|||_e \le C\|v\|_\kappa^p (1 + |t|)^{-p^*} \{(1 + |t|)^{[1-2p\kappa]_+}\}^{1/2} \quad \text{for } t \le 0, \tag{5.27}$$

$$|||(v - v^-)(t)|||_e \le C\|u\|_\nu^q (1 + |t|)^{-q^*} \{(1 + |t|)^{[1-2q\nu]_+}\}^{1/2} \quad \text{for } t \le 0, \tag{5.28}$$

where u^- and v^- are the solutions to the homogeneous wave equation satisfying (1.16) *and* (1.17). *In addition, $[\cdot]_\nu$ and $|||\cdot|||_e$ are defined by* (2.4) *and* (2.5), *respectively.*

(B) *Let ν and κ satisfy* (5.15), (5.10), (5.12) *and* (5.16). *Then we have*

$$|\partial_x^\alpha(u(x,t) - u^-(x,t))| \le C[v]_\kappa^p (1 + r + |t|)^{-(n-1)/2} \Phi_n(r, t; p^*) \tag{5.29}$$

for any $(x,t) \in \boldsymbol{R}^n \times \boldsymbol{R}$ and $|\alpha| \le 2$, and

$$|||(u - u^-)(t)|||_e \le C\|v\|_\kappa^p (1 + |t|)^{-p^*} \quad \text{for } t \le 0. \tag{5.30}$$

(C) *Suppose*

$$qp^* > 1, \quad i.e., \quad \beta > 0. \tag{5.31}$$

Let ν *and* κ *satisfy*

$$1/q < \nu \le p^*, \quad 1/p < \kappa \le q^*. \tag{5.32}$$

Then we have (5.29), (5.30),

$$|\partial_x^\alpha (v(x,t) - v^-(x,t))| \le C[u]_\nu^q (1 + r + |t|)^{-(n-1)/2} \Phi_n(r,t;q^*) \tag{5.33}$$

for any $(x,t) \in \boldsymbol{R}^n \times \boldsymbol{R}$ *and* $|\alpha| \le 2$, *and*

$$|||(v - v^-)(t)|||_e \le C\|u\|_\nu^q (1 + |t|)^{-q^*} \quad \textit{for } t \le 0. \tag{5.34}$$

Here C *is a constant depending only on* ν, κ, p, q *and* A.

Remark. The existence of such ν and κ as in the part (A) and (B) in Theorem 5.1 is guaranteed by Lemma 5.1. In particular, if we take ν as $\nu = p^*$, we can find κ satisfying (5.10) and

$$\frac{1}{p} < \kappa < q^* - 1 + qp^* = \frac{1}{p} + \frac{\Gamma}{p}, \tag{5.35}$$

by Lemma 5.1. Hence, we have (5.29) and (5.30).

Note that if ν and κ satisfy (5.32), then (5.15), (5.10), (5.12) and (5.16) hold. Also remark that if ν and κ satisfy (5.15), (5.10), (5.12) and (5.16), then (5.9) through (5.12) hold.

If (5.31) holds, we can find ν such that $1/q < \nu \le p^*$. Moreover, since $\nu > 1/q$ imply (5.15), there is a number κ satisfying $1/p < \kappa \le q^*$. Furthermore, the first part of Theorem 1.2 follows from the parts (A) and (C) in Theorem 5.1.

In both (5.27) and (5.28), the right hand sides tend to zero as $t \to -\infty$. More precisely, we have the following.

Corollary 5.1. *Let* $n = 2$ *or* $n = 3$. *Suppose that* (1.9), (1.10), (1.18) *and* (1.19) *hold. Let* ν *and* κ *satisfy* (5.8) *through* (5.12).

If $2p\kappa > 1$, (5.30) *holds. While, if* $2p\kappa < 1$, *we have*

$$|||(u - u^-)(t)|||_e \le C\|v\|_\kappa^p (1 + |t|)^{-1/2-\nu} \quad \textit{for } t \le 0. \tag{5.36}$$

Moreover, if $2q\nu > 1$, (5.34) *holds. While, if* $2q\nu < 1$, *we have*

$$|||(v - v^-)(t)|||_e \le C\|u\|_\nu^q (1 + |t|)^{-1/2-\kappa} \quad \textit{for } t \le 0. \tag{5.37}$$

Proof. When $2p\kappa > 1$, (5.30) immediately follows from (5.27). Moreover, when $2p\kappa < 1$, we have $p^* - ([1-2p\kappa]_+)/2 > 1/2 + \nu$ by (5.11). Therefore, (5.36) follows from (5.27).

Furthermore, when $2q\nu > 1$, (5.34) follows from (5.28). On the other hand, $2q\nu < 1$ and (5.12) yield $q^* - ([1-2q\nu]_+)/2 > 1/2 + \kappa$, hence (5.37) follows from (5.28). This completes the proof. □

Remark. When $2p\kappa < 1$, we must have $p^* > 1/2 + \nu > 1/2$, by (5.11). When $2q\nu < 1$, we must have $q^* > 1/2 + \kappa > 1/2$, by (5.12).

We can find κ satisfying $2p\kappa > 1$ and (5.10) through (5.12), when ν satisfy (5.8), (5.9), and (5.13) if $\Gamma > (pq-1)/2$, by Lemma 5.1.

If we assume

$$2qp^* > 1, \quad i.e., \quad 2\beta + 1 > 0, \tag{5.38}$$

we can choose ν verifying $2q\nu > 1$, (5.8), (5.9) and (5.13). On the other hand, if (5.38) does not hold, we have $2q\nu \le 1$, because $\nu \le p^*$.

The following corollary follows from Theorem 5.1 and Corollary 5.1, by setting $p = q$.

Corollary 5.2. *Let $n = 2$ or $n = 3$ and let $p = q$. Suppose that $pp^* > 1$,* (5.8), (1.18) *and* (1.19) *hold.*

(A) *If ν and κ satisfy* (5.10),

$$\nu > p^* - \frac{pp^* - 1}{p-1} = \frac{1-p^*}{p-1}, \tag{5.39}$$

and

$$\frac{1}{p} - \frac{p^* - \nu}{p} < \kappa < p^* - 1 + p\nu, \tag{5.40}$$

then (5.25) *through* (5.28) *with $p = q$ hold.*

(B) *If ν and κ satisfy* (5.10),

$$\nu > p^* - \frac{(p+1)(pp^*-1)}{p^2} = \frac{1}{p} - \frac{pp^*-1}{p^2} = \frac{1}{p}\left(1 + \frac{1}{p} - p^*\right), \tag{5.41}$$

and

$$\frac{1}{p} < \kappa < p^* - 1 + p\nu, \tag{5.42}$$

then (5.29) *and* (5.30) *hold.*

(C) *Let ν and κ satisfy* (5.32). *Then* (5.29), (5.30), (5.33) *and* (5.34) *with $p = q$ hold.*

(D) *Let* $\nu = \kappa$ *and* ν *satisfy* (5.39). *If* $2p\nu > 1$, *we have* (5.30) *and* (5.34) *with* $p = q$. *While, if* $2p\nu < 1$, *we have* (5.36) *and* (5.37) *with* $p = q$.

Remark. If we take $\kappa = \nu$, then (5.40) is equivalent to (5.39).

Here we would like to compare Corollary 5.2 with the previous results in [**20**], [**26**] and [**36**], which concern with the asymptotic behavior of the solution to

$$\partial_t^2 u - \Delta u = F(u) \quad \text{in } \boldsymbol{R}^n \times \boldsymbol{R}, \tag{5.43}$$

such that

$$|||u(t) - u_0(t)|||_e \to 0 \quad \text{as } t \to -\infty, \tag{5.44}$$

where u_0 is the solutions to the homogeneous wave equation satisfying

$$u_0(x, 0) = f(x), \quad \partial_t u_0(x, 0) = g(x) \quad \text{in } \boldsymbol{R}^n. \tag{5.45}$$

Since $u = v$ and $u^- = v^-$ if we choose $f_1 = f_2 = f$ and $g_1 = g_2 = g$, Corollary 5.2 is a natural extension of those previous works in the sense that the parameters ν and κ are taken as $\nu = \kappa = p^*$ in [**26**] and [**36**], while in [**20**], it is assumed that $p\nu \neq 1$.

At the end of this section, we state asymptotic behavior of the solution (u, v) obtained by Theorem 5.1 as $t \to +\infty$.

Theorem 5.2. *Let* $n = 2$ *or* $n = 3$. *Suppose that* (1.9), (1.10), (1.18), (1.19) *and* (5.8) *hold. Let* u^- *and* v^- *be the solutions to the homogeneous wave equation satisfying* (1.16) *and* (1.17). *For a solution* $(u, v) \in X_{\nu,\kappa}$ *of* (5.1) *and* (5.2) *verifying* (5.25) *through* (5.28), *we define*

$$u^+(x, t) = u(x, t) - L_1(F(v))(x, t) \quad \text{in } \boldsymbol{R}^n \times \boldsymbol{R}, \tag{5.46}$$

$$v^+(x, t) = v(x, t) - L_1(G(u))(x, t) \quad \text{in } \boldsymbol{R}^n \times \boldsymbol{R}, \tag{5.47}$$

where we have set

$$L_1(F)(x, t) = \frac{1}{2\pi} \int_t^\infty ds \int_0^{s-t} \frac{\rho}{\sqrt{(s - t)^2 - \rho^2}} \, d\rho \int_{|\omega|=1} F(x + \rho\omega, s) \, dS_\omega \tag{5.48}$$

for $(x, t) \in \boldsymbol{R}^2 \times \boldsymbol{R}$ *and*

$$L_1(F)(x, t) = \frac{1}{4\pi} \int_t^\infty (s - t) \, ds \int_{|\omega|=1} F(x + (s - t)\omega, s) \, dS_\omega \tag{5.49}$$

for $(x, t) \in \boldsymbol{R}^3 \times \boldsymbol{R}$.

(A) *Let* ν *and* κ *satisfy* (5.9) *through* (5.12). *Then* u^+ *and* v^+ *are classical solutions to the homogeneous wave equation satisfying*

$$|\partial_x^\alpha(u(x,t)-u^+(x,t))| \le C[v]_\kappa^p(1+r+|t|)^{-(n-1)/2}\Phi_n(r,-t;\nu), \tag{5.50}$$

$$|\partial_x^\alpha(v(x,t)-v^+(x,t))| \le C[u]_\nu^q(1+r+|t|)^{-(n-1)/2}\Phi_n(r,-t;\kappa) \tag{5.51}$$

for any $(x,t)\in \boldsymbol{R}^n\times\boldsymbol{R}$ *and* $|\alpha|\le 2$, *and*

$$\|\!|(u-u^+)(t)\|\!|_e \le C\|v\|_\kappa^p(1+|t|)^{-p^*}\{(1+|t|)^{[1-2p\kappa]_+}\}^{1/2} \quad \textit{for}\ t\ge 0, \tag{5.52}$$

$$\|\!|(v-v^+)(t)\|\!|_e \le C\|u\|_\nu^q(1+|t|)^{-q^*}\{(1+|t|)^{[1-2q\nu]_+}\}^{1/2} \quad \textit{for}\ t\ge 0. \tag{5.53}$$

(B) *Let* ν *and* κ *satisfy* (5.15), (5.10), (5.12) *and* (5.16). *Then we have*

$$|\partial_x^\alpha(u(x,t)-u^+(x,t))| \le C[v]_\kappa^p(1+r+|t|)^{-(n-1)/2}\Phi_n(r,-t;p^*) \tag{5.54}$$

for any $(x,t)\in \boldsymbol{R}^n\times\boldsymbol{R}$ *and* $|\alpha|\le 2$, *and*

$$\|\!|(u-u^+)(t)\|\!|_e \le C\|v\|_\kappa^p(1+|t|)^{-p^*} \quad \textit{for}\ t\ge 0. \tag{5.55}$$

(C) *Suppose that* (5.31) *holds and that* ν *and* κ *satisfy* (5.32). *Then we have* (5.54), (5.55),

$$|\partial_x^\alpha(v(x,t)-v^+(x,t))| \le C[u]_\nu^q(1+r+|t|)^{-(n-1)/2}\Phi_n(r,-t;q^*) \tag{5.56}$$

for any $(x,t)\in \boldsymbol{R}^n\times\boldsymbol{R}$ *and* $|\alpha|\le 2$, *and*

$$\|\!|(v-v^+)(t)\|\!|_e \le C\|u\|_\nu^q(1+|t|)^{-q^*} \quad \textit{for}\ t\ge 0. \tag{5.57}$$

6. A priori estimate.

The aim of this section is to prove a priori estimates, which are needed to show Theorems 5.1 and 5.2. The following lemma is a consequence of Theorem 1.1.

Lemma 6.1. *Assume that* (1.9) *and* (1.10) *hold. Let* ν *and* κ *satisfy* (5.8) *through* (5.12). *Let* $(u,v)\in X_{\nu,\kappa}$. *Then we have*

$$\|L(|v|^p)\|_\nu \le C_1\|v\|_\kappa^p, \quad \|L(|u|^q)\|_\kappa \le C_1\|u\|_\nu^q, \tag{6.1}$$

where C_1 *is a constant depending only on* p, q, ν *and* κ.

Proof. It follows from (2.7) and (1.4) with $F=|v|^p$ and $F=|u|^q$ that for $\mu>0$

$$|L(|v|^p)(x,t)|(1+r+|t|)^{(n-1)/2}/\Phi_n(r,t;\nu) \le CM_{\nu,\mu}(|v|^p), \tag{6.2}$$

$$|L(|u|^q)(x,t)|(1+r+|t|)^{(n-1)/2}/\Phi_n(r,t;\kappa) \le CM_{\kappa,\mu}(|u|^q). \tag{6.3}$$

Therefore, (6.1) will be established, if we can find a positive number μ such that

$$M_{\nu,\mu}(|v|^p) \le C\|v\|_\kappa^p, \quad M_{\kappa,\mu}(|u|^q) \le C\|u\|_\nu^q, \tag{6.4}$$

because $\Phi_n(r,t;\nu) \le \Phi_n(r,|t|;\nu)$. It follows from (1.11) and (1.13) that

$$\lambda^{(n-1)/2}(1+\lambda+|s|)^{1+\nu}(1+|\lambda-|s||)^{1+\mu}|v(y,s)|^p \le \|v\|_\kappa^p \frac{(1+|\lambda-|s||)^{1+\mu-p\kappa}}{(1+\lambda+|s|)^{p^*-\nu}}$$

and

$$\lambda^{(n-1)/2}(1+\lambda+|s|)^{1+\kappa}(1+|\lambda-|s||)^{1+\mu}|u(y,s)|^q \le \|u\|_\nu^q \frac{(1+|\lambda-|s||)^{1+\mu-q\nu}}{(1+\lambda+|s|)^{q^*-\kappa}},$$

unless $n=2$ and either $\nu \ge 1/2$ or $\kappa \ge 1/2$. By (5.11) and (5.12), we can choose a positive number μ such that

$$\mu < p\kappa - 1 + p^* - \nu, \quad \mu < q\nu - 1 + q^* - \kappa. \tag{6.5}$$

Hence, by (5.8) and (5.10) together with (2.7), we obtain (6.4) with $C=1$. When $n=2$ and either $\nu \ge 1/2$ or $\kappa \ge 1/2$, we get (6.4) less hard. Indeed, if $n=2$, we have $3 < p \le q$ by (1.10), hence $p/2 > 3/2$. This completes the proof. □

Next we introduce a function space $Y_{\nu,\kappa}$ defined by

$$Y_{\nu,\kappa} = \{(u,v) \in C^2(\boldsymbol{R}^n \times \boldsymbol{R}) \times C^2(\boldsymbol{R}^n \times \boldsymbol{R}) : [u]_\nu + [v]_\kappa < +\infty\}, \tag{6.6}$$

where $[\cdot]_\nu$ is defined by (2.4). In what follows, by C we denote the various constants depending only on A, p, q, ν and κ. On the basis of Lemma 6.1, we prove the following.

Proposition 6.1. *Let ν and κ satisfy* (5.8) *through* (5.12). *Assume* (5.3) *through* (5.5) *hold.*

(A) *Let $(u,v) \in X_{\nu,\kappa}(1) \cap Y_{\nu,\kappa}$. Then we have*

$$\|L(F(v))\|_\nu \le AC_1\|v\|_\kappa^p, \quad \|L(G(u))\|_\kappa \le AC_1\|u\|_\nu^q, \tag{6.7}$$

$$\|\partial_x L(F(v))\|_\nu \le ApC_1\|v\|_\kappa^{p-1}\|\partial_x v\|_\kappa, \quad \|\partial_x L(G(u))\|_\kappa \le AqC_1\|u\|_\nu^{q-1}\|\partial_x u\|_\nu, \tag{6.8}$$

where $\|\partial_x u\|_\nu = \sum_{|\alpha|=1}\|\partial_x^\alpha u\|_\nu$ and C_1 is the constant in (6.1). *Moreover, we get*

$$\|\partial_x^2 L(F(v))\|_\nu \le ApC_1\|v\|_\kappa^{p-1}\|\partial_x^2 v\|_\kappa + C\|v\|_\kappa^{p-2}\|\partial_x v\|_\kappa^2, \tag{6.9}$$

$$\|\partial_x^2 L(G(u))\|_\kappa \le AqC_1\|u\|_\nu^{q-1}\|\partial_x^2 u\|_\nu + C\|u\|_\nu^{q-2}\|\partial_x u\|_\nu^2, \tag{6.10}$$

where $\|\partial_x^2 u\|_\nu = \sum_{|\alpha|=2}\|\partial_x^\alpha u\|_\nu$.

(B) *Let* $(u_j, v_j) \in X_{\nu,\kappa}(1) \cap Y_{\nu,\kappa}$ $(j = 1, 2)$. *Then we have*

$$\|L(F(v_1)) - L(F(v_2))\|_\nu \le ApC_1(\|v_1\|_\kappa + \|v_2\|_\kappa)^{p-1}\|v_1 - v_2\|_\kappa, \tag{6.11}$$

$$\|L(G(u_1)) - L(G(u_2))\|_\kappa \le AqC_1(\|u_1\|_\nu + \|u_2\|_\nu)^{q-1}\|u_1 - u_2\|_\nu, \tag{6.12}$$

$$\begin{aligned}\|\partial_x\{L(F(v_2)) - L(F(v_1))\}\|_\nu &\le ApC_1(\|v_1\|_\kappa + \|v_2\|_\kappa)^{p-1}\|\partial_x(v_1 - v_2)\|_\kappa \\ &\quad + C([v_1]_\kappa + [v_2]_\kappa)^{p-1}\|v_1 - v_2\|_\kappa,\end{aligned} \tag{6.13}$$

$$\begin{aligned}\|\partial_x\{L(G(u_2)) - L(G(u_1))\}\|_\kappa &\le AqC_1(\|u_1\|_\nu + \|u_2\|_\nu)^{q-1}\|\partial_x(u_1 - u_2)\|_\nu \\ &\quad + C([u_1]_\nu + [u_2]_\nu)^{q-1}\|u_1 - u_2\|_\nu,\end{aligned} \tag{6.14}$$

$$\begin{aligned}\|\partial_x^2\{L(F(v_2)) - L(F(v_1))\}\|_\nu &\le ApC_1(\|v_1\|_\kappa + \|v_2\|_\kappa)^{p-1}\|\partial_x^2(v_1 - v_2)\|_\kappa \\ &\quad + C_2([v_1]_\kappa + [v_2]_\kappa)^2\|v_1 - v_2\|_\kappa^{p-2} \\ &\quad + C([v_1]_\kappa + [v_2]_\kappa)^{p-1}\sum_{|\alpha|\le 1}\|\partial_x^\alpha(v_1 - v_2)\|_\kappa,\end{aligned} \tag{6.15}$$

$$\begin{aligned}\|\partial_x^2\{L(G(u_2)) - L(G(u_1))\}\|_\kappa &\le AqC_1(\|u_1\|_\nu + \|u_2\|_\nu)^{q-1}\|\partial_x^2(u_1 - u_2)\|_\nu \\ &\quad + C_3([u_1]_\nu + [u_2]_\nu)^2\|u_1 - u_2\|_\nu^{q-2} \\ &\quad + C([u_1]_\nu + [u_2]_\nu)^{q-1}\sum_{|\alpha|\le 1}\|\partial_x^\alpha(u_1 - u_2)\|_\nu,\end{aligned} \tag{6.16}$$

where C_2 *and* C_3 *are constants depending only on* A, p, q, ν *and* κ *such that* $C_2 = 0$ *if* $p > 3$, *and* $C_3 = 0$ *if* $q > 3$.

Proof. It is easy to see from (5.3), (5.4) and (5.5) that for $|u| \le 1$ and $|v| \le 1$

$$|F(v)| \le A|v|^p, \quad |G(u)| \le A|u|^q, \tag{6.17}$$

$$|F'(v)| \le Ap|v|^{p-1}, \quad |G'(u)| \le Aq|u|^{q-1}, \tag{6.18}$$

$$|F''(v)| \le Ap(p-1)|v|^{p-2}, \quad |G''(u)| \le Aq(q-1)|u|^{q-2}. \tag{6.19}$$

Note that we have

$$\|\,|w_1|^{\theta_1}|w_2|^{\theta_2}|w_3|^{\theta_3}\|_\nu \le \|w_1\|_\nu^{\theta_1}\|w_2\|_\nu^{\theta_2}\|w_3\|_\nu^{\theta_3}$$

for $w_i \in C(\boldsymbol{R}^n \times \boldsymbol{R})$ and $\theta_i \in [0, 1]$ with $\theta_1 + \theta_2 + \theta_3 = 1$. Therefore, we get from (6.1)

$$\|L(|w_1|^{\theta_1 p}|w_2|^{\theta_2 p}|w_3|^{\theta_3 p})\|_\nu \le C_1\|w_1\|_\kappa^{\theta_1 p}\|w_2\|_\kappa^{\theta_2 p}\|w_3\|_\kappa^{\theta_3 p}. \tag{6.20}$$

Since $\partial_{x_j} L(F(v)) = L(\partial_{x_j} F(v))$ $(1 \le j \le n)$ and

$$|u(x,t)| \le \|u\|_\nu, \quad |v(x,t)| \le \|v\|_\kappa \quad \text{for } (x,t) \in \boldsymbol{R}^n \times \boldsymbol{R}, \tag{6.21}$$

the statements of the part (A) follow from (6.17) through (6.20).

Next we prove (6.15). When $2 < p \le 3$, we have from (6.18), (6.19), (5.4) and (6.20)

$$\begin{aligned}\|\partial_j\partial_k\{L(F(v_2)) - L(F(v_1))\}\|_\nu \le ApC_1\|v_1\|_\kappa^{p-1}\|\partial_j\partial_k(v_1 - v_2)\|_\kappa \\ + Ap(p-1)C_1(\|v_1\|_\kappa + \|v_2\|_\kappa)^{p-2}\|v_1 - v_2\|_\kappa\|\partial_j\partial_k v_2\|_\kappa \\ + Ap(p-1)C_1\|v_2\|_\kappa^{p-2}\|\partial_j(v_1 - v_2)\|_\kappa\|\partial_k v_1\|_\kappa \\ + Ap(p-1)C_1\|v_2\|_\kappa^{p-2}\|\partial_k(v_1 - v_2)\|_\kappa\|\partial_j v_2\|_\kappa \\ + Ap(p-1)C_1\|v_1 - v_2\|_\kappa^{p-2}\|\partial_j v_1\|_\kappa\|\partial_k v_1\|_\kappa,\end{aligned}$$

which yields (6.15). When $p > 3$, we get (6.15) with $C_2 = 0$, similarly. Since the treatment of others are less hard, we omit the further details. (See also [**12**]). □

7. Proof of Theorems 5.1 and 5.2.

Firstly we show the part (A) in Theorem 5.1. The classical solution (u, v) of (5.1) and (5.2) verifying (5.25) through (5.28) is furnished by a solution of the following system of integral equations:

$$u(x,t) = u^-(x,t) + L(F(v))(x,t) \quad \text{in } \boldsymbol{R}^n \times \boldsymbol{R}, \tag{7.1}$$

$$v(x,t) = v^-(x,t) + L(G(u))(x,t) \quad \text{in } \boldsymbol{R}^n \times \boldsymbol{R}, \tag{7.2}$$

where u^- and v^- are the solutions to the homogeneous wave equation satisfying (1.16) and (1.17), and $L(F(v))(x,t)$ and $L(G(u))(x,t)$ are given by (1.2) and (1.3) with F replaced by $F(v)$ and $G(u)$. To establish this fact, we introduce a sequence $\{(u_m, v_m)\}_{m=0}^\infty$ defined by $u_0 = u^-$, $v_0 = v^-$ and for $m \ge 0$

$$u_{m+1}(x,t) = u_0(x,t) + L(F(v_m))(x,t) \quad \text{in } \boldsymbol{R}^n \times \boldsymbol{R}, \tag{7.3}$$

$$v_{m+1}(x,t) = v_0(x,t) + L(G(u_m))(x,t) \quad \text{in } \boldsymbol{R}^n \times \boldsymbol{R}. \tag{7.4}$$

By (1.20) and (2.4), we have

$$\|u_0\|_\nu + \|v_0\|_\kappa \le C_0\varepsilon, \quad [u_0]_\nu + [v_0]_\kappa \le C, \tag{7.5}$$

provided (5.8) and (5.10) hold. As for $\varepsilon > 0$, we assume that

$$2C_0\varepsilon \leq 1, \tag{7.6}$$

$$ApC_1(4C_0\varepsilon)^{p-1} \leq \frac{1}{2}, \quad AqC_1(4C_0\varepsilon)^{q-1} \leq \frac{1}{2}, \tag{7.7}$$

where C_1 is the constant in Lemma 6.1.

Lemma 7.1. *We assume that* (1.9), (1.10), (1.18), (1.19), (7.6) *and* (7.7) *hold. Then we have for nonnegative integers m*

$$\|u_m\|_\nu + \|v_m\|_\kappa \leq 2(\|u_0\|_\nu + \|v_0\|_\kappa) \leq 1, \tag{7.8}$$

$$\|u_{m+1} - u_m\|_\nu + \|v_{m+1} - v_m\|_\kappa \leq C_4 2^{-m}, \tag{7.9}$$

$$[u_m]_\nu + [v_m]_\kappa \leq C, \tag{7.10}$$

$$\|\partial_x(u_{m+1} - u_m)\|_\nu + \|\partial_x(v_{m+1} - v_m)\|_\kappa \leq C_5 2^{-m} + CC_4 m 2^{-m}, \tag{7.11}$$

and

$$\begin{aligned} \|\partial_x^2(u_{m+1} - u_m)\|_\nu + \|\partial_x^2(v_{m+1} - v_m)\|_\kappa &\leq C_6 2^{-m} + Cm(C_5 2^{-m} + C_4 m 2^{-m}) \\ &\quad + CC_7 m(C_2 2^{-(p-2)m} + C_3 2^{-(q-2)m}), \end{aligned} \tag{7.12}$$

where C_2 and C_3 are the constants in (6.15) *and* (6.16) *respectively, and we have set*

$$C_4 = \|u_1 - u_0\|_\nu + \|v_1 - v_0\|_\kappa, \quad C_5 = \|\partial_x(u_1 - u_0)\|_\nu + \|\partial_x(v_1 - v_0)\|_\kappa,$$
$$C_6 = \|\partial_x^2(u_1 - u_0)\|_\nu + \|\partial_x^2(v_1 - v_0)\|_\kappa, \quad C_7 = C_4^{p-2} + C_4^{q-2}.$$

Proof. First we show (7.8). It is clear that (7.8) holds for $m = 0$, by (7.5) and (7.6). Inductively, suppose that (7.8) holds for some m ($m \geq 0$). Then it follows from (7.3), (7.4), (6.7) and (7.5) that

$$\|u_{m+1}\|_\nu + \|v_{m+1}\|_\kappa \leq (\|u_0\|_\nu + \|v_0\|_\kappa)(1 + 2AC_1(2C_0\varepsilon)^{p-1} + 2AC_1(2C_0\varepsilon)^{q-1}).$$

By virtue of (7.7), we find that (7.8) holds for any nonnegative integer m.

Next we show (7.9). Note that by (7.8), (7.5) and (7.7) we have

$$ApC_1(\|v_m\|_\kappa + \|v_{m-1}\|_\kappa)^{p-1} \leq \frac{1}{2}, \quad AqC_1(\|u_m\|_\nu + \|u_{m-1}\|_\nu)^{q-1} \leq \frac{1}{2} \tag{7.13}$$

for $m \geq 1$. It follows from (7.3), (7.4), (6.11), (6.12) and (7.13) that

$$\|u_{m+1} - u_m\|_\nu + \|v_{m+1} - v_m\|_\kappa \leq \frac{1}{2}(\|u_m - u_{m-1}\|_\nu + \|v_m - v_{m-1}\|_\kappa), \tag{7.14}$$

which implies (7.9).

Next we show (7.10). It follows from (7.3), (7.4), (7.5), (6.8) and (7.13) that

$$\|\partial_x u_{m+1}\|_\nu + \|\partial_x v_{m+1}\|_\kappa \leq C + \frac{1}{2}(\|\partial_x u_m\|_\nu + \|\partial_x v_m\|_\kappa),$$

which gives

$$\|\partial_x u_m\|_\nu + \|\partial_x v_m\|_\kappa \leq C \quad \text{for } m \geq 0. \tag{7.15}$$

In a similar fashion, we see from (6.9), (6.10), (7.5), (7.13), (7.8) and (7.15) that (7.15) with ∂_x replaced by ∂_x^2 holds. Hence we get (7.10).

Next we show (7.11). By (6.13), (6.14), (7.13), (7.10) and (7.9), we have

$$\begin{aligned}
&\|\partial_x(u_{m+1} - u_m)\|_\nu + \|\partial_x(v_{m+1} - v_m)\|_\kappa \\
&\quad \leq \frac{1}{2}(\|\partial_x(u_m - u_{m-1})\|_\nu + \|\partial_x(v_m - v_{m-1})\|_\kappa) + CC_4\left(\frac{1}{2}\right)^{m-1} \quad \text{for } m \geq 1,
\end{aligned}$$

which implies (7.11).

Finally, we show (7.12). By (6.15), (6.16), (7.13), (7.10), (7.9) and (7.11), we have

$$\begin{aligned}
&\|\partial_x^2(u_{m+1} - u_m)\|_\nu + \|\partial_x^2(v_{m+1} - v_m)\|_\kappa \\
&\quad \leq \frac{1}{2}(\|\partial_x^2(u_m - u_{m-1})\|_\nu + \|\partial_x^2(v_m - v_{m-1})\|_\kappa) \\
&\qquad + C\left((C_4 + C_5)\left(\frac{1}{2}\right)^{m-1} + C_4(m-1)\left(\frac{1}{2}\right)^{m-2}\right) \\
&\qquad + CC_2\left(C_4\left(\frac{1}{2}\right)^{m-1}\right)^{p-2} + CC_3\left(C_4\left(\frac{1}{2}\right)^{m-1}\right)^{q-2} \\
&\quad \leq \frac{1}{2}(\|\partial_x^2(u_m - u_{m-1})\|_\nu + \|\partial_x^2(v_m - v_{m-1})\|_\kappa) \\
&\qquad + C\left(C_5\left(\frac{1}{2}\right)^{m} + C_4 m\left(\frac{1}{2}\right)^{m}\right) + CC_7\left(C_2\left(\frac{1}{2}\right)^{(p-2)m} + C_3\left(\frac{1}{2}\right)^{(q-2)m}\right)
\end{aligned}$$

for $m \geq 1$, because $C_2 = 0$ for $p > 3$, and $C_3 = 0$ for $q > 3$. This estimate gives (7.12) and the proof is completed. □

From Lemma 7.1, we see that there is a solution $(u, v) \in X_{\nu,\kappa}(2C_0\varepsilon) \cap Y_{\nu,\kappa}$ of (7.1) and (7.2). In addition, we observe from the proof of (7.14) that such a solution is unique. Moreover, the solution satisfies (5.25) and (5.26). Indeed, when $\alpha = 0$, those asymptotic estimates follow from (6.17), (6.2), (6.3) and (6.4). Analogously, we obtain them for $|\alpha| = 1, 2$.

Next we prove that the solution (u,v) of (7.1) and (7.2) satisfies (5.27) and (5.28). We start with showing that for $(u,v) \in X_{\nu,\kappa}(1)$

$$\|F(v)(s)\|_{L^2(\boldsymbol{R}^n)}^2 \le C\|v\|_\kappa^{2p}(1+|s|)^{-2p^*-2}(1+|s|)^{[1-2p\kappa]_+}, \tag{7.16}$$

$$\|G(u)(s)\|_{L^2(\boldsymbol{R}^n)}^2 \le C\|u\|_\nu^{2q}(1+|s|)^{-2q^*-2}(1+|s|)^{[1-2q\nu]_+}. \tag{7.17}$$

Since $-(n-1)p+(n-1) = -2p^*-2$, it follows from (1.13) and (6.17) that

$$\|F(v)(s)\|_{L^2}^2 \le A^2\omega_n\|v\|_\kappa^{2p} I(s), \tag{7.18}$$

where we have set

$$I(s) = \int_0^\infty (1+r+|s|)^{-2p^*-2}(1+|r-|s||)^{-2p\kappa}\,dr, \tag{7.19}$$

unless $n=2$ and $\kappa \ge 1/2$. We divide $I(s)$ into $I_1(s)$ and $I_2(s)$ which are defined by

$$I_1(s) = \int_0^{2|s|} (1+r+|s|)^{-2p^*-2}(1+|r-|s||)^{-2p\kappa}\,dr, \tag{7.20}$$

$$I_2(s) = \int_{2|s|}^\infty (1+r+|s|)^{-2p^*-2}(1+|r-|s||)^{-2p\kappa}\,dr. \tag{7.21}$$

It is easy to see that for $\kappa > 0$

$$I_2(s) \le C(1+|s|)^{-2p^*-1-2p\kappa}. \tag{7.22}$$

While we have

$$(1+|s|)^{2p^*+2} I_1(s) \le C(1+|s|)^{[1-2p\kappa]_+}. \tag{7.23}$$

These estimates yield (7.16).

When $n=2$ and $\kappa \ge 1/2$, we have

$$\begin{aligned}
\|F(v)(s)\|_{L^2}^2 &\le C\|v\|_\kappa^{2p}\int_0^\infty (1+r+|s|)^{-2p^*-2}(1+|r-|s||)^{-p}(1+\log(1+|r-|s||))^{2p}\,dr \\
&\le C\|v\|_\kappa^{2p}\int_0^\infty (1+r+|s|)^{-2p^*-2}(1+|r-|s||)^{-2}\,dr \\
&\le C\|v\|_\kappa^{2p}(1+|s|)^{-2p^*-2},
\end{aligned}$$

since $p>3$. We thus obtain (7.16). Analogously, we also get (7.17).

Since $u(t,\cdot)-u^-(t,\cdot)$ and $v(t,\cdot)-v^-(t,\cdot)$ can be represented by (8.6) below, with $F=F(v)$ and $F=G(u)$ respectively, it holds that

$$\|\!|(u-u^-)(t)|\!\|_e \le (n+1)\int_{-\infty}^t \|F(v)(s)\|_{L^2(\boldsymbol{R}^n)}\,ds, \tag{7.24}$$

$$\|\!|(v-v^-)(t)|\!\|_e \le (n+1)\int_{-\infty}^t \|G(u)(s)\|_{L^2(\boldsymbol{R}^n)}\,ds, \tag{7.25}$$

provided there are positive constants θ and C satisfying

$$\|F(v)(s)\|_{L^2(\boldsymbol{R}^n)} + \|G(u)(s)\|_{L^2(\boldsymbol{R}^n)} \le C(1+|s|)^{-1-\theta} \quad \text{for } s\in\boldsymbol{R}. \tag{7.26}$$

A proof of (7.24) and (7.25) will be given in the appendix below.

Now it follows from (7.24) and (7.16) that for $t\le 0$

$$\begin{aligned} \|\!|(u-u^-)(t)|\!\|_e &\le C\|v\|_\kappa^p \int_{-\infty}^t (1+|s|)^{-p^*-1}\{(1+|s|)^{[1-2p\kappa]_+}\}^{1/2}\,ds \\ &= C\|v\|_\kappa^p \int_{|t|}^\infty (1+s)^{-p^*-1}\{(1+s)^{[1-2p\kappa]_+}\}^{1/2}\,ds. \end{aligned} \tag{7.27}$$

If $2p\kappa>1$, we easily have (5.27). When $2p\kappa=1$, by integration by parts, we get (5.27). Moreover, if we notice that (5.11) implies $p^*-(1/2)+p\kappa>(1/2)+\nu>0$, we obtain (5.27). Since the proof of (5.28) is similar to that of (5.27) if we use (5.12) instead of (5.11), we omit the details.

Next we prove the uniqueness of the solution. More precisely, we show that a solution $(u,v)\in X_{\nu,\kappa}(2C_0\varepsilon)\cap Y_{\nu,\kappa}$ of (5.1) and (5.2) satisfying (5.27) and (5.28) is unique, provided (5.8) through (5.12), (7.6) and (7.7) hold. Moreover it is enough to show that such a solution of (5.1) and (5.2) must satisfy the integral equations (7.1) and (7.2), since we have already established the uniqueness of the solution of (7.1) and (7.2). To this end, we set

$$w_1 = u-u^- - L(F(v)), \quad w_2 = v-v^- - L(G(u)). \tag{7.28}$$

We easily see that $w_i\in C^2(\boldsymbol{R}^n\times\boldsymbol{R})$ $(i=1,2)$ and $\Box w_i=0$ in $\boldsymbol{R}^n\times\boldsymbol{R}$. Therefore, if we could show

$$w_i(x,0)=\partial_t w_i(x,0)=0 \quad \text{for } x\in\boldsymbol{R}^n, \tag{7.29}$$

we would obtain $w_i(x,t)=0$ for any $(x,t)\in\boldsymbol{R}^n\times\boldsymbol{R}$.

We prove (7.29) only for $i=1$. It follows from the proof of (5.27) that

$$\|\!|L(F(v))(t)|\!\|_e \le C\|v\|_\kappa^p(1+|t|)^{-\theta} \quad \text{for } t\le 0, \tag{7.30}$$

where θ is a positive constant satisfying $\theta<p^*$ and $\theta\le 1/2$. Therefore we have from (5.27)

$$\|\!|w_1(0)|\!\|_e = \|\!|w_1(t)|\!\|_e \le C\|v\|_\kappa^p(1+|t|)^{-\theta} \quad \text{for } t\le 0. \tag{7.31}$$

We thus get

$$\partial_t w_1(x,0) = 0, \quad \nabla w_1(x,0) = 0 \quad \text{for } x \in \boldsymbol{R}^n. \tag{7.32}$$

In particular, we see that $w_1(x,0)$ is a constant. Moreover, since $(u,v) \in X_{\nu,\kappa}(1)$ and $(u^-,v^-) \in X_{p^*,q^*}$, we have from (6.7)

$$|w_1(x,0)| \leq (\|u\|_\nu + \|u^-\|_{p^*} + AC_1\|v\|_\kappa^p)(1+|x|)^{-(n-1)/2},$$

which implies $w_1(x,0) = 0$ for $x \in \boldsymbol{R}^n$, hence (7.29) holds. This completes the proof of the part (A).

Secondly we show the part (B) in Theorem 5.1. Since (5.30) follows from (5.27) by (5.16), it is enough to show (5.29). In view of the proof of Lemma 6.1, one can assume without loss of generality that $\nu < 1/2$ and $\kappa < 1/2$ when $n = 2$. By (6.2) with $\nu = p^*$ and (6.3), it suffices to show that there is a positive number μ such that

$$M_{p^*,\mu}(|v|^p) \leq \|v\|_\kappa^p, \quad M_{\kappa,\mu}(|u|^q) \leq \|u\|_\nu^q. \tag{7.33}$$

Indeed, we can choose a positive number μ such that

$$\mu < p\kappa - 1, \quad \mu < q\nu - 1 + q^* - \kappa, \tag{7.34}$$

by (5.16) and (5.12), hence (7.33) holds. (See the proof of (6.1)).

Finally we show the part (C) in Theorem 5.1. It is enough to show (5.33), since the others are easily handled. By (6.2) with $\nu = p^*$ and (6.3) with $\kappa = q^*$, it suffices to show that there is a positive number μ such that

$$M_{p^*,\mu}(|v|^p) \leq \|v\|_\kappa^p, \quad M_{q^*,\mu}(|u|^q) \leq \|u\|_\nu^q. \tag{7.35}$$

In fact, we can choose a positive number μ such that

$$\mu < p\kappa - 1, \quad \mu < q\nu - 1, \tag{7.36}$$

by (5.32). This completes the proof of Theorem 5.1. □

Next we prove Theorem 5.2. If we note that $L_1(F)(x,t) = L(\hat{F})(x,-t)$ with $\hat{F}(x,t) = F(x,-t)$, we obtain the desired estimates from the proof of Theorem 5.1. We omit further details. □

8. Appendix.

In this section we prove (7.24) and (7.25) in a more general situation. By $u(t)$ we denote a function of $t \in \boldsymbol{R}$ with values in $\mathscr{D}'(\boldsymbol{R}^n)$, the space of distributions on $\boldsymbol{R}^n$. Consider the initial value problem

$$u''(t) - \Delta u(t) = F(t) \quad \text{for } t \in \boldsymbol{R} \tag{8.1}$$

with zero initial data

$$u(t) = u'(t) = 0 \quad \text{at} \;\; t = s, \tag{8.2}$$

where s is an arbitrary real number and $u''(t)$ stands for the second derivative of $u(t)$, and so on. For a function $f \in L^2(\boldsymbol{R}^n)$ we denote by $\hat{f}$ and $\hat{f}^*$, respectively the Fourier transform of f and the inverse Fourier transform of f such that

$$\|\hat{f}\|_{L^2(\boldsymbol{R}^n)} = \|\hat{f}^*\|_{L^2(\boldsymbol{R}^n)} = \|f\|_{L^2(\boldsymbol{R}^n)}. \tag{8.3}$$

Then the following facts are in essence well known.

Proposition 8.1. *Assume that* $F(t) \in C(\boldsymbol{R}; L^2(\boldsymbol{R}^n))$.

i) *Let* $s \in \boldsymbol{R}$ *be fixed. For* $t \in \boldsymbol{R}$ *we define a linear form* $u(t; s)$ *on* $\mathscr{S}(\boldsymbol{R}^n)$ *by*

$$\langle u(t;s), \varphi \rangle = \int_s^t d\tau \int_{\boldsymbol{R}^n} \frac{\sin(t-\tau)|\xi|}{|\xi|} \hat{F}(\xi, \tau) \hat{\varphi}^*(\xi)\, d\xi \quad \textit{for} \;\; \varphi \in \mathscr{S}(\boldsymbol{R}^n), \tag{8.4}$$

where $\mathscr{S}(\boldsymbol{R}^n)$ *stands for the space of rapidly decreasing functions on* $\boldsymbol{R}^n$. *Then* $u(t;s)$, *with* s *regarded as a parameter, is a solution of the initial value problem* (8.1)–(8.2) *such that* $u(t;s) \in C^2(\boldsymbol{R}; H^{-1}(\boldsymbol{R}^n))$. *Moreover the solution is unique in* $C^2(\boldsymbol{R}; \mathscr{D}'(\boldsymbol{R}^n))$.

ii) *Let* $n \geq 2$. *Assume that there are positive constants* θ *and* C *such that*

$$\|F(t)\|_{L^2(\boldsymbol{R}^n)} \leq C(1 + |t|)^{-1-\theta} \quad \textit{for} \;\; t \in \boldsymbol{R}. \tag{8.5}$$

For $t \in \boldsymbol{R}$ *we define a linear form* $u(t)$ *on* $\mathscr{S}(\boldsymbol{R}^n)$ *by*

$$\langle u(t), \varphi \rangle = \int_{-\infty}^t d\tau \int_{\boldsymbol{R}^n} \frac{\sin(t-\tau)|\xi|}{|\xi|} \hat{F}(\xi, \tau) \hat{\varphi}^*(\xi)\, d\xi \quad \textit{for} \;\; \varphi \in \mathscr{S}(\boldsymbol{R}^n). \tag{8.6}$$

Then

$$u(t) \in C^2(\boldsymbol{R}; \mathscr{S}'(\boldsymbol{R}^n)), \tag{8.7}$$

where $\mathscr{S}'(\boldsymbol{R}^n)$ *stands for the space of tempered distributions on* $\boldsymbol{R}^n$, *and we have for each* $t \in \boldsymbol{R}$

$$u'(t) \in L^2(\boldsymbol{R}^n), \quad \nabla u(t) \in L^2(\boldsymbol{R}^n) \tag{8.8}$$

and

$$\|u'(t)\|_{L^2(\boldsymbol{R}^n)} + \|\nabla u(t)\|_{L^2(\boldsymbol{R}^n)} \leq (n+1) \int_{-\infty}^t \|F(\tau)\|_{L^2(\boldsymbol{R}^n)}\, d\tau. \tag{8.9}$$

Proof. The first part i) is well known. (For the uniqueness see for instance [**16**], Lemma 5.1). First we shall prove (8.7). Let $t \in \boldsymbol{R}$ be fixed. Then we claim that $u(t) \in \mathscr{S}'(\boldsymbol{R}^n)$. To see this we take a positive number δ such that $\delta < \theta$ and $\delta < 1$. Then

$$\left|\frac{\sin(t-\tau)|\xi|}{|\xi|}\right| \leq (t-\tau)^{\delta}\frac{1}{|\xi|^{1-\delta}} \tag{8.10}$$

and $|\xi|^{\delta-1}\hat{\varphi}^*(\xi) \in L^2(\boldsymbol{R}^n)$ for $n \geq 2$, hence the integrand in the right hand side of (8.6) is integrable with respect to (ξ, τ) over $\boldsymbol{R}^n \times (-\infty, t]$, according to (8.5). Therefore we have for $\varphi \in \mathscr{S}(\boldsymbol{R}^n)$

$$\langle u(t), \varphi\rangle = \lim_{k\to\infty}\langle u(t;-k), \varphi\rangle,$$

where $u(t;s)$ is given by (8.4). Since $u(t;-k) \in \mathscr{S}'(\boldsymbol{R}^n)$ for all $k = 1, 2, \dots,$ we find by the Banach-Steinhaus' theorem that $u(t) \in \mathscr{S}'(\boldsymbol{R}^n)$.

Now the desired property (8.7) follows easily from the above procedure.

Finally we shall prove (8.8) and (8.9). Let $D_k = -\sqrt{-1}\partial/\partial x_k$ $(k = 1, \dots, n)$. Then we get by virtue of (8.5)

$$|\langle D_k u(t), \varphi\rangle| \leq \|\varphi\|_{L^2(\boldsymbol{R}^n)}\int_{-\infty}^{t}\|F(\tau)\|_{L^2(\boldsymbol{R}^n)}\,d\tau.$$

Hence $D_k u(t) \in L^2(\boldsymbol{R}^n)$ and

$$\|D_k u(t)\|_{L^2(\boldsymbol{R}^n)} \leq \int_{-\infty}^{t}\|F(\tau)\|_{L^2(\boldsymbol{R}^n)}\,d\tau.$$

Analogously we obtain (8.8) and (8.9). □

References

[1] R. Agemi and H. Takamura, The lifespan of classical solutions to nonlinear wave equations in two space dimensions, Hokkaido Math. J., **21** (1992), 517–542.

[2] R. Agemi, Y. Kurokawa and H. Takamura, Critical curve for *p*-*q* systems of nonlinear wave equations in three space dimensions, to appear in J. Differential Equations.

[3] F. Asakura, Existence of a global solution to a semi-linear wave equation with slowly decreasing initial data in three space dimensions, Comm. Partial Differential Equations, **11** (1986), 1459–1487.

[4] Y. Choquet-Bruhat, Global existence for solutions of $\square u = A|u|^p$, J. Differential Equations, **82** (1989), 98–108.

[5] D. Del Santo, Global existence and blow-up for a hyperbolic system in three space dimensions, Rend. Instit. Math. Univ. Trieste, **26** (1997) 115–140.

[6] D. Del Santo, V. Georgiev and E. Mitidieri, Global existence of the solutions and formation of singularities for a class of hyperbolic systems, *in* "Geometric Optics and Related topics" (F. Colombini and N. Lerner ed.), Progress in Nonlinear Differential Equations and Their Applications, Vol. 32, 117–140, Berkhäuser, Boston, 1997.

[7] D. Del Santo and E. Mitidieri, Blow-up of solutions of a hyperbolic system: the critical case, Differential Equations, **34**(9) (1998), 1157–1163.

[8] K. Deng, Nonexistence of global solutions of a nonlinear hyperbolic system, Trans. Amer. Math. Soc., **349** (1997), 1685–1696.

[9] V. Georgiev, H. Lindblad and C. Sogge, Weighted Strichartz estimate and global existence for semilinear wave equation, Amer. J. Math., **119** (1997), 1291–1319.

[10] R. T. Glassey, Finite-time blow-up for solutions of nonlinear wave equations, Math. Z., **177** (1981), 323–340.

[11] R. T. Glassey, Existence in the large for $\square u = F(u)$ in two space dimensions, Math. Z., **178** (1981), 233–261.

[12] F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, Manuscripta Math., **28** (1979), 235–268.

[13] K. Hidano, Nonlinear small data scattering for the wave equation in $\boldsymbol{R}^{4+1}$, J. Math. Soc. Japan, **50** (1998), 253–292.

[14] H. Kubo, On the critical decay and power for semilinear wave equations in odd space dimensions, Discrete Contin. Dynam. Systems, **2** (1996), 173–190.

[15] H. Kubo and K. Kubota, Asymptotic behaviors of radially symmetric solutions of $\square u = |u|^p$ for super critical values p in odd space dimensions, Hokkaido Math. J., **24** (1995), 287–336.

[16] H. Kubo and K. Kubota, Asymptotic behaviors of radially symmetric solutions of $\square u = |u|^p$ for super critical values p in even space dimensions, Japanese J. of Math., **24** (1998), 191–256.

[17] H. Kubo and M. Ohta, Critical blowup for systems of semilinear wave equations in low space dimensions, J. Math. Anal. Appl., **240** (1999), 340–360.

[18] H. Kubo and M. Ohta, Small data blowup for systems of semilinear wave equations with different propagation speeds in three space dimensions, to appear in J. Differential Equations.

[19] K. Kubota, Existence of a global solutions to a semi-linear wave equation with initial data of non-compact support in low space dimensions, Hokkaido Math. J., **22** (1993), 123–180.

[20] K. Kubota and K. Mochizuki, On small data scattering for 2-dimensional semilinear wave equations, Hokkaido Math. J., **22** (1993), 79–97.

[21] H. Lindblad, Blow-up for solutions of $\square u = |u|^p$ with small initial data, Comm. Partial Differential Equations, **15** (1990), 757–821.

[22] H. Lindblad and C. D. Sogge, On existence and scattering with minimal regularity for semilinear wave equations, J. Funct. Anal., **130** (1995), 375–426.

[23] H. Lindblad and C. D. Sogge, Long-time existence for small amplitude semilinear wave equations, Amer. J. Math., **118** (1996), 1047–1135.

[24] K. Mochizuki and T. Motai, The scattering theory for the nonlinear wave equation with small data, J. Math. Kyoto Univ., **25** (1985), 703–715.

[25] K. Mochizuki and T. Motai, The scattering theory for the nonlinear wave equation with small data II, Publ. EIMS, Kyoto Univ., **23** (1987), 771–790.

[26] H. Pecher, Scattering for semilinear wave equations with small data in three space dimensions, Math. Z., **198** (1988), 277–289.

[27] J. Schaeffer, Wave equation with positive nonlinearities, ph. D. Thesis, Indiana Univ. (1983).

[28] J. Schaeffer, The equation $u_{tt} - \Delta u = |u|^p$ for the critical value of p, Proc. Roy. Soc. Edinburgh, **101A** (1985), 31–44.

[29] T. C. Sideris, Nonexistence of global solutions to semilinear wave equations in high dimensions, J. Differential Equations, **52** (1984), 378–406.

[30] W. A. Strauss, Nonlinear scattering theory at low energy, J. Funct. Analysis, **41** (1981), 110–133.

[31] H. Takamura, An elementary proof of the exponential blow-up for semilinear wave equations, Math. Meth. Appl. Sci., **17** (1994), 239–249.

[32] K. Tsutaya, A global existence theorem for semilinear wave equations with data of non compact support in two space dimensions, Comm. Partial Differential Equations, **17** (1992), 1925–1954.

[33] K. Tsutaya, Global existence theorem for semilinear wave equations with non-compact data in two space dimensions, J. Differential Equations, **104** (1993), 332–360.

[34] K. Tsutaya, Global existence theorem and the life span of solutions of semilinear wave equations with data of non compact support in three space dimensions, Funkcialaj Ekvacioj, **37** (1994), 1–18.

[35] K. Tsutaya, Lower bounds for the life span of solutions of semilinear wave equations with data of non compact support, Hokkaido Math. J., **23** (1994), 549–560.

[36] K. Tsutaya, Scattering theory for semilinear wave equations with small data in two space dimensions, Trans. Amer. Math. Soc., **342** (1994), 595–618.

[37] Y. Zhou, Blow up of classical solutions to $\Box u = |u|^{1+\alpha}$ in three space dimensions, J. Partial Differential Equations, **5** (1992), 21–32.

[38] Y. Zhou, Life span of classical solutions to $\Box u = |u|^p$ in two space dimensions, Chinese Ann. Math., **14B** (1993), 225–236.

Hideo KUBO

Department of Applied Mathematics
Faculty of Engineering
Shizuoka University
Hamamatsu 432, Japan

Kôji KUBOTA

Department of Mathematics
Hokkaido University of Education
Hakodate
Hakodate 040, Japan