J. Math. Soc. Japan
Vol. 53, No. 4, 2001

On factorization of the solutions of second order linear differential equations

By Jianhua Zheng and Yuzan He

(Received Oct. 14, 1999)
(Revised Apr. 22, 2000)

Abstract. In this paper, we discuss factorization of the solutions of some linear ordinary differential equations with transcendental entire coefficients. We give a condition which shows that the solutions for some differential equations are prime, for some are factorizable.

1. Introduction.

Let $f(z)$ be a meromorphic function. $f(z)$ is said to be factorizable, if there exist a transcendental mermorphic function $h(\zeta)$ and a transcendental entire function $g(z)$ such that $f(z) = h(g(z))$; $f(z)$ is said to be prime (pseudo-prime, left-prime), if every factorization $f(z) = h(g(z))$ implies that either $h(\zeta)$ is bilinear (a rational function, bilinear) or $g(z)$ is linear (a polynomial). N. Steinmetz ([**9**]) proved that each meromorphic solution of linear ordinary differential equation with rational coefficients is pseudo-prime. A natural question arises: how about the solutions of linear differential equations with transcendental coefficients? In this paper, we shall point out firstly that certain class of entire functions are prime, and then show that the solutions of some linear differential equations with transcendental coefficients are prime. We also discuss the factorization of the solution of some other differential equations and prove that the solutions are factorizable.

Throughout this paper, by $\rho(f)$ we denote the order of $f(z)$. We assume that the reader is familiar with Nevanlinna's theory of meromorphic functions and the meaning of the symbols $T(r, f), N(r, f)$ and etc. For other notation and terminology, the reader is referred to [**6**].

2. A class of prime entire functions.

In this section, we obtain a class of prime entire functions, which is used to discuss the factorization of the solutions of certain differential equation with the coefficients of periodic entire functions.

2000 *Mathematics Subject Classification.* Primary 30D30; Secondary 34M05, 34M45.
Key Words and Phrases. factorization, meromorphic solution, differential equation.
Supported by National Science Foundation of China.

Therefore

$$\frac{T(r,g)}{r} < \frac{\deg_\zeta \psi(\zeta)}{\pi(m-1-o(1))},$$

noting m is arbitrary we have

$$\liminf_{r\to\infty} \frac{T(r,g)}{r} \le \liminf_{\substack{r\to\infty \\ r\bar{\in} E}} \frac{T(r,g)}{r} = 0.$$

Obviously, the set of zeros of $\psi(e^z)$ is the same as the set of all the roots of $g(z) - w_n = 0$, $(n = 1, 2, \ldots)$. Suppose that $\zeta_1, \ldots, \zeta_k$ are the zeros of $\psi(\zeta)$, $\zeta_0 = \max_{1\le j\le k}\{\log|\zeta_j|\} > -\infty$. Then all the zeros of $\psi(e^z)$ must lie on the left hand side of the straight $\{z, \operatorname{Re} z = \zeta_0\}$, and so do all the roots of $g(z) - w_n = 0$. By Lemma 2.3, $g(z)$ is a polynomial of degree ≤ 2.

If $\deg g(z) = 2$ and $g(z) = (\alpha z + \beta)^2 + \gamma$, then set $Z = \alpha z + \beta$ or $z = aZ + b$, $g(z) = g(aZ + b) = Z^2 + \gamma$ is even in Z, and $f(aZ + b)$ is even, writing Z by z, we have

$$\begin{aligned} &\psi(e^{az+b}) \exp(\Phi(az+b) + d(az+b)) \\ &\qquad = \psi(e^{-az+b}) \exp(\Phi(-az+b) + d(-az+b)), \end{aligned}$$

therefore

$$\exp(2adz) = \frac{\psi(e^{-az+b})}{\psi(e^{az+b})} \exp(\Phi(-az+b) - \Phi(az+b)).$$

This implies that $\Phi(-az+b) - \Phi(az+b)$ is a constant, for it is periodic. Therefore $\exp(2adz)$ has period $2\pi i/a$, $2d$ must be an integer. This is clearly a contradiction. Therefore $g(z)$ must be linear. Lemma 2.2 is proved.

Lemma 2.4. *Suppose that* $h(\zeta)$ *in* (2.1) *has only finitely many zeros, then* $h(\zeta)$ *is linear*.

Proof. Since $h(\zeta)$ has only finitely many zeros, then $h(\zeta)$ must be of the form

$$h(\zeta) = C(\zeta)e^{B(\zeta)} \tag{2.2}$$

for a non-constant polynomial $C(\zeta)$ and an entire function $B(\zeta)$. Since $f(z)$ has infinitely many zeros, $g(z)$ must be transcendental. It follows from (2.1) that

$$C(g(z)) = \psi(e^z)e^{M(z)+dz} \tag{2.3}$$

3. Solutions of $w'' + A(e^z)w = 0$.

With respect to representations of solutions of periodic second order linear differential equation

$$w'' + A(e^z)w = 0, \tag{3.1}$$

where $A(t) = \sum_{j=0}^{p} b_j t^j$, $b_p \neq 0$, $p \geq 2$, S. Bank and I. Laine [**2**] proved the following

Theorem A. *Let $f(z)(\not\equiv 0)$ be a solution of* (3.1) *with the property that the exponent of convergence for the zero-sequence of $f(z)$ is finite, then the following is true*:

(I) *If $f(z)$ and $f(z+2\pi i)$ are linearly dependent, then $f(z)$ can be represented in the form*

$$f(z) = \Psi(e^z)\exp\left(\sum_{j=q}^{m} d_j e^{jz} + dz\right). \tag{3.2}$$

(II) *If $f(z)$ and $f(z+2\pi i)$ are linearly independent, then $f(z)$ can be represented in the form*

$$f(z) = \Psi(e^{z/2})\exp\left(\sum_{j=q}^{m} d_j e^{(j+1/2)z} + dz\right), \tag{3.3}$$

where $\Psi(t)$ is a polynomial all of whose roots are simple and nonzero, m and $q \in \boldsymbol{Z}$ with $m \geq q$ and $d, d_q, \dots, d_m \in C$ with $d_q \cdot d_m \neq 0$.

In this section, we shall point out that in (3.2) and (3.3), $m > 0$, $q \geq 0$ and $d = \sqrt{-b_0}$.

Theorem 3.1. *Let $f(z)$ be defined as in Theorem A. If $f(z)$ and $f(z+2\pi i)$ are linearly dependent, then in* (3.2), *$d = \sqrt{-b_0}$, $m = p/2$ and $q \geq 0$.*

Proof. By Theorem A, we have $f(z) = t^d G(t)$, where $t = e^z$,

$$G(t) = \Psi(t)\exp\left(\sum_{j=q}^{m} d_j t^j\right).$$

Since

$$\frac{df(z)}{dz} = \frac{d(t^d G(t))}{dt}\cdot\frac{dt}{dz} = d\cdot t^d G(t) + t^{d+1}G'(t),$$

$$\frac{d^2 f(z)}{dz^2} = d^2 t^d G(t) + (2d+1)t^{d+1}G'(t) + t^{d+2}G''(t),$$

We shall prove $q \geq 0$. In fact, if $q < 0$, then by using the same method as in the above, dividing both sides of (3.5) by s^{-2q} and setting $s \to \infty$, we get $q^2 d_q^2 a_0 = 0$. This is a contradiction. Therefore it must be $q \geq 0$. Since $s^2 H''$ and $sH'(s)$ tend to zero as $s \to \infty$, it follows from (3.5) that $b_0 + d^2 = 0$, and further $d = (-b_0)^{1/2}$. Theorem 3.1 follows. □

Theorem 3.2. *Let $f(z)$ be defined as in Theorem A. If $f(z)$ and $f(z + 2\pi i)$ are linearly independent, then in* (3.3), $d = \sqrt{-b_0}$, $m = (p-1)/2$ *and* $q \geq 0$.

Proof. We set

$$f(z) = t^{2d} G_1(t),$$

where $G_1(t) = \Psi_1(t) \exp(\sum_{j=q}^{m} d_j t^{2j+1})$, $t = e^{z/2}$, where $\Psi_1(t) = a_0 + a_1 t + \cdots + a_n t^n$, $a_0 \neq 0$. From (3.1), it is easy to see that $\Psi_1(t)$ satisfies the following equation

$$\begin{aligned} t^2 \Psi_1'' &+ \left(2 \sum_{j=q}^{m} (2j+1) d_j t^{2j+1} + 4d + 1 \right) t \Psi_1' \\ &+ \left(\sum_{j=q}^{m} 2j(2j+1) d_j t^{2j+1} + \left(\sum_{j=q}^{m} (2j+1) d_j t^{2j+1} \right)^2 \right. \\ &\left. + (4d+1) \sum_{j=q}^{m} (2j+1) d_j t^{2j+1} + 4d^2 + 4A(t^2) \right) \Psi_1 = 0. \qquad (3.6) \end{aligned}$$

Similar to Theorem 3.1, we can also point out that $m > 0$, and then dividing both sides of (3.6) by $\Psi_1(t) t^{4m+2}$ and setting $t \to \infty$, we obtain

$$\frac{A(t^2) f}{t^{4m+1}} \to (2m+1)^2 d_m^2 \neq 0,$$

so that $m = (p-1)/2$.

Set $H_1(s) = \Psi_1(s^{-1})$, we can show that $H_1(s)$ solves the following equation

$$\begin{aligned} s^2 H_1'' &+ \left(1 - 2d - \sum_{j=q}^{m} (2j+1) d_j s^{-(2j+1)} \right) s H_1' \\ &+ \left(\sum_{j=q}^{m} 2j(2j+1) d_j s^{-(2j+1)} + \left(\sum_{j=q}^{m} (2j+1) d_j s^{-(2j+1)} \right)^2 \right. \\ &\left. + (4d+1) \sum_{j=q}^{m} (2j+1) d_j s^{-(2j+1)} + 4d^2 + 4A(s^{-2}) \right) H_1 = 0. \qquad (3.7) \end{aligned}$$

Furthermore, $f(z)$ admits the following representation

$$f(z) = \Psi(e^{z/3})\exp\{-z + S(e^{z/3})\},$$

$$\Psi(\zeta) = \sum_{j=0}^{s} c_j\zeta^j, \quad S(\zeta) = d_0 + d_2\zeta^2, \quad d_2 = -\frac{3}{2}\sqrt[3]{b_2}.$$

Corollary. *Let $f(z)$ be a solution of* (4.1) *with the condition* (4.2), *then $f(z)$ is factorizable. Moreover, one of the factorization is that $f(z) = h(g(z))$ where $g(z) = e^{z/3}$, $h(\zeta) = \psi(\zeta)\zeta^{-3}\exp S(\zeta)$.*

Proof of Theorem 4.1. Let $\hat{d} = d - \frac{r}{q}$, $\hat{s} = s + r$, we may rewrite $f(z)$ as

$$\begin{aligned} f(z) &= \left\{\sum_{j=0}^{\hat{s}} \hat{c}_j e^{zj/q}\right\}\exp\left\{\hat{d}z + \sum_{j=-n}^{m} d_j e^{zj/q}\right\} \\ &= \hat{\Psi}(e^{z/q})\exp\{\hat{d}z + S(e^{z/q})\}. \end{aligned} \tag{4.4}$$

Set $\zeta = e^{z/q}$, then $f(z) = \zeta^{\hat{d}q}G(\zeta)$, $G(\zeta) = \hat{\Psi}(\zeta)\exp S(\zeta)$, where $\hat{\Psi}(\zeta) = \sum_{j=0}^{\hat{s}} \hat{c}_j\zeta^j$, $S(\zeta) = \sum_{j=-n}^{m} d_j\zeta^j$. A computation implies that

$$f'(z) = \hat{d}\zeta^{\hat{d}q}G(\zeta) + \frac{1}{q}\zeta^{\hat{d}q+1}G'(\zeta),$$

$$f''(z) = \hat{d}^2\zeta^{\hat{d}q}G(\zeta) + \frac{1}{q^2}(2\hat{d}q+1)\zeta^{\hat{d}q+1}G'(\zeta) + \frac{1}{q^2}\zeta^{\hat{d}q+2}G''(\zeta),$$

$$\begin{aligned} f'''(z) &= \hat{d}^3\zeta^{\hat{d}q}G(\zeta) + \frac{1}{q^3}(1 + 3\hat{d}q + 3\hat{d}^2q^2)\zeta^{\hat{d}q+1}G'(\zeta) + \frac{3}{q^3}(1+\hat{d}q)\zeta^{\hat{d}q+2}G''(\zeta) \\ &\quad + \frac{1}{q^3}\zeta^{\hat{d}q+3}G'''(\zeta). \end{aligned}$$

Substituting the above expressions into (4.1), we get

$$\begin{aligned} &\zeta^3G'''(\zeta) + 3(\hat{d}q+1)\zeta^2G''(\zeta) - \{Kq^2 - (1+\hat{d}q)^3 + \hat{d}^3q^3\}\zeta G'(\zeta) \\ &\quad + q^3(\hat{d}^3 - K\hat{d} + b_0 + b_1\zeta^2 + b_2\zeta^{2q})G(\zeta) = 0. \end{aligned} \tag{4.5}$$

We may assume that $m \neq 0$, $d_m \neq 0$, then by substituting the representation of $G'(\zeta)/G(\zeta)$, $G''(\zeta)/G(\zeta)$ and $G'''(\zeta)/G(\zeta)$ with respect to ζ into (4.5), we have

$$\begin{aligned} &(md_m)^3\zeta^{3m} + 3(md_m)^2(m-1)d_{m-1}\zeta^{3m-1} + 3(\hat{d}q+1)(md_m)^2\zeta^{2m} \\ &\quad - \{Kq^2 - (1+\hat{d}q)^3 + \hat{d}^3q^3\}md_m\zeta^m \\ &\quad + q^3(\hat{d}^3 - K\hat{d} + b_0 + b_1\zeta^q + b_2\zeta^{2q}) + O(\zeta^{3m-2}) = 0 \end{aligned} \tag{4.6}$$