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Half twists of Hodge structures of CM-type

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Abstract. To a Hodge structure V of weight k with CM by a field K we associate Hodge structures $V_{-n/2}$ of weight $k+n$ for n positive and, under certain circumstances, also for n negative. We show that these 'half twists' come up naturally in the Kuga-Satake varieties of weight two Hodge structures with CM by an imaginary quadratic field.

Introduction.

A Hodge structure of CM-type is a Hodge structure V on which a CM-field K acts. Given a CM-type (a set of certain complex embeddings of K), we give a simple construction of Hodge structures $V_{-n/2}$ of weight $k+n$, for any positive integer n, where k is the weight of V. These half twists of V are *not* related to the Tate twists $V(-n)$ of $V : V_{-n} \not\cong V(-n)$.

In certain circumstances one can also define the half twist $V_{1/2}$, of weight $k-1$. The geometry underlying the half twist in the case of hypersurfaces in projective space is investigated in [**vGI**]. A basic case is Example 2.12 in this paper.

In the first section we recall the basic definitions of Hodge structures and we define the half twist. In the second section we look at half twists from the point of view of representations of $\boldsymbol{C}^*$. We also give a geometrical interpretation of the half twist in case V is a sub-Hodge structure of $H^k(X,\boldsymbol{Q})$ for a smooth projective variety X. In the last section we consider Hodge structures V of weight two with $\dim V^{2,0}=1$ and with a suitable action of an imaginary quadratic field. We show that the first cohomology group of the Kuga-Satake variety associated to such a Hodge structure has a summand which is the half twist of V and that half twists can be used to understand the other summands as well. So, in a certain sense, half twists 'partly generalize' the Kuga-Satake construction which associates weight one Hodge structures to certain weight two Hodge structures. These results were motivated by an example of C. Voisin [**V**] which also inspired the general definition of half twist.

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1. Definitions and basic properties.

1.1. Hodge structures.

Recall that a (rational) Hodge structure of weight k ($\in \boldsymbol{Z}_{\geq 0}$) is a $\boldsymbol{Q}$-vector space V with a decomposition of its complexification $V_{\boldsymbol{C}} := V \otimes_{\boldsymbol{Q}} \boldsymbol{C}$ (where complex conjugation is given by $\overline{v \otimes z} := v \otimes \bar{z}$ for $v \in V$ and $z \in \boldsymbol{C}$):

$$V_{\boldsymbol{C}} = \bigoplus_{p+q=k} V^{p,q}, \quad \text{and} \quad \overline{V^{p,q}} = V^{q,p}, \quad (p, q \in \boldsymbol{Z}_{\geq 0}).$$

Note that we insist on p and q being non-negative integers throughout this paper, so we only consider 'effective' Hodge structures.

1.2. Hodge structures of CM-type.

Let V be a Hodge structure such that V is also a K-vector space for some CM-field K and such that the Hodge decomposition on V is stable under the action of K:

$$xV^{p,q} \subset V^{p,q}, \quad (x \in K,\ p, q \in \boldsymbol{Z}_{\geq 0}).$$

In particular, $K \hookrightarrow \mathrm{End}_{Hod}(V)$. We will then say that V is a Hodge structure of CM-type (with field K).

1.3. CM-types.

To define the half twist we need to fix a CM-type of the field K. Recall that the CM-field K has $2r = [K : \boldsymbol{Q}]$ complex embeddings $K \hookrightarrow \boldsymbol{C}$ and that a CM-type is a subset $\Sigma = \{\sigma_1, \dots, \sigma_r\}$ of distinct embeddings with the property that no two are complex conjugate. Hence if we define as usual $\bar{\Sigma} := \{\bar{\sigma}_1, \dots, \bar{\sigma}_r\}$ then any embedding of K is either in Σ or in $\bar{\Sigma}$.

1.4. The half twists.

Let V be a Hodge structure of CM-type with field K. We consider the eigenspaces of the K-action on the $V^{p,q}$'s:

$$V_{\sigma}^{p,q} := \{v \in V^{p,q} : xv = \sigma(x)v \ \ \forall x \in K\}, \quad \sigma : K \hookrightarrow \boldsymbol{C}.$$

Given a CM-type Σ, we define two subspaces of $V^{p,q}$ whose direct sum is $V^{p,q}$:

$$V_{\Sigma}^{p,q} := \bigoplus_{\sigma \in \Sigma} V_{\sigma}^{p,q}, \quad V_{\bar{\Sigma}}^{p,q} := \bigoplus_{\sigma \in \bar{\Sigma}} V_{\sigma}^{p,q}.$$

The negative half twist of V, denoted by $V_{-1/2}$ $(= V_{\Sigma,-1/2})$, is the decomposition of $V_{\boldsymbol{C}}$ given by the subspaces:

$$V_{-1/2}^{r,s} := V_{\Sigma}^{r-1,s} \oplus V_{\bar{\Sigma}}^{r,s-1}.$$

It is not hard to see that this is a Hodge structure, of CM-type with field K, on V of weight $k+1$ where k is the weight of V. By successively performing the negative half twist one obtains $V_{-n/2}$, a Hodge structure on V of weight $k+n$.

We observe that to define the (positive) half twist one would put:

$$V_{1/2}^{r,s} := V_{\Sigma}^{r+1,s} \oplus V_{\bar{\Sigma}}^{r,s+1},$$

however, now the subspaces $V_{\bar{\Sigma}}^{k,0}$ and $V_{\Sigma}^{0,k}$ of $V_{\boldsymbol{C}}$ do not appear in $(V_{1/2})_{\boldsymbol{C}}$ and therefore this definition does not define a Hodge structure on V (and in general not on any $\boldsymbol{Q}$-subspace of V). In particular, we will define the half twist of V only if $V_{\bar{\Sigma}}^{k,0} = 0$ (the complex conjugate of this space is $V_{\Sigma}^{0,k}$ which is then also trivial).

2. Half twists via representations.

2.1.

Hodge structures can also be defined via representations of $\boldsymbol{C}^*$ (more precisely, algebraic representations of $\mathrm{Res}_{\boldsymbol{C}/\boldsymbol{R}}(G_m)$ in $GL(V_{\boldsymbol{R}})$). We determine the representations corresponding to the half twists. Proposition 2.8 often points out interesting geometry, as in example 2.12. In [**vGI**] more such examples are investigated.

2.2.

The Hodge structure on a $\boldsymbol{Q}$-vector space V defined by an algebraic representation $h : \boldsymbol{C}^* \to GL(V_{\boldsymbol{R}})$ will be denoted by (V, h), its Hodge decomposition is:

$$V^{p,q} := \{v \in V_{\boldsymbol{C}} : h(z)v = z^p \bar{z}^q v\}.$$

The usual algebra constructions on representations can be applied to Hodge structures. In particular, given rational Hodge structures (V, h), (W, h_W) of weight k, k_W their tensor product is the rational Hodge structure of weight $k + k_W$ defined by:

$$h \otimes h_W : \boldsymbol{C}^* \to GL(V_{\boldsymbol{R}} \otimes W_{\boldsymbol{R}}), \quad z \mapsto [v \otimes w \mapsto (h(z)w) \otimes (h_W(z)w)].$$

2.3.

Let (V, h) be a Hodge structure of weight k of CM-type with field K. To identify the $\boldsymbol{C}^*$-representation on $V_{\boldsymbol{R}}$ which defines the half twist, consider the $\boldsymbol{R}$-

linear extension of the action of K on $V_{\boldsymbol{R}}$. This gives an action of $K \otimes_{\boldsymbol{Q}} \boldsymbol{R}$ on $V_{\boldsymbol{R}}$ and recall that $K \otimes_{\boldsymbol{Q}} \boldsymbol{R} \cong \bigoplus_{i=1}^{r} \boldsymbol{C}$ (if you write

$$K = \boldsymbol{Q}[X]/(f) \quad \text{then } K \otimes \boldsymbol{R} \cong \boldsymbol{R}[X]/(f) \cong \prod_{i=1}^{r} \boldsymbol{R}[X]/(f_i),$$

where $f = \prod f_i$ is the decomposition of f in irreducible polynomials f_i of degree 2 in $\boldsymbol{R}[X]$). Choosing a CM-type $\Sigma = \{\sigma_1, \dots, \sigma_r\}$ of K specifies an isomorphism

$$K \otimes \boldsymbol{R} \xrightarrow{\cong} \bigoplus_{j=1}^{r} \boldsymbol{C}, \quad x \otimes t \mapsto (t\sigma_1(x), \dots, t\sigma_r(x)),$$

we will denote the $\boldsymbol{R}$-linear extensions of the σ_i's by the same symbols. The inverse of this isomorphism is denoted by ϕ

$$\phi : \bigoplus_{j=1}^{r} \boldsymbol{C} \xrightarrow{\cong} K \otimes \boldsymbol{R}, \quad \text{with } \sigma_i(\phi(z_1, \dots, z_r)) = z_i \quad (1 \leq i \leq r).$$

Denoting by $x \mapsto \bar{x}$ the complex conjugation on K as well as its $\boldsymbol{R}$-linear extension to $K \otimes_{\boldsymbol{Q}} \boldsymbol{R}$ let $\sigma_{r+i}(x) := \overline{\sigma_i(x)} = \sigma_i(\bar{x})$, hence

$$\sigma_{r+i}(\phi(z_1, \dots, z_r)) = \bar{z}_i \quad (1 \leq i \leq r).$$

We define a homomorphism (depending on the CM-type Σ) by composing the diagonal inclusion Δ with the isomorphism $\phi = \phi_\Sigma$:

$$g_\Sigma : \boldsymbol{C}^* \xrightarrow{\Delta} \prod_{i=1}^{r} \boldsymbol{C}^* \xrightarrow{\phi} (K \otimes_{\boldsymbol{Q}} \boldsymbol{R})^*.$$

Since $(K \otimes_{\boldsymbol{Q}} \boldsymbol{R})^*$ acts on $V_{\boldsymbol{R}}$, we get a representation, also denoted by g_Σ of $\boldsymbol{C}^*$ on $V_{\boldsymbol{R}}$. As V is of CM-type, the actions of K and $h(\boldsymbol{C}^*)$ commute and thus the action of $g_\Sigma(\boldsymbol{C}^*)$ $(\subset (K \otimes_{\boldsymbol{Q}} \boldsymbol{R})^*)$ on $V_{\boldsymbol{R}}$ commutes with the action of $h(\boldsymbol{C}^*)$.

To understand the action of g_Σ we observe that the eigenvalues of $x \in K$ on $V_{\boldsymbol{C}}$ are the $\sigma_i(x)$, $1 \leq i \leq 2r$, each with the same multiplicity. The decomposition in eigenspaces is

$$V_{\boldsymbol{C}} = \bigoplus_{\sigma \in \Sigma \cup \bar{\Sigma}} V_{\boldsymbol{C},\sigma} \quad \text{with } xv = \sigma(x)v \quad (x \in K \otimes_{\boldsymbol{Q}} \boldsymbol{R}, v \in V_{\boldsymbol{C},\sigma}).$$

In particular, if $x = g_\Sigma(z)$ then since $g_\Sigma(z) = \phi(\Delta(z)) = \phi(z, \dots, z)$ we get:

$$g_\Sigma(z)v = \phi(z, \dots, z)v = \begin{cases} zv & v \in \bigoplus_{\sigma \in \Sigma} V_{\boldsymbol{C},\sigma}, \\ \bar{z}v & v \in \bigoplus_{\sigma \in \bar{\Sigma}} V_{\boldsymbol{C},\sigma}. \end{cases}$$

This leads to an alternative, but equivalent, definition of the half twist:

2.4. Definition.

Let (V, h, K) be a Hodge structure of CM-type and let Σ be a CM-type of K. For $n \in \boldsymbol{Z}$, the n-th half twist $V_{-n/2}$ is the $\boldsymbol{C}^*$-representation defined by the homomorphism

$$g_\Sigma^n h : \boldsymbol{C}^* \to GL(V_{\boldsymbol{R}}), \quad z \mapsto g_\Sigma^n(z)h(z).$$

2.5.

For positive n the representation space $V_{-n/2}$ is a Hodge structure of weight $k+n$ where k is the weight of V. Moreover, with the same CM-type, one has:

$$(V_{n/2})_{m/2} = V_{(n+m)/2}.$$

We already observed that the half twist $V_{1/2}$ is a Hodge structure iff the eigenspaces for the K-action $V_\sigma^{k,0}$ are trivial for $\sigma \in \bar{\Sigma}$.

2.6. Tate twists.

The Tate Hodge structure $\boldsymbol{Q}(n)$ $(n \in \boldsymbol{Z})$ is defined by the vector space $\boldsymbol{Q}$ and the homomorphism:

$$h_n : \boldsymbol{C}^* \to GL_1(\boldsymbol{R}), \quad z \mapsto (z\bar{z})^{-n},$$

it has weight $-2n$ and $\boldsymbol{Q}(n)^{p,q} = 0$ unless $p = q = -n$ in which case $\boldsymbol{Q}(n)^{-n,-n} = \boldsymbol{C}$. It is convenient to allow negative weights for this Hodge structure. The n-th Tate twist of V is defined by $V(n) := V \otimes \boldsymbol{Q}(n)$, it is a Hodge structure of weight $k - 2n$ with $V(n)^{p,q} = V^{p+n,q+n}$.

Since the homomorphism h_n acts via scalar multiplication on $V_{\boldsymbol{R}}$, it commutes with the representation $g_\Sigma^m h$ which defines $V_{m/2}$. Therefore one has

$$(V_{m/2})(n) = (V(n))_{m/2}.$$

This is also easy to see from the Hodge decompositions. Note that $V(-1) \neq V_{-1}$ since the representations h_1 and g_Σ^2 are not equivalent.

2.7.

The 'abstractly' defined half twists are in fact sub-Hodge structures of rather natural Hodge structures of CM-type. Recall that to the CM-field K and the CM-type Σ one can associate a weight one Hodge structure on the $\boldsymbol{Q}$-vector space K. These Hodge structures, and the abelian varieties associated to them, have been extensively investigated, cf. [**La**], [**DMOS**]. If we endow K with the trivial Hodge structure of weight zero, $K_{\boldsymbol{C}} = K^{0,0}$, this Hodge structure is just $K_{-1/2}$. One can identify $K_{-1/2}$ with $H^1(A_K, \boldsymbol{Q})$ for an abelian variety A_K with CM by the field K and whose CM-type is the one used to define the (negative) half twist.

Let V be a Hodge structure of CM-type with field K of weight n, then

$V \otimes_{\boldsymbol{Q}} K_{-1/2}$ is a Hodge structure of weight $n+1$ which has an action of the algebra $K \otimes_{\boldsymbol{Q}} K$. The element $x \otimes y \in K \otimes K$ acts as $(x \otimes y)(v \otimes z) = xv \otimes yz$ on $V \otimes_{\boldsymbol{Q}} K_{-1/2}$. We will identify some subspaces of $V \otimes_{\boldsymbol{Q}} K_{-1/2}$ with half twists of V.

2.8. Proposition.

Let (V, h, K) be a Hodge structure of CM-type. Then we have an inclusion of Hodge structures:

$$V_{-1/2} \subset V \otimes_{\boldsymbol{Q}} K_{-1/2},$$

more precisely:

$$V_{-1/2} = \{w \in V \otimes_{\boldsymbol{Q}} K_{-1/2} : (x \otimes 1)w = (1 \otimes x)w, \ \forall x \in K\}.$$

If V admits a half twist, $V_{1/2}(-1)$ is also a sub-Hodge structure of $V \otimes_{\boldsymbol{Q}} K_{-1/2}$:

$$V_{1/2}(-1) = \{w \in V \otimes_{\boldsymbol{Q}} K_{-1/2} : (x \otimes 1)w = (1 \otimes \bar{x})w, \ \forall x \in K\}$$

and the Hodge structure V can be recovered from $V_{1/2}$ by applying the first result to $V_{1/2}$ rather than V:

$$V \subset V_{1/2} \otimes_{\boldsymbol{Q}} K_{-1/2}.$$

PROOF. We give two proofs.

The field K acts on the $\boldsymbol{Q}$-vector space V and the (complex) eigenvalues of the action of $x \in K$ are the $\sigma_j(x) \in \boldsymbol{C}$, each with the same multiplicity. In particular, x, $\bar{x} \in K$ are eigenvalues of the K-linear extension of the action of x on $V \otimes_{\boldsymbol{Q}} K$ and we denote by V' and V'' the corresponding eigenspaces:

$$V \otimes_{\boldsymbol{Q}} K \cong V' \oplus V'' \oplus W, \quad \text{with } (x \otimes 1)w = \begin{cases} (1 \otimes x)w & w \in V', \\ (1 \otimes \bar{x})w & w \in V'', \end{cases}$$

and W is a K-stable complementary subspace. The projections on the summands give isomorphisms of $K = K \otimes 1$-vector spaces $V = V \otimes 1 \to V'$ and $V \to V''$. These are sub-Hodge structures of $V \otimes K_{-1/2}$ since the actions of K and h, g commute. The Hodge structure on $K_{-1/2}$ is defined by the $g(z) \in (K \otimes \boldsymbol{R})^*$ and thus $1 \otimes g(z) = g(z) \otimes 1$ on V' and $1 \otimes g(z) = g(\bar{z}) \otimes 1$ on V''. Note also that $g(tz) = tg(z)$ for $t \in \boldsymbol{R}$. Therefore

$$h(z) \otimes g(z) = h(z)g(\bar{z}) \otimes 1 = z\bar{z}h(z)g(z^{-1}) \otimes 1$$

on V'' which identifies the Hodge structure on V'' with the one on $V_{1/2}(-1)$. The proof of $V' \cong V_{-1/2}$ is similar but easier. The last result follows by applying the first result to $V_{1/2}$ rather than V and using $(V_{1/2})_{-1/2} = V$, note this identifies V with a specific subspace of $V_{1/2} \otimes K_{-1/2}$.

Another proof of the first result is as follows: since for $x \in K$ the map $w \mapsto (1 \otimes x - x \otimes 1)w$ is a $\boldsymbol{Q}$-linear endomorphism of $V \otimes_{\boldsymbol{Q}} K_{-1/2}$, its kernel is a $\boldsymbol{Q}$-vector space and therefore

$$V' := \{w \in V \otimes_{\boldsymbol{Q}} K_{-1/2} : (x \otimes 1)w = (1 \otimes x)w, \ \forall x \in K\}$$

is a $\boldsymbol{Q}$-vector space. Its complexification $V'_{\boldsymbol{C}}$ is the following subspace of $(V \otimes K_{-1/2})_{\boldsymbol{C}}$:

$$V'_{\boldsymbol{C}} = \bigoplus_{\sigma \in \Sigma \cup \bar{\Sigma}} V_\sigma \otimes K_\sigma$$

with V_σ (resp. K_σ) the subspace of $V_{\boldsymbol{C}}$ (resp. $K_{\boldsymbol{C}}$) on which $x \in K$ acts as $\sigma(x)$ ($\in \boldsymbol{C}$). Since $K_{-1/2}^{1,0} = \bigoplus_{\sigma \in \Sigma} K_\sigma$ we get:

$$(V')^{r,s} = \left(\bigoplus_{\sigma \in \Sigma} V_\sigma^{r-1,s} \otimes K_\sigma\right) \oplus \left(\bigoplus_{\sigma \in \bar{\Sigma}} V_\sigma^{r,s-1} \otimes K_\sigma\right)$$

which, since $K_\sigma \cong \boldsymbol{C}$, is just the definition of $V_{-1/2}^{r,s}$ hence $V' \cong V_{-1/2}$. The other statements can be proved in a similar fashion. □

2.9. Geometrical version of the half twist.

The proposition shows that if V is a sub-Hodge structure of $H^k(X, \boldsymbol{Q})$ for some projective variety X, then $V_{1/2}(-1)$ is a sub-Hodge structure of $H^k(X, \boldsymbol{Q}) \otimes H^1(A_K, \boldsymbol{Q})$, which is itself a summand of $H^{k+1}(X \times A_K, \boldsymbol{Q})$.

2.10. Polarizations.

A polarization on the Hodge structure (V, h) of weight k is a bilinear map:

$$\Psi : V \times V \to \boldsymbol{Q}$$

satisfying (for all $v, w \in V_{\boldsymbol{R}}$):

$$\Psi(h(z)v, h(z)w) = (z\bar{z})^k \Psi(v, w)$$

and

$$\Psi(v, h(i)w) \text{ is a symmetric and positive definite form:}$$

$\Psi(v, h(i)w) = \Psi(w, h(i)v)$ for all $v, w \in V_{\boldsymbol{R}}$ and $\Psi(v, h(i)v) > 0$ for all $v \in V_{\boldsymbol{R}} - \{0\}$. The primitive cohomology groups of algebraic varieties are polarized.

A polarized Hodge structure of CM-type with field K is a polarized Hodge structure (V, h, ψ) such that (V, h, K) is of CM-type and such that

$$\Psi(xv, w) = \Psi(v, \bar{x}w), \quad x \in K, \ v, w \in V.$$

2.11. A polarization on the half twist.

The Hodge structure $K_{-1/2}$ has a polarization, cf. [**La**], [**DMOS**]. If V has a polarization, then also $V \otimes K_{-1/2}$ has a polarization (the tensor product polarization) and by restriction one obtains a polarization on $V_{1/2}$.

One can also proceed more explicitly: if Ψ is a polarization on V then one chooses an element $\alpha \in K$ such that $\bar{\alpha} = -\alpha$ and such that for $\sigma \in \Sigma$ the purely imaginary complex numbers $\sigma(\alpha)$ all have positive imaginary part. Then the bilinear form

$$\Psi' : V \times V \to \boldsymbol{Q}, \quad \Psi'(v, w) := \Psi(v, \alpha w)$$

is a polarization on $V_{1/2}$. To verify all the properties it is convenient to split

$$(V_{1/2})_{\boldsymbol{R}} = \bigoplus_{i=1}^{r} (V_{1/2})_i$$

corresponding to the decomposition $K \otimes_{\boldsymbol{Q}} \boldsymbol{R} = \bigoplus_{i=1}^{r} \boldsymbol{C}$.

2.12. Example.

Let $F \in \boldsymbol{C}[X_0, \dots, X_n]$, homogeneous of degree $n+2$, define a smooth variety $Y \subset \boldsymbol{P}^n$. Let

$$X = Zeroes(X_{n+1}^{n+2} - F) \ \ (\subset \boldsymbol{P}^{n+1}), \quad \text{let } V := H^n(X, \boldsymbol{Q})_0$$

be the primitive cohomology group of X. Then V is a vector space over the field K of $n+2$-th roots of unity, where the roots of unity act by multiplication on the variable X_{n+1}.

The vector space $H^{n,0}(X)$ is one dimensional, hence we can apply the half twist to V (for any CM-type which includes the embedding σ of K defined by $xv = \sigma(x)v$ for $v \in H^{n,0}(X)$). Proposition 2.8 implies that

$$V = H^n(X, \boldsymbol{Q})_0 \hookrightarrow V_{1/2} \otimes K_{-1/2},$$

with $V_{1/2}$ and $K_{-1/2}$ Hodge structures of weight $n-1$ and 1 respectively. In this case it is not hard to see geometrically that such Hodge structures exist.

Note that a general line $l \subset \boldsymbol{P}^n$ meets the hypersurface $Y := Zeroes(F)$ (the branch locus of the natural map $X \to \boldsymbol{P}^n$) in $n+2$ distinct points, so these have $n+2-3 = n-1$ moduli. Since the grassmanian of lines in $\boldsymbol{P}^n$ has dimension $2(n-1)$, we find that for a general set of $n+2$ points on $\boldsymbol{P}^1$ there is an $n-1$-dimensional family S of lines $l \subset \boldsymbol{P}^n$ each of which meet Y in $n+2$ points with the same moduli. The union of these lines will be $\boldsymbol{P}^n$. Let C_n be the inverse image of (any) of these lines in X. After replacing S by the desingularization of a finite cover $\tilde{S}$, we then get a dominant, rational map:

$$\pi : \tilde{S} \times C_n \to X.$$

Since a rational map is defined outside a subset of codimension at least 2, the pull-back of the regular n-form on X extends to a regular n-form on $\tilde{S} \times C_n$. Hence we get $H^n(X, \boldsymbol{Q})_0 \hookrightarrow H^{n-1}(\bar{S}, \boldsymbol{Q}) \otimes H^1(C_n, \boldsymbol{Q})$ where $\bar{S}$ is some compactified desingularization of $\tilde{S}$. We will relate this example to Shioda's results on Fermat type hypersurfaces in [**vGI**].

3. Kuga-Satake varieties.

3.1.

To a polarized Hodge structure (V, ψ) of weight 2 with $\dim V^{2,0} = 1$ the construction of Kuga and Satake associates a Hodge structure $(C^+(V), h_s)$ of weight 1 on the even Clifford algebra $C^+(V)$ of the quadratic space (V, ψ) (see [**KS**] and [**vG**] for a detailed construction). It has the property that there is an inclusion of Hodge structures $V \hookrightarrow C^+(V) \otimes C^+(V)$. The (isogeny class of) abelian variety associated to $C^+(V)$ is called the Kuga-Satake variety $KS(V)$ of V, so $H^1(KS(V), \boldsymbol{Q}) = C^+(V)$.

In the remainder of this paper we consider such Hodge structures which are of CM-type with an imaginary quadratic field K. Since $\dim V^{2,0} = 1$, the half twist $V_{1/2}$ is a Hodge structure and has weight one. Our main result is Theorem 3.10 which shows that $V_{1/2}$ is a summand of $(C^+(V), h_s)$. We also determine the other summands and relate them to half twists.

This completes the results of C. Voisin in [**V**], she already found that two summands of $C^+(V)$, S_0 of dimension 2 and S_1 with $\dim S_1 = \dim V$, such that $V \hookrightarrow S_0 \otimes S_1$. We will identify S_1 with $V_{1/2}$ in Theorem 3.10. To find the simple summands of $(C^+(V), h_s)$ we use the Mumford-Tate group of the Hodge structure V.

3.2. The Mumford-Tate group.

Recall that the Special Mumford-Tate group $SMT(V)$ of a polarized Hodge structure (V, h, ψ) is the smallest algebraic subgroup G of $GL(V)$, defined over $\boldsymbol{Q}$, for which $h(z) \in G(\boldsymbol{R})$ $(\subset GL(V)(\boldsymbol{R}))$ for all $z \in \boldsymbol{C}$ with $z\bar{z} = 1$.

The simple summands of the Hodge structure $(C^+(V), h_s)$ are then the irreducible subrepresentations of $SMT(V)$ in $C^+(V)$. It takes some rather long computations to determine these summands though.

The following lemma recalls the basic facts on the Mumford Tate group in this situation.

3.3. Lemma.

Let (V, h, ψ) be a weight 2 rational polarized Hodge structure of CM-type by an imaginary quadratic field $K = \boldsymbol{Q}(\phi)$, with $\bar{\phi} = -\phi$ and $\phi^2 = -d$.

i) *The $\boldsymbol{Q}$-bilinear map*:

$$H : V \times V \to K, \quad H(v, w) := \psi(v, w) + \phi^{-1}\psi(v, \phi w)$$

is a hermitian form on the $K := \boldsymbol{Q}(\phi)$*-vector space* V (*so* $\overline{H(v,w)} = H(w,v)$ *and* H *is* K*-linear in the second variable*).

ii) *We have*

$$SMT(V) \subset U(H) := \{g \in GL(V) : H(gv, gw) = H(v,w),\ \ g\phi = \phi g\},$$

the unitary group of the K*-vector space* V *with the hermitian form* H.

iii) *The* $\boldsymbol{R}$*-linear extension of the hermitian form* H *has signature* $(1/2\dim V^{1,1}, \dim V^{2,0})$ *on the complex vector space* $V \otimes_{\boldsymbol{Q}} \boldsymbol{R}$.

PROOF. Since the weight is even, ψ is symmetric and:

$$\begin{aligned} H(w,v) &= \psi(w,v) + \phi^{-1}\psi(w,\phi v) \\ &= \psi(v,w) + \phi^{-1}\psi(\phi v, w) \\ &= \psi(v,w) + (d\phi)^{-1}\psi(\phi^2 v, \phi w) \\ &= \psi(v,w) - \phi^{-1}\psi(v,\phi w) \\ &= \overline{H(v,w)}. \end{aligned}$$

The K-linearity in the second factor follows from:

$$\begin{aligned} H(v,\phi w) &= \psi(v,\phi w) + \phi^{-1}\psi(v,\phi^2 w) \\ &= \psi(v,\phi w) - d\phi^{-1}\psi(v,w) \\ &= \phi(\psi(v,w) + \phi^{-1}\psi(v,\phi w)) \\ &= \phi H(v,w). \end{aligned}$$

Using that $\psi(h(z)v, h(z)w) = (z\bar{z})^2\psi(v,w)$ and that $\phi h(z) = h(z)\phi$ we get:

$$\begin{aligned} H(h(z)v, h(z)w) &= \psi(h(z)v, h(z)w) + \phi^{-1}\psi(h(z)v, \phi h(z)w) \\ &= (z\bar{z})^2\psi(v,w) + \phi^{-1}\psi(h(z)v, h(z)\phi w) \\ &= (z\bar{z})^2(\psi(v,w) + \phi^{-1}\psi(v,\phi w)) \\ &= (z\bar{z})^2 H(v,w), \end{aligned}$$

hence $SMT(V) \subset U(H)$.

We write:

$$V \otimes_{\boldsymbol{Q}} \boldsymbol{R} = V_1 \oplus V_2 \quad \text{with } V_1 \otimes_{\boldsymbol{R}} \boldsymbol{C} = V^{1,1},\ \ V_2 \otimes_{\boldsymbol{R}} \boldsymbol{C} = V^{2,0} \oplus V^{0,2}.$$

Then ψ is negative definite on V_2 and positive definite on V_1. Since ϕ and $h(z)$ commute, ϕ maps the eigenspaces $V^{p,q}$ into themselves hence $\phi(V_i) \subset V_i$. As $\phi^2 = -d$ with $d > 0$, ϕ does not have real eigenvalues and we can choose a $\boldsymbol{R}$-basis $f_1, \phi f_1, \dots, f_r, \phi f_r$ of V_2 (and similarly for V_1). Since $\psi(f_i, \phi f_i) = d^{-1}\psi(\phi f_i, \phi^2 f_i) = -\psi(\phi f_i, f_i) = -\psi(f_i, \phi f_i)$, we may assume this basis to be orthonormal.

Since $f_1, \dots, f_r$ is a $\boldsymbol{C} = K \otimes_{\boldsymbol{Q}} \boldsymbol{R}$-basis of V_2 and $H(f_i, f_j) = \psi(f_i, f_j)$, we see that H is negative definite on V_2 (and H is positive definite on V_1). □

3.4. Remark.

If (V, h, ψ) and $K = \boldsymbol{Q}(\phi)$ are as in section 3.1, then the Lie group $U(H)(\boldsymbol{R})$ of real points of $U(H)$ is ismomorphic to $U(1, m-1)$.

For any $g \in U(H)(\boldsymbol{R})$ the Hodge structure (V, h^g) with

$$h^g : \boldsymbol{C}^* \to GL(V)(\boldsymbol{R}), \quad z \mapsto gh(z)g^{-1}$$

is also polarized by the same ψ and has $\phi \in \mathrm{End}_{Hod}(V, h^g)$. This implies that $SMT(V, h^g) = U(H)$ for any general $g \in U(H)(\boldsymbol{R})$. One can show that the moduli space of such Hodge structures is isomorphic to the complex $m-1$-ball $U(1, m-1)/(U(1) \times U(m-1))$.

The inclusion $U(H) \subset SO(\psi)$ induces $U(1, m-1) \subset SO(2, 2m-2)$, this well-known inclusion is used for example to restrict modular forms from othogonal groups to unitary groups.

3.5.

The universal cover of the orthogonal group $SO(\psi)$ has a natural spin representation on the even Clifford algebra $C^+(V)$. The following proposition describes the spin representation over $\boldsymbol{Q}$ for the quadratic forms under consideration. Over the complex numbers these results are very well known, but over a number field the situation is a bit delicate. The proposition will be used to decompose the spin representation as a representation of $U(H)$, the Mumford Tate group of a general Hodge structure of CM-type (V, h, ψ, K) under consideration.

3.6. Proposition.

Let (V, h, ψ) be a polarized weight 2 Hodge structure with $\dim V^{2,0} = 1$ which is of CM-type for an imaginary quadratic field K. Then there is a $\boldsymbol{Q}$-basis of V such that

$$\psi(x, x) = \sum_{i=1}^{m} d_i x_i^2 + d \sum_{i=1}^{m} d_i x_{m+i}^2, \quad \text{with } d_1 < 0,\ d_2, \dots, d_m > 0.$$

The representation of $so(2m)$ on $C^+(Q)$ decomposes as $C^+(V) = S^{2^{m-2}}$ where S is a $so(\psi)$-representation of dimension 2^{m+1} whose irreducible components are:

$$S \cong \begin{cases} S_+ \oplus S_-, & \mathrm{End}_{so(\psi)}(S_\pm) \cong D, \quad S_\pm \otimes \boldsymbol{C} \cong \Gamma_\pm^2 \quad \text{if } m \equiv 0 \ (4), \\ S_1^2, & \mathrm{End}_{so(\psi)}(S_1) \cong K, \quad S_1 \otimes \boldsymbol{C} \cong \Gamma_+ \oplus \Gamma_- \quad \text{if } m \equiv 1,3 \ (4), \end{cases}$$

with D a skew field of degree 4 over $\boldsymbol{Q}$, and Γ_+, Γ_- are the two half-spin representations of $so(2m) \otimes_{\boldsymbol{Q}} \boldsymbol{C}$, each of which has dimension 2^{m-1}.

In case $m \equiv 2$ (4) there are two possibilities. If the equation $-\prod d_i = x^2 + dy^2$ has a solution $(x, y) \in \boldsymbol{Q}^2$, then

$$S \cong S_+^2 \oplus S_-^2, \quad \mathrm{End}_{so(\psi)}(S_\pm) \cong \boldsymbol{Q}, \quad S_\pm \otimes \boldsymbol{C} \cong \Gamma_\pm$$

(*the split case*). *In case this equation has no solution we have*

$$S \cong S_+ \oplus S_-, \quad \mathrm{End}_{so(\psi)}(S_\pm) \cong D, \quad S_\pm \otimes \boldsymbol{C} \cong \Gamma_\pm^2$$

(*and D is a skew field of degree 4 over $\boldsymbol{Q}$, we call this the non-split case*).

Proof. To find this basis of V, choose $e_1 \in V$ with $\psi(e_1, e_1) \neq 0$ and let $d_1 := \psi(e_1, e_1)$, $e_{m+1} := \phi e_1$. Then $\psi(e_{m+1}, e_{m+1}) = d\psi(e_1, e_1) = dd_1$ and $d\psi(e_1, e_{m+1}) = \psi(\phi e_1, (-d)e_1) = -d\psi(e_1, e_{m+1})$, hence $\psi(e_1, e_{m+1}) = 0$. Next we take $e_2 \in \langle e_1, e_{m+1}\rangle^\perp$ etc. Since the signature of ψ is $(2-, (2n-2)+)$, we may assume $d_1 < 0$ and $d_2, \dots, d_m > 0$.

We recall that over K the spin representation of $so(\psi)_K := so(\psi) \otimes_{\boldsymbol{Q}} K$ on $C^+(V)_K := C^+(V) \otimes_{\boldsymbol{Q}} K$ decomposes as a direct sum of 2^{m-1} copies of $\Gamma_+ \oplus \Gamma_-$ and then we take Galois invariants to find the irreducible $so(\psi)$-representations over $\boldsymbol{Q}$.

Let $V_K := V \otimes_{\boldsymbol{Q}} \boldsymbol{Q}(\sqrt{-d})$, and consider the following K-basis of V_K:

$$f_i := (1/2dd_i)(\sqrt{-d}e_i + e_{m+i}), \quad f_{m+i} := 1/2(-\sqrt{-d}e_i + e_{m+i}), \quad (1 \le i \le m)$$

where we wrote $\sqrt{-d}e_i$ for $e_i \otimes \sqrt{-d}$. One verifies that:

$$\psi\Big(\sum y_j f_j, \sum y_j f_j\Big) = \sum_{i=1}^m y_i y_{m+i}.$$

Since $\psi(f_j, f_j) = 0$, $\psi(f_j + f_k, f_j + f_k) = 1$ if $|j - k| = m$ and is zero otherwise, we have in $C(V)_K$:

$$f_j^2 = 0, \quad f_i f_{m+i} + f_{m+i} f_i = 1, \quad f_j f_k = -f_k f_j \quad \text{if } |j-k| \neq 0, m.$$

We denote conjugation on K by a '$^-$', this acts on $C(V)_K$ via the second factor and:

$$\bar{f}_i = (1/dd_i) f_{m+i}, \quad \bar{f}_{m+i} = d_i d f_i.$$

Let

$$f := f_{m+1} f_{m+2} \cdots f_{2m}, \quad \text{then } \bar{f} = \bar{f}_{m+1} \cdots \bar{f}_{2m} = \left(d^m \prod_i d_i\right) f_1 \cdots f_m.$$

In $C(V)_K$ we have:

$$f\bar{f}f = \delta f, \quad \bar{f}f\bar{f} = \delta\bar{f}, \quad \text{with } \delta := (-1)^{m(m-1)/2} d^m \prod_{i=1}^{m} d_i,$$

the second is just the conjugate of the first which is an easy computation:

$$\begin{aligned} f\bar{f}f &= \left(d^m \prod_i d_i\right)(f_{m+1}\cdots f_{2m})(f_1\cdots f_m)(f_{m+1}\cdots f_{2m}) \\ &= \delta(f_{m+1}\cdots f_{2m})(f_m f_{m-1}\cdots f_2 f_1)(f_{m+1}\cdots f_{2m}) \\ &= \delta(f_{m+1}\cdots f_{2m})(f_m\cdots f_2)(1 - f_{m+1}f_1)(f_{m+2}\cdots f_{2m}) \\ &= \delta(f_{m+1}\cdots f_{2m})(f_m\cdots f_3)(1 - f_{m+2}f_2)(f_{m+3}\cdots f_{2m}) - 0 \\ &= \cdots \\ &= \delta f. \end{aligned}$$

Now we consider the left $C(V)_K$-modules generated by f and $\bar{f}$. These modules are isomorphic, in fact the relations we just proved imply that

$$R_{\bar{f}} : C(V)_K f \to C(V)_K \bar{f}, \quad xf \mapsto xf\bar{f}$$

is an isomorphism of left $C(V)_K$-modules with inverse

$$R_f : C(V)_K \bar{f} \to C(V)_K f, \quad y\bar{f} \mapsto y\bar{f}f.$$

In [**FH**], Chapter 20, an inclusion $so(\psi)_K \hookrightarrow C^+(V)_K$ $(\subset C(V)_K)$ is constructed and it is shown ([**FH**], 20.19 and 20.20) that, as an $so(\psi)_K$-module, $C(V)_K f$ is isomorphic to the direct sum of the two half-spin representations:

$$C(V)_K f \cong \Gamma_{+,K} \oplus \Gamma_{-,K}, \quad \Gamma_{\pm,K} \otimes_K \boldsymbol{C} \cong \Gamma_{\pm}.$$

Moreover, $C(V)_K^+$, the even Clifford algebra, is isomorphic to a product of two matrix algebras:

$$C(V)_K^+ \cong M_{2^{m-1}}(K) \times M_{2^{m-1}}(K),$$

this implies that the spin representation of $so(\psi)_K$ on $C^+(V)_K$ is isomorphic to 2^{m-1} copies of $\Gamma_{+,K} \oplus \Gamma_{-,K}$.

Before considering the situation over $\boldsymbol{Q}$ we recall that the center of $C^+(V)$ is $\boldsymbol{Q} \oplus \boldsymbol{Q}z$ with $z := e_1 e_2 \cdots e_{2m}$ and that $z^2 = (-1)^m d^m \prod_i d_i^2$ (cf. [**L**], §5.2). Since

$\sqrt{(-1)^m d^m \prod_i d_i^2} \in K$, the center of $C(V)_K^+$ is $K \times K$ and one can verify that $\Gamma_{\pm,K}$ are the two eigenspaces of z in $C(V)_K f$. We also observe that $C(V)_K f \cap C(V)_K \bar{f} = \{0\}$, in fact if $af = b\bar{f}$ then $af\bar{f} = b\bar{f}^2 = 0$, hence $0 = af\bar{f}f = \delta af$, hence $af = 0$.

The subspace $C(V)_K f$ is not defined over $\boldsymbol{Q}$ in general, but the direct sum

$$S_K := C(V)_K f \oplus C(V)_K \bar{f}$$

obviously is defined over $\boldsymbol{Q}$, that is $S_K = S \otimes_{\boldsymbol{Q}} K$ for some $\boldsymbol{Q}$-vector space $S \subset C(V)$, in fact

$$S = C(V)(f + \bar{f}) + C(V)\sqrt{-d}(f - \bar{f}).$$

Moreover, S is a representation space for $so(\psi)$ $(\subset so(\psi)_K)$.

To decompose S into irreducible components we determine $A := \mathrm{End}_{so(\psi)}(S)$, the endomorphisms of S which commute with $so(\psi)$. Since $S_K \cong \Gamma_+^2 \oplus \Gamma_-^2$, we have $A_K := A \otimes_{\boldsymbol{Q}} K \cong M_2(K) \times M_2(K)$, hence $\dim_{\boldsymbol{Q}} A = 8$. It is clear that A_K is generated by the center of $C^+(V)_K$ and the maps R_f and $R_{\bar{f}}$. To determine A it suffices to find the invariants under conjugation in A_K.

Obviously the center of $C^+(V)$ lies in A. Moreover, the maps

$$\alpha, \beta : S \to S, \quad \alpha : x \mapsto x(f + \bar{f}), \quad \beta : x \mapsto x\sqrt{-d}(f - \bar{f})$$

commute with $so(\psi)$ (which acts from the left whereas α and β act from the right). Note we have:

$$(f + \bar{f})^2 = f^2 + f\bar{f} + \bar{f}f + \bar{f}^2 = f\bar{f} + \bar{f}f = \delta,$$

the last equality holds since in S_K we have:

$$(af + b\bar{f})(f\bar{f} + \bar{f}f) = 0 + af\bar{f}f + b\bar{f}f\bar{f} + 0 = \delta(af + b\bar{f})$$

and similarly $(\sqrt{-d}(f - \bar{f}))^2 = d(f\bar{f} + \bar{f}f) = d\delta$. Moreover, $\alpha\beta = -\beta\alpha$. Thus the $\boldsymbol{Q}$-algebra generated by α and β is the quaternion algebra $D := (\delta, d\delta)$ and

$$A \cong D \otimes_{\boldsymbol{Q}} \boldsymbol{Q}(z), \quad \text{with } D = (\delta, d\delta).$$

Since $(\alpha\beta)^2 = -\alpha^2\beta^2 = -d\delta^2$, D contains a copy of the field $\boldsymbol{Q}(\sqrt{-d}) \cong K$.

The center of $C^+(V)$ is $\boldsymbol{Q}(z)$ with $z^2 = (-d)^m \prod_i d_i^2$, hence:

$$\boldsymbol{Q}(z) \cong \boldsymbol{Q} \times \boldsymbol{Q} \quad \text{if } m \equiv 0 \ (2), \quad \boldsymbol{Q}(z) \cong K \quad \text{if } m \equiv 1 \ (2).$$

As $d > 0$, $d_1 < 0$ and $d_2, \ldots, d_m > 0$, the sign of δ is the sign of $(-1)^{m(m-1)/2}(-1)$. Thus δ, $d\delta$ are both negative if $m \equiv 0, 1 \ (4)$ so $D \otimes_{\boldsymbol{Q}} \boldsymbol{R}$ is isomorphic to the algebra of quaternions, a skew field, and thus D is also a skew field.

Hence in case $m \equiv 0 \ (4)$ we have $A \cong D \times D$ with a skew field D, therefore S splits up in two components S_+ and S_- with $\mathrm{End}_{so(\psi)}(S_\pm) \cong D$ and since $D \otimes_{\boldsymbol{Q}} \boldsymbol{C} \cong M_2(\boldsymbol{C})$ the $S_\pm \otimes_{\boldsymbol{Q}} \boldsymbol{C}$ are both direct sums of two copies of one irreducible $so(\psi)_{\boldsymbol{C}}$ representation.

In case $m \equiv 1 \ (4)$, $\boldsymbol{Q}(z) \cong K$ and D contains a copy of K hence $A \cong M_2(K)$ and thus S is isomorphic to the sum of two isomorphic representations, irreducible over $\boldsymbol{Q}$, each of which, after tensoring with K, is the direct sum of two non-isomorphic representations.

In case $m \equiv 2 \ (4)$, we have $A \cong D \times D$, but the structure of D depends on the d_i. In fact, $D \cong (-d, \delta)$ (using $\alpha\beta$ and α as generators) and $\delta = -d_1 d_2 \cdots d_m d^{2k}$ (with $2k = m$), we also have $D \cong (-d, n)$ with $n := -\prod_i d_i \in \boldsymbol{Z}_{>0}$. Hence $D \cong M_2(\boldsymbol{Q})$ iff $n = x^2 + dy^2$ for some $x, y \in \boldsymbol{Q}^2$.

In case $m \equiv 3 \ (4)$ we have, as in the case $m \equiv 1 \ (4)$, that $A \cong M_2(K)$. □

3.7.

For the general (V, h, ψ, K) we consider, the Mumford Tate group is $U(H)$. Thus the simple factors of the Hodge structure $(C^+(V), h_s)$ associated to (V, h, ψ) are exactly the irreducible subrepresentations of the Lie algebra $u(H)$ $(\subset so(\psi))$ of $U(H)$ in $C^+(V)$ (which are defined over $\boldsymbol{Q}$). We now determine the restriction of the $so(\psi)$ representation S from Proposition 3.6 to $u(H)$.

3.8. Proposition.

Let S be the 2^{m+1}-dimensional $so(\psi)$-representation defined in Proposition 3.6. The $u(H)$-representation S decomposes as follows:

$$S \cong S_0 \oplus S_1 \oplus \cdots \oplus S_m, \quad S_i \cong S_{m-i}, \quad \dim_{\boldsymbol{Q}} S_i = 2\binom{m}{i}.$$

The S_i are irreducible $u(H)$-representations except S_l if $2l = m$, $l \equiv 2 \ (4)$ and we are in the split case, in that case $S_l \cong (S_l')^2$ and S_l' is irreducible.

The S_i are K-vector spaces and:

$$\mathrm{End}_{u(H)}(S_i) = \begin{cases} K & (2i \neq m) \\ D & (2i = m), \end{cases}$$

with D a quaternion algebra (*a skew field except for the split case*) *which contains K, but* $\mathrm{End}_{u(H)}(S_l') = \boldsymbol{Q}$.

Proof. First we determine the inclusion $u(H)_K \hookrightarrow so(\psi)_K$ and the restriction of the half-spin representations $\Gamma_{\pm,K}$ to $u(H)_K$.

Extending the scalars from $\boldsymbol{Q}$ to K, the endomorphism ϕ of V has two eigenspaces in $V \otimes_{\boldsymbol{Q}} K$:

$$V_K := V \otimes_{\boldsymbol{Q}} K = V_+ \oplus V_-, \quad (\phi \otimes 1)v = \pm(1 \otimes \phi)v \quad (v \in V_\pm).$$

Each of the eigenspaces is isotropic for the K-linear extension of ψ to V_K, in fact,

$$\begin{aligned}\psi(v,w) &= d^{-1}\psi((\phi\otimes 1)v,(\phi\otimes 1)w)\\ &= d^{-1}\psi(\pm(1\otimes\phi)v,\pm(1\otimes\phi)w)\\ &= d^{-1}\phi^2\psi(v,w)\\ &= -\psi(v,w).\end{aligned}$$

The actions of $u(H)$ and K on V commute, hence $u(H)$ acts on the eigenspaces V_+ and V_-, each of which has dimension m. In particular we have a Lie algebra map $u(H)\hookrightarrow gl(V_+)$, which, for dimension reasons, gives an isomorphism: $u(H)_K\cong gl(V_+)$. Since ψ is preserved, this fixes the map $u(H)\to gl(V_-)$ (in fact ψ gives a duality $V_-\xrightarrow{\cong} V_+^*$) and we get (with respect to the basis $f_1,\dots,f_{2m}$ of the proof of Prop. 3.6):

$$u(H)_K\cong gl(V_+)\to so(\psi)_K,\quad A\mapsto\begin{pmatrix}A & 0\\ 0 & -{}^tA\end{pmatrix}.$$

This inclusion $gl(V_+)\hookrightarrow so(\psi)$ is the same as the one obtained from composing the isomorphism $gl(V_+)\cong \mathrm{end}(V_+)\cong V_+^*\otimes V_-\cong V_-\otimes V_+$ and the inclusion (cf. [**FH**], formula (20.4)) $V_-\otimes V_+\hookrightarrow\bigwedge^2 V_K\cong so(\psi)$. From [**FH**], 20.15 we obtain the restrictions of the half-spin representations $\Gamma_{\pm,K}$ to $u(H)_K$:

$$\Gamma_{+,K}\cong\bigoplus_i \Lambda^{2i}V_+,\quad \Gamma_{-,K}\cong\bigoplus_i \Lambda^{2i+1}V_+,$$

(this is actually only an isomorphism of $su(H)_K$-representations).

To find the irreducible representations of $su(H)$ which are defined over $\boldsymbol{Q}$ we need to know how the conjugation on K acts. For this we use the following $C(V)$-modules ([**FH**], 20.12):

$$C(V)f\cong\bigoplus_{i=0}^m \Lambda^i V_+,\quad C(V)\bar f\cong\bigoplus_{i=0}^m \Lambda^i V_-.$$

The $f_{i_1}\cdots f_{i_r}f$ with $1\le i_1<\cdots<i_r\le m$ are a basis of $C(V)_Kf$, and these correspond to the elements $f_{i_1}\wedge\cdots\wedge f_{i_r}\in\Lambda^rV_+$. Their conjugate is, up to a constant, $f_{i_1+m}\cdots f_{i_r+m}\bar f$ which corresponds to $f_{i_1+m}\wedge\cdots\wedge f_{i_r+m}\in\Lambda^rV_-\cong\Lambda^{m-r}V_+$. In particular, we get conjugation-invariant subspaces

$$S_{i,K}=(\Lambda^iV_+)f\oplus(\Lambda^{m-i}V_+)\bar f\ \ (\subset S_K=C(V)_Kf\oplus C(V)_K\bar f)$$

for $0\le i\le m$. Thus $S_{i,K}=S_i\otimes_{\boldsymbol{Q}}K$ for a subspace $S_i\in C(V)$ which is invariant under $su(H)$. If $i \neq m - i$, the two summands are not isomorphic as $gl(V_+)_K$-representations, hence S_i is an irreducible $su(H)$-representation (if $W \subset S_i$ is an invariant subspace, then $W \otimes_{\mathbf{Q}} K$ is a $gl(V_+)_K$ and Galois invariant subspace of $S_{i,K}$ and hence $W = \{0\}$ or $W = S_i$). Since $\mathrm{End}_{sl(V_+)}(S_{i,k}) = K^2$ and S_i is irreducible, $B := \mathrm{End}_{su(H)}(S_i)$ is a field, of degree two over $\mathbf{Q}$. As $B \otimes_{\mathbf{Q}} K \cong K^2$ we get $B \cong K$. Note that $S_i \cong S_{m-i}$.

The interesting case is when m is even and $2l = m$. Both summands of $S_{l,K}$ are in the same half spin representation (hence in the same eigenspace of the center of $C^+(V)_K$). The maps R_f and $R_{\bar{f}}$ generate $\mathrm{End}_{sl(V_+)}(S_{l,K}) \cong M_2(K)$, hence $\mathrm{End}_{su(H)}(S_l) = (\delta, d\delta)$ (see the proof of Prop. 3.6). If $d \equiv 0$ (4) this quaternion algebra is a skew field and hence S_l is irreducible. If $d \equiv 2$ (4), S_l is irreducible in the non-split case and in the split case is isomorphic to $(S_l')^2$ with $S_{l,K}' \cong \Lambda^l V_+$. Thus we always have $\mathrm{End}_{su(H)}(S_l) \cong D$, and $K \subset D$, but D is not a skew field in the split case (then $D \cong M_2(\mathbf{Q})$). Since $S_{l,K}'$ is irreducible we have $\mathrm{End}_{su(H)}(S_l') \cong \mathbf{Q}$. □

3.9.

The previous propositions show that the Kuga-Satake Hodge structure $(C^+(V), h_s)$ associated to (V, h, ψ, K) decomposes as

$$C^+(V) \cong (S_0 \oplus S_1 \oplus \cdots \oplus S_m)^{2^{m-2}}$$

with $m = \dim_K V$ and they give the decomposition in simple factors as well as the endomorphism rings of the S_i in the generic case.

The Hodge structure on the S_i can be obtained as follows. Since V is K-vector space, the exterior products $\bigwedge_K^i V$ are well-defined and, as $\mathbf{Q}$-vector spaces, they have dimension $2\binom{m}{i}$, which is just the dimension of the summand S_i. Weil already pointed out that there is a natural inclusion

$$\textstyle\bigwedge_K^i V \hookrightarrow \bigwedge^i V,$$

and the $\bigwedge_K^i V$ are sub-Hodge structures of $\bigwedge^i V$ of weight $2i$. Combining Tate and half twists of these, one obtains weight 1 Hodge structures which are the summands of the Kuga-Satake Hodge structure. Note that the moduli of V, which has weight two, and of S_1 (with its K-action) which has weight one, are both the $m-1$-ball. Part of this theorem was already proved by Voisin in [**V**].

3.10. Theorem.

Let (V, h, ψ) be a polarized weight 2 Hodge structure with $\dim V^{2,0} = 1$ *which is of CM-type for an imaginary quadratic field K.*

Then the Hodge structure on the summand S_i (*see Proposition* 3.8) *of the Kuga-Satake Hodge structure is*:

$$S_i \cong \left(\bigwedge_K^i V\right)(i-1)_{1/2}.$$

In particular, $S_1 = V_{1/2}$ *and* S_0 *is the CM-type Hodge structure of weight one on* K. *Moreover*:

$$V \hookrightarrow S_0 \otimes S_1.$$

The dimensions of the eigenspaces for the K*-action on* $S_i^{1,0}$ *are*:

$$\dim S_{i,\sigma}^{1,0} = \binom{m-1}{i-1}, \quad \dim S_{i,\bar{\sigma}}^{1,0} = \binom{m-1}{i}, \quad \text{with } m = \dim_K V.$$

Proof. Let $V_{\boldsymbol{C}} = V_+ \oplus V_-$ be the decomposition in eigenspaces for the K-action, combining this with the Hodge decomposition we get:

$$V_{\boldsymbol{C}} = V_+^{2,0} \oplus V_+^{1,1} \oplus V_-^{0,2} \oplus V_-^{1,1}.$$

We choose a basis $e_1, \dots, e_m$ of V_+ and $e_{m+1}, \dots, e_n$ of V_- such that

$$h(z) = diag(z^2, 1 \dots, 1, z^{-2}, 1, \dots, 1) \ \ (\in SO(\psi)(\boldsymbol{C})) \quad (z \in S^1),$$

so, for example, $V^{0,2} = V_-^{0,2} = \langle e_{m+1} \rangle$. Then $h(z)$ lies in the 1-parameter subgroup generated by $H_1 := 1/2(e_1 \wedge e_{m+1}) \in \bigwedge^2 V_{\boldsymbol{C}} \cong so(\psi)_{\boldsymbol{C}}$. From the proof of [**FH**], 20.15, one finds that H_1 multiplies $w := e_{i_1} \wedge \cdots \wedge e_{i_k}$, $1 \le i_1 < \cdots < i_k \le m$, by $+1/2$ if $i_1 = 1$ and else by $-1/2$. Hence $h_s(z)$ multiplies w by z if $i_1 = 1$ and else by $\bar{z}$. Since:

$$S_{i,\boldsymbol{C}} = \bigwedge^i V_+ \oplus \bigwedge^{m-i} V_+ = \left(V_+^{2,0} \otimes \bigwedge^{i-1} V_+^{1,1}\right) \oplus \left(\bigwedge^i V_+^{1,1}\right)$$
$$\oplus \left(V_+^{2,0} \otimes \bigwedge^{m-i-1} V_+^{1,1}\right) \oplus \left(\bigwedge^{m-i} V_+^{1,1}\right)$$

the action of $h_s(z)$ is by $diag(z, \bar{z}, z, \bar{z})$. The isomorphism $\bigwedge^{m-i} V_+ \cong \bigwedge^i V_+^* \cong \bigwedge^i V_-$ induces

$$V_+^{2,0} \otimes \bigwedge^{m-i-1} V_+^{1,1} \cong \bigwedge^i V_-^{1,1}, \qquad \bigwedge^{m-i} V_+^{1,1} \cong V_-^{2,0} \otimes \bigwedge^{i-1} V_-^{1,1}.$$

Therefore the Hodge decomposition of $S_{i,\boldsymbol{C}} \cong \bigwedge^i V_+ \oplus \bigwedge^i V_-$ is given by:

$$S_i^{1,0} = \left(V_+^{2,0} \otimes \bigwedge^{i-1} V_+^{1,1}\right) \oplus \left(\bigwedge^i V_-^{1,1}\right).$$

The eigenspace decomposition of $S_i^{1,0}$ is now obvious and we see that $S_1 = V_{1/2}$.

On the other hand, following Weil, we have

$$\left(\bigwedge\nolimits_K^i V\right) \otimes_{\boldsymbol{Q}} \boldsymbol{C} = \left(\bigwedge\nolimits^i V_+\right) \oplus \left(\bigwedge\nolimits^i V_-\right) \hookrightarrow \bigoplus_j \left(\bigwedge\nolimits^{i-j} V_+ \otimes \bigwedge\nolimits^j V_-\right) = \left(\bigwedge\nolimits^i V\right) \otimes \boldsymbol{C}.$$

The Hodge structure on $\bigwedge_K^i V$ induced from this inclusion is

$$\left(\bigwedge\nolimits_K^i V\right) \otimes_{\boldsymbol{Q}} \boldsymbol{C} = \left(V_+^{2,0} \otimes \bigwedge\nolimits^{i-1} V_+^{1,1}\right) \oplus \left(\bigwedge\nolimits^i V_+^{1,1} \oplus \bigwedge\nolimits^i V_-^{1,1}\right) \oplus \left(\bigwedge\nolimits^{i-1} V_+^{1,1} \otimes V_-^{0,2}\right),$$

thus the Hodge numbers are $(2,0)+(i-1,i-1)=(i+1,i-1)$, (i,i) and $(i-1,i+1)$. Therefore if we Tate twist $(i-1)$-times and then do a half twist we obtain a Hodge structure of weight one which is just the one obtained from h_s.

The inclusion $V \subset V_{1/2} \otimes K(-1)_{1/2} = S_1 \otimes S_0$ follows from Proposition 2.8. □

3.11. Example.

We construct, geometrically, a 9 dimensional family of polarized Hodge structures with $h^{2,0}=1$, $h^{1,1}=18$ with CM by the field $K \cong \boldsymbol{Q}(\sqrt{-3})$. For a Hodge structure V of this family we identify the Hodge structures S_0 and S_1 as in Theorem 3.10 and we give a geometrical realisation of the inclusion $V \hookrightarrow S_0 \otimes S_1$.

For $a_1, \dots, a_{12} \in \boldsymbol{C}$ we define an (isotrivial) elliptic surface S over $\boldsymbol{P}^1$ by the Weierstrass model:

$$S : Y^2 = X^3 + \prod_{i=1}^{12}(t-a_i), \quad S \to \boldsymbol{P}^1, \quad (X, Y, t) \mapsto t.$$

Since S has twelve fibers which are cuspidal it is a K3 surface (and $\omega := Y^{-1}\,\mathrm{d}X \wedge \mathrm{d}t$ is a no where zero holomorphic 2-form on S).

The orthogonal complement in $H^2(S, \boldsymbol{Q})$ of the classes of a fiber and the section at infinity is a sub-Hodge structure V of dimension 20 in with $V^{2,0}=1$ and the field $K = \boldsymbol{Q}(\sqrt{-3})$ acts on V via the automorphism $(X, Y, t) \mapsto (\zeta^2 X, Y, t)$ with a primitive 6-th root of unity ζ.

Define curves C, C', of genus 25 and 1 by:

$$C : y^6 = \prod_{i=1}^{12}(x-a_i); \quad C' : v^2 = u^3 - 1.$$

Both of these curves have automorphisms of order 6:

$$\psi : C \to C, \quad (x, y) \mapsto (x, \zeta y); \quad \psi' : C' \to C', \quad \psi' : (u, v) \mapsto (\zeta^2 u, -v).$$

The surface S is the (minimal model of the desingularisation of the) quotient of $C \times C'$ by the automorphism $\phi = (\psi^{-1}, \psi')$ of order 6, the quotient map is given by

$$\pi : S = C \times C' \to S, \quad ((x, y), (u, v)) \mapsto (X, Y, t) = (y^2 u, y^3 v, x).$$

To define V_1, consider the following rational 1-forms on C:

$$\omega_{a,b} := x^a y^b \frac{\mathrm{d}x}{y^5}, \quad \text{note } \psi^* : \omega_{a,b} \mapsto \zeta^{b+1} \omega_{a,b}.$$

It is easy to check that the $\omega_{a,b}$ with $a, b \geq 0$ and $a + 2b \leq 8$ are a basis of $H^0(C, \omega_C)$. In particular, the eigenspace of ψ^* with eigenvalue ζ has dimension 9 (and is spanned by the $\omega_{a,0}$ with $0 \leq a \leq 8$) whereas the eigenspace with eigenvalue $\zeta^{-1} = \zeta^5$ has dimension 1 (and is spanned by $\omega_{0,4}$).

Let $V_1 \subset H^1(C, \boldsymbol{Q})$ be the $\boldsymbol{Q}$-subspace on which the eigenvalues of ζ are primitive 6-th roots of unity. Then $\dim V_1 = 20$, and the associated abelian variety is of Weil type $(1, 9)$. Let $V_0 := H^1(C', \boldsymbol{Q})$, note ψ' acts on $H^0(C', \omega_{C'}) = \langle \omega' := \mathrm{d}u/v \rangle$ as ζ^{-1}.

The pull-back π^* maps V into the ϕ-invariants in $H^1(C, \boldsymbol{Q}) \otimes H^1(C', \boldsymbol{Q})$ and it is easy to verify that these invariants are exactly the ϕ-invariants in $V_1 \otimes V_0$. For dimension reasons we then have:

$$V \cong \pi^* V = (V_0 \otimes_{\boldsymbol{Q}} V_1)^{\langle \phi \rangle}.$$

Since $V_0 \cong K_{-1/2}$, the half twist of this identity gives $V_{1/2} \subset K \otimes V_1 \cong V_1^{\oplus 2}$, which implies that the half twist of V is just V_1:

$$V_{1/2} \cong V_1$$

and that π^* is a geometrical realization of the Kuga-Satake correspondence.

The parameter space of 20 dimensional Hodge structures with CM by K and $V^{2,0} = 1$ is (a quotient of) the 9-ball (cf. 3.4). The K3 surfaces in this example are parametrized by 12 points in $\boldsymbol{P}^1$, Deligne and Mostov ([**DM**]) actually showed that the geometrical quotient $(\boldsymbol{P}^1)^{12}//PGL(2)$ is a 9-ball quotient. See [**Va**] for old and new results on this moduli space.

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