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A method of energy estimates in L^∞ and its application to porous medium equations

Dedicated to Professor Kyûya MASUDA on the occasion of his 60-th birthday

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Abstract. The existence of time local C^∞-solutions is shown for Cauchy problem of the porous medium equations. Our arguments rely on the "L^∞-energy method" developed in our previous paper [**16**] and a new method based on the theory of evolution equations in the L^2-framework which enables us to handle with perturbations which can be decomposed into monotone parts and small parts in Sobolev spaces of higher order.

1. Introduction.

In this paper, we are concerned with Cauchy problem for the following nonlinear parabolic equations:

$$(\mathrm{P})\begin{cases} u_t = (u^\ell u_x)_x, & (x,t) \in \boldsymbol{R} \times [0,\infty), \\ u(x,0) = u_0(x), & x \in \boldsymbol{R}. \end{cases} \tag{1.1}$$

This equation is widely known as the porous medium equations, which describes the isentropic flow of an ideal gas through a homogeneous porous medium and other physical phenomena such as in gas dynamics and plasma physics, (see Aronson [**1**]). It is well known that (P) possesses self-similar special solutions constructed by Barenblatt [**5**], and that (P) admits a unique (time) global weak solution, which is proved by Oleinik-Kalashnikov-Chzhou [**15**]. After these pioneering works, enormous number of studies in various aspects were devoted to this equation.

As for the regularity of weak solution u, Hölder continuity with respect to x

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and t is established, (see Aronson [**2**], Kruzhkov [**12**], Gilding [**10**], and Caffarelli-Friedman [**7**]).

Furthermore, higher regularity is known for the so-called pressure of gas given by $v = ((\ell+1)/\ell)u^{\ell}$. In fact, v enjoys Lipschitz continuity in x and t, (see Aronson [**2**], DiBenedetto [**9**], Bénilan [**6**], and Aronson-Caffarelli [**3**] and Caffarelli-Vazquez-Wolanski [**8**]), and if the space dimension is one, then v becomes C^{∞} on each side of the (moving) interface after the "waiting time", (see Aronson-Vazquez [**4**]).

However, as to the derivative estimates of solution u itself, little is investigated except in our previous result [**16**], where a time local solution is constructed in $W^{1,\infty}(\boldsymbol{R}^N)$. Our main concern here is the existence of smooth (say C^{∞}) solution of (P). In studying this kind of problem, it should be recalled that by the result of Kalashnikov [**11**], we can not expect the global existence of classical solution for (P). So we are led to the very natural and basic problem whether (P) admits a time local C^{∞}-solution or not. Our goal in this paper is to give an affirmative answer to this important open problem left unsolved for a long time. The precise statement of our main result is given in the next section. To achieve our aim, we first introduce approximate equations for (P). In order to construct global C^{∞}-solutions of approximate equations, we introduce a new method based on the theory of evolution equations in the L^2-framework which enables us to handle with perturbations which can be decomposed into monotone parts and small parts in Sobolev spaces of higher order. Furthermore, to establish a priori bounds for solutions of approximate equations, we expand the "L^{∞}-energy method", which is developed in [**16**]. We shall carry out these procedures in §4 and §5. For this purpose, some lemmas are prepared in §3, and the proof of main theorem is given in the last section.

2. Main Theorem.

Our basic assumptions imposed on the parameter ℓ and the initial data u_0 are the following (A.1) and (A.2).

(A.1) ℓ is an even natural number.

(A.2) $u_0(x) \in \bigcap_{m=0}^{\infty} H^m(\boldsymbol{R})$.

Then our main result is stated as follows.

Theorem. *Let* (A.1) *and* (A.2) *be satisfied, then there exists a positive number* T_0 *depending on* $\|u_0\|_{L^{\infty}(\boldsymbol{R})}$ *and* $\|u_{0x}\|_{L^{\infty}(\boldsymbol{R})}$ *such that Cauchy problem* (P) *has a unique solution* $u \in C^{\infty}([0,T_0]\times\boldsymbol{R})$ *such that*

$$\sup_{0\le t\le T_0} \|u(\cdot,t)\|_{L^{\infty}(\boldsymbol{R})} \le \|u_0\|_{L^{\infty}(\boldsymbol{R})}. \tag{2.1}$$

Moreover T_0 can be chosen as a monotone decreasing function of $\|u_{0x}\|_{L^\infty(\boldsymbol{R})}$ such that T_0 tends to 0 as $\|u_{0x}\|_{L^\infty(\boldsymbol{R})}$ tends to $+\infty$.

As an immediate consequence of this theorem, we can derive the following observation.

Corollary 2.1. *A solution $u \in C^\infty([0,T) \times \boldsymbol{R})$ of* (P) *can be continued as a C^∞-solution to the right of $t = T$, if and only if $\|u_x(\cdot,t)\|_{L^\infty(\boldsymbol{R})}$ is bounded on $[0,T)$. Furthermore, if u can not be continued as a C^∞-solution to the right of $t = T$, then it holds that*

$$\lim_{t \uparrow T} \|u_x(\cdot,t)\|_{L^\infty(\boldsymbol{R})} = +\infty. \tag{2.2}$$

Remark 2.2. Since $(u^\ell u_x)_x = (1/(\ell+1))(u^{\ell+1})_{xx}$ and the function $r \mapsto |r|^\ell r$ $(\ell > 0)$ belongs to $C^\infty(\boldsymbol{R})$ if and only if ℓ is an even integer, it seems rather plausible to assume (A.1) for the argument in the C^∞-category.

3. Some Lemmas.

In this section, we shall prepare several lemmas which will be often used in the next section, the main parts of our arguments. We first fix some notations which will appear frequently in what follows.

We use the simplified notations:

Notations

(1) $D = \partial/\partial x,\ D^m = (\partial/\partial x)^m,\ D^0 = I_d$.

(2) $L^r = L^r(\boldsymbol{R}),\ \|\cdot\|_{L^r} = \|\cdot\|_{L^r(\boldsymbol{R})},\ (1 \le r \le \infty)$.

(3) $\|\cdot\|_{H^n} = \|\cdot\|_{H^n(\boldsymbol{R})},\ (n \in \boldsymbol{N}),\ \|\cdot\|_{H^0} = \|\cdot\|_{L^2(\boldsymbol{R})}$.

(4) $(u,v) = (u,v)_{L^2(\boldsymbol{R})},\ \|u\| = \|u\|_{L^2}$.

Let $A = -D^2$ and put $H_k = H^{2k}(\boldsymbol{R})$. We define the inner product of H_k by

(5) $(u,v)_{H_k} = (u,v)_{H^{2k}} = (u,v) + (A^k u, A^k v),\ k \in \boldsymbol{N}$.

We first note the following property.

Lemma 3.1. *It holds that*

$$\|D^j u\|_{L^2} \le \|u\|_{H_k} \quad \textit{for all } u \in H_k,\ 0 \le j \le 2k,\ k \in \boldsymbol{N}. \tag{3.1}$$

Proof. From the definition of the topology of H_k, the cases $j = 0$ and $j = 2k$ are obvious. In order to verify the other cases $1 \le j \le 2k-1$, it suffices to derive the following inequalities.

$$\|D^j u\| \le \|D^n u\|^{j/n} \cdot \|u\|^{1-j/n} \quad \text{for all } u \in H^n,\ n \in \boldsymbol{N},\ 1 \le j \le n-1. \tag{3.2}$$

Indeed, (3.2) with $n = 2k$ yields (3.1), since $\|D^{2k}u\|^{j/2k} \cdot \|u\|^{1-j/2k} \leq \|u\|_{H_k}$.

We are going to prove (3.2) by induction.

Since $\|Du\|^2 = (Du, Du) = (-D^2u, u) \leq \|D^2u\|\|u\|$, (3.2) holds true with $n = 2$.

Assume that (3.2) hold true with $n = m-1$ for all $1 \leq j \leq m-2$, $(m \geq 3)$. Then, by using (3.2) with $n = m-1$, and $j = m-2$, we get

$$\begin{aligned}\|D^{m-1}u\|^2 &= -\int_{\boldsymbol{R}} D^m u D^{m-2}\,dx \leq \|D^m u\| \cdot \|D^{m-2}u\| \\ &\leq \|D^m u\| \cdot \|D^{m-1}u\|^{(m-2)/(m-1)} \cdot \|u\|^{1/(m-1)},\end{aligned}$$

whence follows

$$\|D^{m-1}u\| \leq \|D^m u\|^{(m-1)/m} \cdot \|u\|^{1/m}, \tag{3.3}$$

which implies that (3.2) holds with $n = m$, $j = m-1$.

For any $1 \leq j \leq m-2$, (3.2) with $n = m-1$ and (3.3) assure

$$\begin{aligned}\|D^j u\| &\leq \|D^{m-1}u\|^{j/(m-1)} \cdot \|u\|^{1-j/(m-1)} \\ &\leq (\|D^m u\|^{(m-1)/m} \cdot \|u\|^{1/m})^{j/(m-1)} \cdot \|u\|^{1-j/(m-1)} = \|D^m u\|^{j/m} \cdot \|u\|^{1-j/m}.\end{aligned}$$

This completes the proof. □

The following two lemmas are standard results from embedding theorems.

Lemma 3.2. *The following inequalities hold.*

$$\|u\|_{L^\infty} \leq \sqrt{2}\|u\|_{L^2}^{1/2} \cdot \|u_x\|_{L^2}^{1/2} \quad \text{for all } u \in H^1(\boldsymbol{R}), \tag{3.4}$$

$$\|D^j u\|_{L^\infty} \leq \sqrt{2}\|u\|_{H_k} \quad \text{for all } u \in H_k \text{ with } k \geq 1 \text{ and } 0 \leq j \leq 2k-1. \tag{3.5}$$

Proof. By the density argument, we have only to show (3.4) for $u \in C_0^\infty(\boldsymbol{R})$. Since $(1/2)(d/dx)(u(x))^2 = u(x) \cdot u_x(x)$, integrating this identity on $(-\infty, x)$, we have

$$u(x)^2 = 2\int_{-\infty}^{x} u(x)u_x(x)\,dx \leq 2\|u\|_{L^2}\|u_x\|_{L^2},$$

which gives (3.4).

Then, applying (3.4) for $u = D^j u$ and Lemma 3.1, we get

$$\|D^j u\|_{L^\infty} \leq \sqrt{2}\|D^j u\|_{L^2}^{1/2} \cdot \|D^{j+1}u\|_{L^2}^{1/2} \leq \sqrt{2}\|u\|_{H_k}.$$ □

Lemma 3.3. *It holds that*

$$\|u\|_{L^4}^4 \leq 2\|u\|_{L^2}^3 \cdot \|u_x\|_{L^2} \quad \text{for all } u \in H^1(\boldsymbol{R}). \tag{3.6}$$

Proof. By using Hölder inequality and (3.4), we obtain

$$\|u\|_{L^4}^4 \le \|u\|_{L^2} \cdot \|u\|_{L^4}^2 \cdot \|u\|_{L^\infty} \le \sqrt{2}\|u\|_{L^2}^{3/2} \cdot \|u\|_{L^4}^2 \cdot \|u_x\|_{L^2}^{1/2},$$

whence follows (3.6). □

The following lemmas play an important role in establishing the L^∞-estimates of solutions.

Lemma 3.4. *Let Ω be a domain in $\boldsymbol{R}^N$ and suppose that there exist $r_0 \ge 1$ and $C_r > 0$ with $\lim_{r\to\infty} C_r = C_\infty < +\infty$ such that*

$$\|u\|_{L^r(\Omega)} \le C_r \quad \textit{for all } r \in [r_0, \infty). \tag{3.7}$$

Then u belongs to $L^\infty(\Omega)$ and satisfies

$$\|u\|_{L^\infty(\Omega)} \le C_\infty. \tag{3.8}$$

Proof. Let $\Omega_k = \Omega \cap \{x \in \boldsymbol{R}^N; \|x\| < k\}$ and let $u_n(x) = |u_x(x)| \cdot \operatorname{sign} u(x)$ with $|u_n(x)| = \min(n, |u(x)|)$. Noting that $u_n \in L^\infty(\Omega_k)$ and $\|u_n\|_{L^r(\Omega_k)} \le C_r$, we find that $\lim_{r\to\infty} \|u_n\|_{L^r(\Omega_k)} = \|u_n\|_{L^\infty(\Omega_k)} \le C_\infty$ for all n and k, (see Theorem 1 of Yosida [**18**], p34). Since $u_n(x) \to u(x)$ a.e. x in Ω_k as $n \to \infty$ and $|u_n(x)| \le C_\infty$, we get $\|u\|_{L^\infty(\Omega_k)} \le C_\infty$ for all k. Therefore, for any $\varepsilon > 0$, there exist null sets $e_k \subset \Omega_k$ such that $|u(x)| \le C_\infty + \varepsilon$ for all $x \in \Omega_k \backslash e_k$ and k. Hence $|u(x)| \le C_\infty + \varepsilon$ for all $x \in \Omega \backslash e$, $e = \bigcup_{k=1}^\infty e_k$, which assures $u \in L^\infty(\Omega)$ and $\|u\|_{L^\infty(\Omega)} \le C_\infty$. □

Lemma 3.5. *Let Ω be a domain in $\boldsymbol{R}^N$. Suppose that $w \in L^1(0, T; L^r(\Omega))$ for all $r \in [r_0, \infty]$, $v(0) = v_0 \in L^\infty(\Omega)$ and $v \in W^{1,1}(0, T; L^r(\Omega))$ for all $r \in [r_0, \infty)$. If it holds that*

$$\frac{d}{dt}\|v(t)\|_{L^r(\Omega)} \le \|w(t)\|_{L^r(\Omega)} \quad \textit{for all } r \in [r_0, \infty) \textit{ and a.e. } t \in [0, T]. \tag{3.9}$$

Then we have

$$\|v\|_{L^\infty(0,T;L^\infty(\Omega))} \le \|v_0\|_{L^\infty(\Omega)} + \|w\|_{L^1(0,T;L^\infty(\Omega))}. \tag{3.10}$$

Proof. Integrating (3.9) on $[0, t]$ and using Young's inequality, we get

$$\begin{aligned}
\|v(t)\|_{L^r(\Omega)} &\le \|v_0\|_{L^r(\Omega)} + \|w\|_{L^1(0,T;L^r(\Omega))} \\
&\le \|v_0\|_{L^\infty(\Omega)}^{(r-r_0)/r} \cdot \|v_0\|_{L^{r_0}(\Omega)}^{r_0/r} + \int_0^t \|w(s)\|_{L^\infty(\Omega)}^{(r-r_0)/r} \cdot \|w(s)\|_{L^{r_0}(\Omega)}^{r_0/r}\, ds \\
&\le \frac{r-r_0}{r}\|v_0\|_{L^\infty(\Omega)} + \frac{r_0}{r}\|v_0\|_{L^{r_0}(\Omega)} \\
&\quad + \frac{r-r_0}{r}\|w\|_{L^1(0,T;L^\infty(\Omega))} + \frac{r_0}{r}\|w\|_{L^1(0,T;L^{r_0}(\Omega))}.
\end{aligned}$$

Hence, by letting $r \mapsto \infty$ and applying Lemma 3.4, we obtain (3.10). □

In the next section, we shall establish a priori estimates for higher derivatives of solutions. To carry out this, we need the following lemmas.

Lemma 3.6. *For any $u \in H^{n+2}(\boldsymbol{R})$, it holds that*

$$D^n(u^\ell D^2u) = I_n^1 + I_n^2 + I_n^3 + R_n^1 + R_n^2 \quad \text{for } n \geq 2, \tag{3.11}$$

where

$$I_n^1 = u^\ell D^{n+2}u,$$

$$I_n^2 = {}_nC_1 \ell u^{\ell-1} DuD^{n+1}u,$$

$$I_n^3 = \{{}_nC_2 \ell(\ell-1)u^{\ell-2}(Du)^2 + ({}_nC_2+1)\ell u^{\ell-1}D^2u\}D^nu, \quad (n \geq 3),$$

$$I_2^3 = \ell u^{\ell-1}(D^2u)^2,$$

$$R_n^1 = \sum_{i=3}^{n-1} {}_nC_i D^i(u^\ell)D^{n-i+2}u \quad \text{for } n \geq 4 \text{ and } R_2^1 = R_3^1 = 0,$$

$$R_n^2 = \sum_{i=1}^{n-1} {}_{n-1}C_i D^i(\ell u^{\ell-1})D^{n-i}uD^2u.$$

Furthermore we have

$$\sup_{2 \leq r \leq \infty} (\|DR_n^1\|_{L^r} + \|DR_n^2\|_{L^r}) \leq 2(\ell+1)^{n+1}(M_{n,\infty})^{\ell+1}, \tag{3.12}$$

$$\|DR_n^1\|_{L^2} + \|DR_n^2\|_{L^2} \leq 2(\ell+1)^{n+1}(M_{n-1,\infty})^\ell M_n \quad \text{for } n \geq 3, \tag{3.13}$$

where

$$M_{m,\infty} = \sup\{\|D^ju\|_{L^r}; 2 \leq r \leq \infty, 0 \leq j \leq m\},$$

$$M_m = \sup\{\|D^ju\|_{L^2}; 0 \leq j \leq m\}.$$

Proof. By Leibniz's formula, we get

$$D^n(u^\ell D^2u) = \sum_{i=0}^n E_i, \quad E_i = {}_nC_i D^i(u^\ell) \cdot D^{n-i+2}u.$$

It is clear that $I_n^1 = E_0$, $I_n^2 = E_1$ and $R_n^1 = \sum_{i=3}^{n-1} E_i$, $(n \geq 4)$.

Since

$$E_n = D^{n-1}(\ell u^{\ell-1}Du)D^2u = \ell u^{\ell-1}D^nuD^2u + R_n^2,$$

we find

$$E_2 = \ell u^{\ell-1}(D^2u)^2 + \ell(\ell-1)u^{\ell-2}(Du)^2D^2u, \quad (n=2),$$

$$E_2 + E_n = I_n^3 + R_n^2, \quad (n \geq 3).$$

Hence (3.11) is derived.

In order to establish the L^∞-estimate for $D^i(u^\ell)$, we first note that the number of ways of distributing D^i to u^ℓ, denoted by $A_{i,\ell}$, is given by

$$A_{i,\ell} = \ell^i,$$

since the number of ways for operating D to $u^\ell = \underbrace{u \cdot u \cdots u}_{\ell}$ is ℓ. Then we obtain

$$\|D^i(u^\ell)\|_{L^\infty} \leq \ell^i \cdot M_{i,\infty}^\ell. \tag{3.14}$$

Hence, by (3.14),

$$\begin{aligned}
\|D^i(u^\ell)\|_{L^2} &= \|D^{i-1}(\ell u^{\ell-1}Du)\|_{L^2} \\
&\leq \|\ell u^{\ell-1}D^iu\|_{L^2} + \ell \sum_{j=1}^{i-1} {}_{i-1}C_j\|D^j(u^{\ell-1})D^{i-j}u\|_{L^2} \\
&\leq \ell M_{0,\infty}^{\ell-1}M_i + \ell \sum_{j=1}^{i-1} {}_{i-1}C_j(\ell-1)^jM_{j,\infty}^{\ell-1}M_{i-j} \\
&\leq \ell M_iM_{i-1,\infty}^{\ell-1}\left(1 + \sum_{j=1}^{i-1} {}_{i-1}C_j(\ell-1)^j\right) \\
&= \ell M_iM_{i-1,\infty}^{\ell-1} \sum_{j=0}^{i-1} {}_{i-1}C_j(\ell-1)^j \\
&\leq \ell^i M_{i-1,\infty}^{\ell-1}M_i.
\end{aligned} \tag{3.15}$$

Therefore

$$\begin{aligned}
\|DR_n^1\|_{L^r} &\leq \sum_{i=3}^{n-1} {}_nC_i\{\|D^{i+1}(u^\ell)\|_{L^\infty} \cdot \|D^{n-i+2}u\|_{L^r} + \|D^i(u^\ell)\|_{L^\infty} \cdot \|D^{n-i+3}u\|_{L^r}\} \\
&\leq \sum_{i=0}^{n} {}_nC_i(\ell^{i+1} + \ell^i) \cdot M_{n,\infty}^{\ell+1} \\
&= (\ell+1)M_{n,\infty}^{\ell+1} \cdot \sum_{i=0}^{n} {}_nC_i\ell^i \\
&= (\ell+1)^{n+1} \cdot M_{n,\infty}^{\ell+1},
\end{aligned}$$

and

$$\begin{aligned}
\|DR_n^2\|_{L^r} &\le \sum_{i=1}^{n-1} {}_{n-1}C_i(\|D^{i+1}(\ell u^{\ell-1})\|_{L^\infty} \cdot \|D^{n-i}u\|_{L^\infty} \cdot \|D^2 u\|_{L^r} \\
&\quad + \|D^i(\ell u^{\ell-1})\|_{L^\infty} \cdot \{\|D^{n-i+1}u\|_{L^\infty}\|D^2 u\|_{L^r} + \|D^{n-i}u\|_{L^\infty}\|D^3 u\|_{L^r}\}) \\
&\le \sum_{i=0}^{n-1} {}_{n-1}C_i\{\ell(\ell-1)^{i+1} + \ell(\ell-1)^i\} \cdot M_{n,\infty}^{\ell+1} \\
&\le \ell^2 M_{n,\infty}^{\ell+1} \cdot \sum_{i=0}^{n-1} {}_{n-1}C_i(\ell-1)^i \\
&= \ell^{n+1} \cdot M_{n,\infty}^{\ell+1}.
\end{aligned}$$

Thus (3.12) is verified.

Similarly, by virtue of (3.14) and (3.15), we find

$$\begin{aligned}
\|DR_n^1\|_{L^2} &\le \sum_{i=3}^{n-1} {}_nC_i\{\|D^{i+1}(u^\ell)\|_{L^2} \cdot \|D^{n-i+2}u\|_{L^\infty} + \|D^i(u^\ell)\|_{L^\infty} \cdot \|D^{n-i+3}u\|_{L^2}\} \\
&\le \sum_{i=0}^{n-1} {}_nC_i(\ell^{i+1} + \ell^i) \cdot M_{n-1,\infty}^\ell \cdot M_n \\
&= (\ell+1)^{n+1} M_{n-1,\infty}^\ell \cdot M_n, \\
\|DR_n^2\|_{L^2} &\le \sum_{i=1}^{n-1} {}_{n-1}C_i(\|D^{i+1}(\ell u^{\ell-1})\|_{L^2} \cdot \|D^{n-i}u\|_{L^\infty} \cdot \|D^2 u\|_{L^\infty} \\
&\quad + \|D^i(\ell u^{\ell-1})\|_{L^\infty} \cdot \{\|D^{n-i+1}u\|_{L^2}\|D^2 u\|_{L^\infty} + \|D^{n-i}u\|_{L^\infty}\|D^3 u\|_{L^2}\}) \\
&\le \sum_{i=1}^{n-1} {}_{n-1}C_i(\ell(\ell-1)^{i+1} M_{i,\infty}^{\ell-1} M_{i+1} M_{n-i,\infty} M_{2,\infty} \\
&\qquad + \ell(\ell-1)^i \cdot M_{i,\infty}^{\ell-1}(M_{n-i+1}M_{2,\infty} + M_{n-i,\infty}M_3)).
\end{aligned}$$

Therefore for $n \ge 3$,

$$\begin{aligned}
\|DR_n^2\|_{L^2} &\le \sum_{i=1}^{n-1} {}_{n-1}C_i M_n M_{n-1,\infty}^\ell (\ell(\ell-1)^{i+1} + \ell(\ell-1)^i) \\
&= \ell^{n+1} \cdot M_{n-1,\infty}^\ell M_n.
\end{aligned}$$

whence follows (3.12) and (3.13). □

LEMMA 3.7. *For any $u \in H^{n+1}(\boldsymbol{R})$, it holds that*

$$D^n(\ell u^{\ell-1}(Du)^2) = J_n^1 + J_n^2 + S_n^1 + S_n^2 + S_n^3 + S_n^4 \quad \text{for } n \geq 3, \tag{3.16}$$

where

$$J_n^1 = 2\ell u^{\ell-1} Du D^{n+1} u,$$

$$J_n^2 = \{(2n+1)\ell(\ell-1)u^{\ell-2}(Du)^2 + 2n\ell u^{\ell-1} D^2 u\} D^n u,$$

$$S_n^1 = \sum_{i=2}^{n-1} {}_nC_i D^i(\ell u^{\ell-1}) D^{n-i}((Du)^2),$$

$$S_n^2 = 2\ell u^{\ell-1} \sum_{i=2}^{n-2} {}_{n-1}C_i D^{i+1} u D^{n-i+1} u \quad \text{for } n \geq 4 \text{ and } S_2^3 = 0,$$

$$S_n^3 = 2n\ell(\ell-1)u^{\ell-2} Du \sum_{i=1}^{n-2} {}_{n-2}C_i D^{i+1} u D^{n-i} u,$$

$$S_n^4 = \ell(\ell-1) \sum_{i=1}^{n-1} {}_{n-1}C_i D^i(u^{\ell-2}) D^{n-i} u (Du)^2.$$

Furthermore, we have

$$\sup_{2 \leq r \leq \infty} \sum_{j=1}^4 \|S_n^j\|_{L^r} \leq 2n\ell^2(\ell+1)^n (M_{n-1,\infty})^{\ell+1}, \tag{3.17}$$

$$\sum_{j=1}^4 \|S_n^j\|_{L^2} \leq 2n\ell^2(\ell+1)^n (M_{n^*,\infty})^{\ell} M_{n-1}, \tag{3.18}$$

where $n^* = \max(3, n-2)$ and $M_{m,\infty}$, M_m are the constants defined in Lemma 3.5.

PROOF. Leibniz's formula gives

$$D^n(\ell u^{\ell-1}(Du)^2) = \sum_{i=0}^n F_i, \quad F_i = {}_nC_i D^i(\ell u^{\ell-1}) D^{n-i}((Du)^2).$$

Obviously

$$S_n^1 = \sum_{i=2}^{n-1} F_i.$$

Furthermore,

$$F_0 = \ell u^{\ell-1} \cdot 2D^{n-1}(DuD^2u)$$
$$= 2\ell u^{\ell-1}\left\{DuD^{n+1}u + \sum_{i=1}^{n-1} {}_{n-1}C_i D^{i+1}u \cdot D^{n-i+1}u\right\}$$
$$= J_n^1 + 2n\ell u^{\ell-1} \cdot D^n u D^2 u + S_n^2,$$
$$F_1 = n\ell(\ell-1)u^{\ell-2}Du \cdot 2D^{n-2}(DuD^2u)$$
$$= 2n\ell(\ell-1)u^{\ell-2}(Du)^2 D^n u + S_n^3,$$
$$F_n = D^{n-1}(\ell(\ell-1)u^{\ell-2}Du)(Du)^2$$
$$= \ell(\ell-1)u^{\ell-2}D^n u(Du)^2 + S_n^4.$$

Thus (3.16) is derived.

Moreover, by virtue of (3.14), we get

$$\|S_n^1\|_{L^r} \le \sum_{i=2}^{n-1} {}_nC_i \|D^i(\ell u^{\ell-1})\|_{L^\infty} \cdot \|D^{n-i}((Du)^2)\|_{L^r}$$
$$\le \ell \sum_{i=2}^{n-1} {}_nC_i \cdot (\ell-1)^i M_{i,\infty}^{\ell-1} \cdot 2^{n-i} M_{n-1,\infty}^2$$
$$\le \ell \sum_{i=0}^{n} {}_nC_i(\ell-1)^i \cdot 2^{n-i} \cdot M_{n-1,\infty}^{\ell+1}$$
$$= \ell(\ell+1)^n \cdot M_{n-1,\infty}^{\ell+1},$$
$$\|S_n^2\|_{L^r} \le 2\ell\|u\|_{L^\infty}^{\ell-1} \cdot \sum_{i=2}^{n-2} {}_{n-1}C_i M_{n-1,\infty}^2$$
$$\le 2^n \ell \cdot M_{n-1,\infty}^{\ell+1},$$
$$\|S_n^3\|_{L^r} \le 2n\ell(\ell-1)\|u\|_{L^\infty}^{\ell-2}\|Du\|_{L^\infty} \cdot \sum_{i=1}^{n-2} {}_{n-2}C_i M_{n-1,\infty}^2$$
$$\le 2^{n-1} n\ell(\ell-1) \cdot M_{n-1,\infty}^{\ell+1},$$
$$\|S_n^4\|_{L^r} \le \ell(\ell-1) \cdot \sum_{i=1}^{n-1} {}_{n-1}C_i(\ell-2)^i \cdot M_{n-1,\infty}^{\ell+1}$$
$$= \ell(\ell-1)^n \cdot M_{n-1,\infty}^{\ell+1}.$$

Hence, these estimates assure (3.17). Moreover, we have

$$
\begin{aligned}
\|S_n^1\|_{L^2} &\le \sum_{i=2}^{n-2} {}_nC_i \|D^i(\ell u^{\ell-1})\|_{L^\infty} \cdot \|D^{n-i}(Du)^2\|_{L^2} \\
&\quad + {}_nC_{n-1}\|D^{n-1}(\ell u^{\ell-1})\|_{L^2}\|D((Du)^2)\|_{L^\infty} \\
&\le \ell \sum_{i=2}^{n-2} {}_nC_i(\ell-1)^i M_{i,\infty}^{\ell-1} \cdot 2^{n-i} M_{n-i,\infty} M_{n-i+1} \\
&\quad + n\ell(\ell-1)^{n-1} M_{n-2,\infty}^{\ell-2} M_{n-1} \cdot 2M_{1,\infty}M_{2,\infty} \\
&\le \ell \sum_{i=2}^{n-2} {}_nC_i(\ell-1)^i \cdot 2^{n-i} \cdot M_{n-2,\infty}^{\ell} M_{n-1} + 2n\ell(\ell-1)^{n-1} M_{n-2,\infty}^{\ell-1} M_{2,\infty} M_{n-1} \\
&\le \ell \sum_{i=2}^{n-1} {}_nC_i(\ell-1)^i \cdot 2^{n-i} M_{n^*,\infty}^{\ell} M_{n-1} \\
&\le \ell(\ell+1)^n M_{n^*}^{\ell} M_{n-1},
\end{aligned}
$$

$$
\begin{aligned}
\|S_n^2\|_{L^2} &\le 2\ell\|u\|_{L^\infty}^{\ell-1} \cdot \sum_{i=2}^{n-3} {}_{n-1}C_i \|D^{i+1}u\|_{L^\infty}\|D^{n-i+1}u\|_{L^2} \\
&\quad + 2\ell\|u\|_{L^\infty}^{\ell-1}\, {}_{n-1}C_{n-2}\|D^{n-1}u\|_{L^2}\|D^3u\|_{L^\infty} \\
&\le 2\ell\|u\|_{L^\infty}^{\ell-1} \cdot \sum_{i=2}^{n-3} {}_{n-1}C_i M_{n-2,\infty} M_{n-1} + 2\ell(n-1)\|u\|_{L^\infty}^{\ell-1} M_{3,\infty} M_{n-1} \\
&\le 2\ell M_{0,\infty}^{\ell-1} \cdot \sum_{i=2}^{n-2} {}_{n-1}C_i M_{n-1} M_{n^*} \\
&\le 2^n \ell \cdot M_{n^*,\infty}^{\ell} M_{n-1},
\end{aligned}
$$

$$
\begin{aligned}
\|S_n^3\|_{L^2} &\le 2n\ell(\ell-1)\|u\|_{L^\infty}^{\ell-2}\|Du\|_{L^\infty} \cdot \sum_{i=1}^{n-3} {}_{n-2}C_i \|D^{i+1}u\|_{L^\infty}\|D^{n-i}u\|_{L^2} \\
&\quad + 2n\ell(\ell-1)\|u\|_{L^\infty}^{\ell-2}\|Du\|_{L^\infty}\|D^{n-1}u\|_{L^2}\|D^2u\|_{L^\infty} \\
&\le 2n\ell(\ell-1) \cdot M_{n-2,\infty}^{\ell} \sum_{i=1}^{n-3} {}_{n-2}C_i M_{n-1} + 2n\ell(\ell-1) M_{n-2,\infty}^{\ell-1} M_{2,\infty} M_{n-1} \\
&= 2n\ell(\ell-1) M_{n^*,\infty}^{\ell} M_{n-1} \sum_{i=1}^{n-2} {}_{n-2}C_i \\
&= 2n\ell(\ell-1) M_{n^*,\infty}^{\ell} M_{n-1} 2^{n-2} \\
&= 2^{n-1} n\ell(\ell-1) M_{n^*,\infty}^{\ell} M_{n-1},
\end{aligned}
$$

$$\begin{aligned}
\|S_n^4\|_{L^2} &\le \ell(\ell-1)\cdot\sum_{i=1}^{n-2} {}_{n-1}C_i\|D^i(u^{\ell-2})\|_{L^\infty}\|D^{n-i}u\|_{L^2}\|(Du)^2\|_{L^\infty}\\
&\quad + \ell(\ell-1)\|D^{n-1}(u^{\ell-2})\|_{L^2}\|(Du)^3\|_{L^\infty}\\
&\le \ell(\ell-1)\cdot\sum_{i=1}^{n-2} {}_{n-1}C_i(\ell-2)^i M_{i,\infty}^{\ell-2}M_{n-1}M_{1,\infty}^2\\
&\quad + \ell(\ell-1)(\ell-2)^{n-1}M_{n-2,\infty}^{\ell-3}M_{n-1}M_{1,\infty}^3\\
&\le \ell(\ell-1)M_{n-2,\infty}^{\ell}M_{n-1}\cdot\sum_{i=1}^{n-2} {}_{n-1}C_i(\ell-2)^i\\
&\quad + \ell(\ell-1)M_{n-2,\infty}^{\ell}M_{n-1}\cdot(\ell-2)^{n-1}\\
&= \ell(\ell-1)^n\cdot M_{n-2,\infty}^{\ell}M_{n-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{j=1}^{4}\|S_n^j\|_{L^2} &\le \ell(\ell+1)^n M_{n^*,\infty}^{\ell}M_{n-1} + 2^n\ell\cdot M_{n^*,\infty}^{\ell}M_{n-1}\\
&\quad + 2^{n-1}n\ell(\ell-1)M_{n^*,\infty}^{\ell}M_{n-1} + \ell(\ell-1)^n\cdot M_{n-2,\infty}^{\ell}M_{n-1}\\
&\le (\ell(\ell+1)^n + 2^n\ell + 2^{n-1}n\ell(\ell-1) + \ell(\ell-1)^n)M_{n^*,\infty}^{\ell}M_{n-1}\\
&= \ell((\ell+1)^n + 2^n + 2^{n-1}n(\ell-1) + (\ell-1)^n)M_{n^*,\infty}^{\ell}M_{n-1}\\
&\le \ell((\ell+1)^n + 2^n n + (\ell-1)(2^{n-1}n + (\ell-1)^{n-1}))M_{n^*,\infty}^{\ell}M_{n-1}\\
&\le \ell((\ell+1)^n + 2^n n)(\ell-1+1)M_{n^*,\infty}^{\ell}M_{n-1}\\
&= \ell^2((\ell+1)^n + 2^n n)M_{n^*,\infty}^{\ell}M_{n-1}\\
&\le 2n\ell^2(\ell+1)^n M_{n^*,\infty}^{\ell}M_{n-1}.
\end{aligned}$$

Hence, these estimates assure (3.18). □

4. Approximate Equations.

In order to approximate the original problem (P), we have introduced the following equations:

$$(\mathrm{P})^{\varepsilon}\begin{cases} u_t = (u^\ell + \varepsilon)u_{xx} + \ell u^{\ell-1}(u_x)^2, & (x,t)\in \boldsymbol{R}\times(0,\infty),\\ u(x,0) = u_0(x), & x\in\boldsymbol{R}.\end{cases}$$

The purpose of this section is to show the existence of global smooth solutions for $(\mathrm{P})^{\varepsilon}$, which reads

Proposition 4.1. *Let* (A.1) *and* (A.2) *be satisfied. Then, for every* $T > 0$ *and* $\varepsilon > 0$, $(\mathrm{P})^\varepsilon$ *has a unique solution* u_ε *belonging to* $C^\infty([0,T] \times \boldsymbol{R})$.

By the standard argument, it is easy to see that Proposition 4.1 can be derived from the following fact.

Proposition 4.2. *Let* (A.1) *and* (A.2) *be satisfied. Then, for every* $T > 0$, $k \in \boldsymbol{N}$ $(k \geq 2)$ *and* $\varepsilon > 0$, $(\mathrm{P})^\varepsilon$ *has a unique solution* u_ε *belonging to* $\mathscr{B}_T^k := \{v \in C([0,T]; H^{2k+1}(\boldsymbol{R})); v_{xx}, v_t \in L^2([0,T]; H^{2k}(\boldsymbol{R}))\}$.

The proof of Proposition 4.2 is divided into three steps in the following subsections 4.1, 4.2 and 4.3.

4.1. Approximation for leading term.

As the first step, we consider the partial approximation which consists only of leading terms.

$$(\mathrm{P})_0^\varepsilon \begin{cases} u_t = (u^\ell + \varepsilon) u_{xx} + f(x,t), & (x,t) \in \boldsymbol{R} \times (0,\infty), \\ u(x,0) = u_0(x), & x \in \boldsymbol{R}. \end{cases}$$

Our aim here is to show the following fact.

Lemma 4.3. *For given* $f \in L^2(0,T; H^{2k}(\boldsymbol{R}))$ *and* $u_0 \in H^{2k+1}(\boldsymbol{R})$, $(\mathrm{P})_0^\varepsilon$ *has a unique solution u belonging to* $\mathscr{B}_T^k$, *the same class of solutions given in Proposition* 4.2.

By putting $A = -(\partial/\partial x)^2$ and $H_k = H^{2k}(\boldsymbol{R})$, we rewrite $(\mathrm{P})_0^\varepsilon$ as evolution equations in H_k:

$$(\mathrm{P})_0^\varepsilon \begin{cases} (d/dt)u(t) + \varepsilon A u(t) + u^\ell A u(t) = f(t), & 0 \leq t \leq T, \\ u(0) = u_0. \end{cases}$$

At a glance, it is easily seen that $u^\ell A u$ can be regarded as a monotone perturbation for $\varepsilon A u$ in $L^2(\boldsymbol{R})$. However, the chief difficulty of this equation lies in the facts that $u^\ell A u$ does not behave as a monotone perturbation anymore in higher order spaces H_k $(k \geq 1)$, and that $u^\ell A u$ is not a small perturbation for $\varepsilon A u$ even in $H_0 = L^2(\boldsymbol{R})$.

In order to get over the first difficulty, we shall show that $u^\ell A u$ can be decomposed into the sum of monotone perturbations and small perturbations.

To avoid the second difficulty, we introduce the following auxiliary equations with two parameters $\varepsilon > 0$ and $\lambda \in [0,1]$.

$$(\mathrm{P})^{\varepsilon,\lambda} \begin{cases} (d/dt)u(t) + \varepsilon A u(t) + \lambda u^\ell A u(t) = h(t) + f(t), & 0 \leq t \leq T, \\ u(0) = u_0. & \text{in } H_k. \end{cases}$$

For any f fixed in $L^2(0,T; H_k)$ and given $h \in L^2(0,T; H_k)$, denote by u^h the unique solution of $(\mathrm{P})^{\varepsilon,\lambda}$ belonging to $\mathscr{B}_T^k$. For any $\eta > 0$, we can define an operator $\mathscr{F}_\eta^\lambda$ by

$$\mathscr{F}_\eta^\lambda : h \mapsto u^h \mapsto -\eta(u^h)^\ell \cdot Au^h.$$

To prove Lemma 4.3, it suffices to establish the following fact on $\mathscr{F}_\eta^\lambda$.

Lemma 4.4. *There exist a positive number R and a (sufficiently small) positive number η_0 depending on $\|u_0\|_{H^{2k+1}}, \varepsilon, R$ and T but not on λ such that for every $\eta \in (0, \eta_0]$ and $\lambda \in [0, 1]$, $\mathscr{F}_\eta^\lambda$ becomes a contraction from $K_R^T := \{v \in L^2(0, T; H_k); \|v\|_{L^2(0,T;H_k)} \le R\}$ into itself, provided that $(\mathrm{P})^{\varepsilon,\lambda}$ admits a unique solution in $\mathscr{B}_T^k$.*

In fact, the following argument shows that Lemma 4.3 is a direct consequence of Lemma 4.4.

Proof of Lemma 4.3. We first choose $m \in \boldsymbol{N}$ and $\eta_1 \in (0, \eta_0]$ such that $m\eta_1 = 1$. Since εA becomes a self-adjoint operator in H_k with $D(A) = H_{k+1} = H^{2(k+1)}(\boldsymbol{R})$, the standard result of the theory of evolution equations says that for every $h \in K_R^T$, $(\mathrm{P})^{\varepsilon,\lambda}$ with $\lambda = 0$ admits a unique solution u^h in $\mathscr{B}_T^k$ (see Tanabe [**17**]). Then Lemma 4.4 assures that $\mathscr{F}_{\eta_1}^0$ has a fixed point $h_0 \in K_R^T$, in other words, u^{h_0} satisfies

$$\begin{aligned}\frac{d}{dt}u^{h_0}(t) + \varepsilon Au^{h_0}(t) &= h_0(t) + f(t)\\ &= \mathscr{F}_{\eta_1}^0(h_0) + f(t)\\ &= -\eta_1(u^{h_0})^\ell Au^{h_0} + f(t).\end{aligned}$$

Hence $u^{h_0}(t)$ gives a unique solution of $(\mathrm{P})^{\varepsilon,\lambda}$ with $\lambda = \eta_1$ and $h = 0$. This observation implies that $(\mathrm{P})^{\varepsilon,\eta_1}$ admits a unique solution in $\mathscr{B}_T^k$. Therefore, applying Lemma 4.4 again with $\lambda = \eta_1$, we find that $\mathscr{F}_{\eta_1}^{\eta_1}$ has a fixed point $h_1 \in K_R^T$. Then, by the same argument as above, it is easily seen that $u^{h_1}(t)$ gives a unique solution of $(\mathrm{P})^{\varepsilon,\lambda}$ with $\lambda = 2\eta_1$ and $h = 0$. Thus we can repeat this procedure for $\lambda = k\eta_1$, up to $k = m$, and find that $(\mathrm{P})^{\varepsilon,\lambda}$ with $\lambda = m\eta_1 = 1$ and $h = 0$, nothing but $(\mathrm{P})_0^\varepsilon$, admits a unique solution in $\mathscr{B}_T^k$. □

In order to derive Lemma 4.4, we need to establish a series of a priori estimates for solutions of $(\mathrm{P})^{\varepsilon,\lambda}$.

Lemma 4.5. *Let u be a solution of $(\mathrm{P})^{\varepsilon,\lambda}$ belonging to $\mathscr{B}_T^k$. Then there exist numbers $\{M_m\}_{m=0}^{2k}$, $\{M_{m,\infty}\}_{m=0}^{2k-1}$ such that*

$$M_m \le M_{m,\infty} \le M_{m+1}, \quad 0 \le m \le 2k-1, \tag{4.1}$$

$$\sup_{0\le t\le T} \|D^m u(t)\|_{L^2} \le M_m, \quad 0 \le m \le 2k, \tag{4.1$_m$}$$

$$\sup_{\substack{0\le t\le T\\ 2\le r\le\infty}} \|D^m u(t)\|_{L^r} \le M_{m,\infty}, \quad 0 \le m \le 2k-1. \tag{4.1$_{m,\infty}$}$$

Here M_m and $M_{m,\infty}$ do not depend on ε but on u_0, f, h and other parameters, more precisely,

$$\begin{aligned}
M_0 &= M_0(\|u_0\|_{L^2}, \|f+h\|_{L^1(0,T;L^2)}),\\
M_m &= M_m(\|D^m u_0\|_{L^2}, \|D^m(f+h)\|_{L^1(0,T;L^2)}, m, \ell, M_{m-1,\infty}), \quad 1 \le m \le 2k,\\
M_{0,\infty} &= M_{0,\infty}\left(\sup_{2\le r\le\infty} \|u_0\|_{L^r}, \sup_{2\le r\le\infty} \|f+h\|_{L^1(0,T;L^r)}\right),\\
M_{m,\infty} &= M_{m,\infty}\left(\sup_{2\le r\le\infty} \|D^m u_0\|_{L^r}, \sup_{2\le r\le\infty} \|D^m(f+h)\|_{L^1(0,T;L^r)}, m, \ell, M_{m-1,\infty}\right)\\
&\qquad m \ne 2,\\
M_{2,\infty} &= \sqrt{2} M_2^{1/2} \cdot M_3^{1/2}.
\end{aligned}$$

Furthermore, the following estimate holds.

$$\sup_{0\le t\le T} \|Du(t)\|_{H_k} + \left(\varepsilon \int_0^T \|D^2 u(t)\|_{H_k}^2\, dt\right)^{1/2} \le M_{2k+1}^{\varepsilon}, \tag{4.2$_k$}$$

where $M_{2k+1}^{\varepsilon} = M_{2k+1}^{\varepsilon}(M_{2k}, M_{2,\infty}, \varepsilon, k, \ell, \|Du_0\|_{H_k}, \|f+h\|_{L^2(0,T;H_k)})$.

Proof. We are going to verify $(4.1)_m$ in several steps, *i.e.*, the cases $m = 0, 1, 2$ and $m \ge 3$. For the sake of simplicity, throughout the present paper, we denote by C_m positive numbers depending only on ℓ and m. We also denote by $\mathbf{M}_m$ (or $\mathbf{M}_{m,\infty}$) positive numbers depending only on ℓ, m and M_m (or $M_{m,\infty}$). These numbers C_m and $\mathbf{M}_m$ will in general have different values in different places.

(The case m = 0)
Multiply $(\mathrm{P})^{\varepsilon,\lambda}$ by $|u|^{r-2}u$ and integrate over $\boldsymbol{R}$, then the integration by parts gives

$$\begin{aligned}
\|u\|_{L^r}^{r-1} \cdot \frac{d}{dt}\|u\|_{L^r} &+ \varepsilon(r-1)\int |u|^{r-2}(Du)^2\,dx + \lambda(\ell+r-1)\int |u|^{\ell+r-2}(Du)^2\,dx\\
&= \int (f+h)|u|^{r-2}u\,dx\\
&\le \|f+h\|_{L^r} \cdot \|u\|_{L^r}^{r-1}.
\end{aligned}$$

Hence, we deduce $(4.1)_0$ and $(4.1)_{0,\infty}$ with

$$M_0 = \|u_0\|_{L^2} + \|f+h\|_{L^1(0,T;L^2)},$$
$$M_{0,\infty} = \sup_{2\le r\le\infty}\{\|u_0\|_{L^r} + \|f+h\|_{L^1(0,T;L^r)}\},$$

we obtain $(4.1)_m$ and $(4.1)_{m,\infty}$ with $m=0$.

(The case m = 1)
Multiplication of $(\mathrm{P})^{\varepsilon,\lambda}$ by $-D(|Du|^{r-2}Du) = -(r-1)|Du|^{r-2}(D^2u)^2$ gives

$$\|Du\|_{L^r}^{r-1}\cdot\frac{d}{dt}\|Du\|_{L^r} + \varepsilon(r-1)\int|Du|^{r-2}(D^2u)^2\,dx + \lambda(r-1)\int u^\ell|Du|^{r-2}(D^2u)^2\,dx$$
$$= -\int D(f+h)D(|Du|^{r-2}Du)\,dx$$
$$\le \|D(f+h)\|_{L^r}\cdot\|Du\|_{L^r}^{r-1},$$

whence follows (4.1), $(4.1)_1$ and $(4.1)_{1,\infty}$ with

$$M_1 = \max(M_{0,\infty}, \|Du_0\|_{L^2} + \|D(f+h)\|_{L^1(0,T;L^2)}),$$
$$M_{1,\infty} = \max\left(M_{0,\infty}, \sup_{2\le r\le\infty}\{\|Du_0\|_{L^r} + \|D(f+h)\|_{L^1(0,T;L^r)}\}\right).$$

(The case m = 2)
The argument similar to those above does not work well for the case $m=2$. So we here try to derive $(4.1)_{2,\infty}$ via the L^2-estimates for D^2u and D^3u. Multiplication of $D^2(\mathrm{P})^{\varepsilon,\lambda}$ by D^2u gives

$$\frac{1}{2}\frac{d}{dt}\|D^2u(t)\|_{L^2}^2 + \varepsilon\|D^3u(t)\|_{L^2}^2 - \lambda(D^2(u^\ell D^2u), D^2u)_{L^2}$$
$$= (D^2(f+h), D^2u)_{L^2}$$
$$\le \frac{1}{2}\|D^2(f+h)\|_{L^2}^2 + \frac{1}{2}\|D^2u(t)\|_{L^2}^2. \tag{4.3}$$

Here, applying the integration by parts, we get

$$-(D^2(u^\ell D^2u), D^2u)_{L^2} = \int u^\ell(D^3u)^2\,dx + \ell\int u^{\ell-1}DuD^2uD^3u\,dx,$$

and

$$-\ell\int u^{\ell-1}DuD^2uD^3u\,dx \le \frac{1}{2}\int u^\ell(D^3u)^2\,dx + \frac{\ell^2}{2}\int u^{\ell-2}(Du)^2(D^2u)^2\,dx$$
$$\le \frac{1}{2}\int u^\ell(D^3u)^2\,dx + \mathbf{M}_{1,\infty}\|D^2u\|_{L^2}^2.$$

Substituting these relations in (4.3), we obtain

$$\frac{d}{dt}\|D^2u(t)\|_{L^2}^2 \leq (\mathbf{M}_{1,\infty}+1)\|D^2u(t)\|_{L^2}^2 + \|D^2(f+h)\|_{L^2(0,T;L^2)}^2.$$

Then Gronwall's inequality yields

$$\sup_{0\leq t\leq T}\|D^2u(t)\|_{L^2} \leq M_2 := (\|f+h\|_{L^2(0,T;H_1)}^2 + \|u_0\|_{H_1}^2)^{1/2}\cdot e^{(\mathbf{M}_{1,\infty}+1)(T/2)}. \quad (4.4)$$

Next, we calculate $(D^3(\mathrm{P})^{\varepsilon,\lambda}, D^3u)$ to get

$$\frac{1}{2}\frac{d}{dt}\|D^3u(t)\|_{L^2}^2 + \varepsilon\|D^4u(t)\|_{L^2}^2 - \lambda(D^3(u^\ell D^2u), D^3u)_{L^2}$$
$$\leq \|D^3(f+h)\|_{L^2}\cdot\|D^3u(t)\|_{L^2}. \quad (4.5)$$

Here, by Lemma 3.6 with $n=2$, we have

$$-(D^3(u^\ell D^2u), D^3u)_{L^2} = \int D^2(u^\ell D^2u)D^4u\,dx$$
$$= \int u^\ell(D^4u)^2\,dx + I_2^2 + I_2^3 + R_2^2, \quad (4.6)$$

where

$$I_2^2 = 2\ell\int u^{\ell-1}DuD^3uD^4u\,dx,$$
$$I_2^3 = \ell\int u^{\ell-1}(D^2u)^2D^4u\,dx,$$
$$R_2^2 = \ell(\ell-1)\int u^{\ell-2}(Du)^2D^2uD^4u\,dx.$$

On the other hand, we obtain

$$|I_2^2| \leq \frac{1}{4}\int u^\ell(D^4u)^2\,dx + 4\ell^2\int u^{\ell-2}(Du)^2(D^3u)^2\,dx$$
$$\leq \frac{1}{4}\int u^\ell(D^4u)^2\,dx + \mathbf{M}_{1,\infty}\|D^3u\|_{L^2}^2, \quad (4.7)$$

and by Lemma 3.3 and (4.4),

$$|I_2^3| \leq \frac{1}{4}\int u^\ell(D^4u)^2\,dx + \ell^2\int u^{\ell-2}(D^2u)^4\,dx$$
$$\leq \frac{1}{4}\int u^\ell(D^4u)^2\,dx + \mathbf{M}_{1,\infty}M_2^3\cdot\|D^3u\|_{L^2}. \quad (4.8)$$

Furthermore, the integration by parts for R_2^2 gives

$$R_2^2 = -\ell(\ell-1)\left[\int(\ell-2)u^{\ell-3}(Du)^3 D^2 u D^3 u\,dx + 2\int u^{\ell-2} Du (D^2u)^2 D^3 u\,dx + \int u^{\ell-2}(Du)^2(D^3u)^2\,dx\right].$$

Then, by virtue of Lemma 3.3 and (4.4),

$$|R_2^2| \le \mathbf{M}_{1,\infty} M_2 \|D^3 u\|_{L^2} + \mathbf{M}_{1,\infty} M_2^{3/2} \|D^3 u\|_{L^2}^{3/2} + \mathbf{M}_{1,\infty} \|D^3 u\|_{L^2}^2. \tag{4.9}$$

Thus, from (4.5) to (4.9) we derive

$$\begin{aligned}\frac{d}{dt}\|D^3 u(t)\|_{L^2} &\le \mathbf{M}_{1,\infty}(\|D^3u\|_{L^2} + M_2^{3/2}\|D^3 u\|_{L^2}^{1/2} + M_2^3 + M_2) + \|D^3(f+h)\|_{L^2} \\ &\le \mathbf{M}_{1,\infty}\|D^3 u\|_{L^2} + \mathbf{M}_{1,\infty}(M_2^3 + M_2) + \|D^3(f+h)\|_{L^2}.\end{aligned}$$

Then $(4.1)_3$ holds with

$$\begin{aligned}M_3 = \max[2M_2, (\|D^3 u_0\|_{L^2} + \|D^3(f+h)\|_{L^1(0,T;L^2)} \\ + \mathbf{M}_{1,\infty}(M_2^3 + M_2)) \times \exp\{\mathbf{M}_{1,\infty} T\}].\end{aligned} \tag{4.10}$$

Furthermore, we define $M_{2,\infty}$ by $M_{2,\infty} = \sqrt{2} M_2^{1/2} M_3^{1/2}$, then $2M_2 \le M_{2,\infty} \le M_3$ holds and by (3.4) in Lemma 3.2, we find

$$\sup_{0 \le t \le T_0} \|D^2 u(t)\|_{L^\infty} \le \sqrt{2} M_2^{1/2} M_3^{1/2} = M_{2,\infty}.$$

Since

$$\begin{aligned}\sup_{2\le r\le\infty}\|D^2u\|_{L^r} &\le \sup_{2\le r\le\infty}(\|D^2u\|_{L^\infty}^{(r-2)/r}\cdot\|D^2u\|_{L^2}^{2/r}) \\ &\le \sup_{2\le r\le\infty}(M_{2,\infty}^{(r-2)/r}\cdot M_2^{2/r}) \le M_{2,\infty},\end{aligned}$$

(4.1) and $(4.1)_{m,\infty}$ hold with $m = 2$.

(The case $3 \le m \le 2k-1$)

Multiply $D^{m-1}(\mathrm{P})^{\varepsilon,\lambda}$ by $-D(|D^m u|^{r-2} D^m u)$, then by Lemma 3.6 with $n = m-1$, we get

$$\begin{aligned}\|D^m u\|_{L^r}^{r-1}\cdot\frac{d}{dt}\|D^m u\|_{L^r} + \varepsilon(r-1)\int|D^m u|^{r-2}(D^{m+1}u)^2\,dx \\ = \lambda I_{m-1} + \|D^m(f+h)\|_{L^r}\cdot\|D^m u\|_{L^r}^{r-1},\end{aligned} \tag{4.11}$$

$$\begin{aligned}I_{m-1} &= -\int D^{m-1}(u^\ell D^2 u)\cdot D(|D^m u|^{r-2}D^m u)\,dx \\ &= \int(I_{m-1}^1 + I_{m-1}^2 + I_{m-1}^3 + R_{m-1}^1 + R_{m-1}^2)\cdot(-D(|D^m u|^{r-2}D^m u))\,dx.\end{aligned}$$

We are going to estimate these 5 terms.

$$\bar{I}^1_{m-1} = \int I^1_{m-1} \cdot (-D(|D^m u|^{r-2} D^m u))\,dx$$
$$= -(r-1)\int u^\ell |D^m u|^{r-2}(D^{m+1}u)^2\,dx \le 0, \tag{4.12}$$

$$\bar{I}^2_{m-1} = \int I^2_{m-1} \cdot (-D(|D^m u|^{r-2} D^m u))\,dx$$
$$= -\frac{r-1}{r}(m-1)\int \ell u^{\ell-1} Du \cdot D(|D^m u|^r)\,dx$$
$$= \frac{(r-1)(m-1)\ell}{r}\int (u^{\ell-1}D^2u + (\ell-1)u^{\ell-2}(Du)^2)\cdot |D^m u|^r\,dx$$
$$\le \mathbf{M}_{2,\infty}\|D^m u\|^r_{L^r}. \tag{4.13}$$

For the case $m \ge 4$, we get

$$\bar{I}^3_{m-1} = \int I^3_{m-1} \cdot (-D(|D^m u|^{r-2} D^m u))\,dx$$
$$= \int \{{}_{m-1}C_2 \ell(\ell-1)u^{\ell-2}(Du)^2 + ({}_{m-1}C_2 + 1)\ell u^{\ell-1} D^2 u\} D^m u \cdot |D^m u|^{r-2} D^m u\,dx$$
$$+ \int D({}_{m-1}C_2 \ell(\ell-1)u^{\ell-2}(Du)^2) D^{m-1}u \cdot |D^m u|^{r-2} D^m u\,dx$$
$$+ \int D(({}_{m-1}C_2 + 1)\ell u^{\ell-1} D^2 u) D^{m-1}u \cdot |D^m u|^{r-2} D^m u\,dx.$$

Then it is easy to obtain

$$\bar{I}^3_{m-1} \le \mathbf{M}_{2,\infty}\|D^m u\|^r_{L^r} + \mathbf{M}_{m-1,\infty}\|D^m u\|^{r-1}_{L^r}, \quad (m \ge 4). \tag{4.14$_m$}$$

As for the case $m = 3$, it holds

$$\bar{E}_2 = \bar{I}^3_{3-1} + \bar{R}^2_{3-1}$$
$$= \int \{\ell u^{\ell-1}(D^2u)^2 + \ell(\ell-1)u^{\ell-2}(Du)^2 D^2 u\} \cdot \{-D(|D^3u|^{r-2}D^3u)\}\,dx$$
$$= \int \{2\ell u^{\ell-1} D^2 u + \ell(\ell-1)u^{\ell-2}(Du)^2\} D^3 u |D^3 u|^{r-2} D^3 u\,dx$$
$$+ \int \{\ell(\ell-1)u^{\ell-2} Du (D^2u)^2 + \ell(\ell-1)(\ell-2)u^{\ell-3}(Du)^3 D^2 u$$
$$+ 2\ell(\ell-1)u^{\ell-2} Du (D^2u)^2\} \cdot |D^3u|^{r-2} D^3 u\,dx.$$

Hence, (4.11), (4.12) and (4.13) yield

$$\frac{d}{dt}\|D^3u\|_{L^r} \le \mathbf{M}_{2,\infty}(\|D^3u\|_{L^r} + 1) + \|D^3(f+h)\|_{L^r}.$$

Then $(4.3)_{m=3}$ holds with

$$M_{3,\infty} = \max(M_3, \overline{M}_{3,\infty}),$$

$$\overline{M}_{3,\infty} = \left[\mathbf{M}_{2,\infty} + \sup_{2\le r\le\infty}\{\|D^3u_0\|_{L^r} + \|D^3(f+h)\|_{L^1(0,T;L^r)}\}\right]\cdot e^{\mathbf{M}_{2,\infty}T}.$$

Furthermore, for the case $m \ge 4$, we note that (3.12) implies

$$\left|\int (R^1_{m-1} + R^2_{m-1})\cdot(-D(|D^mu|^{r-2}D^mu))\,dx\right| \le 2(\ell+1)^m \cdot M^{\ell+1}_{m-1,\infty}\cdot\|D^mu\|^{r-1}_{L^r}. \tag{4.15}$$

Thus, in view of (4.11), (4.12), (4.13), $(4.14)_m$ and (4.15), we obtain

$$\frac{d}{dt}\|D^mu\|_{L^r} \le C_m\mathbf{M}_{2,\infty}\|D^mu\|_{L^r} + \mathbf{M}_{m-1,\infty} + \|D^m(f+h)\|_{L^r}.$$

Therefore $(4.1)_m$ and $(4.1)_{m,\infty}$ are valid with

$$M_m = \max(M_{m-1,\infty}, \overline{M}_{m,2}), \quad M_{m,\infty} = \max\left(M_{m-1,\infty}, \sup_{2\le r\le\infty}\overline{M}_{m,r}\right),$$

$$\overline{M}_{m,r} = [\mathbf{M}_{m-1,\infty} + \|D^mu_0\|_{L^r} + \|D^m(f+h)\|_{L^1(0,T;L^r)}]\cdot e^{C_m\mathbf{M}_{2,\infty}T}.$$

Now we are going to verify $(4.2)_k$. To do this, we take the inner product of H_k between $(\mathrm{P})^{\varepsilon,\lambda}$ and Au to get

$$\frac{1}{2}\frac{d}{dt}\|Du\|^2_{H_k} + \varepsilon\|Au\|^2_{H_k} = -\lambda(u^\ell Au, Au)_{H_k} + (f+h, Au)_{H_k}, \tag{4.16}$$

where

$$-\lambda(u^\ell Au, Au)_{H_k} = -\lambda(u^\ell Au, Au)_{L^2} - \lambda(A^k(u^\ell Au), A^{k+1}u)_{L^2}$$

$$= -\lambda\int u^\ell(Au)^2\,dx + I_{2k}(2).$$

Lemma 3.6 with $n = 2k$ gives

$$I_{2k}(2) = -(-1)^{2k+2}\lambda\int D^{2k}(u^\ell D^2u)\cdot D^{2k+2}u\,dx$$

$$\le -\lambda\int u^\ell(D^{2k+2}u)^2\,dx + \bar{I}^2_{2k}(2) + \bar{I}^3_{2k}(2) + \bar{R}^1_{2k}(2) + \bar{R}^2_{2k}(2).$$

Here we obtain

$$\begin{aligned}
\bar{I}_{2k}^2(2) &= -\int {}_{2k}C_1 \ell u^{\ell-1} Du D^{2k+1} u D^{2k+2} u \, dx \\
&\le 2k\ell M_{1,\infty}^{\ell} \|Du\|_{H_k} \|Au\|_{H_k} \\
&\le \frac{\varepsilon}{4} \|Au\|_{H_k}^2 + \frac{1}{\varepsilon} C_k \mathbf{M}_{1,\infty} \|Du\|_{H_k}^2, \\
\bar{I}_{2k}^3(2) &= -\int ({}_{2k}C_2 \ell(\ell-1) u^{\ell-2} (Du)^2 + ({}_{2k}C_2 + 1)\ell u^{\ell-1} D^2 u) \cdot D^{2k} u D^{2k+2} u \, dx \\
&\le 2k^2 \ell^2 M_{2,\infty}^{\ell} M_{2k} \|Au\|_{H_k} \\
&\le \frac{\varepsilon}{4} \|Au\|_{H_k}^2 + \frac{1}{\varepsilon} C_k \mathbf{M}_{2,\infty} M_{2k}^2,
\end{aligned}$$

and by (3.13)

$$\begin{aligned}
\bar{R}_{2k}^1(2) + \bar{R}_{2k}^2(2) &= -\int (R_{2k}^1 + R_{2k}^2) \cdot D^{2k+2} u \, dx \\
&\le 2(\ell+1)^{2k+1} \cdot M_{2k}^{\ell+1} \cdot \|D^{2k+1} u\|_{L^2} \\
&\le \mathbf{M}_{2k} + \|Du\|_{H_k}^2.
\end{aligned}$$

Then, by substituting these estimates in (4.16), we have

$$\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \|Du\|_{H_k}^2 + \frac{\varepsilon}{4} \|Au\|_{H_k}^2 \\
&\quad \le \left(\frac{1}{\varepsilon} C_k \mathbf{M}_{1,\infty} + 1 \right) \|Du\|_{H_k}^2 + \frac{1}{\varepsilon} \mathbf{M}_{2,\infty} \mathbf{M}_{2k} + \mathbf{M}_{2k} + \frac{1}{\varepsilon} \|f + h\|_{H_k}^2.
\end{aligned}$$

Thus Gronwall's inequality assures $(4.2)_k$. □

In showing that $\mathscr{F}_\eta^\lambda$ becomes a contraction, we need to investigate how the solution u of $(\mathrm{P})^{\varepsilon,\lambda}$ depends on h. In fact, we get the following estimates.

Lemma 4.6. *Let $f \in L^2(0,T;H_k)$ and $h_1, h_2 \in K_R^T := \{v \in L^2(0,T;H_k);\ \|v\|_{L^2(0,T;H_k)} \le R\}$. Let u_1 and u_2 be solutions of $(\mathrm{P})^{\varepsilon,\lambda}$ belonging to $\mathscr{B}_T^k$ with h replaced by h_1 and h_2 respectively. Then there exist constants G_1 and G_2 depending only on R, k, ℓ and $1/\varepsilon$ such that*

$$\begin{aligned}
&\sup_{0 \le t \le T} (\|u_1(t) - u_2(t)\|_{H_k}^2 + \|D(u_1(t) - u_2(t))\|_{H_k}^2) \\
&\qquad \le G_1 e^{G_2 T} \|h_1 - h_2\|_{L^2(0,T;H_k)}^2, \qquad (4.17)
\end{aligned}$$

$$\varepsilon \int_0^T \|\Delta(u_1 - u_2)\|_{H_k}^2 \, dt \le G_1 e^{G_2 T} \|h_1 - h_2\|_{L^2(0,T;H_k)}^2, \quad (k \ge 1). \qquad (4.18)$$

Proof. As in the proof of Lemma 4.5, we adopt the expedient notations $C_m, \mathbf{M}_m$ and $\mathbf{M}_{m,\infty}$ to mean positive numbers with the dependence $C_m(\ell, m)$, $\mathbf{M}_m(\ell, m, M_m)$ and $\mathbf{M}_{m,\infty}(\ell, m, M_{m,\infty})$.

Since $u_1, u_2 \in \mathscr{B}_T^k$, we note by Lemma 4.5 that u_1 and u_2 satisfy estimates $(4.1)_m$ and $(4.1)_{m,\infty}$ and $(4.2)_k$. It is easy to see that $w = u_1 - u_2$ satisfies

$$w_t + \varepsilon Aw + \lambda u_1^\ell Aw + \lambda w d_\ell Au_2 = \delta h, \tag{4.19}$$

where

$$d_\ell = u_1^{\ell-1} + u_1^{\ell-2}u_2 + \cdots + u_1 u_2^{\ell-2} + u_2^{\ell-1}, \quad \delta h = h_1 - h_2. \tag{4.20}$$

Then, by taking the inner product of H_k between (4.19) and w, we have

$$\frac{1}{2}\frac{d}{dt}\|w\|_{H_k}^2 + \varepsilon\|Dw\|_{H_k}^2 + \delta I_1 + \delta I_2 + \delta I_3 + \delta I_4 \le \frac{1}{2}\|w\|_{H_k}^2 + \frac{1}{2}\|\delta h\|_{H_k}^2, \tag{4.21}$$

$$\delta I_1 = \lambda(u_1^\ell Aw, w), \quad \delta I_2 = \lambda(A^k(u_1^\ell Aw), A^k w),$$

$$\delta I_3 = \lambda(w d_\ell Au_2, w), \quad \delta I_4 = \lambda(A^k(w d_\ell Au_2), A^k w).$$

Here it is easy to get

$$|\delta I_1| \le \|u_1^\ell\|_{L^\infty}\|D^2 w\|_{L^2}\|w\|_{L^2} \le \mathbf{M}_{0,\infty}\|w\|_{H_k}^2, \tag{4.22}$$

$$\begin{aligned} |\delta I_3| &\le \|w\|_{L^2}\|d_\ell\|_{L^\infty}\|Au_2\|_{L^\infty}\|w\|_{L^2} \\ &\le \mathbf{M}_{2,\infty}\|w\|_{H_k}^2. \end{aligned} \tag{4.23}$$

Furthermore by virtue of (3.1), (3.5), (3.14) and (3.15) and the argument similar to that in the proof of Lemma 3.6, we obtain

$$\begin{aligned} |\delta I_2| &\le \int |D^{2k}(u_1^\ell D^2 w) \cdot D^{2k} w|\, dx \\ &\le \int |u_1^\ell D^{2k+2} w \cdot D^{2k} w|\, dx + C_k \int |u_1^{\ell-1} D u_1 D^{2k+1} w \cdot D^{2k} w|\, dx \\ &\quad + C_k \sum_{i=2}^{2k-1} \int |D^i(u_1^\ell) D^{2k-i+2} w \cdot D^{2k} w|\, dx + \int |D^{2k}(u_1)^\ell D^2 w \cdot D^{2k} w|\, dx \\ &\le \mathbf{M}_{0,\infty}\|Aw\|_{H_k} \cdot \|w\|_{H_k} + C_k \mathbf{M}_{1,\infty}\|Dw\|_{H_k} \cdot \|w\|_{H_k} \\ &\quad + \mathbf{M}_{2k-1,\infty}\|w\|_{H_k}^2 + \ell^{2k} M_{2k-1,\infty}^{\ell-1} M_{2k}\|D^2 w\|_{L^\infty} \cdot \|w\|_{H_k} \\ &\le \mathbf{M}_{2k}(\|Aw\|_{H_k} + \|Dw\|_{H_k} + \|w\|_{H_k})\|w\|_{H_k}, \end{aligned} \tag{4.24}$$

$$
\begin{aligned}
|\delta I_4| \le & \int |D^{2k}(wd_\ell D^2 u_2)\cdot D^{2k}w|\,dx \\
\le & \int |wd_\ell D^{2k+2}u_2 \cdot D^{2k}w|\,dx + C_k \int |D(wd_\ell)D^{2k+1}u_2\cdot D^{2k}w|\,dx \\
& + C_k \sum_{i=2}^{2k} \int |D^i(wd_\ell)D^{2k-i+2}u_2\cdot D^{2k}w|\,dx. \\
\le & \mathbf{M}_{0,\infty}\|A^{k+1}u_2\|_{L^2}\|w\|_{L^\infty}\|D^{2k}w\|_{L^2} \\
& + C_k(\|Dw\|_{L^\infty}\|d_\ell\|_{L^\infty} + \|w\|_{L^\infty}\|Dd_\ell\|_{L^\infty})\|D^{2k+1}u_2\|_{L^2}\|D^{2k}w\|_{L^2} \\
& + C_k\sum_{i=3}^{2k}\|D^i(wd_\ell)\|_{L^2}\|D^{2k-i+2}u_2\|_{L^\infty}\|D^{2k}w\|_{L^2} \\
& + C_k\|D^2(wd_\ell)\|_{L^\infty}\|D^{2k}u_2\|_{L^2}\|D^{2k}w\|_{L^2} \\
\le & \mathbf{M}_{0,\infty}\|A^{k+1}u_2\|_{L^2}\|w\|_{H_k}^2 + C_k\mathbf{M}_{1,\infty}M_{2k+1}^{\varepsilon}\|w\|_{H_k}^2 \\
& + C_k\sum_{i=3}^{2k}\|D^i(wd_\ell)\|_{L^2}M_{2k-1,\infty}\|w\|_{H_k} + C_k\|D^2(wd_\ell)\|_{L^\infty}M_{2k}\|w\|_{H_k}.
\end{aligned}
$$

By the same verification for (3.14) and (3.15), we find that

$$
\begin{aligned}
\|D^j d_\ell\|_{L^\infty} &\le \ell(\ell-1)^j M_{2k-1,\infty}^{\ell-1} \quad \text{for } 0\le j\le 2k-1, \\
\|D^j d_\ell\|_{L^2} &\le \ell(\ell-1)^j M_{2k-1,\infty}^{\ell-2}M_{2k} \\
&\le \ell(\ell-1)^j M_{2k}^{\ell-1} \quad \text{for } 0\le j\le 2k.
\end{aligned}
$$

Hence, by (3.1),

$$
\begin{aligned}
\|D^i(wd_\ell)\|_{L^2} &= \left\|\sum_{j=0}^{i} {}_iC_j D^j d_\ell\cdot D^{i-j}w\right\|_{L^2} \\
&\le \sum_{j=0}^{i} {}_iC_j \ell(\ell-1)^j M_{2k-1,\infty}^{\ell-1}\|w\|_{H_k} \\
&\le \ell^{i+1}M_{2k}^{\ell-1}\|w\|_{H_k}, \quad (0\le i\le 2k-1), \qquad (4.25)_i
\end{aligned}
$$

$$\|D^{2k}(wd_\ell)\|_{L^2} = \left\|\sum_{j=0}^{2k} {}_{2k}C_j D^j d_\ell \cdot D^{2k-j} w\right\|_{L^2}$$

$$\leq \sum_{j=0}^{2k-1} {}_{2k}C_j \ell(\ell-1)^j M_{2k-1,\infty}^{\ell-1} \|w\|_{H_k} + \|D^{2k} d_\ell\|_{L^2} \|w\|_{L^\infty}$$

$$\leq \left(\sum_{j=0}^{2k-1} {}_{2k}C_j \ell(\ell-1)^j + \sqrt{2}\ell(\ell-1)^{2k}\right) M_{2k}^{\ell-1} \|w\|_{H_k}$$

$$\leq \sqrt{2}\ell^{2k+1} M_{2k}^{\ell-1} \|w\|_{H_k}, \tag{4.25$_{2k}$}$$

$$\|D^i(wd_\ell)\|_{L^\infty} = \left\|\sum_{j=0}^{i} D^j d_\ell \cdot D^{i-j} w\right\|_{L^\infty}$$

$$\leq \sum_{j=0}^{i} {}_iC_j \ell(\ell-1)^j M_{2k-1,\infty}^{\ell-1} \|D^{i-j} w\|_{L^\infty}$$

$$\leq \sqrt{2}\ell^{i+1} M_{2k-1}^{\ell-1} \|w\|_{H_k}. \tag{4.25$_{i,\infty}$}$$

Then

$$\sum_{i=3}^{2k} \|D^i(wd_\ell)\|_{L^2} + \|D^2(wd_\ell)\|_{L^\infty}$$

$$\leq \left(\sqrt{2}\sum_{i=3}^{2k} \ell^{i+1} + \sqrt{2}\ell^{2+1}\right) M_{2k}^{\ell-1} \|w\|_{H_k}$$

$$\leq 2\sqrt{2}k\ell^{2k+1} M_{2k}^{\ell-1} \|w\|_{H_k}. \tag{4.26}$$

Therefore,

$$|\delta I_4| \leq \mathbf{M}_{0,\infty} \|A^{k+1} u_2\|_{L^2} \|w\|_{H_k}^2 + C_k \mathbf{M}_{1,\infty} M_{2k+1}^{\varepsilon} \|w\|_{H_k}^2 + \mathbf{M}_{2k} \|w\|_{H_k}^2.$$

Consequently, in view of (4.21)–(4.26), we deduce

$$\frac{1}{2}\frac{d}{dt}\|w\|_{H_k}^2 + \varepsilon\|Dw\|_{H_k}^2$$

$$\leq \frac{\varepsilon}{8}\|Aw\|_{H_k}^2 + \frac{1}{2}\|Dw\|_{H_k}^2 + \frac{1}{2}\|\delta h\|_{H_k}^2$$

$$+ \left(\frac{1}{\varepsilon}\mathbf{M}_{2k} + \mathbf{M}_{2k} + \mathbf{M}_{2,\infty} + C_k \mathbf{M}_{1,\infty} + \mathbf{M}_{2k+1}^{\varepsilon} + \mathbf{M}_{0,\infty}\|A^{2k+1} u_2\|_{L^2}\right)\|w\|_{H_k}^2. \tag{4.27}$$

Now we are going to establish the same type of estimate for Dw in H_k. To do this, we take the inner product of H_k between (4.19) and Aw to get

$$\frac{1}{2}\frac{d}{dt}\|Dw\|_{H_k}^2+\varepsilon\|Aw\|_{H_k}^2+\delta I_5+\delta I_6+\delta I_7+\delta I_8\le\frac{\varepsilon}{8}\|Aw\|_{H_k}^2+\frac{2}{\varepsilon}\|\delta h\|_{H_k}^2,\quad(4.28)$$

$$\delta I_5=\lambda(u_1^\ell Aw,Aw),\quad \delta I_6=\lambda(A^k(u_1^\ell Aw),A^{k+1}w),$$

$$\delta I_7=\lambda(wd_\ell Au_2,Aw),\quad \delta I_8=\lambda(A^k(wd_\ell Au_2),A^{k+1}w).$$

Then it is easy to see that

$$\delta I_5\ge 0,\quad |\delta I_7|\le \mathbf{M}_{2,\infty}\|w\|_{L^2}\|Aw\|_{L^2}\le\mathbf{M}_{2,\infty}\|w\|_{H_k}^2.\quad(4.29)$$

Furthermore, by much the same arguments as for (4.24) and (4.25), we can derive

$$\begin{aligned}-\delta I_6&=-\lambda\int u_1^\ell(A^{k+1}w)^2\,dx+\lambda C_k\int \ell u_1^{\ell-1}Du_1D^{2k+1}w\cdot A^{k+1}w\,dx\\&\quad+C_k\sum_{i=2}^{2k-1}\int D^i(u_1^\ell)D^{2k-i+2}w\cdot A^{k+1}w\,dx+\int D^{2k}(u_1^\ell)D^2w\cdot A^{k+1}w\,dx\\&\le C_k\mathbf{M}_{1,\infty}\|D^{2k+1}w\|_{L^2}\cdot\|A^{k+1}w\|_{L^2}+C_k\sum_{i=2}^{2k-1}M_{i,\infty}^\ell\|w\|_{H_k}\cdot\|A^{k+1}w\|_{L^2}\\&\quad+\|D^{2k}(u_1^\ell)\|_{L^2}\cdot\|D^2w\|_{L^\infty}\cdot\|A^{k+1}w\|_{L^2}\\&\le C_k\mathbf{M}_{1,\infty}\|Dw\|_{H_k}\cdot\|Aw\|_{H_k}+\mathbf{M}_{2k-1,\infty}\|w\|_{H_k}\|Aw\|_{H_k}\\&\quad+\sqrt{2}\ell^{2k}M_{2k-1,\infty}^{\ell-1}M_{2k}\|w\|_{H_k}\|Aw\|_{H_k}\\&\le C_k\mathbf{M}_{1,\infty}\|Dw\|_{H_k}\|Aw\|_{H_k}+\mathbf{M}_{2k}\|w\|_{H_k}\|Aw\|_{H_k},\end{aligned}\quad(4.30)$$

$$\begin{aligned}|\delta I_8|&\le\int|wd_\ell D^{2k+2}u_2\cdot A^{k+1}w|\,dx+C_k\int|D(wd_\ell)D^{2k+1}u_2\cdot A^{k+1}w|\,dx\\&\quad+C_k\sum_{i=2}^{2k}\int|D^i(wd_\ell)D^{2k-i+2}u_2\cdot A^{k+1}w|\,dx\\&\le\mathbf{M}_{0,\infty}\|w\|_{L^\infty}\|A^{k+1}u_2\|_{L^2}\|A^{k+1}w\|_{L^2}+C_k[\|Dw\|_{L^\infty}\|d_\ell\|_{L^\infty}+\|w\|_{L^\infty}\|D(d_\ell)\|_{L^\infty}]\\&\quad\times\|D^{2k+1}u_2\|_{L^2}\|A^{k+1}w\|_{L^2}+C_k\sum_{i=3}^{2k}\|D^i(wd_\ell)\|_{L^2}\|D^{2k-i+2}u_2\|_{L^\infty}\|A^{k+1}w\|_{L^2}\\&\qquad+C_k\|D^2(wd_\ell)\|_{L^\infty}\|D^{2k}u_2\|_{L^2}\|A^{k+1}w\|_{L^2}\\&\le\mathbf{M}_{0,\infty}\|A^{k+1}u_2\|_{L^2}\|w\|_{H_k}\|Aw\|_{H_k}+C_k\mathbf{M}_{1,\infty}M_{2k+1}^\varepsilon\|w\|_{H_k}\|Aw\|_{H_k}\\&\quad+\mathbf{M}_{2k}\|w\|_{H_k}\|Aw\|_{H_k}+\mathbf{M}_{2,\infty}\|w\|_{H_k}M_{2k}\|Aw\|_{H_k}\\&\le(\mathbf{M}_{0,\infty}\|A^{k+1}u_2\|_{L^2}+C_k\mathbf{M}_{1,\infty}M_{2k+1}^\varepsilon+\mathbf{M}_{2k}(\mathbf{M}_{2,\infty}+1))\|w\|_{H_k}\|Aw\|_{H_k}.\end{aligned}\quad(4.31)$$

Thus, in view of (4.28)–(4.31), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|Dw\|_{H_k}^2 + \varepsilon\|Aw\|_{H_k}^2 \\
&\quad\le \frac{\varepsilon}{8}\|Aw\|_{H_k}^2 + \frac{2}{\varepsilon}\|\delta h\|_{H_k}^2 \\
&\qquad + (\mathbf{M}_{0,\infty}\|A^{k+1}u_2\|_{L^2} + C_k\mathbf{M}_{1,\infty}M_{2k+1}^{\varepsilon} + \mathbf{M}_{2k})\|w\|_{H_k}\|Aw\|_{H_k} \\
&\qquad\quad + C_k\mathbf{M}_{1,\infty}\|Dw\|_{H_k}\|Aw\|_{H_k} + \mathbf{M}_{2,\infty}\|w\|_{H_k}^2 \\
&\quad\le \frac{3\varepsilon}{8}\|Aw\|_{H_k}^2 + \frac{2}{\varepsilon}\|\delta h\|_{H_k}^2 + \frac{1}{\varepsilon}\mathbf{M}_{2k}\|Dw\|_{H_k}^2 \\
&\qquad + \left(1+\frac{1}{\varepsilon}\right)\mathbf{M}_{2k}(\|A^{2k+1}u_2\|_{L^2}^2 + (M_{2k+1}^{\varepsilon})^2 + \mathbf{M}_{2,\infty} + 1)\|w\|_{H_k}^2. \qquad (4.32)
\end{aligned}$$

Therefore, combining (4.27) with (4.32), we find that there exists a constant K_1 depending only on $1/\varepsilon, k, \ell, \mathbf{M}_{2,\infty}, \mathbf{M}_{2k}, M_{2k+1}^{\varepsilon}$ such that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|w\|_{H_k}^2 + \|Dw\|_{H_k}^2) + \frac{\varepsilon}{2}\|Aw\|_{H_k}^2 \\
&\qquad \le K_1(\|w\|_{H_k}^2 + \|Dw\|_{H_k}^2)(\|A^{k+1}u_2\|_{L^2}^2 + 1) + \left(\frac{2}{\varepsilon}+\frac{1}{2}\right)\|\delta h\|_{H_k}^2.
\end{aligned}$$

Hence, since $\|A^{k+1}u_2\|_{L^2}^2$ belong to $L^1(0,T)$ by $(4.2)_k$, Gronwall's inequality yields (4.17) and (4.18). □

Now we are ready to prove Lemma 4.4.

Proof of Lemma 4.4. We choose a positive number such that

$$R^2 = \|u_0\|_{H^{2k+1}}^2 + \|f\|_{L^2(0,T;H_k)}^2 + 1. \qquad (4.33)$$

Let $h \in K_R^T := \{v \in L^2(0,T;H_k); \|v\|_{L^2(0,T;H_k)} \le R\}$ and let u be a unique solution of $(\mathrm{P})^{\varepsilon,\lambda}$ belonging to $\mathscr{B}_T^k$. Then, Lemma 4.5 assures that there exist numbers $M_{2k} = M_{2k}(k,\ell,R)$ and $M_{2k+1}^{\varepsilon} = M_{2k+1}^{\varepsilon}(k,\ell,R,M_{2k},\varepsilon)$ such that

$$\sup_{\substack{0\le t\le T\\ 2\le r\le\infty\\ 0\le m\le 2k-1}}\|D^m u(t)\|_{L^r} + \sup_{\substack{0\le t\le T\\ 2\le m\le 2k}}\|D^m u(t)\|_{L^2} \le M_{2k}, \qquad (4.34)$$

$$\sup_{0\le t\le T}\|Du(t)\|_{H_k} + \sqrt{\varepsilon}\|D^2u(t)\|_{L^2(0,T;H_k)} \le M_{2k+1}^{\varepsilon}. \qquad (4.35)$$

We are going to show below that $\mathscr{F}_\eta^{\lambda}$ maps K_R^T into itself for a sufficiently small η. We first note that

$$\begin{aligned}
&\|\mathscr{F}_\eta^{\lambda}(h)\|_{L^2(0,T;H_k)} = \eta(\|u^\ell Au\|_{L^2(0,T;L^2)} + \|A^k(u^\ell Au)\|_{L^2(0,T;L^2)}), \\
&\|u^\ell Au\|_{L^2(0,T;L^2)} = M_{2k}^{\ell+1}\sqrt{T}.
\end{aligned}$$

Moreover, by using (4.34), (4.35), Lemma 3.6, (3.14) and (3.15), we get

$$
\begin{aligned}
&\|A^k(u^\ell Au)\|_{L^2(0,T;L^2)}\\
&\quad\le \|u^\ell D^{2k+2}u\|_{L^2(0,T;L^2)} + \|{}_{2k}C_1\ell u^{\ell-1}DuD^{2k+1}u\|_{L^2(0,T;L^2)}\\
&\qquad + \sum_{i=2}^{2k-1} {}_{2k}C_i\|D^i(u^\ell)D^{2k-i+2}u\|_{L^2(0,T;L^2)} + \|D^{2k}(u^\ell)D^2u\|_{L^2(0,T;L^2)}\\
&\quad\le \frac{1}{\sqrt{\varepsilon}}M_{2k}^\ell M_{2k+1}^\varepsilon + \sqrt{T}\left({}_{2k}C_1\ell M_{2k}^\ell M_{2k+1}^\varepsilon + \sum_{i=2}^{2k-1}{}_{2k}C_i\ell^i M_{2k}^{\ell+1} + \ell^{2k}M_{2k}^{\ell+1}\right)\\
&\quad\le \frac{1}{\sqrt{\varepsilon}}M_{2k}^\ell M_{2k+1}^\varepsilon + (\ell+1)^{2k}M_{2k}^\ell(M_{2k}+M_{2k+1}^\varepsilon)\sqrt{T}.
\end{aligned}
$$

Thus we find

$$\|\mathscr{F}_\eta^\lambda(h)\|_{L^2(0,T;H_k)} \le \eta P_1(R),$$

$$P_1(R) = M_{2k}^{\ell+1}\sqrt{T} + \left(\frac{1}{\sqrt{\varepsilon}}M_{2k}^\ell M_{2k+1}^\varepsilon + (\ell+1)^{2k}M_{2k}^\ell(M_{2k}+M_{2k+1}^\varepsilon)\sqrt{T}\right).$$

Here $\mathscr{F}_\eta^\lambda$ maps K_R^T into itself for all η such that $\eta \le R/(P_1(R))$.

Next, we are going to show that $\mathscr{F}_\eta^\lambda$ becomes a contraction, let $h_1, h_2 \in K_T^R$ and let u_1 and u_2 be the solutions of $(\mathrm{P})^{\varepsilon,\lambda}$ with h replaced by h_1 and h_2 respectively, then we get

$$\|\mathscr{F}_\eta^\lambda(h_1) - \mathscr{F}_\eta^\lambda(h_2)\|_{H_k} \le \eta(\|u_1^\ell A(u_1-u_2)\|_{H_k} + \|(u_1^\ell - u_2^\ell)Au_2\|_{H_k}). \tag{4.36}$$

Using the same notations $w = u_1 - u_2, d_\ell$ and the same argument as in the proof of Lemma 4.6, we obtain

$$
\begin{aligned}
\|u_1^\ell Aw\|_{H_k} &= \|u_1^\ell Aw\|_{L^2} + \|A^k(u_1^\ell Aw)\|_{L^2}\\
&\le M_{2k}^\ell\|Aw\|_{L^2} + \Big(\|u_1^\ell A^{k+1}w\|_{L^2} + \|2k\ell u_1^{\ell-1}Du_1D^{2k+1}w\|_{L^2}\\
&\qquad + \sum_{i=2}^{2k-1}{}_{2k}C_i\|D^i(u_1^\ell)D^{2k-i+2}w\|_{L^2} + \|D^{2k}(u_1^\ell)D^2w\|_{L^2}\Big)\\
&\le M_{2k}^\ell\|Aw\|_{H_k} + 2k\ell M_{2k}^\ell\|Dw\|_{H_k}\\
&\qquad + \sum_{i=2}^{2k-1}{}_{2k}C_i\ell^i M_{2k}^\ell\|w\|_{H_k} + \sqrt{2}\ell^{2k}M_{2k}^\ell\|w\|_{H_k}\\
&\le M_{2k}^\ell(\|Aw\|_{H_k} + 2k\ell\|Dw\|_{H_k} + \sqrt{2}(\ell+1)^{2k}\|w\|_{H_k}),
\end{aligned}
\tag{4.37}
$$

$$
\begin{aligned}
&\|(u_1^\ell - u_2^\ell)Au_2\|_{H_k} \\
&\quad = \|wd_\ell Au_2\|_{H_k} \\
&\quad = \|wd_\ell Au_2\|_{L^2} + \|A^k(wd_\ell Au_2)\|_{L^2} \\
&\quad \le \ell M_{2k}^{\ell-1}\|w\|_{L^\infty}\|Au_2\|_{L^2} + \|wd_\ell A^{k+1}u_2\|_{L^2} + \|2kD(wd_\ell)D^{2k+1}u_2\|_{L^2} \\
&\qquad + \sum_{i=2}^{2k-1} {}_{2k}C_i\|D^i(wd_\ell)D^{2k-i+2}u_2\|_{L^2} + \|D^{2k}(wd_\ell)D^2u_2\|_{L^2} \\
&\quad \le \sqrt{2}\ell M_{2k}^\ell\|w\|_{H_k} + \sqrt{2}\ell M_{2k}^{\ell-1}\|A^{k+1}u_2\|_{L^2}\|w\|_{H_k} + 2\sqrt{2}k\ell^2 M_{2k}^{\ell-1}M_{2k+1}^\varepsilon\|w\|_{H_k} \\
&\qquad + \sqrt{2}\sum_{i=2}^{2k-1} {}_{2k}C_i\ell^{i+1}M_{2k}^{\ell-1}\|w\|_{H_k}M_{2k} + \sqrt{2}\ell^{2k+1}M_{2k}^{2\ell-1}\|w\|_{H_k}M_{2k} \\
&\quad \le \|w\|_{H_k}\left[M_{2k}^\ell\left(\sqrt{2}\ell + \sqrt{2}\left(\sum_{i=2}^{2k-1} {}_{2k}C_i\ell^{i+1} + \ell^{2k+1}\right)\right)\right. \\
&\qquad\qquad \left. + \sqrt{2}\ell M_{2k}^{\ell-1}\|A^{k+1}u_2\|_{L^2} + 2\sqrt{2}k\ell^2 M_{2k}^{\ell-1}M_{2k+1}^\varepsilon\right] \\
&\quad \le \|w\|_{H_k}M_{2k}^{\ell-1}\sqrt{2}\ell[(1+(\ell+1)^{2k})M_{2k} + \|A^{k+1}u_2\|_{L^2} + 2k\ell M_{2k+1}^\varepsilon]. \qquad (4.38)
\end{aligned}
$$

Then, by substituting estimates (4.17) and (4.18) in (4.37) and (4.38), we find that there exists a number $P_2(R) > 0$ depending only on R, k, ℓ, ε and T such that

$$
\begin{aligned}
&\|\mathscr{F}_\eta^\lambda(h_1) - \mathscr{F}_\eta^\lambda(h_2)\|_{L^2(0,T;H_k)} \\
&\qquad \le \eta e^{G_2T}P_2(R)\|h_1 - h_2\|_{L^2(0,T;H_k)}.
\end{aligned}
$$

Therefore, for every $\eta \in (0, \eta_0)$ with $\eta_0 = \min(R/(P_1(R)), 1/(2P_2(R)e^{G_2T}))$, $\mathscr{F}_\eta^\lambda$ becomes a contraction from K_R^T into itself. □

4.2. Approximate equations: local existence.

In this subsection, we are going to show that approximate equations $(\mathrm{P})^\varepsilon$ admit local solutions.

Lemma 4.7. *Let $u_0 \in H^{2k+1}(\boldsymbol{R})$, $k \in \boldsymbol{N}$ $(k \ge 2)$, then there exists a positive number T_0 depending only on ε, k, ℓ and $\|u_0\|_{H^{2k+1}}$ such that $(\mathrm{P})^\varepsilon$ has a unique solution u belonging to $\mathscr{B}_{T_0}^k$.*

To prove this lemma, we shall apply the arguments similar to those in the proof of Lemma 4.3. We introduce the following auxiliary equations.

$$
(\mathrm{P})_h^\varepsilon \begin{cases} (d/dt)u(t) + \varepsilon Au(t) + u^\ell Au(t) = h(t), \quad 0 \le t \le T, \\ u(0) = u_0. \end{cases}
$$

Lemma 4.3 assures that for any $h \in L^2(0,T;H_k)$ and $u_0 \in H^{2k+1}(\boldsymbol{R})$, $(\mathrm{P})_h^\varepsilon$ has a unique solution u belonging to $\mathscr{B}_T^k$. So we can define an operator $\mathscr{S}$ by

$$\mathscr{S} : h \mapsto u \mapsto \ell(u)^{\ell-1}(u_x)^2.$$

Therefore, to prove Lemma 4.7, it suffices to show that $\mathscr{S}$ becomes a contraction from $K_R^{T_0} := \{v \in L^2(0,T_0;H_k); \|v\|_{L^2(0,T_0;H_k)} \le R\}$ into itself for suitable R and T_0.

Proof of Lemma 4.7. We choose $R > 0$ such that

$$R^2 = \|u_0\|_{H^{2k+1}}^2 + 1.$$

Let $h \in K_R^{T_0}$ with $0 < T_0 \le T$, and let u be a unique solution of $(\mathrm{P})_h^\varepsilon$ belonging to $\mathscr{B}_{T_0}^k$. Then, Lemma 4.5 says that there exist numbers $M_{2k} = M_{2k}(k,\ell,R)$ and $M_{2k+1}^\varepsilon = M_{2k+1}^\varepsilon(k,\ell,R,M_{2k},\varepsilon)$ such that (4.34) and (4.35) hold true. We easily note that

$$\|\mathscr{S}(h)\|_{H_k} = \|\ell u^{\ell-1}(Du)^2\|_{L^2} + \|D^{2k}(\ell u^{\ell-1}(Du)^2)\|_{L^2},$$

$$\|\ell u^{\ell-1}(Du)^2\|_{L^2} \le (\ell\|u\|_{L^\infty}^{\ell-1}\|Du\|_{L^4}^2) \le \ell M_{2k}^{\ell+1}.$$

Moreover, by Lemma 3.7, we get

$$\|D^{2k}(\ell u^{\ell-1}(Du)^2)\|_{L^2} \le \|J_{2k}^1\|_{L^2} + \|J_{2k}^2\|_{L^2} + \sum_{j=1}^4 \|S_{2k}^j\|_{L^2},$$

$$\|J_{2k}^1\|_{L^2} = \|2\ell u^{\ell-1}DuD^{2k+1}u\|_{L^2} \le 2\ell M_{2k}^\ell M_{2k+1}^\varepsilon,$$

$$\begin{aligned}\|J_{2k}^2\|_{L^2} &= \|((4k+1)\ell(\ell-1)u^{\ell-2}(Du)^2 + 4k\ell u^{\ell-1}D^2u)D^{2k}u\|_{L^2}\\ &\le (4k+1)\ell^2 M_{2k}^{\ell+1},\end{aligned}$$

$$\sum_{j=1}^4 \|S_{2k}^j\|_{L^2} \le 4k\ell^2(\ell+1)^{2k}M_{2k}^{\ell+1}.$$

Hence, we obtain

$$\|\mathscr{S}(h)\|_{L^2(0,T;H_k)} \le \sqrt{T_0}Q_1(R), \tag{4.39}$$

$$Q_1(R) = M_{2k}^\ell \ell[2M_{2k+1}^\varepsilon + ((4k+2)\ell + 4k\ell(\ell+1)^{2k})M_{2k}].$$

Let $h_1, h_2 \in K_R^{T_0}$ and let u_1 and u_2 be the unique solutions of $(\mathrm{P})_h^\varepsilon$ with h replaced by h_1 and h_2 respectively. Then, by using the notations $w = u_1 - u_2$ and

$$d_{\ell-1} = u_1^{\ell-2} + u_1^{\ell-3}u_2 + \cdots + u_1u_2^{\ell-3} + u_2^{\ell-2},$$

we have

$$\mathscr{S}(h_1) - \mathscr{S}(h_2) = \ell u_1^{\ell-1}D(u_1+u_2)Dw + \ell d_{\ell-1}(Du_2)^2 w.$$

Hence

$$\|\mathscr{S}(h_1) - \mathscr{S}(h_2)\|_{H_k} \le \|\ell u_1^{\ell-1} D(u_1+u_2)Dw\|_{L^2} + \|D^{2k}(\ell u_1^{\ell-1} D(u_1+u_2)Dw)\|_{L^2}$$
$$+ \|\ell d_{\ell-1}(Du_2)^2 w\|_{L^2} + \|D^{2k}(\ell d_{\ell-1}(Du_2)^2 w)\|_{L^2}.$$

It is easy to see

$$\begin{aligned}\|\ell u_1^{\ell-1} D(u_1+u_2)Dw\|_{L^2} + \|\ell d_{\ell-1}(Du_2)^2 w\|_{L^2} &\le 2\ell M_{2k}^{\ell}\|w\|_{H_k} + \ell(\ell-1)M_{2k}^{\ell}\|w\|_{H_k} \\ &\le \ell(\ell+1)M_{2k}^{\ell}\|w\|_{H_k}. \end{aligned} \tag{4.40}$$

Furthermore we obtain, by (3.14)

$$\begin{aligned}
&\|D^{2k}(\ell u_1^{\ell-1} D(u_1+u_2)Dw)\|_{L^2} \\
&\quad \le \|\ell u_1^{\ell-1} D(u_1+u_2)D^{2k+1}w\|_{L^2} + \sum_{i=1}^{2k-2} {}_{2k}C_i \|D^i(\ell u_1^{\ell-1}D(u_1+u_2))D^{2k-i+1}w\|_{L^2} \\
&\qquad + \|{}_{2k}C_{2k-1}D^{2k-1}(\ell u_1^{\ell-1}D(u_1+u_2))D^2 w\|_{L^2} + \|D^{2k}(\ell u_1^{\ell-1}D(u_1+u_2))Dw\|_{L^2} \\
&\quad \le 2\ell M_{2k}^{\ell}\|D^{2k+1}w\|_{L^2} + \ell\sum_{i=1}^{2k-2} {}_{2k}C_i \sum_{j=0}^{i} {}_iC_j \|D^j(u_1^{\ell-1})\|_{L^\infty}\|D^{i-j+1}(u_1+u_2)\|_{L^\infty}\|w\|_{H_k} \\
&\qquad + 2k\ell\sum_{i=0}^{2k-1} {}_{2k-1}C_i\|D^i(u_1^{\ell-1})\|_{L^\infty}\|D^{2k-i}(u_1+u_2)\|_{L^2}\|D^2 w\|_{L^\infty} \\
&\qquad + \ell\sum_{i=0}^{2k-1} {}_{2k}C_i\|D^i(u_1^{\ell-1})\|_{L^\infty}\|D^{2k+1-i}(u_1+u_2)\|_{L^2}\|Dw\|_{L^\infty} \\
&\qquad + \ell\|D^{2k}(u_1^{\ell-1})\|_{L^2}\|D(u_1+u_2)\|_{L^\infty}\|Dw\|_{L^\infty} \\
&\quad \le 2\ell M_{2k}^{\ell}\|D^{2k+1}w\|_{L^2} + \ell\sum_{i=1}^{2k-2} {}_{2k}C_i\sum_{j=0}^{i} {}_iC_j(\ell-1)^j M_{2k}^{\ell-1}2M_{2k}\|w\|_{H_k} \\
&\qquad + 2k\ell\sum_{i=0}^{2k-1} {}_{2k-1}C_i(\ell-1)^i M_{2k}^{\ell-1}2M_{2k}\sqrt{2}\|w\|_{H_k} \\
&\qquad + \ell\sum_{i=0}^{2k} {}_{2k}C_i(\ell-1)^i M_{2k}^{\ell-1}2(M_{2k}+M_{2k+1}^{\varepsilon})\sqrt{2}\|w\|_{H_k} \\
&\quad \le 2\ell M_{2k}^{\ell}\|Dw\|_{H_k} \\
&\qquad + 2\ell M_{2k}^{\ell-1}\|w\|_{H_k}\left[\left(\sum_{i=1}^{2k-2} {}_{2k}C_i\ell^i + 2\sqrt{2}k\ell^{2k-1}\right)M_{2k} + \sqrt{2}\ell^{2k}(M_{2k}+M_{2k+1}^{\varepsilon})\right] \\
&\quad \le 2\ell M_{2k}^{\ell}\|Dw\|_{H_k} + 2\sqrt{2}\ell M_{2k}^{\ell-1}(M_{2k}+M_{2k+1}^{\varepsilon})(\ell+1)^{2k}\|w\|_{H_k}.
\end{aligned} \tag{4.41}$$

Here, by the same argument as in the proof of Lemma 4.6, we find

$$
\begin{aligned}
&\|D^i(d_{\ell-1}(Du_2)^2)\|_{L^\infty} \\
&\quad\le \|(D^i d_{\ell-1})(Du_2)^2\|_{L^\infty} + \sum_{j=1}^{i} {}_iC_j\|2D^{j-1}(Du_2D^2u_2)D^{i-j}(d_{\ell-1})\|_{L^\infty} \\
&\quad\le (\ell-1)(\ell-2)^i M_{2k}^{\ell-2}M_{2k}^2 + \sum_{j=1}^{i} {}_iC_j 2\cdot 2^{j-1}M_{2k}^2(\ell-1)(\ell-2)^{i-j}M_{2k}^{\ell-2} \\
&\quad\le (\ell-1)\ell^i M_{2k}^{\ell}, \quad (0\le i\le 2k-2).
\end{aligned}
$$

Similarly we get

$$
\begin{aligned}
\|D^{2k-1}(d_{\ell-1}(Du_2)^2)\|_{L^2} &\le (\ell-1)\ell^{2k-1}M_{2k}^{\ell}, \\
\|D^{2k}(d_{\ell-1}(Du_2)^2)\|_{L^2} &\le (\ell-1)\ell^{2k}M_{2k}^{\ell-1}(M_{2k}+M_{2k+1}^{\varepsilon}).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|D^{2k}(\ell d_{\ell-1}(Du_2)^2)w\|_{L^2} \\
&\quad\le \sum_{i=0}^{2k-2} {}_{2k}C_i\|D^i(d_{\ell-1}(Du_2)^2)D^{2k-i}w\|_{L^2} \\
&\qquad + {}_{2k}C_{2k-1}\|D^{2k-1}(d_{\ell-1}(Du_2)^2)Dw\|_{L^2} + \|D^{2k}(d_{\ell-1}(Du_2)^2)w\|_{L^2} \\
&\quad\le \ell(\ell-1)\Bigg(\sum_{i=0}^{2k-2} {}_{2k}C_i\ell^i M_{2k}^{\ell}\|w\|_{H_k} + 2k\ell^{2k-1}M_{2k}^{\ell}\|D^2w\|_{L^\infty} \\
&\qquad\qquad + \ell^{2k}M_{2k}^{\ell-1}(M_{2k}+M_{2k+1}^{\varepsilon})\|w\|_{L^\infty}\Bigg) \\
&\quad\le \sqrt{2}\ell(\ell-1)(\ell+1)^{2k}M_{2k}^{\ell-1}(M_{2k}+M_{2k+1}^{\varepsilon})\|w\|_{H_k}. \qquad (4.42)
\end{aligned}
$$

Thus, by substituting (4.17) in (4.40)–(4.42), we find that there exists a number $Q_2(R)$ depending only on R, k, ℓ and ε such that

$$
\|\mathscr{S}(h_1)-\mathscr{S}(h_2)\|_{L^2(0,T_0;H_k)} \le \sqrt{T_0}e^{G_2T_0}Q_2(R)\|h_1-h_2\|_{L^2(0,T_0;H_k)}. \qquad (4.43)
$$

In view of (4.39) and (4.43), we set

$$
T_0 = \min\left(1, \left(\frac{R}{Q_1(R)}\right)^2, \left(\frac{R}{2Q_2(R)e^{G_2T}}\right)^2\right),
$$

then $\mathscr{S}$ becomes a contraction from $K_R^{T_0}$ into itself. Therefore there exists a fixed point h_0 of $\mathscr{S}$ in $K_R^{T_0}$ and it is clear that the solution of $(\mathrm{P})_h^\varepsilon$ with h replaced by h_0 gives the unique solution of $(\mathrm{P})^\varepsilon$. □

4.3. Approximate equations: global existence.

In this subsection, we are going to show that the local solutions of $(\mathrm{P})^\varepsilon$ constructed in the previous subsection can be continued globally. As we observed in the Proof of Lemma 4.7, the solution $u(t)$ on $[0, T_0)$ can be continued to the right of $t = T_0$ if $\|u(t)\|_{H^{2k+1}}$ is bounded on $[0, T_0)$. Therefore, in order to prove the existence of global solutions of $(\mathrm{P})^\varepsilon$, we have only to establish the a priori bound for the H^{2k+1}-norm of solutions. In fact, our main results in this subsection are as follows.

Lemma 4.8. *Let $u_0 \in H^{2k+1}(\boldsymbol{R})$ with $k \in \boldsymbol{N}$ $(k \geq 2)$, then $(\mathrm{P})^\varepsilon$ has a unique global solution u such that $u \in \mathscr{B}_T^k$ for all $T > 0$.*

This lemma is a direct consequence of Lemma 4.7 and the following Lemma 4.9.

Lemma 4.9. *Let u be a solution of $(\mathrm{P})^\varepsilon$ belonging to $\mathscr{B}_T^k$. Then there exist numbers $\{L_m\}_{m=0}^{2k+1}$, $\{L_{m,\infty}\}_{m=0}^{2k}$ such that*

$$L_m \leq L_{m,\infty} \leq L_{m+1}, \quad 0 \leq m \leq 2k, \tag{4.44}$$

$$\sup_{0 \leq t \leq T} \|D^m u(t)\|_{L^2} \leq L_m, \quad 0 \leq m \leq 2k+1, \tag*{$(4.45)_m$}$$

$$\sup_{\substack{0 \leq t \leq T \\ 2 \leq r \leq \infty}} \|D^m u(t)\|_{L^r} \leq L_{m,\infty}, \quad 0 \leq m \leq 2k. \tag*{$(4.45)_{m,\infty}$}$$

Here L_m and $L_{m,\infty}$ do not depend on ε explicitly except $L_{1,\infty}$, more precisely,

$$L_0 = L_0(\|u_0\|_{L^2}), \quad L_{0,\infty} = L_{0,\infty}\left(\sup_{2 \leq r \leq \infty} \|u_0\|_{L^r}\right),$$

$$L_1 = L_{1,\infty} = L_{1,\infty}\left(\sup_{2 \leq r \leq \infty} \|Du_0\|_{L^r}, \ell, L_{0,\infty}, \varepsilon\right),$$

$$L_2 = L_2(\|D^2 u_0\|_{L^2}, \ell, L_{1,\infty}),$$

$$L_3 = L_3(\|D^3 u_0\|_{L^2}, \ell, L_{1,\infty}, L_2),$$

$$L_{2,\infty} = \sqrt{2} L_2^{1/2} L_3^{1/2},$$

$$L_m = L_m(\|D^m u_0\|_{L^2}, \ell, L_{m-1,\infty}),$$

$$L_{m,\infty} = L_{m,\infty}\left(\sup_{2 \leq r \leq \infty} \|D^m u_0\|_{L^r}, \ell, L_{m-1,\infty}\right), \quad (m \geq 3).$$

PROOF. We repeat the same type of arguments as in the proof of Lemma 4.5. We here denote by $\mathbf{L}_m$ (or $\mathbf{L}_{m,\infty}$) positive numbers depending only on ℓ, m and L_m (or $L_{m,\infty}$), which will have different values in different places.

(The case m = 0)
Multiplication of $(\mathrm{P})^\varepsilon$ by $|u|^{r-2}u$ gives

$$\|u\|_{L^r}^{r-1}\frac{d}{dt}\|u\|_{L^r} + \varepsilon(r-1)\int |u|^{r-2}(Du)^2\,dx + (\ell+r-1)\int |u|^{\ell+r-2}(Du)^2\,dx$$
$$= \ell\int |u|^{\ell+r-2}(Du)^2\,dx,$$

whence follows

$$\|u\|_{L^r}^{r-1}\frac{d}{dt}\|u\|_{L^r} \le 0.$$

Then we get

$$\sup_{0\le t\le T}\|u\|_{L^r} \le \|u_0\|_{L^r} \quad \text{for all } r\in[2,\infty], \tag{4.46}$$

which yields $(4.45)_m$ and $(4.45)_{m,\infty}$ with $m=0$.

(The case m = 1)
The direct energy method as in the proof of Lemma 4.5 does not work for this case. However, we can apply the argument of Oleinik and Kruzhkov [**14**] based on the change of variables and the maximum principle to get a priori bound of $\|Du\|_{L^\infty}$. For example, Theorem 11.16 of Lieberman [**13**] assures that there exists a constant $C_{1,\infty}$ depending only on $\|Du_0\|_{L^\infty}, \ell, L_{0,\infty}$ and ε such that

$$\sup_{0\le t\le T}\|Du(t)\|_{L^\infty} \le C_{1,\infty}. \tag{4.47}$$

On the other hand, multiplication of $(\mathrm{P})^\varepsilon$ by $-D^2u$ and the integration by parts yield

$$\frac{1}{2}\frac{d}{dt}\|Du\|_{L^2}^2 + \int(\varepsilon+u^\ell)(D^2u)^2\,dx = \ell(\ell-1)\int u^{\ell-2}(Du)^4\,dx$$
$$\le \mathbf{L}_{0,\infty}C_{1,\infty}^2\|Du\|_{L^2}^2.$$

Hence, by Gronwall's inequality and the inequality $\|u\|_{L^r} \le \|u\|_{L^2}^{2/r}\|u\|_{L^\infty}^{(r-2)/r}$, we deduce $(4.45)_m$ and $(4.45)_{m,\infty}$ with $m=1$.

(The case m = 2)
Multiplication of $D^2(\mathrm{P})^{\varepsilon}$ by D^2u with the integration by parts gives

$$\frac{1}{2}\frac{d}{dt}\|D^2u\|_{L^2}^2+\varepsilon\|D^3u\|_{L^2}^2=(D^2(u^{\ell}D^2u),D^2u)_{L^2}+(D^2(\ell u^{\ell-1}(Du)^2),D^2u)_{L^2}$$
$$=-\int u^{\ell}(D^3u)^2\,dx-3\ell\int u^{\ell-1}DuD^2uD^3u\,dx$$
$$-\ell(\ell-1)\int u^{\ell-2}(Du)^3D^3u\,dx.$$

Here we get

$$-3\ell\int u^{\ell-1}DuD^2uD^3u\,dx\le\frac{3}{4}\int u^{\ell}(D^3u)^2\,dx+3\ell^2\int u^{\ell-2}(Du)^2(D^2u)^2\,dx$$
$$\le\frac{3}{4}\int u^{\ell}(D^3u)^2\,dx+\mathbf{L}_{1,\infty}\|D^2u\|_{L^2}^2$$

and by Lemma 3.2

$$-\ell(\ell-1)\int u^{\ell-2}(Du)^3D^3u\,dx=\ell(\ell-1)\int D(u^{\ell-2}(Du)^3)D^2u\,dx$$
$$\le\mathbf{L}_{1,\infty}\cdot\|D^2u\|_{L^2}^2.$$

Hence it holds that

$$\frac{d}{dt}\|D^2u\|_{L^2}\le\mathbf{L}_{1,\infty}\|D^2u\|_{L^2}.$$

Therefore, by Gronwall's inequality, $(4.45)_m$ with $m=2$ is assured with $L_2=\max(L_{1,\infty},\exp(\mathbf{L}_{1,\infty}T)\|D^2u_0\|_{L^2})$.

Next, we calculate $(D^3(\mathrm{P})^{\varepsilon},D^3u)$ to get

$$\frac{1}{2}\frac{d}{dt}\|D^3u\|_{L^2}^2+\varepsilon\|D^4u\|_{L^2}^2=(D^3(u^{\ell}D^2u),D^3u)_{L^2}+(D^3(\ell u^{\ell-1}(Du)^2),D^3u)_{L^2}.$$

By exactly the same arguments as for (4.6)–(4.9), we can obtain

$$(D^3(u^{\ell}D^2u),D^3u)_{L^2}\le-\frac{1}{2}\int u^{\ell}(D^4u)^2\,dx+\mathbf{L}_2(\|D^3u\|_{L^2}+\|D^3u\|_{L^2}^2).\tag{4.48}$$

On the other hand, Lemma 3.7 with $n=3$ yields

$$(D^3(\ell u^{\ell-1}(Du)^2),D^3u)_{L^2}=\bar{J}_3^1+\bar{J}_3^2+\bar{S},\quad\bar{S}=\bar{S}_1+\bar{S}_2+\bar{S}_3,$$

where

$$\bar{J}_3^1 = \int 2\ell u^{\ell-1} Du D^4 u D^3 u \, dx,$$

$$\bar{J}_3^2 = \int \{7\ell(\ell-1)u^{\ell-2}(Du)^2 + 6\ell u^{\ell-1} D^2 u\}(D^3 u)^2 \, dx,$$

$$\bar{S}_1 = \int 9\ell(\ell-1)(\ell-2)u^{\ell-3}(Du)^3 D^2 u D^3 u \, dx,$$

$$\bar{S}_2 = \int 12\ell(\ell-1)u^{\ell-2} Du (D^2 u)^2 D^3 u \, dx,$$

$$\bar{S}_3 = \int \ell(\ell-1)(\ell-2)(\ell-3)u^{\ell-4}(Du)^5 D^3 u \, dx.$$

Here we have

$$\bar{J}_3^1 \le \frac{1}{4}\int u^\ell (D^4 u)^2 \, dx + \mathbf{L}_{1,\infty}\|D^3 u\|_{L^2}^2,$$

$$\bar{J}_3^2 \le \mathbf{L}_{1,\infty}\|D^3 u\|_{L^2}^2 + 3\ell \int u^{\ell-1} D^3 u D((D^2 u)^2) \, dx,$$

and by (3.6),

$$\begin{aligned} 3\ell \int u^{\ell-1} D^3 u D((D^2 u)^2) \, dx &= -3\ell \int u^{\ell-1}(D^2 u)^2 D^4 u \, dx - \frac{1}{4}\bar{S}_2 \\ &\le \frac{1}{4}\int u^\ell (D^4 u)^2 \, dx + 9\ell^2 \int u^{\ell-2}(D^2 u)^4 \, dx - \frac{1}{4}\bar{S}_2 \\ &\le \frac{1}{4}\int u^\ell (D^4 u)^2 \, dx + \mathbf{L}_2\|D^3 u\|_{L^2} - \frac{1}{4}\bar{S}_2. \end{aligned}$$

Furthermore, by (3.4),

$$\bar{S}_1 + \bar{S}_2 \le \mathbf{L}_{1,\infty} L_2 \|D^3 u\|_{L^2}(1 + L_2 + \|D^3 u\|_{L^2}),$$

$$\bar{S}_3 \le \mathbf{L}_{1,\infty}\|D^3 u\|_{L^2}.$$

Consequently, we deduce

$$(D^3(\ell u^{\ell-1}(Du)^2), D^3 u)_{L^2} \le \frac{1}{2}\int u^\ell (D^4 u)^2 \, dx + \mathbf{L}_2\|D^3 u\|_{L^2}(\|D^3 u\|_{L^2} + 1). \quad (4.49)$$

Thus, in view of (4.48) and (4.49), we obtain

$$\frac{d}{dt}\|D^3 u\|_{L^2} \le \mathbf{L}_2(\|D^3 u\|_{L^2} + 1).$$

Then $(4.45)_m$ with $m = 3$ holds with

$$L_3 = \max(2L_2, (\|D^3 u_0\|_{L^2} + \mathbf{L}_2)\exp(\mathbf{L}_2 T)).$$

Now we can apply the same verification for $(4.1)_{m,\infty}$ with $m = 2$ to derive $(4.45)_{m,\infty}$ with $m = 2$.

(The case $3 \le m \le 2k$)
We multiply $D^{m-1}(\mathrm{P})^\varepsilon$ by $-D(|D^m u|^{r-2} D^m u)$ to get

$$\|D^m u\|_{L^r}^{r-1} \frac{d}{dt}\|D^m u\|_{L^r} + \varepsilon(r-1)\int |D^m u|^{r-2}(D^{m+1}u)^2\,dx = I_{m-1} + J_m,$$

$$I_{m-1} = -\int D^{m-1}(u^\ell D^2 u)\cdot D(|D^m u|^{r-2}D^m u)\,dx,$$

$$J_m = \int D^m(\ell u^{\ell-1}(Du)^2)|D^m u|^{r-2}D^m u\,dx.$$

We first note that exactly the same arguments as for (4.12)–(4.15) give

$$I_{m-1} \le -(r-1)\int u^\ell |D^m u|^{r-2}(D^{m+1}u)^2\,dx + \|D^m u\|_{L^r}^{r-1}(\mathbf{L}_{2,\infty}\|D^m u\|_{L^r} + \mathbf{L}_{m-1,\infty}). \tag{4.50}$$

Making use of Lemma 3.7 with $n = m$, we get

$$\begin{aligned} J_m &= \bar{J}_m^1 + \bar{J}_m^2 + \sum_{i=1}^4 \bar{S}_m^i, \\ \bar{J}_m^i &= \int J_m^i |D^m u|^{r-2}D^m u\,dx \quad (i = 1,2), \\ \bar{S}_m^i &= \int S_m^i |D^m u|^{r-2}D^m u\,dx \quad (i = 1,2,3,4). \end{aligned} \tag{4.51}$$

It is easy to see

$$\bar{J}_m^2 \le \mathbf{L}_{2,\infty}\|D^m u\|_{L^r}^r, \tag{4.52}$$

and by (3.17)

$$\sum_{i=1}^4 \bar{S}_m^i \le \mathbf{L}_{m-1,\infty}\|D^m u\|_{L^r}^{r-1}. \tag{4.53}$$

Moreover, by Schwarz's inequality, we have

$$\bar{J}_m^1 = 2\ell \int u^{\ell-1} Du D^{m+1}u |D^m u|^{r-2} D^m u \, dx$$
$$\leq \frac{1}{4} \int u^\ell |D^m u|^{r-2} (D^{m+1}u)^2 \, dx + \mathbf{L}_{1,\infty} \|D^m u\|_{L^r}^r. \tag{4.54}$$

Hence we deduce

$$\frac{d}{dt} \|D^m u\|_{L^r} \leq \mathbf{L}_{2,\infty} \|D^m u\|_{L^r} + \mathbf{L}_{m-1,\infty}.$$

Now it is clear that there exist numbers L_m and $L_{m,\infty}$ satisfying $(4.45)_m$ and $(4.45)_{m,\infty}$ for all $3 \leq m \leq 2k$.

(The case m = 2k + 1)
Multiplying $A^k(\mathrm{P})^\varepsilon$ by $A^{k+1}u$, we get

$$\frac{1}{2} \frac{d}{dt} \|D^{2k+1}u\|_{L^2}^2 + \varepsilon \|A^{k+1}u\|_{L^2}^2 = I_{2k} + J_{2k+1}, \tag{4.55}$$

where

$$I_{2k} = -\int D^{2k}(u^\ell D^2 u) D^{2k+2} u \, dx,$$
$$J_{2k+1} = \int D^{2k+1}(u^{\ell-1}(Du)^2) D^{2k+1} u \, dx.$$

Then Lemma 3.6 with $n = 2k$ gives

$$I_{2k} = -\int u^\ell (D^{2k+2}u)^2 \, dx + \bar{I}_{2k}^2 + \bar{I}_{2k}^3 + \bar{R}_{2k}^1 + \bar{R}_{2k}^2,$$
$$\bar{I}_{2k}^2 = \int {}_{2k}C_1 \ell u^{\ell-1} Du D^{2k+1} u D^{2k+2} u \, dx,$$
$$\bar{I}_{2k}^3 = \int \{ {}_{2k}C_2 \ell(\ell-1) u^{\ell-2} (Du)^2 + ({}_{2k}C_2 + 1) \ell u^{\ell-1} D^2 u \} D^{2k} u D^{2k+2} u \, dx,$$
$$\bar{R}_{2k}^i = \int R_{2k}^i \cdot D^{2k+2} u \, dx, \quad (i = 1, 2).$$

By the integration by parts and (3.13), we easily find

$$\bar{I}_{2k}^3 \leq \mathbf{L}_{2,\infty} \|D^{2k+1}u\|_{L^2}^2 + \mathbf{L}_{3,\infty} \|D^{2k}u\|_{L^2} \|D^{2k+1}u\|_{L^2},$$
$$\bar{R}_{2k}^1 + \bar{R}_{2k}^2 \leq \mathbf{L}_{2k} \|D^{2k+1}u\|_{L^2}.$$

Moreover, by Schwarz's inequality,

$$\bar{I}_{2k}^2 \leq \frac{1}{4}\int u^{\ell}(D^{2k+2}u)^2\,dx + \mathbf{L}_{1,\infty}\|D^{2k+1}u\|_{L^2}^2.$$

On the other hand, by Lemma 3.7 with $n = 2k+1$, we get

$$\begin{aligned}
J_{2k+1} &= \int D^{2k+1}(\ell u^{\ell-1}(Du)^2)D^{2k+1}u\,dx \\
&\leq \int 2\ell u^{\ell-1}DuD^{2k+2}uD^{2k+1}u\,dx \\
&\quad + \int((4k+3)\ell(\ell-1)u^{\ell-2}(Du)^2 + 2(2k+1)\ell u^{\ell-1}D^2u)(D^{2k+1}u)^2\,dx \\
&\quad + \int\sum_{j=1}^{4} S_{2k+1}^{j}D^{2k+1}u\,dx \\
&\leq \frac{1}{4}\int u^{\ell}(D^{2k+2}u)^2\,dx + \|D^{2k+1}u\|_{L^2}(\mathbf{L}_{2,\infty}\|D^{2k+1}u\|_{L^2} + \mathbf{L}_{2k-1,\infty}L_{2k}).
\end{aligned}$$

Thus we deduce

$$\frac{d}{dt}\|D^{2k+1}u\|_{L^2} \leq \mathbf{L}_{2,\infty}\|D^{2k+1}u\|_{L^2} + \mathbf{L}_{2k},$$

whence follows $(4.25)_m$ with $m = 2k+1$. □

5. Proof of Theorem.

In this section, we give a proof of our main theorem. To do this, it suffices to observe that the following theorem holds true.

THEOREM 5.1. *Let $u_0 \in H^{2k+1}(\boldsymbol{R})$ with $k \in \boldsymbol{N}$ $(k \geq 2)$, then there exists a positive number T_0 depending only on $\ell, \|u_0\|_\infty$ and $\|(u_0)_x\|_\infty$ such that* (P) *has a unique solution u belonging to $\mathscr{C}_{T_0}^k := \{v \in C([0,T_0];H^{2k}(\boldsymbol{R}));\ v \in L^\infty(0,T_0;H^{2k+1}(\boldsymbol{R})),$ $v_t \in L^2(0,T_0;H^{2k}(\boldsymbol{R})), v^{\ell}D^2v \in L^2(0,T_0;H^{2k}(\boldsymbol{R}))\}$, such that*

$$\sup_{0\leq t\leq T_0}\|u(\cdot,t)\|_{L^\infty(\mathbf{R})} \leq \|u_0\|_{L^\infty(\mathbf{R})}. \tag{5.1}$$

Moreover T_0 can be chosen as a monotone decreasing function of $\|(u_0)_x\|_{L^\infty}$ such that T_0 tends to 0 as $\|(u_0)_x\|_{L^\infty}$ tends to ∞.

As an immediate consequence of this theorem, the following corollary holds.

COROLLARY 5.2. *A solution u of* (P) *in $[0,T)$ belonging to $\mathscr{C}_{T_0}^k$ for all $T_0 \in [0,T)$ can be continued as a solution of* (P) *belonging to $\mathscr{C}_{T_1}^k$ for some $T_1 > T$,* *if and only if* $\|u_x(\cdot,t)\|_{L^\infty(\boldsymbol{R})}$ *is bounded on* $[0,T)$. *Furthermore, if* u *can not be continued as a solution of* (P) *belonging to* $\mathscr{C}^k_{T_1}$ *for some* $T_1 > T$, *then it holds that*

$$\lim_{t\uparrow T}\|u_x(\cdot,t)\|_{L^\infty(\boldsymbol{R})} = +\infty. \tag{5.2}$$

To prove Theorem 5.1, we prepare the following lemma.

LEMMA 5.3. *Let* $u_0 \in H^{2k+1}(\boldsymbol{R})$ *with* $k \in \boldsymbol{N}$ $(k \geq 2)$ *and let* u *be the unique global solution of* $(\mathrm{P})^\varepsilon$ *belonging to* $\mathscr{B}^k_T$ *for all* $T > 0$ (*whose existence is assured by Lemma* 4.8). *Then there exists a positive number* T_0 *depending only on* ℓ, $\|u_0\|_\infty$ *and* $\|(u_0)_x\|_\infty$ *not on* ε *such that* (4.44), $(4.45)_m$, $(4.45)_{m,\infty}$ *hold true with* T *replaced by* T_0 *and constants* L_m $(0 \leq m \leq 2k+1)$ *and* $L_{m,\infty}$ $(0 \leq m \leq 2k)$ *are independent of* ε.

Furthermore it holds that

$$\varepsilon\int_0^{T_0}\|D^{2k+2}u\|^2_{L^2}\,dt \leq L_{2k+1}, \tag{5.3}$$

$$\int_0^{T_0}\int u^\ell(D^{2k+2}u(t))^2\,dxdt \leq L_{2k+1}, \tag{5.4}$$

$$\int_0^{T_0}\|u_t\|^2_{H^{2k}}\,dt \leq L_{2k+1}. \tag{5.5}$$

Here T_0 *can be chosen as a monotone decreasing function of* $\|(u_0)_x\|_{L^\infty}$ *such that* T_0 *tends to* 0 *as* $\|(u_0)_x\|_{L^\infty}$ *tends to* ∞.

PROOF. Recalling the proof of Lemma 4.9, we find that if we establish the a priori bound for $\sup_{0\leq t\leq T_0}\|Du(t)\|_{L^\infty}$ for some $T_0 > 0$, then (4.44), $(4.45)_m$ and $(4.45)_{m,\infty}$ hold true with $T = T_0$ and constants L_m and $L_{m,\infty}$ do not depend on ε. Furthermore, in view of the arguments for the case $m = 2k+1$ in the proof of Lemma 4.9, we easily see that (5.4) holds true. Hence (5.3) is also derived form (4.55). Moreover, since

$$\begin{aligned}
\|u_t\|_{H_k} &= \|u_t\|_{L^2} + \|D^{2k}u_t\|_{L^2}\\
&\leq \|D(u^\ell Du) + \varepsilon D^2u\|_{L^2} + \|D^{2k+1}(u^\ell Du) + \varepsilon D^{2k+2}u\|_{L^2}\\
&\leq (\ell+1)L^\ell_{1,\infty}L_2 + \varepsilon L_2 + \left(\int (u^\ell D^{2k+2}u)^2\,dx\right)^{1/2} + \sqrt{\varepsilon}\|\sqrt{\varepsilon}D^{2k+2}u\|_{L^2}\\
&\quad + \sum_{i=1}^{2k+1}{}_{2k+1}C_i\|D^i(u^\ell)D^{2k+2-i}u\|_{L^2},
\end{aligned}$$

it is easy to obtain (5.5).

Now we are going to derive the a priori bound of $\|Du(t)\|_{L^\infty}$. Multiplying (P) by $-D(|Du|^{r-2}Du) = -(r-1)|Du|^{r-2}D^2u$, we get, by (4.46),

$$\begin{aligned}
&\|Du\|_{L^r}^{r-1}\frac{d}{dt}\|Du\|_{L^r} + (r-1)\int u^\ell |Du|^{r-2}(D^2u)^2\,dx \\
&= -(r-1)\int \ell u^{\ell-1}(Du)^2|Du|^{r-2}D^2u\,dx \\
&= -\frac{r-1}{r+1}\int \ell u^{\ell-1}D(|Du|^r Du)\,dx \\
&= \frac{r-1}{r+1}\int \ell(\ell-1)u^{\ell-2}|Du|^{r+2}\,dx \\
&\le \ell(\ell-1)\|u_0\|_{L^\infty}^{\ell-2}\|Du\|_{L^\infty}^2\|Du\|_{L^r}^r. \qquad (5.6)
\end{aligned}$$

Hence

$$\|Du(t)\|_{L^r} \le \|Du_0\|_{L^r} + \ell(\ell-1)\|u_0\|_{L^\infty}^{\ell-2}\int_0^t \|Du(s)\|_{L^\infty}^2\|Du(s)\|_{L^r}\,ds.$$

Noting that $\|Du\|_{L^r} \le \|Du\|_{L^\infty}^{(r-2)/r}\|Du\|_{L^2}^{2/r}$ and letting r tends to ∞, we find by lemma 3.4 that

$$\|Du(t)\|_{L^\infty} \le \|Du_0\|_{L^\infty} + \ell(\ell-1)\|u_0\|_{L^\infty}^{\ell-2}\int_0^t \|Du(s)\|_{L^\infty}^3\,ds. \qquad (5.7)$$

Here we define T_0 by

$$T_0 = \frac{1}{\ell(\ell-1)\|u_0\|_{L^\infty}^{\ell-2}(\|Du_0\|_{L^\infty}+2)^3}. \qquad (5.8)$$

Then the following estimate holds

$$\|Du(t)\|_{L^\infty} \le \|Du_0\|_{L^\infty} + 2 =: K_0 \quad \text{for all } t \in [0, T_0]. \qquad (5.9)$$

Indeed, suppose that (5.9) does not hold, then there exists a number $t_1 \in [0, T_0]$ such that $\|Du(t_1)\|_{L^\infty} > K_0$. Since $\|Du_0\|_{L^\infty} < K_0$ and $\|Du(t)\|_{L^\infty}$ is a continuous function, there exists $t_0 \in (0, t_1]$ such that $\|Du(t_0)\|_{L^\infty} = K_0$ and $\|Du(t)\| < K_0$ for all $t \in [0, t_0)$. Hence by (5.7) and the definition of T_0, we obtain

$$\begin{aligned}
K_0 = \|Du(t_0)\|_{L^\infty} &\le \|Du_0\|_{L^\infty} + \ell(\ell-1)\|u_0\|_{L^\infty}^{\ell-2}K_0^3T_0 \\
&\le \|Du_0\|_{L^\infty} + 1,
\end{aligned}$$

which leads to a contradiction. Thus the a priori bound for $\|Du(t)\|_{L^\infty}$ on $[0, T_0]$ is derived. □

Proof of Theorem 5.1. Let u_ε be the global solution of $(\mathrm{P})^\varepsilon$ belonging to $\mathscr{B}_T^k$. Then, by Lemma 5.3, we know that $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^\infty(0,T_0;H^{2k+1}(\boldsymbol{R}))$ and (5.3)–(5.5) hold good with $u = u_\varepsilon$ for all $\varepsilon > 0$. Now we are going to show below that $\{u_\varepsilon\}_{\varepsilon>0}$ forms a Cauchy sequence in $C([0,T_0];H^2(\boldsymbol{R}))$. For any $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, we denote $u_1 = u_{\varepsilon_1}$, $u_2 = u_{\varepsilon_2}$ and $w = u_1 - u_2$. Then w satisfies

$$
\begin{aligned}
w_t &- \varepsilon_1 D^2 u_1 + \varepsilon_2 D^2 u_2 \\
&= \frac{1}{\ell+1} D^2(d_{\ell+1} w) \qquad (5.10)\\
&= u_1^\ell D^2 w + d_\ell D^2 u_2 w + \ell u_1^{\ell-1} D(u_1+u_2) Dw + \ell (Du_2)^2 d_{\ell-1} w, \qquad (5.11)
\end{aligned}
$$

where $d_\ell = u_1^{\ell-1} + u_1^{\ell-2}u_2 + \cdots + u_1 u_2^{\ell-2} + u_2^{\ell-1}$.

Multiplication of (5.10) by w gives

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 &\le (\varepsilon_1\|D^2u_1\|_{L^2} + \varepsilon_2\|D^2u_2\|_{L^2})\|w\|_{L^2} + \frac{1}{\ell+1}\int d_{\ell+1} w D^2 w\,dx \\
&\le (\varepsilon_1+\varepsilon_2) L_2 \|w\|_{L^2} + L_{1,\infty}^\ell \|w\|_{L^2}\|D^2w\|_{L^2}. \qquad (5.12)
\end{aligned}
$$

We differentiate (5.11) once and multiply it by $-D^3 w$, then we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|D^2w\|_{L^2}^2 \\
&\le (\varepsilon_1\|D^4u_1\|_{L^2} + \varepsilon_2\|D^4u_2\|_{L^2})\|D^2w\|_{L^2} - \int u_1^\ell (D^3w)^2\,dx - \int \ell u_1^{\ell-1} Du_1 D^2 w D^3 w\,dx \\
&\quad + \int D^2(d_\ell D^2u_2 w) D^2 w\,dx - \int \ell u_1^{\ell-1} D(u_1+u_2) D^2 w D^3 w\,dx \\
&\quad + \int D(\ell(\ell-1) u_1^{\ell-2} Du_1 D(u_1+u_2) Dw) D^2 w\,dx \\
&\quad + \int D(\ell u_1^{\ell-1} D^2(u_1+u_2) Dw) D^2 w\,dx + \int D(\ell (D^2u_2)^2 d_{\ell-1} w) D^2 w\,dx \\
&\le (\varepsilon_1+\varepsilon_2) L_4 \|D^2w\|_{L^2} - \int u_1^\ell (D^3w)^2\,dx + \frac{1}{4}\int u_1^\ell (D^3w)^2\,dx \\
&\quad + \int \ell^2 u_1^{\ell-2} (Du_1)^2 (D^2w)^2\,dx + \|D^2(d_\ell D^2u_2 w)\|_{L^2}\|D^2w\|_{L^2} + \frac{1}{4}\int u_1^\ell (D^3w)^2\,dx \\
&\quad + \int \ell^2 u_1^{\ell-2} (D(u_1+u_2))^2 (D^2w)^2\,dx \\
&\quad + \ell(\ell-1)\|D(u_1^{\ell-2} Du_1 D(u_1+u_2) Dw)\|_{L^2}\|D^2w\|_{L^2} \\
&\quad + \ell\|D(u_1^{\ell-1} D^2(u_1+u_2) Dw)\|_{L^2}\|D^2w\|_{L^2} \\
&\quad + \ell\|D((D^2u_2)^2 d_{\ell-1} w)\|_{L^2}\|D^2w\|_{L^2}.
\end{aligned}
$$

Then it is easy to see that there exists a constant C_ℓ depending only on ℓ such that

$$\frac{1}{2}\frac{d}{dt}\|D^2 w\|_{L^2}^2 \le (\varepsilon_1 + \varepsilon_2)L_4\|D^2 w\|_{L^2} + C_\ell L_4^\ell \|D^2 w\|_{L^2}^2. \tag{5.13}$$

Hence, by (5.12), (5.13) and Gronwall's inequality, we obtain

$$\|w\|_{H^2} \le 2(\varepsilon_1 + \varepsilon_2)L_4 e^{(C_\ell + 1)L_4^\ell t} \quad \forall t \in [0, T_0].$$

Thus $\{u_\varepsilon\}_{\varepsilon>0}$ forms a Cauchy sequence in $C([0, T_0]; H^2(\boldsymbol{R}))$.

Here we note that $u_\varepsilon^\ell D^2 u_\varepsilon$ and $\ell u_\varepsilon^{\ell-1}(Du_\varepsilon)^2$ are also bounded in $L^2(0, T_0; H^{2k}(\boldsymbol{R}))$ since u_ε is bounded in $L^\infty(0, T_0; H^{2k+1}(\boldsymbol{R}))$ and satisfies (5.4). Therefore, in view of (5.3)–(5.5), we find that there exists a sequence $\varepsilon_n \to 0$ such that $\{u_n\} = \{u_{\varepsilon_n}\}$ satisfies

$$\begin{aligned}
&u_n \to u && \text{strongly} && \text{in } C([0, T_0]; H^2(\boldsymbol{R})),\\
&u_n \rightharpoonup u && \text{weakly} && \text{in } L^2(0, T_0; H^{2k+1}(\boldsymbol{R})),\\
& && \text{and weakly star} && \text{in } L^\infty(0, T_0; H^{2k+1}(\boldsymbol{R})),\\
&(u_n)_t \rightharpoonup u_t && \text{weakly} && \text{in } L^2(0, T_0; H^{2k}(\boldsymbol{R})),\\
&u_n^\ell D^2 u_n \rightharpoonup g && \text{weakly} && \text{in } L^2(0, T_0; H^{2k}(\boldsymbol{R})),\\
&\ell u_n^{\ell-1}(Du_n)^2 \rightharpoonup \chi && \text{weakly} && \text{in } L^2(0, T_0; H^{2k}(\boldsymbol{R})),\\
&\varepsilon_n D^2 u_n \to 0 && \text{strongly} && \text{in } L^2(0, T_0; H^{2k}(\boldsymbol{R})).
\end{aligned}$$

On the other hand, since the convergence of u_n to u in $C([0, T_0]; H^2(\boldsymbol{R}))$ implies that u_n converges to u in $L^\infty(0, T_0; L^\infty(\boldsymbol{R}))$, it is clear that

$$\begin{aligned}
&u_n^\ell D^2 u_n \to u^\ell D^2 u && \text{strongly in } L^2(0, T_0; L^2(\boldsymbol{R})),\\
&\ell u_n^{\ell-1}(Du_n)^2 \to \ell u^{\ell-1}(Du)^2 && \text{strongly in } L^2(0, T_0; L^2(\boldsymbol{R})),
\end{aligned}$$

whence follow $g = u^\ell D^2 u$ and $\chi = \ell u^{\ell-1}(D^2 u)$. Consequently u belongs to $W^{1,2}(0, T_0; H^{2k}(\boldsymbol{R}))$, which implies that $u \in C([0, T_0]; H^{2k}(\boldsymbol{R}))$. Then u turns out to be the desired solution in Theorem 5.1. □

Now we are ready to prove our main theorem.

Proof of Theorem. Since $u_0 \in \bigcap_{m=0}^{\infty} H^m(\boldsymbol{R})$, Theorem 5.1 says that solution u belongs to $\mathscr{C}_{T_0}^k$ for all k. Therefore $u_t \in L^2(0, T_0; H^m(\boldsymbol{R}))$ for all $m \in \boldsymbol{N}$. Noting that $u_{tt} = D^2(u^\ell u_t)$, we know $u_{tt} \in L^2(0, T_0; H^m(\boldsymbol{R}))$ for all $m \in \boldsymbol{N}$, which implies $u_t \in C([0,T_0]; H^m(\boldsymbol{R}))$ for all $m \in \boldsymbol{N}$. Repeating this procedure, we easily find that $D_t^j u \in C([0,T_0]; H^m(\boldsymbol{R}))$ for all $j, m \in \boldsymbol{N}$. Then the standard argument assures that $u \in C^\infty([0,T_0] \times \boldsymbol{R})$. □

Concluding remarks.
(0) Our arguments can cover also porous medium equations with external forces. For example, for any $u_0 \in H^{2k+1}(\boldsymbol{R})$ and $f \in L^2(0,T; H^{2k}(\boldsymbol{R}))$, the assertion of Lemma 4.8 holds true also for the equation: $u_t = (u^\ell + \varepsilon)u_{xx} + \ell u^{\ell-1}(u_x)^2 + f(x,t)$; $u(x,0) = u_0(x)$. Therefore, under additional assumption

$$\text{(A.3)} \quad f \in \bigcap_{m=0}^{\infty} L^2(0,T; H^m(\boldsymbol{R})) \cap C^\infty([0,T] \times \boldsymbol{R}),$$

the non-autonomous equations: $u_t = (u^\ell u_x)_x + f(x,t)$; $u(x,0) = u_0(x)$, admit unique local C^∞-solutions.

(1) Consider the following parabolic equation governed by the leading term with the external force f:

$$(\mathrm{P})_0 \begin{cases} u_t = u^\ell u_{xx} + f(x,t), & (x,t) \in \boldsymbol{R} \times [0,\infty), \\ u(x,0) = u_0(x), & x \in \boldsymbol{R}. \end{cases}$$

Then Lemma 4.3 assures that for every $f \in L^2(0,T; H^{2k}(\boldsymbol{R}))$ and $u_0 \in H^{2k+1}(\boldsymbol{R})$, the approximate equation $(\mathrm{P})_0^\varepsilon$ of $(\mathrm{P})_0$ admits a unique solution u belonging to $\mathscr{B}_T^k$. Moreover, in parallel with (4.46) and (5.6), *i.e.*, multiplying $(\mathrm{P})_0^\varepsilon$ by $|u|^{r-2}u$ and $-D(|Du|^{r-2}Du)$, we now have

$$\begin{aligned} &\|u\|_{L^r}^{r-1} \frac{d}{dt}\|u\|_{L^r} + \varepsilon(r-1)\int |u|^{r-2}(Du)^2\,dx + (\ell + r - 1)\int |u|^{\ell+r-2}(Du)^2\,dx \\ &\quad \le \|f\|_{L^r}\|u\|_{L^r}^{r-1}, \\ &\|Du\|_{L^r}^{r-1}\frac{d}{dt}\|Du\|_{L^r} + (r-1)\int(\varepsilon + u^\ell)|Du|^{r-2}(D^2u)^2\,dx \le \|Df\|_{L^r}\|Du\|_{L^r}^{r-1}. \end{aligned}$$

Hence, we obtain the a priori estimate:

$$\begin{aligned} &\sup_{0\le t\le T}\{\|u(t)\|_{L^\infty} + \|Du(t)\|_{L^\infty}\} \\ &\qquad \le \|u_0\|_{L^\infty} + \|Du_0\|_{L^\infty} + \int_0^T (\|f\|_{L^\infty} + \|Df\|_{L^\infty})\,ds. \qquad (5.14) \end{aligned}$$

Thus, by the same arguments as in the proofs of Lemma 4.5 and Theorem 5.1, we conclude that for every $f \in L^2(0,T; H^{2k}(\boldsymbol{R}))$ and $u_0 \in H^{2k+1}(\boldsymbol{R})$, $(\mathrm{P})_0$ has a unique (global) solution u belonging to $\mathscr{C}_T^k$. Furthermore, if $f \in C^\infty([0,T] \times \boldsymbol{R}) \cap \bigcap_{m=1}^\infty L^2(0,T; H^m(\boldsymbol{R}))$, then the solution u of $(\mathrm{P})_0$ belongs to $C^\infty([0,T] \times \boldsymbol{R})$.

(2) It is also possible to treat the initial boundary value problems in our framework. For example, for homogeneous Dirichlet problem denoted by $(\mathrm{P})_{\mathrm{D}}$, and homogeneous Neumann problem denoted by $(\mathrm{P})_{\mathrm{N}}$, in some interval $I \subset \boldsymbol{R}$, the same arguments as above with obvious modifications show that $(\mathrm{P})_{\mathrm{D}}$ and $(\mathrm{P})_{\mathrm{N}}$ have the (time) local C^{∞}-solutions, provided that $u_0 \in \bigcap_{m=0}^{\infty} H^m(\boldsymbol{I})$ satisfies the following compatibility conditions $(\mathrm{C})_{\mathrm{D}}$ and $(\mathrm{C})_{\mathrm{N}}$ respectively:

$$(\mathrm{C})_{\mathrm{D}} \quad D^{2j-2}u|_{\partial I} = 0 \quad \text{for all } j \in \boldsymbol{N},$$

$$(\mathrm{C})_{\mathrm{N}} \quad D^{2j-1}u|_{\partial I} = 0 \quad \text{for all } j \in \boldsymbol{N}.$$

(3) Our framework can work also for the multi-dimension cases with some modifications which contain much more heavy calculations than those in the one-dimensional case. However, for the higher dimensional cases, the existence time T_0 depends on up to the second derivatives of the initial data, i.e., $T_0 = T_0(\|u_0\|_{W^{2,\infty}})$.

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