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Galois points on quartic surfaces

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Abstract. Let S be a smooth hypersurface in the projective three space and consider a projection of S from $P \in S$ to a plane H. This projection induces an extension of fields $k(S)/k(H)$. The point P is called a Galois point if the extension is Galois. We study structures of quartic surfaces focusing on Galois points. We will show that the number of the Galois points is zero, one, two, four or eight and the existence of some rule of distribution of the Galois points.

1. Introduction.

Let k be an algebraically closed field of characteristic zero. We fix it as the ground field of our discussion. Let S be a smooth hypersurface of degree d in the projective three space $\boldsymbol{P}^3 = \boldsymbol{P}^3(k)$, where we assume that $d \geq 4$. Let $K = k(S)$ be the rational function field of S. A subfield K_m is said to be a maximal rational subfield if it is rational, i.e., a purely transcendental extension of k, and is not contained in any other rational subfield. It seems interesting to study the structure of the extension K/K_m. If we know it, we will be able to classify of all the subfields of K. Because, by Zariski-Castelnuovo's theorem any subfield (which is not k) of K_m is rational. So that it is sufficient to study what fields exist between K and K_m. Let L be the Galois closure of K/K_m, then we need to study the structure of the Galois group $\mathrm{Gal}(L/K_m)$.

For that reason, the study we have to do first is to find when the extension is Galois (cf. [**6**]). Here the meaning "when" is a little ambiguous, it will become clear if we consider the model of K as follows. For each point $P \in S$, let $\pi_P : S \cdots \to H$ be a projection of S from P to a plane H. This rational map induces the extension of fields $K/k(H)$. We know that the degree of irrationality of S is $d-1$ or $d-2$ (cf. [**1**], [**10**]), hence $k(H)$ is a maximal rational subfield. Clearly the structure of this extension does not depend on H, but on P, so that we write K_P instead of $k(H)$. Therefore, the above question is equivalent to say for which point $P \in S$ the extension K/K_P becomes Galois. The study following the above method has been done for curves of degrees 4 and 5 (cf. [**6**], [**7**]).

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However we have to note here that not all maximal rational subfields are obtained as the projection above, i.e., there are many maximal rational subfields which cannot be obtained from the projections.

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2. Statement of results.

We use the same notation as is used in Section 1.

Definition 1. A point $P \in S$ is called a Galois point if the extension K/K_P is Galois.

Let Σ be the set of lines on S passing through P. Then $S' = S \backslash (\Sigma \cup P)$ becomes a covering of U of degree $d-1$ by $\pi'_P = \pi_P|_{S'}$, where $U = \boldsymbol{P}^2 \backslash \{$a finitely many points$\}$. Hence P is a Galois point if and only if π'_P is a Galois covering. First, we want to know the set of Galois points.

Theorem 1. *Suppose that $d \geq 4$. Then the number of Galois points is finite. If S is general in the class of surfaces with degree d, then the number is zero.*

Let $\delta = \delta(S)$ denote the number of the Galois points. Note that δ is invariant under projective transformations of S.

Here we mention a note, which will be often used later.

Note 2.1. If H_P is a general plane among the ones passing through P, then $S \cap H_P$ is a smooth curve.

Hereafter we restrict ourselves to the case where $d = 4$ and we assume that k is the field of complex numbers. We want to know the exact value δ and the place where the Galois points exist. First we find necessary conditions that a point to be a Galois one.

Let $(X : Y : Z : W)$ be homogeneous coordinates on $\boldsymbol{P}^3$. Then we have the following standard form for the equation of S if S has a Galois point.

Theorem 2. *If there exists a Galois point P on S, then S is projectively equivalent to the surface given by the equation $ZW^3 + G(X, Y, Z) = 0$, where G is a quartic form and $P = (0:0:0:1)$.*

Let T_P denote the tangent plane of S at P.

Corollary 2.2. *If P is a Galois point, then $T_P \cap S$ consists of four distinct lines.*

Let F be the homogeneous defining equation of S and $H(F)$ be the Hessian of F. Then we have another necessary condition.

Proposition 2.3. *If P is a Galois point, then $H(F)(P) = 0$.*

Let P be a point on S with $P = (0:0:0:1)$ and put $x = X/W$, $y = Y/W$, $z = Z/W$ and $f(x, y, z) = F(X, Y, Z, W)/W^4 = \sum_{i=1}^{4} f_i$, where f_i is a homogeneous part of f with degree i ($i = 1, 2, 3, 4$). Using these expressions, we have the following criterion that a point to be a Galois point.

Proposition 2.4. *Under the notation above, a point P is a Galois point if and only if $f_2^2 = 3f_1 f_3$.*

In the paper [**6**], we have studied Galois points on quartic curves. The following proposition is also useful for checking whether a point is Galois or not.

Proposition 2.5. *Suppose that H_P is a general plane passing through P and let $C = S \cap H_P$ be a quartic curve. Then a point P is a Galois point of S if and only if it is a Galois point of C.*

For a Galois point P, take three lines $\{l_i\}$ from $S \cap T_P$ and consider a divisor $D = l_1 + l_2 + l_3$. The rational map associated with the complete linear system $|D|$ gives S a structure of a fiber space, i.e., we have the following.

Lemma 2.6. *If there exists a Galois point on S, then S has a structure of an elliptic surface.*

Definition 2. We call the surface with the structure defined in Lemma 2.6 an elliptic surface associated with the Galois point.

Note that there are four possibilities for the choice of the lines $S \cap T_P$, hence there are four elliptic surfaces associated with the Galois point. Observing the singular fibers of the elliptic surface, we obtain the following.

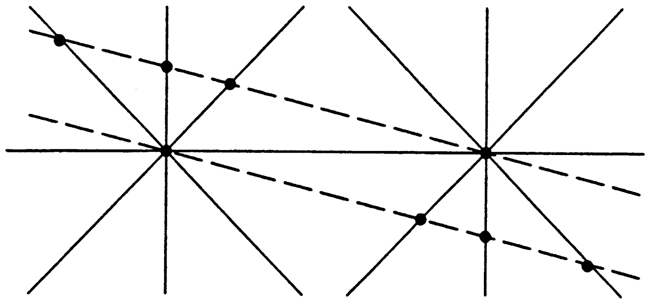
Theorem 3. *If S is a quartic surface, then $\delta(S) = 0, 1, 2, 4$ or 8. Especially, $\delta(S) = 8$ if and only if S is projectively equivalent to the surface S_8 given by the equation $XY^3 + ZW^3 + X^4 + Z^4 = 0$.*

Moreover, the distribution of the Galois points are illustrated as follows, where the dots indicate Galois points, the lines indicate the lines on S and the broken lines indicate the lines in $\boldsymbol{P}^3$ but not on S.

(1) $\delta = 2$

(2) $\delta = 4$

(3) $\delta = 8$



By Lemma 2.2 *there exist four lines on S passing through each Galois point, but we omit to illustrate here some of them.*

Note that the coordinates of the Galois points on S_8 are $(0:0:0:1)$, $(0:0:\zeta:1)$, $(0:0:\zeta^3:1)$, $(0:0:\zeta^5:1)$, $(0:1:0:0)$, $(\zeta:1:0:0)$, $(\zeta^3:1:0:0)$ and $(\zeta^5:1:0:0)$, where ζ is a primitive sixth root of unity. Furthermore there exist some rules between Galois points and lines on S, for the details, see Lemma 3.10.

Corollary 2.7. *If* $\delta(S) = 2$ *or* 8, *then S has a structure of an elliptic surface whose singular fibers are all of type IV* (*in the sense of Kodaira's notation in* [**4**]).

A quartic surface is a $K3$ surface, and it is known that the maximum number of lines lying on a quartic surface is 64 (cf. [**9**]). The surface S_8 in Theorem 3 is the most special one among quartic surfaces as we see below.

Remark 2.8. The surface S_8 has the following properties:
(a) The number of lines on S_8 is 64.
(b) The surface S_8 is a singular $K3$ surface (cf. [**2**]).

For each value of δ in Theorem 3, there are many examples taking the value as follows.

Example 2.9. (1) If S is (i) a general quartic surface, or (ii) the Fermat quartic given by the equation $X^4 + Y^4 + Z^4 + W^4 = 0$, or (iii) the surface given by the equation $X^3Y + Y^3Z + Z^3W + W^3X = 0$, then $\delta(S) = 0$.

(2) If S is the surface given by the equation (i) $ZW^3 + G(X, Y, Z) = 0$, where G is a general quartic form, or (ii) $ZW^3 + X^4 + Y^4 + YZ^3 = 0$, then $\delta(S) = 1$.

(3) Suppose that S is given by the equation $XY^3 + ZW^3 + H(X, Z) = 0$, where $H(X, Z) = \sum_{i=0}^{4} c_i X^i Z^{4-i}$, $c_0 c_4 \neq 0$. Then,

(a) if at least one of $c_i (i = 1, 2, 3)$ is not zero, then $\delta(S) = 2$,

(b) if $c_1 = c_2 = c_3 = 0$, then $\delta(S) = 8$.

(4) If S is the surface given by the equation (i) $ZW^3 + Z^4 + H(X, Y) = 0$, where H is a general quartic form, or (ii) $ZW^3 + X^4 + Y^4 + Z^4 = 0$, then $\delta(S) = 4$.

Remark 2.10. Although there are many lines on the Fermat quartic, indeed there are 48 pieces lines on it, there exists no Galois point.

3. Proofs and some other results.

First we prove Theorem 1. Let P be a Galois point and σ be an element of $\mathrm{Gal}(K/K_P)$. Then σ induces a birational transformation of S over $k(H) \cong k(\boldsymbol{P}^2)$, which turns out an isomorphism, since S is the minimal model of the field K. We claim that σ is a restriction of a projective transformation of $\boldsymbol{P}^3$. This assertion is a well known fact in the case where $d \geq 5$, so that we prove it when $d = 4$.

Let H_P be a plane passing through P. If it is general, then $C = S \cap H_P$ is a smooth quartic curve by Note 2.1. Let l_P be a line in H_P passing through P. If l_P is general, then $C \cap l_P$ consists of four distinct points $\{P, P_1, P_2, P_3\}$. By definition σ induces a permutaiton of the set $\{P_1, P_2, P_3\}$. Hence we infer that $\sigma(C) = C$, especially we have that $\sigma(P) = P$. This implies that $\sigma^*(f) \in H^0(S, \mathcal{O}_S(H_P))$ if $f \in H^0(S, \mathcal{O}_S(H_P))$. Thus σ induces an element of $\mathrm{Aut}(H^0(S, \mathcal{O}_S(H_P)))$. Since $H^0(S, \mathcal{O}_S(H_P)) \cong H^0(\boldsymbol{P}^3, \mathcal{O}(H_P))$, σ is a restriction of a projective transformation of $\boldsymbol{P}^3$. We denote it by $M(\sigma) \in PGL(4, k)$.

Definition 3. We call σ an automorphism belonging to the Galois point P and $M(\sigma)$ the representation of σ.

Let $\mathcal{L}(S)$ denote the set of automorphisms of S induced by the projective transformations which leave S invariant. Suppose that σ and σ' are automorphisms belonging to Galois points P and P' respectively. Then, it is easy to see that $\sigma \neq \sigma'$ if P and P' are distinct points, hence $M(\sigma) \neq M(\sigma')$. Thus we infer readily Theorem 1 from the following lemma (cf. [**5**]).

Lemma 3.1. *The group $\mathcal{L}(S)$ has a finite order if $d \geq 3$. If S is generic, then $\mathcal{L}(S)$ consists of only an identity element.*

Next we investigate the structure of the covering $\pi_P' : S' \to U$.

Let Ψ be the discriminant determined by the projection π_P. Let us express Ψ explicitly using a suitable affine coordinates as follows. First we take homogeneous coordinates $(X : Y : Z : W)$ on $\boldsymbol{P}^3$ satisfying the following conditions:

(1) $P = (0:0:0:1)$

(2) The plane given by $Z = 0$ is the tangent plane of S at P.

(3) The plane given by $X = 0$ is not a tangent plane at any point of S.

(4) The number of lines passing through P and touching S at $S \cap \{W = 0\}$ is finite.

(5) The line given by the equations $X = Y = 0$ does not touch S.

We use the notation in the previous sections and consider the projection π_P restricted to the affine part $W \neq 0$. Let μ be the blowing up of $\boldsymbol{A}^3 = \boldsymbol{P}^3 \backslash \{W = 0\}$ with center P. Then in an affine part, μ can be expressed as $\mu(x, s, t) = (x, sx, tx)$. Since the structure of the extension K/K_P does not depend on the choice of planes H, we may assume that $\pi_P(x, sx, tx) = (s, t)$. Thus $\tilde{\pi}_P = \pi_p \cdot \mu$ maps (x, s, t) to (s, t). The extension of fields is not changed if we take $\tilde{\pi}_P$ instead of π_P. The defining equation (of the affine part) of the proper transform of S is

$$f^{\star}(x, s, t) = \frac{f(x, sx, tx)}{x} \in k[x, s, t].$$

Let $\psi = \psi(s, t)$ be the discriminant of $f^{\star}(x, s, t)$ with respect to x. Then Ψ is obtained by homogenizing ψ and we have that $\deg \Psi = \deg \psi$ by the choice of coordinates $(1) \sim (5)$. Let $I_R(X, Y)$ denote the intersection number of X and Y at R and let $(X, Y) = \sum_R I_R(X, Y)$. We will consider the intersection numbers on $\boldsymbol{P}^2$, $\boldsymbol{P}^3$ or S, and use the same notation.

LEMMA 3.2. $\deg \Psi = d^2 - d - 2$.

PROOF. Let Γ be the divisor of Ψ on $\boldsymbol{P}^2$ and (Γ, l) be the intersection number of Γ and a line l on $\boldsymbol{P}^2$. Then we have that $\deg \psi = (\Gamma, l)$. If $H = \pi_P^{-1}(l)$, then $C = S \cap H$ is a smooth curve of degree d if l is general by Note 2.1. Using Hurwitz's theorem, we infer that the degree of the discriminant for the smooth curve C is $d^2 - d - 2$ (cf. [**6**]). $\square$

Hereafter we assume that $d = 4$.

Let P be a Galois point. Then $\mathrm{Gal}(K/K_P)$ is the cyclic group of order three and let σ be a generator of it.

LEMMA 3.3. *The subvariety* $\mathscr{F}(\sigma) = \{Q \in S \mid \sigma(Q) = Q\}$ *contains a curve.*

PROOF. Let Σ be the set of four lines $S \cap T_P$. Then we have that $\sigma(\Sigma) = \Sigma$ and $\pi' : S' \to U$ is a triple Galois covering. By Lemma 3.2 we have that $\deg \Gamma = 10$. Therefore π' is ramified along $\pi^{-1}(U \cap \Gamma)$, thus $\mathscr{F}(\sigma)$ contains a curve. $\square$

When $A = (a_{ij})$ is a diagonal matrix of size four and $a_{ii} = a_i$ $(i = 1, 2, 3, 4)$, we denote it by $(a_1 \dotplus a_2 \dotplus a_3 \dotplus a_4)$. Let $M(\sigma) \in PGL(4, k)$ be the representation of σ. Since $\sigma^3 = id$, the matrix $M(\sigma)$ is similar to $(\omega \dotplus \omega^i \dotplus \omega^j \dotplus 1)$, where ω is a primitive cubic root of 1 and $0 \le i \le j \le 2$. By taking a suitable projective change of coordinates, we may assume that $M(\sigma)$ is expressed as above. From Lemma 3.3 we infer that σ must fix a hyperplane. This implies that three eigen values of $M(\sigma)$ coincide, hence we have that $i = j = 0$ or $i = j = 1$. Consequently we may assume that $i = j = 1$. We express F as $\sum_{i=0}^{4} F_i W^{4-i}$ where $F_i \in k[X, Y, Z]$ is a homogeneous polynomial of degree $i (0 \le i \le 4)$. Since $\sigma \in \mathscr{L}(S)$, we have that $F^{\sigma} = \lambda F$ for some $\lambda \in k \backslash \{0\}$. Whence we can conclude easily that F has an expression as $F_1 W^3 + F_4$. Since $F_1 \ne 0$, this form can be transformed to the standard one by a projective transformation. Thus we complete the proof of Theorem 2.

Suppose that $P = (0 : 0 : 0 : 1)$ is a Galois point. Then the equation of S can be given by $ZW^3 + G(X, Y, Z) = 0$. The equation of the tangent plane T_P is $Z = 0$. Since S is smooth, the form $G(X, Y, 0)$ has no multiple factor, this proves Corollary 2.2.

The proof of Proposition 2.3 is easy from the following lemma.

Lemma 3.4 (§7, [**8**]). *Let $\bar{f}$ be the restriction of f to the affine tangent plane of S at P. Then the Taylor expansion of $\bar{f}$ at P starts with a nondegenerate quadratic form if and only if $H(F)(P) \ne 0$.*

Next we prove Proposition 2.4. If P is a Galois point, then making use of Theorem 2, we easily obtain $f_2^2 = 3 f_1 f_3$. Conversely we assume this relation. As we have defined above, $f^{\star}$ can be expressed as $f^{\star}(x, s, t) = f_4(1, s, t)x^3 + f_3(1, s, t)x^2 + f_2(1, s, t)x + t$. Then we have that $K = k(x, s, t)$, where $f^{\star}(x, s, t) = 0$. Since $L_P = k(x', x, s, t)$, where x' is another root of $f^{\star}(x', s, t) = 0$, we can write $L_P = k(x, s, t, u)$, where

$$u^2 = (f_3 + f_4)^2 + 4 f_1 f_4 = f_2^2 - 2 f_1 f_2 - 3 f_1^2 - 4 f_1 f_3.$$

Thus, if $f_2^2 = 3 f_1 f_3$, then u^2 becomes a complete square in $k(x, s, t)$. Hence K/K_P is a Galois extension.

The proof of Proposition 2.5 may be clear if we consider the branching divisor. The point P is a Galois point if and only if the divisor Γ is two times of some divisor. Since H_P is a general plane passing through P and the discriminant of the projection of C from P to a line is obtained by restricting Ψ, the assertion may be clear.

When P is a Galois point, let $\sum_{i=0}^{3} l_i$ be the four lines $S \cap T_P$ and put $D = l_1 + l_2 + l_3$. The complete linear system $|D|$ is obtained as follows. Consider the set $\mathscr{H} = \{H_\lambda \mid H_\lambda \supset l_0, H_\lambda \text{ is a plane}\}$. Then $S \cap H_\lambda$ can be written as a divisor $l_0 + C_\lambda$, where C_λ is a curve. Taking off the fixed part l_0

from the linear system $\{S \cap H_\lambda \mid H_\lambda \in \mathscr{H}\}$, we obtain a base points free linear system, which coincides with $|D|$. Thus we obtain the following lemma.

LEMMA 3.5. *We have that* $D^2 = 0$, $(D, l_0) = 3$, $\dim H^0(S, \mathcal{O}(D)) = 2$ *and the complete linear system* $|D|$ *has no base point.*

Consequently we obtain an elliptic surface $f = f_{|D|} : S \to \boldsymbol{P}^1$ in Lemma 2.6.

Now we proceed with the proof of Theorem 3. The elliptic fibration $f : S \to \boldsymbol{P}^1$ has a singular fiber D, which is of type IV.

LEMMA 3.6. *The automorphism* σ *preserves each fiber of* f, *i.e.,* $\sigma(F_a) = F_a$, *where* $F_a = f^{-1}(a)$ *for* $a \in \boldsymbol{P}^1$. *Especially, a smooth fiber is an elliptic curve with an automorphism of order three.*

PROOF. Since σ is determined from the projection, we have that $\sigma(l_0 + C_\lambda) = l_0 + C_\lambda$ and $\sigma(l_0) = l_0$, where $S \cap H_\lambda = C_\lambda + l_0$ as above. Hence we obtain $\sigma(C_\lambda) = C_\lambda$. □

Note that $l_0/\langle\sigma\rangle$ is the base curve $\boldsymbol{P}^1$ of the elliptic fibration. Thus $f|_{l_0} : l_0 \to \boldsymbol{P}^1$ is a triple Galois covering with two branching points which are fixed points of $\sigma|_{l_0}$, i.e., l_0 is a triple section of f. Hence we infer the following.

LEMMA 3.7. *The elliptic fibration* $f : S \to \boldsymbol{P}^1$ *has at most one singular fiber* D' *besides* D *satisfying that* D' *is of type IV and* $D' \cap l_0$ *consists of one point. Especially, if* l *is a line on* S, *then the number of Galois points on* l *is at most two.*

LEMMA 3.8. *If* P' *is another Galois point and* σ' *is an automorphism belonging to* P', *then* $\sigma(P')$ *is also a Galois point and* $\sigma\sigma'\sigma^{-1}$ *is an automorphism belonging to* $\sigma(P')$.

PROOF. Put $\sigma'' = \sigma\sigma'\sigma^{-1}$ and $\sigma(P') = P''$. Suppose that l is a line passing through P'' and $I_Q(S, l) \geq 2$ for some point $Q \in S$. Then we have that $I_Q(S, l) = I_{\sigma^{-1}(Q)}(S, l')$, where $l' = \sigma^{-1}(l)$. Since l' passes through the Galois point P', we have that $I_Q(S, l) \geq 3$, this means that P'' is a Galois point. Since σ' is an automorphism belonging to P', we have that $\sigma'(l') = l'$. Hence we have that $\sigma''(l) = l$, this implies that σ'' is an automorphism belonging to P''. □

LEMMA 3.9. *Suppose that* P *and* P' *are two Galois points and the line* l *passing through these points does not lie on* S. *Then in Lemma* 3.8 *we have that* $\sigma(P') \neq P'$, *hence there exist two more Galois points* $\sigma(P')$ *and* $\sigma^2(P')$.

PROOF. In case $I_P(l, S) \geq 2$, the line l is contained in $T_P(S)$, hence it lies on S by Lemma 2.2. Therefore we have that $I_P(l, S) = 1$. Suppose that $\sigma(P') = P'$. Then we have that $I_{P'}(l, S) = 3$. By the same reasoning as above we have that l must lie on S, which is a contradiction. □

Suppose that $\delta = \delta(S) \geq 2$ and take another Galois point P'. Then one of the following cases takes place.

(i) There exists a unique i satisfying $l_i \ni P'$ $(i = 0, 1, 2, 3)$, or

(ii) There does not exist i satisfying $l_i \ni P'$.

In the case (i) we may assume that $i = 0$ and consider the elliptic fiber space $f : S \to \boldsymbol{P}^1$ associated with the Galois point P with the singular fiber $l_1 + l_2 + l_3$. By Lemma 2.2, $S \cap T_{P'}$ can be expressed as $\sum_{i=0}^{3} l_i'$, where l_i' $(i = 0, 1, 2, 3)$ is a line on S. Since there exist just four lines on S passing through P', we may assume that $l_0 = l_0'$. Thus $D' = l_1' + l_2' + l_3'$ is a singular fiber of f, especially $D \cap D' = \emptyset$.

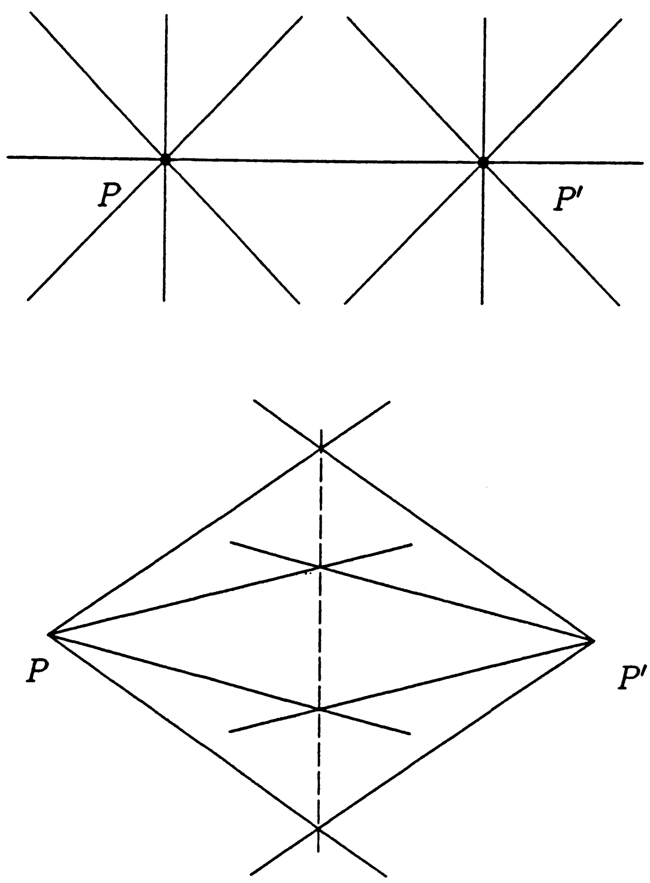
On the contrary in the case (ii), put $l = T_P \cap T_{P'}$. Since the degree of S is four, we infer that l does not lie on S, and $P \notin l$ and $P' \notin l$.

Let $\mathscr{X}$ be the set consisting of the Galois points and the lines on S passing through at least one Galois point. Combining the results obtained above, we conclude the following properties of distribution of the Galois points.

Lemma 3.10. *The set $\mathscr{X}$ has the following properties.*

(P1) *For each point of $\mathscr{X}$, there exist four lines of $\mathscr{X}$ passing through it.*

(P2) *For any two points P and P' of $\mathscr{X}$, the configuration of lines are illustrated as follows, where a line indicates a line of $\mathscr{X}$ and a broken line indicates a line in $\boldsymbol{P}^3$ but not belonging to $\mathscr{X}$.*



(P3) *For each line l of $\mathscr{X}$, there exist one or two points of $\mathscr{X}$ lying on l.*

(P4) *For each point P of $\mathscr{X}$, there exists an automorphism σ of S belonging to P, which has the following properties*:

(a) $\sigma(P) = P$.

(b) *σ has an order three.*

(c) *σ induces a permutation of elements in $\mathscr{X}$.*

(d) *If $\sum_{i=0}^{3} l_i$ is the lines passing through P, i.e., it is $T_P \cap S$, then $\sigma(l_i) = l_i$ $(0 \leq i \leq 3)$ and $\sigma|_{l_i}$ is an automorphism of l_i with an order three and fixes two points.*

(P5) *If there does not exist a line of $\mathscr{X}$ passing through two points P and P' of $\mathscr{X}$, then there exist two more points of $\mathscr{X}$. These four points are collinear, but the line passing through them is not an element of $\mathscr{X}$.*

In the case where $\delta \geq 2$, there are two lines on S not meeting each other as we have seen above (P2). Referring to Proposition 1 in [**10**], we obtain the following.

Corollary 3.11. *If $\delta(S) \geq 2$, then the degree of irrationality of S is two.*

Let us prove Theorem 3 by examining the following cases separately.

(1) For each line l of $\mathscr{X}$, there exists just one point of $\mathscr{X}$ on l.

(2) There exists a line l of $\mathscr{X}$ on which there exist two points P and P' of $\mathscr{X}$.

Take one point P of $\mathscr{X}$ and consider the associated elliptic surface $f : S \to \boldsymbol{P}^1$. Here we assume that k is the field of complex numbers. Then the topological Euler characteristic of S is 24.

In the case (1), for each point $Q(\neq P)$ of $\mathscr{X}$, if we choose a suitable a line l from the irreducible components of $S \cap T_Q$, then l meets l_0 and does not meet D by (P2). That is, l is contained in a singular fiber of f. Suppose that $\delta \geq 5$. Then four Galois points $P = P_1, P_2, P_3 = \sigma(P_2)$ and $P_4 = \sigma(P_3)$ are collinear and suppose that $Q = Q_1$ is another Galois point. Then we can find two more Galois points $Q_2 = \sigma(Q)$ and $Q_3 = \sigma(Q_2)$. Next we consider the automorphism σ_2 belonging to P_2. Then we can find new Galois points $\sigma_2(Q_1)$, $\sigma_2^2(Q_1)$, $\sigma_2(Q_2)$, $\sigma_2^2(Q_2)$, $\sigma_2(Q_3)$ and $\sigma_2^2(Q_3)$. Next we consider the automorphism belonging to the Galois point P_3. Then we can find new Galois points, etc. In this way, continuing these processes, we will be able to find more than 24 pieces of Galois points. This contradicts to the Euler characteristic of S. Thus in this case $\delta = 1$ or 4.

In the case (2), first we prove the following.

Lemma 3.12. *Suppose that there exists a line l on S satisfying that there exist two Galois points on l. Then the defining equation of S can be given by the equation $XY^3 + ZW^3 + H(X,Z) = 0$, where $H(X,Z)$ is a quartic form. The* *coordinates of Galois points are* $(0:0:0:1)$ *and* $(0:1:0:0)$ *and the equations of the line are given by* $X = Z = 0$. *Especially each singular fiber of the elliptic surface associated with the Galois points is of type IV.*

Proof. By Theorem 2 we have the standard form $ZW^3 + G(X, Y, Z)$. Since $G(X, Y, 0)$ factors into four distinct linear forms, we can transform $G(X, Y, 0)$ to $X \cdot G_3(X, Y)$, where G_3 is a cubic form. We may assume that two Galois points P and P' lie on the line given by the equations $l_0 : X = Z = 0$. Since the automorphism σ belonging to P fixes the point at infinity $W = 0$, we see that $P = (0:0:0:1)$ and $P' = (0:1:0:0)$ by Lemma 3.7. The curve $C = S \cap \{W = 0\}$ is a smooth quartic curve given by the equation $G(X, Y, Z) = 0$ on the plane $W = 0$.

Since P' lies on the plane given by $W = 0$, the point P' is also a Galois point of the quartic curve C. This assertion can be proved by similar argument of the proof of Proposition 2.5.

By the way, G can be written as $\sum_{i=1}^{4} G_i(X, Z) Y^{4-i}$, hence $G(X, Y, Z)/Y^4 = g(x, z) = \sum_{i=1}^{4} g_i(x, z)$.

It is easy to see that $g_1(x, 0) \neq 0$, hence we can transform g to the expression whose linear part is x. So that we may assume that $g_1 = x$. The similar assertion to Proposition 2.4 holds true for quartic curves, i.e., we have that $g_2^2 = 3xf_3$. This implies that g_2 and g_3 are divisible by x, from which we infer that, by taking projective transformations, G can be expressed as $XY^3 + H(X, Z)$. Since each fiber of f is obtained by cutting S by $\alpha X + \beta Z = 0$, we infer easily the last assertion. $\square$

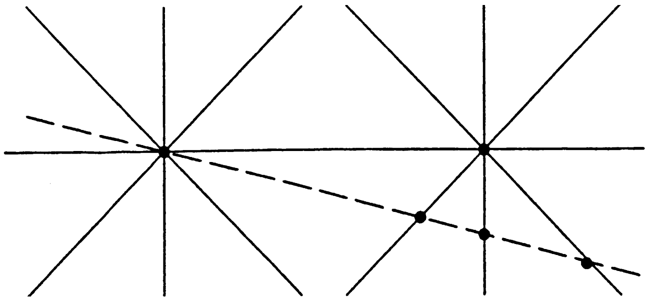
Claim 1. *There is no Galois point not lying on* $(T_P \cup T_{P'}) \cap S$.

Proof. Suppose the contrary. Then, let Q be such a point. By the property (P5) there exist three points $Q = Q_1, Q_2 = \sigma(Q_1)$ and $Q_3 = \sigma(Q_2)$, which are collinear. Corresponding to each point Q_i $(i = 1, 2, 3)$, there exists a line m_i meeting l_0 and does not meet D by (P2). By Lemma 3.12 these three lines make a singular fiber of type IV. Moreover, take an automorphism σ' belonging to P'. Considering $\sigma'(Q_i)$ and ${\sigma'}^2(Q_i)$ $(i = 1, 2, 3)$ and using the property (P3) and Lemma 3.12, we obtain a singular fiber containing $m_1 + m_2 + m_3$, which cannot appear as a fiber of any elliptic fiber space (cf. [**4**]). This is a contradiction. $\square$

Therefore we conclude that $\delta \leq 8$ in view of (P3).

Claim 2. *In the case* (2) *we have that* $\delta = 2$, 5 *or* 8.

Proof. In case $\delta \geq 3$, we use (P5) and Claim 1. As we see from the illustration below, we conclude that $\delta = 5$ or 8.



□

We now prove that the surface with $\delta = 5$ cannot exist. Let $H(X, Z)$ be expressed as $\sum_{i=0}^{4} a_i X^{4-i} Z^i$ in Lemma 3.12, where $a_0 \neq 0$. Suppose that $\delta \geq 5$. Then $R = (\xi : 1 : 0 : 0)$, where $a_0 \xi^3 + 1 = 0$, is also a Galois point by Claim 1 and R is on the plane given by $W = 0$. Putting $u = X/Y, v = Z/Y, w = W/Y$ and $h = F/Y^4$, we have that $h(u, v, w) = u + vw^3 + h_4(u, v)$. Moreover, putting $u' = u - \xi$, we have that $h'(u', v, w) = h(u' + \xi, v, w)$. Here we make use of Proposition 2.4. Then we obtain that $a_1 = a_2 = a_3 = 0$. This implies that $\delta = 8$. Combining the assertions obtained above, we see that $\delta(S) = 8$ if and only if $S = S_8$. Thus we complete the proof of Theorem 3.

Remark 3.13. Suppose that σ and σ' are the automorphisms belonging to Galois points P and P' respectively. Then, $\sigma\sigma' = \sigma'\sigma$ if and only if the line l passing through P and P' lies on S.

Proof. If l lies on S, then by Lemma 3.12 we may assume that $M(\sigma) = (\omega \dotplus \omega \dotplus \omega \dotplus 1)$ and $M(\sigma') = (\omega^i \dotplus 1 \dotplus \omega^i \dotplus \omega^i)$, where $i = 1$ or 2. Especially σ and σ' are commutative. Conversely, if $\sigma\sigma' = \sigma'\sigma$, then $M(\sigma)$ and $M(\sigma')$ can be diagonalized simultaneously. Hence in view of Lemma 3.3, σ and σ' have the same projective representation as above. By using the action of $M(\sigma)$ and $M(\sigma')$ on the defining equation of S, we obtain the same defining equation as in Lemma 3.12, this implies that l lies on S. □

Remark 3.14. Let G be the group generated by the automorphisms belonging to the Galois points on S_8. We will show in the forthcoming paper [**3**] that G has an order 288 and some other properties.

We mention the methods to check Example 2.9. By Proposition 2.3 all the Galois points exist on the curve given by the equations $F = H(F) = 0$. Proposition 2.5 may be helpful for checking some example such as the Fermat quartic. Next, we use the distribution rule of the Galois points in Lemma 3.10. Using Proposition 2.4, we will be able to find all the Galois points.

Finally we raise problems.

Problem 3.15. (1) Find the degrees of irrationality for the surfaces with $\delta = 1$.

(2) Let $\tilde{S}$ be the nonsingular projective model of the Galois closure of K/K_P, where K is the function field of a quartic surface S. Suppose that P is not a Galois point. Then is it true that the Kodaira dimension of $\tilde{S}$ is two? Moreover, find several geometric invariants of it as we have done in the case of quartic curves (cf. [**6**]).

(3) Describe the configuration of $\mathscr{X}$ for the surface S_8.

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