J. Math. Soc. Japan
Vol. 53, No. 3, 2001

On Hankel transformation, convolution operators and multipliers on Hardy type spaces*

By Jorge J. Betancor and Lourdes Rodríguez-Mesa

(Received May 6, 1999)
(Revised Feb. 28, 2000)

Abstract. In this paper we study the Hankel transformation on Hardy type spaces. We also investigate Hankel convolution operators and Hankel multipliers on these Hardy spaces.

1. Introduction and preliminaries.

The Hankel transform is defined by ([**29**])

$$h_\mu(\phi)(y) = \int_0^\infty (xy)^{-\mu} J_\mu(xy)\phi(x)x^{2\mu+1}\,dx,$$

where J_μ denotes the Bessel function of the first kind and order μ. We will assume throughout this paper that $\mu > -1/2$.

For every $1 \le p < \infty$, we consider the space L_μ^p constituted by all those Lebesgue measurable functions ϕ on $(0, \infty)$ such that

$$\|\phi\|_p = \left\{\int_0^\infty |\phi(x)|^p\,d\gamma(x)\right\}^{1/p} < \infty.$$

Here $d\gamma(x)$ denotes the measure $(x^{2\mu+1}/2^\mu\Gamma(\mu+1))\,dx$. By L_μ^∞ we understand the space $L_\infty((0,\infty), dx)$ of the essentially (respect to the Lebesgue measure on $(0,\infty)$) bounded functions on $(0,\infty)$.

It is clear that h_μ defines a continuous mapping from L_μ^1 into L_μ^∞. Herz [**18**, Theorem 3] established that h_μ can be extended to L_μ^p as a continuous mapping from L_μ^p into $L_\mu^{p'}$, for every $1 \le p \le 2$. Here p' denotes the conjugate of p (that is, $p' = p/(p-1)$).

In [**2**, Lemma 3.1] we proved by using the Marcinkiewicz interpolation Theorem the following L^p-inequality that is a Pitt type inequality for the Hankel transformation [**13**, Corollary 7.4].

2000 *Mathematics Subject Classification*. 46F12.
Key Words and Phrases. Hankel transform, Hankel multipliers, Hankel convolution operators.
*Partially supported by DGICYT Grant PB 97-1489 (Spain).

Theorem A. *Let* $1 < p \le 2$. *For every* $\phi \in L^p_\mu$ *we have*

$$\int_0^\infty x^{2(\mu+1)(p-2)} |h_\mu(\phi)(x)|^p \, d\gamma(x) \le C \int_0^\infty |\phi(x)|^p \, d\gamma(x), \tag{1}$$

where C *is a suitable positive constant depending only on* p.

Our first objective in this paper is to give a sense to the inequality (1) when $0 < p \le 1$. Note that in general (1) is not true when $p = 1$. Indeed, define

$$\phi(x) = \begin{cases} 1, & x \in (0,1) \\ 0, & \text{otherwise.} \end{cases}$$

Then according to [**12**, p. 22 (6)], $h_\mu(\phi)(y) = y^{-\mu-1} J_{\mu+1}(y)$, $y \in (0, \infty)$. Moreover there exists $K > 0$ such that

$$|z^{-\mu-1} J_{\mu+1}(z)| \ge \frac{1}{2^{\mu+2}\Gamma(\mu+2)}, \quad \text{for every } z \in (0, K).$$

Hence, we have

$$\int_0^K \frac{dx}{x} \le 2^{\mu+2}\Gamma(\mu+2) \int_0^K |h_\mu(\phi)(x)| \frac{dx}{x}. \tag{2}$$

Suppose now that (1) holds for $p = 1$ and for every $\phi \in L^1_\mu$. Then, since $\phi \in L^1_\mu$, we can write

$$\int_0^\infty |h_\mu(\phi)(x)| \frac{dx}{x} \le C \int_0^1 d\gamma(x) = \frac{C}{2^{\mu+1}\Gamma(\mu+2)}, \tag{3}$$

for a certain $C > 0$. By combining (2) and (3) it concludes that

$$\int_0^K \frac{dx}{x} \le C.$$

Thus we get a contradiction.

To study the inequality (1) when $0 < p \le 1$, inspired in celebrated and well-known results concerning to Fourier transforms ([**7**] and [**13**, Chapter III]), we need to introduce new Hardy type function spaces. The Hankel translation ([**19**]) plays an important role in the definition of our atomic Hardy spaces.

Haimo [**17**] and Hirschman [**19**] investigated a convolution operation and a translation operator associated to the Hankel transformation. If $f, g \in L^1_\mu$, the Hankel convolution $f \sharp g$ of f and g is defined by

$$(f \sharp g)(y) = \int_0^\infty f(x)(\tau_y g)(x) \, d\gamma(x), \quad y \in (0, \infty)$$

where the Hankel translation τ_y, $y \in (0,\infty)$, is given by

$$(\tau_y g)(x) = \int_0^\infty D_\mu(x,y,z) g(z)\, d\gamma(z), \quad x, y \in (0,\infty),$$

being

$$D_\mu(x,y,z) = \frac{2^{3\mu-1}\Gamma(\mu+1)^2}{\Gamma(\mu+1/2)\sqrt{\pi}} (xyz)^{-2\mu} A(x,y,z)^{2\mu-1}, \quad x,y,z \in (0,\infty),$$

and where $A(x,y,z)$ denotes the area of a triangle having sides with lengths x, y and z, when such a triangle exists, and $A(x,y,z) = 0$, otherwise.

In [**17**] and [**19**] the Hankel convolution and the Hankel translation were studied on the L^p_μ-spaces. More recently, in [**4**] and [**25**] the $\sharp$-convolution and the operator τ_y, $y \in (0,\infty)$, have been studied in spaces of generalized functions with exponential and slow growth.

We now define our atomic Hardy spaces. Firstly we introduce a class of fundamental functions that we will call atoms. Let $0 < p \leq 1$. A Lebesgue measurable function a on $(0,\infty)$ is a p-atom when a satisfies the following conditions

(i) there exists $\alpha \in (0,\infty)$ such that $a(x) = 0$, $x \geq \alpha$;
(ii) $\|a\|_2 \leq \gamma((0,\alpha))^{1/2-1/p}$, where $\alpha \in (0,\infty)$ is given in (i);
(iii) $\int_0^\alpha x^{2j} a(x)\, d\gamma(x) = 0$, for every $j = 0, 1, \dots, r$,

where $r = [(\mu+1)(1-p)/p]$. Here by $[x]$ we denote the integer part of x.

By S_e we represent the function space that consists of all those even functions ϕ belonging to the Schwartz space S. S_e is endowed with the topology induced in it by S. As usual S'_e denotes the dual space of S_e. S'_e is equipped with the weak $*$ topology.

Let $0 < p \leq 1$. Our Hardy type space $\mathscr{H}_{p,\mu}$ is constituted by all those $f \in S'_e$ that can be represented by

$$f = \sum_{j=0}^\infty \lambda_j \tau_{x_j} a_j \tag{4}$$

being $x_j \in (0,\infty)$, $\lambda_j \in \boldsymbol{C}$ and a_j is a p-atom, for every $j \in \boldsymbol{N}$, where $\sum_{j=0}^\infty |\lambda_j|^p < \infty$ and the series in (4) converges in S'_e.

We define on $\mathscr{H}_{p,\mu}$ the quasinorm $\| \, \|_{p,\mu}$ by

$$\|f\|_{p,\mu} = \inf \left(\sum_{j=0}^\infty |\lambda_j|^p \right)^{1/p},$$

where the infimum is taken over all those sequences $(\lambda_j)_{j=0}^\infty \subset \boldsymbol{C}$ such that f is given by (4) for certain $x_j \in (0,\infty)$ and p-atoms a_j, $j \in \boldsymbol{N}$.

By proceeding in a standard way (see [**14**], for instance) we can see that defining the metric $d_{p,\mu}$ on $\mathcal{H}_{p,\mu}$ by

$$d_{p,\mu}(f,g) = \|f-g\|_{p,\mu}^p, \quad f,g \in \mathcal{H}_{p,\mu},$$

$\mathcal{H}_{p,\mu}$ is a complete, metric linear space. Moreover, $\mathcal{H}_{p,\mu}$ is a quasiBanach space.

Recently, Bloom and Xu [**6**] have defined Hardy spaces on Chébli-Trimèche hypergroups. $((0,\infty),\sharp)$ is a Chébli-Trimèche hypergroup that it is usually called Bessel-Kingman hypergroup ([**28**]). Our Hardy type space is different from the one considered by Bloom and Xu [**6**]. We would like to thank to Bloom and Xu for turning our attention to their paper [**6**].

In Section 2 we study the Hankel transformation on the Hardy type space $\mathcal{H}_{p,\mu}$. In particular we establish the following extension of Theorem A to $0<p\le 1$.

Theorem 1.1. *Let $0<p\le 1$. Then there exists $C>0$ such that*

$$\int_0^\infty |h_\mu(f)(x)|^p x^{2(\mu+1)(p-2)}\,d\gamma(x) \le C\|f\|_{p,\mu}^p,$$

for every $f\in\mathcal{H}_{p,\mu}$.

Note that the inequality showed in Theorem 1.1 can be seen as a Paley type inequality for Hankel transforms [**13**, p. 55]. In [**22**] Y. Kanjin has recently obtained, for other variant of the Hankel transformation, an inequality similar to the one established in Theorem 1.1 that holds on classical Hardy spaces.

In [**5**] we investigated Hankel convolution operators on L_μ^p and weighted L_μ^p spaces. There the following result was proved.

Theorem B ([**5**, Theorem 1.1]). *Let $1<p<\infty$. Assume that k is a locally integrable function on $(0,\infty)$ and define the operator T_k by $T_kf=k\sharp f$. If the following two conditions*

(i) *there exists $C_p>0$ such that $\|T_kf\|_p\le C_p\|f\|_p$, $f\in L_\mu^p$;*

(ii) *there exist two positive constants A and B such that for every $x,y\in(0,\infty)$*

$$\int_{|x-z|>B|y-x|} |(\tau_x k)(z)-(\tau_y k)(z)|\,d\gamma(z)\le A,$$

hold, then for every $1<q<p$ there exists $C_q>0$ for which

$$\|T_kf\|_q\le C_q\|f\|_q, \quad f\in L_\mu^q,$$

and there exists $C_1>0$ being

$$\gamma(\{x \in (0,\infty) : |T_k f(x)| > \lambda\}) \le \frac{C_1}{\lambda}\|f\|_1, \quad \lambda > 0 \text{ and } f \in L^1_\mu.$$

In Section 3 we study the Hankel convolution operators on $\mathscr{H}_{p,\mu}$. If $m \in L^\infty_\mu$ then m defines a Hankel multiplier M_m through

$$M_m f = h_\mu(m h_\mu f).$$

In particular, if $m \in L^1_\mu$ and $h_\mu(m) \in L^1_\mu$, M_m coincides with the convolution operator $T_{h_\mu(m)}$ ([**19**, Theorem 2d]). Gosselin and Stempak [**15**] obtained a Hankel version of the celebrated Mihlin-Hörmander Fourier multiplier Theorem. Recently the authors [**5**, Theorems 1.2 and 1.4] and Kapelko [**21**] have extended the multiplier theorem of Gosselin and Stempak in different ways. In Section 4, inspired in the ideas included in the papers of Coifman [**8**] and Miyachi [**26**], we study Hankel multipliers in the space $\mathscr{H}_{1,\mu}$.

Throughout the paper C always will denote a suitable positive constant not necessarily the same in each occurrence.

2. The Hankel transformation of $\mathscr{H}_{p,\mu}$.

In this section we study the Hankel transformation on the Hardy type spaces $\mathscr{H}_{p,\mu}$. Here we prove, as a main result, Theorem 1.1. Our results can be seen as a Hankel version of celebrated properties concerning Fourier transforms of classical Hardy spaces ([**7**], [**9**] and [**13**]).

Firstly we establish useful estimates for the Hankel transform of p-atoms.

Lemma 2.1. *Let $0 < p \le 1$. Then, for every p-atom, we have*
(i) $|h_\mu(a)(y)| \le C y^{2(r+1)} \|a\|_2^{-A}$, $y \in (0,\infty)$,
where $A = \{2(r+1)p + 2(\mu+1)(p-1)\}/\{(\mu+1)(2-p)\}$.
(ii) $|h_\mu(a)(y)| \le C\|a\|_2^{2(p-1)/(p-2)}$, $y \in (0,\infty)$.

Proof. Let a be a p-atom. Assume that $\alpha \in (0,\infty)$ is such that $a(x) = 0$, $x \ge \alpha$ and

$$\|a\|_2 \le \gamma((0,\alpha))^{1/2-1/p}. \tag{5}$$

(i) Since $\int_0^\infty a(x)x^{2j}\,d\gamma(x) = 0$, for every $j \in \boldsymbol{N}$, $0 \le j \le r = [(\mu+1)(1-p)/p]$, we can write

$$\begin{aligned} h_\mu(a)(y) &= \int_0^\alpha (xy)^{-\mu} J_\mu(xy) a(x) x^{2\mu+1}\,dx \\ &= \int_0^\alpha \left((xy)^{-\mu} J_\mu(xy) - \sum_{j=0}^r c_{j,\mu}(xy)^{2j}\right) a(x) x^{2\mu+1}\,dx, \quad y \in (0,\infty), \end{aligned}$$

where $c_{j,\mu} = (-1)^j/\{2^{\mu+2j}\Gamma(\mu+j+1)j!\}$, $j = 0,\dots,r$.

Hence, according to [**23**, (2.2)], from (5) it follows

$$\begin{aligned}|h_\mu(a)(y)| &\le Cy^{2(r+1)}\int_0^\alpha |a(x)|x^{2(r+1)}\,d\gamma(x)\\ &\le Cy^{2(r+1)}\|a\|_2\left(\int_0^\alpha x^{4(r+1)}\,d\gamma(x)\right)^{1/2}\\ &\le Cy^{2(r+1)}\|a\|_2\alpha^{2(r+1)+\mu+1} \le Cy^{2(r+1)}\|a\|_2^{-A}, \quad y \in (0,\infty),\end{aligned}$$

being $A = \{2(r+1)p + 2(\mu+1)(p-1)\}/\{(\mu+1)(2-p)\}$.

(ii) By taking into account that the function $z^{-\mu}J_\mu(z)$ is bounded on $(0,\infty)$, we can write

$$\begin{aligned}|h_\mu(a)(y)| &\le C\int_0^\alpha |a(x)|x^{2\mu+1}\,dx \le C\|a\|_2\alpha^{\mu+1}\\ &\le C\|a\|_2^{2(p-1)/(p-2)}, \quad y \in (0,\infty).\end{aligned}$$ □

As a consequence of Lemma 2.1 we prove the following essential property.

Proposition 2.1. *Let $0 < p \le 1$. If a is a p-atom then*

$$|h_\mu(\tau_x a)(y)| \le Cy^{2(\mu+1)(1/p-1)}, \quad x, y \in (0,\infty).$$

Proof. Let a a p-atom. Assume firstly that $y^{2(r+1)}\|a\|_2^{-A} \le \|a\|_2^{2(p-1)/(p-2)}$, where $y \in (0,\infty)$ and, as in Lemma 2.1, $A = \{2(r+1)p + 2(\mu+1)(p-1)\}/\{(\mu+1)(2-p)\}$. Then, from Lemma 2.1, (i), it infers that

$$|h_\mu(a)(y)| \le Cy^{2(r+1)}\|a\|_2^{-A} \le Cy^{2(\mu+1)(1/p-1)}, \quad y \in (0,\infty).$$

On the other hand, if $y^{2(r+1)}\|a\|_2^{-A} \ge \|a\|_2^{2(p-1)/(p-2)}$ then Lemma 2.1, (ii), leads to

$$|h_\mu(a)(y)| \le C\|a\|_2^{2(p-1)/(p-2)} \le Cy^{2(\mu+1)(1/p-1)}, \quad y \in (0,\infty).$$

Thus we have proved that

$$|h_\mu(a)(y)| \le Cy^{2(\mu+1)(1/p-1)}, \quad y \in (0,\infty). \tag{6}$$

According to [**25**, (2.1)]

$$h_\mu(\tau_x a)(y) = 2^\mu \Gamma(\mu+1)(xy)^{-\mu}J_\mu(xy)h_\mu(a)(y), \quad x, y \in (0,\infty). \tag{7}$$

Note that here C is a positive constant that is not depending on $x, y \in (0,\infty)$. Thus the proof of proposition is finished. □

The Hankel transformation h_μ is an automorphism of S_e ([**1**, Satz 5] and [**11**, p. 81]). The transformation h_μ is defined on the dual space S_e' by transposition. That is, if $f \in S_e'$, $h_\mu f$ is the element of S_e' defined by

$$\langle h_\mu f, \phi\rangle = \langle f, h_\mu \phi\rangle, \quad \phi \in S_e.$$

Thus, as it is well-known, h_μ is an automorphism of S_e'. Hence, if $f \in \mathscr{H}_{p,\mu}$, with $0 < p \le 1$ and f admits the representation (4) where $x_j \in (0,\infty)$, $\lambda_j \in \boldsymbol{C}$ and a_j is a p-atom, for every $j \in \boldsymbol{N}$, and $\sum_{j=0}^\infty |\lambda_j|^p < \infty$, then, according to (7),

$$h_\mu(f)(y) = 2^\mu \Gamma(\mu+1) \sum_{j=0}^\infty \lambda_j (x_j y)^{-\mu} J_\mu(x_j y) h_\mu(a_j)(y), \quad y \in (0,\infty). \tag{8}$$

Moreover, since $\sum_{j=0}^\infty |\lambda_j| \le (\sum_{j=0}^\infty |\lambda_j|^p)^{1/p}$, from Proposition 2.1 it deduces that $h_\mu f$ is a continuous function on $(0,\infty)$ and that

$$|h_\mu(f)(y)| \le C\left(\sum_{j=0}^\infty |\lambda_j|^p\right)^{1/p} y^{2(\mu+1)(1/p-1)}, \quad y \in (0,\infty).$$

Hence we can conclude that

$$y^{-2(\mu+1)(1/p-1)} |h_\mu(f)(y)| \le C\|f\|_{p,\mu}, \quad y \in (0,\infty). \tag{9}$$

From (9) it infers the following weak type inequality for the Hankel transformation h_μ.

Proposition 2.2. *Let $0 < p \le 1$. There exists $C > 0$ such that for every $f \in \mathscr{H}_{p,\mu}$*

$$\gamma(\{y \in (0,\infty) : |h_\mu(f)(y)| y^{2(\mu+1)(1-2/p)} > \lambda\}) \le C \frac{\|f\|_{p,\mu}^p}{\lambda^p}, \quad \lambda \in (0,\infty).$$

Proof. Let $f \in \mathscr{H}_{p,\mu}$ and $\lambda \in (0,\infty)$. By (9) it follows

$$\begin{aligned}\gamma(\{y \in (0,\infty) : |h_\mu(f)(y)| y^{2(\mu+1)(1-2/p)} > \lambda\}) &\le \int_0^{(C\|f\|_{p,\mu}/\lambda)^{p/(2\mu+2)}} d\gamma(y) \\ &\le C\frac{\|f\|_{p,\mu}^p}{\lambda^p}. \qquad \square\end{aligned}$$

To establish Theorem 1.1 next lemma is fundamental.

Lemma 2.2. *Let $0 < p \le 1$. There exists $C > 0$ such that, for every p-atom,*

$$\int_0^\infty |h_\mu(a)(y)|^p y^{2(\mu+1)(p-2)} \, d\gamma(y) \le C.$$

Proof. Let a be a p-atom. Assume that $R > 0$. By virtue of Lemma 2.1, (i), since $r > \{(\mu+1)(1-p)/p\} - 1$, we can write

$$\int_0^R |h_\mu(a)(y)|^p y^{2(\mu+1)(p-2)}\,d\gamma(y) \le C\int_0^R y^{2(r+1)p+2(\mu+1)(p-2)}\,d\gamma(y)\|a\|_2^{-Ap}$$
$$\le C(R\|a\|_2^{p/[(\mu+1)(p-2)]})^{2[(r+1)p+(\mu+1)(p-1)]}. \tag{10}$$

Also, according to [**18**, Theorem 3], Hölder's inequality leads to

$$\int_R^\infty |h_\mu(a)(y)|^p y^{2(\mu+1)(p-2)}\,d\gamma(y)$$
$$\le \left\{\int_0^\infty |h_\mu(a)(y)|^2\,d\gamma(y)\right\}^{p/2}\left\{\int_R^\infty y^{-4(\mu+1)}\,d\gamma(y)\right\}^{(2-p)/2}$$
$$\le C\|a\|_2^p R^{-(\mu+1)(2-p)}. \tag{11}$$

By taking now $R = \|a\|_2^{p/[(\mu+1)(2-p)]}$, from (10) and (11) we conclude that

$$\int_0^\infty |h_\mu(a)(y)|^p y^{2(\mu+1)(p-2)}\,d\gamma(y) \le C. \qquad \square$$

Now we prove Theorem 1.1.

Proof of Theorem 1.1. Let $0 < p \le 1$ and $f \in \mathscr{H}_{p,\mu}$. Assume that f is given by (4). Then $h_\mu(f)$ admits the representation (8) for certain $x_j \in (0,\infty)$, $\lambda_j \in \boldsymbol{C}$ and a_j p-atom, for each $j \in \boldsymbol{N}$, and being $\sum_{j=0}^\infty |\lambda_j|^p < \infty$.

According to Lemma 2.2 and since the function $z^{-\mu}J_\mu(z)$ is bounded on $(0,\infty)$, we can write

$$\int_0^\infty |h_\mu(f)(y)|^p y^{2(\mu+1)(p-2)}\,d\gamma(y) \le C\sum_{j=0}^\infty |\lambda_j|^p \int_0^\infty |h_\mu(a_j)(y)|^p y^{2(\mu+1)(p-2)}\,d\gamma(y)$$
$$\le C\sum_{j=0}^\infty |\lambda_j|^p.$$

Hence,

$$\int_0^\infty |h_\mu(f)(y)|^p y^{2(\mu+1)(p-2)}\,d\gamma(y) \le C\|f\|_{p,\mu}^p.$$

Thus the proof is finished. $\square$

A Hankel version of the Hardy inequality appears when we take $p = 1$ in Theorem 1.1.

Corollary 2.1. *There exists* $C > 0$ *such that*

$$\int_0^\infty |h_\mu(f)(y)| \frac{dy}{y} \le C\|f\|_{1,\mu},$$

for every $f \in \mathscr{H}_{1,\mu}$.

Finally, from a Paley-Wiener type theorem for the Hankel transform due to Griffith [**16**], we can deduce a characterization of the distributions in $\mathscr{H}_{p,\mu}$ through Hankel transforms.

Let a be a p-atom. Assume that $\alpha \in (0, \infty)$ is such that $a(x) = 0$, $x \ge \alpha$, and $\|a\|_2 \le \gamma((0,\alpha))^{1/2-1/p}$. Then, according to [**18**, Theorem 3], it follows,

$$\|h_\mu(a)\|_2 = \|a\|_2 \le \gamma((0,\alpha))^{1/2-1/p}.$$

Moreover, by taking into account well-known properties of the Bessel functions [**31**, §5.1 (6) and (7)] we can write

$$\Delta_\mu^j h_\mu(a)(0) = 0, \quad j = 0, \dots, r,$$

where $\Delta_\mu = x^{-2\mu-1}(d/dx)x^{2\mu+1}(d/dx)$ and $r = [(\mu+1)(1-p)/p]$.

Also, by [**16**], $h_\mu(a)$ is an even and entire function such that

$$|h_\mu(a)(z)| \le Ce^{\alpha|Im\, z|}, \quad z \in \boldsymbol{C}.$$

To simplify we will say that an even and entire function A is p-normalized and of exponential type $\alpha \in (0, \infty)$ when A satisfies the following conditions.

(i) $\|A\|_2 \le \gamma((0,\alpha))^{1/2-1/p}$,

(ii) $\Delta_\mu^j A(0) = 0$, $j = 0, 1, \dots, r$, being Δ_μ and r as above, and

(iii) $|A(z)| = O(e^{\alpha|\mathrm{Im}\, z|})$, as $|z| \to \infty$.

Hence, in other words, we have proved that if a is a p-atom then $h_\mu(a)$ is p-normalized and of exponential type α, for some $\alpha \in (0, \infty)$.

Conversely, suppose that an even and entire function A is p-normalized and of exponential type $\alpha \in (0, \infty)$. Then Griffith's Theorem [**16**] implies that $h_\mu(A)(x) = 0$, $x \ge \alpha$, and that

$$\|h_\mu(A)\|_2 \le \gamma((0,\alpha))^{1/2-1/p}.$$

Moreover, $h_\mu(h_\mu(A)) = A$ and $\Delta_\mu^j A(0) = (-1)^j \int_0^\alpha x^{2j} h_\mu(A)(x)\, d\gamma(x) = 0$, $j = 0, \dots, r$.

Thus by taking into account (7) we can conclude the following characterization of the distributions in $\mathscr{H}_{p,\mu}$.

Proposition 2.3. *Let* $0 < p \leq 1$. *A distribution* $f \in S_e'$ *is in* $\mathscr{H}_{p,\mu}$ *if, and only if, there exist* $x_j \in (0,\infty)$, $\lambda_j \in \boldsymbol{C}$ *and a p-normalized and of exponential type* α_j *function* A_j, $\alpha_j \in (0,\infty)$, *for every* $j \in \boldsymbol{N}$, *such that*

$$h_\mu(f)(y) = \sum_{j=0}^{\infty} \lambda_j (x_j y)^{-\mu} J_\mu(x_j y) A_j(y), \quad y \in (0,\infty),$$

and that $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$.

3. Hankel convolution operators in the spaces $\mathscr{H}_{p,\mu}$.

In this section we study Hankel convolution operators defined by

$$T_k f = k \sharp f,$$

where k is a locally integrable function on $(0,\infty)$, on the Hardy type spaces $\mathscr{H}_{p,\mu}$.

According to [**1**] and [**11**] the topology of S_e is also generated by the family $\{\gamma_{m,n}\}_{m,n\in\boldsymbol{N}}$ of seminorms, where

$$\gamma_{m,n}(\phi) = \sup_{x\in(0,\infty)} \left| x^m \left(\frac{1}{x}\frac{d}{dx}\right)^n \phi(x) \right|, \quad \phi \in S_e, \ m,n \in \boldsymbol{N}.$$

Also, Sánchez [**27**] proved that if $\eta_{m,n}^\mu$ is the seminorm of S_e defined by

$$\eta_{m,n}^\mu(\phi) = \sup_{x\in(0,\infty)} |x^m \Delta_\mu^n \phi(x)|, \quad \phi \in S_e, \ m,n \in \boldsymbol{N},$$

where $\Delta_\mu = x^{-2\mu-1}(d/dx)x^{2\mu+1}(d/dx)$, then $\{\eta_{m,n}^\mu\}_{m,n\in\boldsymbol{N}}$ generates the topology of S_e. Hence, from [**25**, Proposition 4.2] we can deduce characterizations of the Hankel convolution operators on S_e and S_e'.

Our first result is an extension of Theorem B.

Proposition 3.1. *Let k be a locally integrable function on* $(0,\infty)$. *Assume that the following two conditions*

(i) T_k *defines a bounded linear operator from* L_μ^2 *into itself.*

(ii) *There exist two positive constants A and B such that*

$$\int_{|x-z|>B|y-x|} |(\tau_x k)(z) - (\tau_y k)(z)|\, d\gamma(z) \leq A, \quad x,y \in (0,\infty),$$

and, for a certain $c > 1$,

$$\int_{cR}^{\infty} |(\tau_x k)(z) - k(z)|\, d\gamma(z) \leq A, \quad x \in (0,R) \ \text{and} \ R \in (0,\infty),$$

hold. Then T_k defines a bounded linear mapping from $\mathscr{H}_{1,\mu}$ *into* L_μ^1.

Proof. Let a be a 1-atom. We choose $\alpha > 0$ such that $a(x) = 0$, $x \geq \alpha$, and $\|a\|_2 \leq \gamma((0,\alpha))^{-1/2}$. We can write

$$\int_0^\infty |(T_k a)(x)|\,d\gamma(x) = \left(\int_0^{c\alpha} + \int_{c\alpha}^\infty\right)|(T_k a)(x)|\,d\gamma(x) = I_1 + I_2.$$

Here $c > 1$ is the one given in (ii).

Since T_k is a bounded operator from L^2_μ into itself, Hölder's inequality leads to

$$\int_0^{c\alpha} |(T_k a)(x)|\,d\gamma(x) \leq \left\{\int_0^\infty |(T_k a)(x)|^2\,d\gamma(x)\right\}^{1/2}\left\{\int_0^{c\alpha} d\gamma(x)\right\}^{1/2}$$
$$\leq C\|a\|_2\alpha^{\mu+1} \leq C.$$

Also, by taking into account that $\int_0^\infty a(y)\,d\gamma(y) = 0$, the condition (ii) allows us to write

$$\begin{aligned}\int_{c\alpha}^\infty |(T_k a)(x)|\,d\gamma(x) &= \int_{c\alpha}^\infty \left|\int_0^\infty (\tau_x k)(y)a(y)\,d\gamma(y)\right| d\gamma(x)\\ &= \int_{c\alpha}^\infty \left|\int_0^\infty [(\tau_x k)(y) - k(x)]a(y)\,d\gamma(y)\right| d\gamma(x)\\ &\leq \int_0^\alpha |a(y)| \int_{c\alpha}^\infty |(\tau_y k)(x) - k(x)|\,d\gamma(x) d\gamma(y) \leq C\int_0^\alpha |a(y)|\,d\gamma(y)\\ &\leq C\|a\|_2\left\{\int_0^\alpha d\gamma(y)\right\}^{1/2} \leq C.\end{aligned}$$

Hence, it concludes that

$$\|T_k a\|_1 \leq C.$$

Note that the positive constant C is not depending on the 1-atom a.

Moreover, according to (7), [**19**, Theorem 2d] and [**30**, p. 16],

$$\|T_k(\tau_x a)\|_1 = \|k\sharp\tau_x a\|_1 = \|\tau_x(k\sharp a)\|_1 \leq \|k\sharp a\|_1 \leq C, \quad \text{for every } x \in (0,\infty). \tag{12}$$

Let now f be in $\mathscr{H}_{1,\mu}$. Then $f \in S'_e$ and

$$f = \sum_{j=0}^\infty \lambda_j \tau_{x_j} a_j, \tag{13}$$

where $\lambda_j \in \boldsymbol{C}$, $x_j \in (0,\infty)$ and a_j is a 1-atom, for every $j \in \boldsymbol{N}$, and $\sum_{j=0}^\infty |\lambda_j| < \infty$.

Series in (13) converges in L^1_μ. In fact, it is sufficient to note that, according to again [**30**, p. 16]

$$\|\tau_x a\|_1 \le \|a\|_1 \le 1,$$

for every $x \in (0, \infty)$ and every 1-atom a. Hence $f \in L^1_\mu$.

By virtue of Theorem B, $T_k f$ is in weak-L^1_μ and

$$T_k f = \sum_{j=0}^{\infty} \lambda_j T_k \tau_{x_j} a_j. \tag{14}$$

By (12) the series in (14) converges in L^1_μ and

$$\|T_k f\|_1 \le C \sum_{j=0}^{\infty} |\lambda_j|.$$

Hence,

$$\|T_k f\|_1 \le C\|f\|_{1,\mu},$$

and then the proof is finished. □

The following result can be established by proceeding as in the proof of Proposition 3.1.

Proposition 3.2. *Let k be a locally integrable function on $(0, \infty)$. Assume that the following three conditions are satisfied.*

(i) *T_k defines a bounded linear operator from L^2_μ into itself.*

(ii) *T_k defines a bounded linear operator from L^1_μ into S'_e.*

(iii) *There exist $A > 0$ and $c > 1$ such that*

$$\int_{cR}^{\infty} |(\tau_x k)(z) - k(z)|\, d\gamma(z) \le A, \quad x \in (0, R) \text{ and } R \in (0, \infty).$$

Then T_k is a bounded linear mapping from $\mathscr{H}_{1,\mu}$ into L^1_μ.

Proof. It is sufficient to proceed as in the proof of Proposition 3.1. Here, the condition (ii) replaces to the $(1,1)$ weak type for the operator T_k that it is used in the proof of Proposition 3.1. □

We now describe some sets of functions that define Hankel convolution operators between Hardy type spaces $\mathscr{H}_{p,\mu}$. The corresponding results for the usual convolution operator on classical Hardy spaces were established by Colzani [**10**].

Proposition 3.3. *Let $0 < p \le q \le 1$. Assume that, for every $n \in \boldsymbol{N}$, $x_n, \varepsilon_n \in (0, \infty)$, and g_n is a function that satisfies the following properties*

(i) $g_n(x) = 0,\ x \ge 2^{-n}$;

(ii) $\|g_n\|_1 \le \varepsilon_n 2^{2(\mu+1)(1/q-1/p)n}$; *and*

(iii) $\|t^{2(\mu+1)(1/p-1)} h_\mu(g_n)\|_2 \le \varepsilon_n 2^{2(\mu+1)(1/q-1/2)n}$.

Suppose also that there exists $C > 0$ such that $x_n \le C2^{-n}$, $n \in \boldsymbol{N}$, and $\sum_{n=0}^{\infty} \varepsilon_n^q < \infty$ and define $k = \sum_{n=0}^{\infty} \tau_{x_n} g_n$. Then T_k defines a bounded linear mapping from $\mathscr{H}_{p,\mu}$ into $\mathscr{H}_{q,\mu}$.

Proof. Note firstly that, according to [**30**, p. 16]

$$\|\tau_{x_n} g_n\|_1 \le \|g_n\|_1 \le \varepsilon_n, \quad n \in \boldsymbol{N}.$$

Hence the series defining k converges in L_μ^1 and $k \in L_\mu^1$.

Let a be a p-atom. By [**19**, Theorem 2b and Theorem 2d] and by (7), we can write

$$T_k a = \sum_{n=0}^{\infty} \tau_{x_n}(a \sharp g_n).$$

Let $n \in \boldsymbol{N}$.

Suppose that $a(x) = 0$, $x \ge \alpha$ and that $\|a\|_2 \le \gamma((0,\alpha))^{1/2-1/p}$, where $\alpha > 0$. Then $(\tau_{x_n}(a \sharp g_n))(x) = 0$, $x \ge \alpha + 2^{-n} + x_n$. Indeed, we have

$$(\tau_y g_n)(z) = \int_{|y-z|}^{y+z} D_\mu(y,z,u) g_n(u)\, d\gamma(u) = 0, \quad |y - z| \ge 2^{-n}.$$

Hence,

$$(a \sharp g_n)(y) = \int_0^\alpha a(z)(\tau_y g_n)(z)\, d\gamma(z) = 0, \quad y \ge \alpha + 2^{-n},$$

and then,

$$(\tau_{x_n}(a \sharp g_n))(x) = \int_{|x_n - x|}^{x_n + x} D_\mu(x_n, x, y)(a \sharp g_n)(y)\, d\gamma(y) = 0, \quad x \ge \alpha + 2^{-n} + x_n.$$

Moreover, since $\int_0^\infty x^{2j} a(x)\, d\gamma(x) = 0$, $j = 0, \dots, r$, being $r = [(\mu+1)(1-p)/p]$, we have that

$$\int_0^\infty x^{2j}(a \sharp g_n)(x)\, d\gamma(x) = 0, \quad j = 0, \dots, r.$$

Indeed, let $j = 0, \dots, r$. Fubini's Theorem leads to

$$\begin{aligned}
&\int_0^\infty x^{2j}(a \sharp g_n)(x)\, d\gamma(x) \\
&\quad = \int_0^\infty \int_0^\infty \int_0^\infty x^{2j} a(y) g_n(z) D_\mu(x,y,z)\, d\gamma(z) d\gamma(y) d\gamma(x) \\
&\quad = \int_0^\infty a(y) \int_0^\infty g_n(z) \int_0^\infty x^{2j} D_\mu(x,y,z)\, d\gamma(x) d\gamma(z) d\gamma(y). \qquad (15)
\end{aligned}$$

We now evaluate the integral

$$\int_0^\infty x^{2j} D_\mu(x,y,z)\,d\gamma(x), \quad y,z \in (0,\infty).$$

Let $y,z \in (0,\infty)$. We can write, for certain $a_{i,j} \in \boldsymbol{R}$, $i = 0,\dots,j$,

$$\begin{aligned}
&\int_0^\infty x^{2j} D_\mu(x,y,z)\,d\gamma(x)\\
&\quad = \lim_{t\to 0^+} 2^\mu \Gamma(\mu+1) \int_0^\infty x^{2j}(xt)^{-\mu} J_\mu(xt) D_\mu(x,y,z)\,d\gamma(x)\\
&\quad = \lim_{t\to 0^+} (-1)^j 2^\mu \Gamma(\mu+1) \Delta_{\mu,t}^j \int_0^\infty (xt)^{-\mu} J_\mu(xt) D_\mu(x,y,z)\,d\gamma(x)\\
&\quad = \lim_{t\to 0^+} (-1)^j 2^{2\mu} \Gamma(\mu+1)^2 \Delta_{\mu,t}^j [(yt)^{-\mu} J_\mu(yt)(zt)^{-\mu} J_\mu(zt)]\\
&\quad = (-1)^j 2^{2\mu} \Gamma(\mu+1)^2 \lim_{t\to 0^+} \sum_{i=0}^j a_{i,j} t^{2i} \left(\frac{1}{t}\frac{d}{dt}\right)^{i+j} [(yt)^{-\mu} J_\mu(yt)(zt)^{-\mu} J_\mu(zt)]\\
&\quad = (-1)^j 2^{2\mu} \Gamma(\mu+1)^2 \lim_{t\to 0^+} \sum_{i=0}^j a_{i,j} t^{2i} \sum_{l=0}^{i+j} \binom{i+j}{l} (yt)^{-\mu-l} J_{\mu+l}(yt)\\
&\qquad\qquad \times (-y^2)^l (zt)^{-\mu-(i+j-l)} J_{\mu+i+j-l}(zt)(-z^2)^{i+j-l}\\
&\quad = \Gamma(\mu+1)^2 a_{0,j} \sum_{l=0}^j \binom{j}{l} \frac{y^{2l}}{2^j \Gamma(\mu+l+1)} \frac{z^{2(j-l)}}{\Gamma(\mu+j-l+1)}.
\end{aligned}$$

Hence, by (15)

$$\begin{aligned}
&\int_0^\infty x^{2j}(a\natural g_n)(x)\,d\gamma(x) = \frac{\Gamma(\mu+1)^2 a_{0,j}}{2^j}\\
&\quad \cdot \sum_{l=0}^j \binom{j}{l} \frac{1}{\Gamma(\mu+l+1)\Gamma(\mu+j-l+1)} \int_0^\infty a(y) y^{2l}\,d\gamma(y) \int_0^\infty g_n(z) z^{2(j-l)}\,d\gamma(z) = 0.
\end{aligned}$$

By proceeding in a similar way to above we obtain

$$\begin{aligned}
&\int_0^\infty x^{2j}(\tau_{x_n}(a\natural g_n))(x)\,d\gamma(x) = \frac{\Gamma(\mu+1)^2 a_{0,j}}{2^j}\\
&\qquad \cdot \sum_{l=0}^j \binom{j}{l} \frac{x_n^{2(j-l)}}{\Gamma(\mu+l+1)\Gamma(\mu+j-l+1)} \int_0^\infty y^{2l}(a\natural g_n)(y)\,d\gamma(y) = 0.
\end{aligned}$$

We conclude that, for some $\beta_n > 0$, $\tau_{x_n}(a\sharp g_n)/\beta_n$ is a q-atom. We shall now determinate β_n.

Firstly let us consider that $\alpha \geq 2^{-n}$. According to [**19**, Theorem 2b], it follows

$$\begin{aligned}\|a\sharp g_n\|_2 &\leq \|a\|_2\|g_n\|_1 \leq \gamma((0,\alpha))^{1/2-1/p}\varepsilon_n 2^{-2n(\mu+1)(1/p-1/q)}\\ &\leq C\varepsilon_n\gamma((0,\alpha+2^{-n}))^{1/2-1/q}.\end{aligned}$$

Here C is not depending on n or a.

Assume now that $\alpha < 2^{-n}$. By taking into account that $\int_0^\infty y^{2j}a(y)\,d\gamma(y) = 0$, $j = 0,\dots,r$, being $r = [(\mu+1)(1-p)/p]$, we have

$$a\sharp g_n(x) = \int_0^\infty a(y)\left[(\tau_x g_n)(y) - \sum_{l=0}^{r}\frac{\Gamma(\mu+1)(\Delta_\mu^l g_n)(x)y^{2l}}{2^{2l}l!\Gamma(l+\mu+1)}\right]d\gamma(y), \quad x \in (0,\infty).$$

Hence, since h_μ is an isometry on L_μ^2 and by taking into account (7), it infers

$$\begin{aligned}\|a\sharp g_n\|_2 &\leq \int_0^\infty |a(y)|\left\|\tau_y g_n - \sum_{l=0}^{r}\frac{y^{2l}\Gamma(\mu+1)}{2^\mu l!\Gamma(\mu+l+1)}\Delta_\mu^l g_n\right\|_2 d\gamma(y)\\ &= \int_0^\infty |a(y)|\left\|\left(2^\mu\Gamma(\mu+1)(xy)^{-\mu}J_\mu(xy) - \sum_{l=0}^{r}\frac{(-1)^l\Gamma(\mu+1)(xy)^{2l}}{2^{2l}l!\Gamma(\mu+l+1)}\right)h_\mu(g_n)\right\|_2 d\gamma(y).\end{aligned}$$

Moreover, by [**23**, (2.2)] it follows

$$\begin{aligned}\|a\sharp g_n\|_2 &\leq C\int_0^\infty |a(y)|y^{2(\mu+1)(1/p-1)}\|x^{2(\mu+1)(1/p-1)}h_\mu(g_n)(x)\|_2\,d\gamma(y)\\ &\leq C\int_0^\alpha |a(y)|y^{2(\mu+1)(1/p-1)}\,d\gamma(y)\varepsilon_n 2^{2(\mu+1)(1/q-1/2)n}\\ &\leq C\|a\|_2\left\{\int_0^\alpha y^{4(\mu+1)(1/p-1)}\,d\gamma(y)\right\}^{1/2}\varepsilon_n 2^{2(\mu+1)(1/q-1/2)n}\\ &\leq C\alpha^{2(\mu+1)(1/2-1/p)}\alpha^{2(\mu+1)(1/p-1)+(\mu+1)}\varepsilon_n 2^{2(\mu+1)(1/q-1/2)n}\\ &= C\varepsilon_n 2^{2(\mu+1)(1/q-1/2)n} \leq C\varepsilon_n\gamma((0,\alpha+2^{-n}))^{1/2-1/q},\end{aligned}$$

where again C is not depending on n or a.

Now, since there exists $C > 0$ such that $x_n \leq C2^{-n}$, for every $n \in \boldsymbol{N}$, by [**30**, p. 16] it has

$$\|\tau_{x_n}(a\sharp g_n)\|_2 \leq \|a\sharp g_n\|_2 \leq C\varepsilon_n\gamma((0,\alpha+2^{-n}+x_n))^{1/2-1/q}.$$

Then $\beta_n = C\varepsilon_n$, where C does not depend on n or a.

Thus we conclude that $T_k a \in \mathscr{H}_{\mu,q}$ and $\|T_k a\|_{q,\mu} \leq C\{\sum_{n=0}^\infty \varepsilon_n^q\}^{1/q}$.

Let now $f \in \mathscr{H}_{\mu,p}$, being

$$f = \sum_{j=0}^{\infty} \lambda_j \tau_{y_j} a_j,$$

where $y_j \in (0,\infty)$, $\lambda_j \in \boldsymbol{C}$ and a_j is a p-atom, for every $j \in \boldsymbol{N}$, and such that $\sum_{j=0}^{\infty} |\lambda_j|^p < \infty$. Since the last series converges in L_μ^1 and $k \in L_\mu^1$, by taking into account [**19**, Theorem 2b]

$$T_k f = \sum_{j=0}^{\infty} \lambda_j \tau_{y_j} T_k a_j.$$

Then we obtain that

$$\|T_k f\|_{q,\mu} \le C \left(\sum_{n=0}^{\infty} |\varepsilon_n|^q \right)^{1/q} \|f\|_{p,\mu},$$

and the proof is completed. □

4. Hankel multipliers on Hardy type spaces $\mathscr{H}_{1,\mu}$.

In this section we study Hankel multipliers on Hardy type spaces $\mathscr{H}_{1,\mu}$. Let m be a measurable bounded function on $(0,\infty)$. According to [**18**, Theorem 3] the operator M_m defined by

$$M_m f = h_\mu(m h_\mu(f))$$

is linear and bounded from L_μ^2 into itself. In [**5**], [**15**] and [**21**] Hankel versions of Mihlin-Hörmander multiplier theorem have been obtained. Here we establish a Mihlin-Hörmander theorem for Hankel multipliers in a certain subspace of $\mathscr{H}_{1,\mu}$. Note firstly that, according to (9), if $f \in \mathscr{H}_{p,\mu}$, $0 < p \le 1$, then $M_m f$ is in S_e' and it is defined by

$$\langle M_m f, \phi \rangle = \int_0^\infty m(y) h_\mu(f)(y) h_\mu(\phi)(y)\, d\gamma(y), \quad \phi \in S_e.$$

Moreover, we have,

$$|\langle M_m f, \phi \rangle| \le C \|f\|_{p,\mu} \int_0^\infty y^{2(\mu+1)(1/p-1)} |h_\mu(\phi)(y)|\, d\gamma(y), \quad \phi \in S_e.$$

Hence M_m is a bounded operator from $\mathscr{H}_{p,\mu}$ into S_e'.

To establish our Hankel multiplier theorem that it is inspired in the results about Fourier multipliers due to Miyachi [**26**], we need to introduce a subspace of $\mathscr{H}_{1,\mu}$.

We say that a measurable function a on $(0,\infty)$ is a $(1,\infty)$-atom when a is a 1-atom and $\|a\|_\infty \le \gamma((0,\alpha))^{-1}$, where $\alpha \in (0,\infty)$ is such that $\phi(x)=0,\ x\ge\alpha$. Note that if $\|a\|_\infty \le \gamma((0,\alpha))^{-1}$ and $\phi(x)=0,\ x\ge\alpha$, where $\alpha\in(0,\infty)$, then

$$\|a\|_2 \le \|a\|_\infty \gamma((0,\alpha))^{1/2} \le \gamma((0,\alpha))^{-1/2}.$$

The space $\mathscr{H}_{1,\mu}^\infty$ consists of all those $f\in L_\mu^1$ being

$$f=\sum_{j=0}^\infty \lambda_j \tau_{x_j} a_j, \tag{16}$$

where the series converges in S_e' and $\lambda_j\in\boldsymbol{C}$, $x_j\in(0,\infty)$ and a_j is a $(1,\infty)$-atom, for every $j\in\boldsymbol{N}$, and being $\sum_{j=0}^\infty |\lambda_j|<\infty$. Note that the series in (16) also converges in L_μ^1.

We define on $\mathscr{H}_{1,\mu}^\infty$ the topology induced by the quasinorm $\|\ \|_{1,\mu}^\infty$ defined by

$$\|f\|_{1,\mu}^\infty = \inf\left\{\sum_{j=0}^\infty |\lambda_j|\right\},\quad f\in\mathscr{H}_{1,\mu}^\infty,$$

where the infimum is taken over all those absolutely convergent complex sequences $(\lambda_j)_{j=1}^\infty$ for which the representation (16) holds for some $x_j\in(0,\infty)$ and $(1,\infty)$-atoms $a_j,\ j\in\boldsymbol{N}$.

It is not hard to see that $\mathscr{H}_{1,\mu}^\infty$ is contained in $\mathscr{H}_{1,\mu}$, and the topology of $\mathscr{H}_{1,\mu}^\infty$ is weaker than the one induced in it by $\mathscr{H}_{1,\mu}$.

We now establish our Hankel version of Mihlin-Hörmander theorem on Hardy type spaces.

Theorem 4.1. *Assume that $a\ge0$, $b\ge0$, $k\in\boldsymbol{N}$, $k>(\mu+1)/2$ and $0<b-a(2k+\mu+1)<2$. Suppose also that $m\in C^k(0,\infty)$ is a bounded measurable function on $(0,\infty)$ such that*

$$\left|\left(\frac{1}{y}\frac{d}{dy}\right)^l m(y)\right| \le y^{-b}(Ay^{a-1})^{2l},\quad 0\le l\le k, \tag{17}$$

where $A\ge1$ and $m(x)=0$, $0<x<\delta$, for certain $\delta>0$. Then the Hankel multiplier M_m defines a bounded operator from $\mathscr{H}_{1,\mu}^\infty$ into L_μ^1.

Proof. To see that M_m defines a bounded operator from $\mathscr{H}_{1,\mu}^\infty$ into L_μ^1 it is sufficient to prove that there exists $C>0$ such that

$$\|M_m a\|_1 \le C \tag{18}$$

for every $(1,\infty)$-atom.

Indeed, let $f \in L^2_\mu \cap \mathscr{H}^\infty_{1,\mu}$. Assume that $f = \sum_{j=1}^\infty \lambda_j \tau_{x_j} a_j$, in S'_e, where $\lambda_j \in \boldsymbol{C}$, $x_j \in (0,\infty)$ and a_j is an $(1,\infty)$-atom, for every $j \in \boldsymbol{N}$, and being $\sum_{j=0}^\infty |\lambda_j| < \infty$. Then

$$M_m f = h_\mu(m h_\mu(f)) = \sum_{j=0}^\infty \lambda_j M_m(\tau_{x_j} a_j)$$

is in S'_e. Moreover, the last series converges in L^1_μ. Indeed, since M_m commutes with Hankel translations, from (18) it deduces

$$\sum_{j=n}^{l} |\lambda_j| \|M_m(\tau_{x_j} a_j)\|_1 \leq C \sum_{j=n}^{l} |\lambda_j|, \quad n, l \in \boldsymbol{N}, n > l.$$

Hence, since L^1_μ-convergence implies S'_e-convergence, we have

$$h_\mu(m h_\mu(f))(x) = \sum_{j=0}^\infty \lambda_j M_m(\tau_{x_j}(a_j))(x), \quad \text{a.e. } x \in (0,\infty)$$

and

$$\|M_m f\|_1 \leq C \sum_{j=0}^\infty |\lambda_j|.$$

Thus we conclude that

$$\|M_m f\|_1 \leq C \|f\|^\infty_{1,\mu}.$$

Since $L^2_\mu \cap \mathscr{H}^\infty_{1,\mu}$ is a dense subspace of $\mathscr{H}^\infty_{1,\mu}$, M_m can be extended to $\mathscr{H}^\infty_{1,\mu}$ as a bounded operator from $\mathscr{H}^\infty_{1,\mu}$ into L^1_μ.

We now prove (18). Suppose that $m(x) = 0$, $x \in (0,1)$. Otherwise we can proceed in a similar way. Let a be a $(1,\infty)$-atom and assume that $a(x) = 0$, $x \geq \alpha$, and $\|a\|_\infty \leq \gamma((0,\alpha))^{-1}$. Since $\|a\|_2 \leq \gamma((0,\alpha))^{-1/2}$ and M_m is bounded from L^2_μ into itself, Hölder's inequality leads to

$$\int_0^{2\alpha} |M_m a(x)|\, d\gamma(x) \leq C \left\{ \int_0^{2\alpha} |M_m a(x)|^2\, d\gamma(x) \right\}^{1/2} \alpha^{\mu+1} \leq C. \tag{19}$$

We choose a function $\phi \in C^\infty(0,\infty)$ such that $\phi(x) = 0$, $x \notin (1/2, 2)$ and $\sum_{j=-\infty}^\infty \phi(x/2^j) = 1$, $x \in (0,\infty)$ (see [**20**]). Since $m(x) = 0$, $x \in (0,1)$, we can write

$$m(x) = \sum_{j=0}^\infty m_j(x), \quad x \in (0,\infty),$$

where $m_j(x) = m(x)\phi(x/2^j)$, $x \in (0,\infty)$ and $j \in \boldsymbol{N}$.

To simplify in the sequel we write M_j instead of M_{m_j}, $j \in \boldsymbol{N}$.

Let $j \in \boldsymbol{N}$. Since $m_j \in L^2_\mu$, we have that ([**3**, Lemma 2.1])

$$M_j a = k_j \sharp a$$

where $k_j = h_\mu(m_j)$.

It is not hard to see that

$$\begin{aligned} |M_j a(x)| &\leq \int_0^\alpha |(\tau_x k_j)(y)|\,|a(y)|\,d\gamma(y) \leq \|a\|_\infty \int_0^\alpha |(\tau_x k_j)(y)|\,d\gamma(y) \\ &\leq C\alpha^{-2(\mu+1)} \int_0^\alpha |(\tau_x k_j)(y)|\,d\gamma(y), \quad x \in (0,\infty). \end{aligned} \tag{20}$$

On the other hand, since $\int_0^\alpha a(x)\,d\gamma(x) = 0$, according to [**24**, p. 256], it has

$$M_j a(x) = \int_0^\alpha a(y)(R_1(y)k_j)(x)\,d\gamma(y), \quad x \in (0,\infty), \tag{21}$$

where for a measurable function f on $(0,\infty)$,

$$(R_1(y)f)(x) = \int_0^y \theta(y,\sigma)\tau_\sigma(\varDelta_\mu f)(x)\sigma^{2\mu+1}\,d\sigma$$

being

$$\theta(y,\sigma) = \begin{cases} \dfrac{y^{-2\mu} - \sigma^{-2\mu}}{2\mu}, & 0 < \sigma < y \\ 0, & \text{otherwise.} \end{cases}$$

For every $l, s \in \boldsymbol{N}$, $0 \leq l \leq k$, by (17), Leibniz's rule leads to

$$\begin{aligned} \left|\left(\frac{1}{x}\frac{d}{dx}\right)^l [x^{2s} m_j(x)]\right| &\leq C \sum_{i=0}^l 2^{-2j(l-i)+2sj} \sup_{2^{j-1} \leq x \leq 2^{j+1}} \left|\left(\frac{1}{x}\frac{d}{dx}\right)^i m_j(x)\right| \\ &\leq C 2^{j(2s-b)}[A^2 2^{2j(a-1)}]^l, \quad x \in (0,\infty). \end{aligned}$$

Hence we obtain

$$\begin{aligned} &\left\|(A^{-2}2^{2j(1-a)})^l \left(\frac{1}{x}\frac{d}{dx}\right)^l [x^{2s} m_j(x)]\right\|_2 \\ &\quad \leq C 2^{j(2s-b)}(\gamma(2^{j-1}, 2^{j+1}))^{1/2} \leq C 2^{j(2s-b+\mu+1)}, \quad l, s \in \boldsymbol{N}, 0 \leq l \leq k. \end{aligned}$$

By taking into account now that $\varDelta_\mu^i = \sum_{h=0}^i c_{h,i} x^{2h}((1/x)(d/dx))^{i+h}$, where $c_{h,i}$ is a suitable positive constant for every $h = 0, \ldots, i$ and $i \in \boldsymbol{N}$, a straightforward manipulation allows us to conclude

$$
\begin{aligned}
&\|(1+A^{-2}2^{2j(1-a)}x^2)^l \Delta_\mu^s k_j\|_2 \\
&\quad \le C\sum_{i=0}^{l}\sum_{h=0}^{i} A^{-2i}2^{2j(1-a)i}\left\| x^{2h}\left(\frac{1}{x}\frac{d}{dx}\right)^{i+h}[x^{2s}m_j(x)]\right\|_2 \\
&\quad \le C\sum_{i=0}^{l}\sum_{h=0}^{i} A^{-2i}2^{2j(1-a)i+2jh}\left\| \left(\frac{1}{x}\frac{d}{dx}\right)^{i+h}[x^{2s}m_j(x)]\right\|_2 \\
&\quad \le CA^{2l}2^{j(2s-b+\mu+1)}2^{2jal}, \quad l,s\in \boldsymbol{N}, 0\le l\le k. \qquad (22)
\end{aligned}
$$

By invoking Hölder's and Minkowski's inequalities, [**30**, p. 16] and (22) it follows

$$
\begin{aligned}
&\int_{2\alpha}^{\infty}\int_0^{\alpha} |(\tau_x k_j)(y)|\,d\gamma(y)d\gamma(x) \\
&\le \left\{\int_{2\alpha}^{\infty}\left(\int_0^{\alpha}|(\tau_x k_j)(y)|\,d\gamma(y)(1+A^{-2}2^{2j(1-a)}x^2)^k\right)^2 d\gamma(x)\right\}^{1/2} \\
&\quad \cdot\left\{\int_0^{\infty}(1+A^{-2}2^{2j(1-a)}x^2)^{-2k}\,d\gamma(x)\right\}^{1/2} \\
&\le C(A^{-1}2^{j(1-a)})^{-(\mu+1)} \\
&\quad \cdot\left\{\int_{2\alpha}^{\infty}\left(\int_0^{\alpha}\int_{|x-y|}^{x+y} D_\mu(x,y,z)|k_j(z)|\,d\gamma(z)d\gamma(y)(1+A^{-2}2^{2j(1-a)}x^2)^k\right)^2 d\gamma(x)\right\}^{1/2} \\
&\le C(A^{-1}2^{j(1-a)})^{-(\mu+1)} \\
&\quad \cdot\left\{\int_{2\alpha}^{\infty}\left(\int_0^{\alpha}\int_{|x-y|}^{x+y}(1+A^{-2}2^{2j(1-a)}z^2)^k|k_j(z)|D_\mu(x,y,z)\,d\gamma(z)d\gamma(y)\right)^2 d\gamma(x)\right\}^{1/2} \\
&= C(A^{-1}2^{j(1-a)})^{-(\mu+1)} \\
&\quad \cdot\left\{\int_{2\alpha}^{\infty}\left(\int_0^{\alpha}[\tau_x(1+A^{-2}2^{2j(1-a)}z^2)^k|k_j(z)|](y)\,d\gamma(y)\right)^2 d\gamma(x)\right\}^{1/2} \\
&\le C(A^{-1}2^{j(1-a)})^{-(\mu+1)}\int_0^{\alpha}\|\tau_y[(1+A^{-2}2^{2j(1-a)}z^2)^k|k_j|]\|_2\,d\gamma(y) \\
&\le C(A^{-1}2^{j(1-a)})^{-(\mu+1)}\alpha^{2(\mu+1)}\|(1+A^{-2}2^{2j(1-a)}z^2)^k k_j\|_2 \\
&\le C(A^{-1}2^{j(1-a)})^{-(\mu+1)}A^{2k}2^{j(\mu+1-b)}2^{2jak}\alpha^{2(\mu+1)} \\
&\le CA^{2k+\mu+1}2^{j(a(2k+\mu+1)-b)}\alpha^{2(\mu+1)}. \qquad (23)
\end{aligned}
$$

By proceeding in a way similar to above (see [**24**, p. 256]), it has

$$\int_{2\alpha}^{\infty}\int_0^{\alpha}|(R_1(y)k_j)(x)|\,d\gamma(y)d\gamma(x)$$

$$\leq C(A^{-1}2^{j(1-a)})^{-(\mu+1)}\int_0^{\alpha}\int_0^{y}\|\tau_\sigma[(1+A^{-2}2^{2j(1-a)}z^2)^k|\Delta_\mu k_j|]\|_2\theta(y,\sigma)\,d\gamma(\sigma)d\gamma(y)$$

$$\leq C(A^{-1}2^{j(1-a)})^{-(\mu+1)}A^{2k}2^{j(2-b+\mu+1)}2^{2jak}\int_0^{\alpha}\int_0^{y}\theta(y,\sigma)\,d\gamma(\sigma)d\gamma(y).$$

Since $\int_0^y \theta(y,\sigma)\,d\gamma(\sigma) \leq Cy^2$, $y \in (0,\infty)$, we conclude that

$$\int_{2\alpha}^{\infty}\int_0^{\alpha}|(R_1(y)k_j)(x)|\,d\gamma(y)d\gamma(x)$$
$$\leq CA^{2k+\mu+1}2^{j(a(2k+\mu+1)-b+2)}\alpha^{2(\mu+2)}. \tag{24}$$

By combining (20), (21), (23) and (24) it obtains

$$\int_{2\alpha}^{\infty}|M_j a(x)|\,d\gamma(x) \leq CA^{2k+\mu+1}2^{j(a(2k+\mu+1)-b)},$$

and

$$\int_{2\alpha}^{\infty}|M_j a(x)|\,d\gamma(x) \leq C\alpha^2 A^{2k+\mu+1}2^{j(a(2k+\mu+1)-b+2)}.$$

Now, we choose $j_0 \in \boldsymbol{N}$ such that $2^{j_0}\alpha \leq 1 < 2^{j_0+1}\alpha$, provided that $\alpha \leq 1$, and we take $j_0 = -1$, when $\alpha > 1$. Since $\sum_{j=0}^{n} M_j a$ converges to $M_m a$, as $n \to \infty$, in L_μ^2, we can write

$$\int_{2\alpha}^{\infty}|M_m a(x)|\,d\gamma(x)$$

$$\leq \sum_{j=0}^{\infty}\int_{2\alpha}^{\infty}|M_j a(x)|\,d\gamma(x)$$

$$\leq C\left(\sum_{j=0}^{j_0}\alpha^2 A^{2k+\mu+1}2^{j(a(2k+\mu+1)-b+2)} + \sum_{j=j_0+1}^{\infty}A^{2k+\mu+1}2^{j(a(2k+\mu+1)-b)}\right)$$

$$\leq C(\alpha^2 A^{2k+\mu+1}2^{j_0(a(2k+\mu+1)-b+2)} + A^{2k+\mu+1}2^{j_0(a(2k+\mu+1)-b)})$$

because $0 < b - a(2k+\mu+1) < 2$. Then, since $b > a(2k+\mu+1)$, it obtains

$$\int_{2\alpha}^{\infty} |M_m a(x)|\, d\gamma(x) \leq CA^{2k+\mu+1}. \tag{25}$$

By combining (19) and (25) we obtain (18). Thus the proof is finished. □

References

[1] G. Altenburg, Bessel-Transformationen in Räumen von Grundfunktionen über dem Intervall $\Omega = (0, \infty)$ und deren Dualräumen, Math. Nachr., **108** (1982), 197–218.

[2] J. J. Betancor and L. Rodríguez-Mesa, Lipschitz-Hankel spaces and partial Hankel integrals, Integral Transforms and Special Functions, **7**(1–2) (1998), 1–12.

[3] J. J. Betancor and L. Rodríguez-Mesa, Lipschitz-Hankel spaces, partial Hankel integrals and Bochner-Riesz means, Arch. Math., **71** (1998), 115–122.

[4] J. J. Betancor and L. Rodríguez-Mesa, Hankel convolution on distributions spaces with exponential growth, Studia Mathematica, **121**(1) (1996), 35–52.

[5] J. J. Betancor and L. Rodríguez-Mesa, Weighted inequalities for Hankel convolution operators, to appear in Illinois J. Maths., **44** (2) (2000), 230–245.

[6] W. Bloom and X. Xu, Hardy spaces on Chébli-Trimèche hypergroups, Methods of Functional Analysis and Topology, **3**(2) (1997), 1–26.

[7] R. R. Coifman, A real variable characterization of H^p, Studia Mathematica, **51** (1974), 269–274.

[8] R. R. Coifman, Characterization of Fourier transforms of Hardy spaces, Proc. Nat. Acad. Sci. USA, **71**(10) (1974), 4133–4134.

[9] R. R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, Bull. Amer. math. Soc., **83**(4) (1977), 569–645.

[10] L. Colzani, Convolution operators on H^p spaces, Inst. Lombardo Rend. Sc., A **116** (1982), 149–156.

[11] S. J. L. van Eijndhoven and J. de Graaf, Some results on Hankel invariant distribution spaces, Proc. Kon. Ned. Akad. van Wetensch., A **86**(1) (1983), 77–87.

[12] A. Erdélyi et al. Tables of integral transforms, Vol. II, McGraw-Hill, New York, 1954.

[13] J. García-Cuerva and J. L. Rubio de Francia, Weighted norm inequalities and related topics, North Holland, 1985.

[14] P. J. Gloor, Oscillatory singular integral operators on Hardy spaces, Ph. D. Thesis, Department of Mathematics, Indiana University, 1996.

[15] J. Gosselin and K. Stempak, A weak-type estimate for Fourier-Bessel multipliers, Proc. Amer. Math. Soc., **106**(3) (1989), 655–662.

[16] J. L. Griffith, Hankel transforms of functions zero outside a finite interval, J. Proc. Roy. Soc. New S. Wales, **86** (1995), 109–115.

[17] D. T. Haimo, Integral equations associated with Hankel convolutions, Trans. Amer. Math. Soc., **116** (1965), 330–375.

[18] C. S. Herz, On the mean inversion of Fourier and Hankel transforms, Proc. Nat. Acad. Sci. USA, **40** (1954), 996–999.

[19] I. I. Hirschman, Jr., Variation diminishing Hankel transforms, J. Analyse Math., **8** (1960/61), 307–336.

[20] L. Hörmander, Estimates for translation invariant oprators in L^p spaces, Acta Math., **104** (1960), 93–140.

[21] R. Kapelko, A multiplier theorem for the Hankel transform, Revista Matemática Complutense, **11**(2) (1998), 1–8.

[22] Y. Kanjin, On Hardy-type inequalities and Hankel transforms, Mh. Math., **127** (1999), 311–319.

[23] H. Kober, Hankelsche Transformationen, Quart. J. Math., Oxford Ser. **8** (1937), 186–199.

[24] J. Löfström and J. Peetre, Approximation theorems connected with generalized translations, Math. Ann., **181** (1969), 255–268.
[25] I. Marrero and J. J. Betancor, Hankel convolution of generalized functions, Rendiconti di Matematica, **15** (1995), 351–380.
[26] A. Miyachi, On some Fourier multipliers for $H^p(\boldsymbol{R}^n)$, J. Fac. Sci., Univ. Tokyo, Sect. I A **27** (1980), 157–179.
[27] A. M. Sánchez, La transformación integral generalizada de Hankel-Schwartz, Ph. D. Thesis, Departamento de Análisis Matemático, Universidad de La Laguna, 1987.
[28] A. L. Schwartz, The structure of the algebra of Hankel transforms and the algebra of Hankel-Stieltjes transforms, Canad. J. Math., **XXIII** (2) (1971), 236–246.
[29] A. L. Schwartz, An inversion theorem for Hankel transform, Proc. Amer. Math. Soc., **22** (1969), 713–719.
[30] K. Stempak, La théorie de Littlewood-Paley pour la transformation de Fourier-Bessel, C.R. Acad. Sci. Paris, **303** Serie I (1) (1986), 15–18.
[31] A. H. Zemanian, Generalized integral transformations, Interscience Publishers, New York, 1968.

Jorge J. Betancor
Departamento de Análisis Matemático
Universidad de La Laguna
38271 – La Laguna. Tenerife
Islas Canarias. España.
E-mail: jbetanco@ull.es

Lourdes Rodríguez-Mesa
Departamento de Análisis Matemático
Universidad de La Laguna
38271 – La Laguna. Tenerife
Islas Canarias. España.
E-mail: lrguez@ull.es