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Lusternik-Schnirelmann type invariants for Menger manifolds

By Kazuhiro KAWAMURA

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Abstract. We study the n-dimensional category $\mathrm{cat}_n(X)$ of a compact space X, a counterpart to Lusternik-Schnirelmann category in the context of n-homotopy theory, and prove Menger manifold analogues of results due to Montejano and Wong for Hilbert cube manifolds.

THEOREM. *For any compact connected Menger manifold M, we have that*

$$\mathrm{cat}_{n-1}(M) = \mathrm{gcat}_{\lambda^n}(M) - 1 = \mathrm{gcat}_{\lambda^n}(\mathrm{Ker}_{n-1}(M)),$$

where $\mathrm{gcat}_{\lambda^n}(M)$

$$= \min\left\{ k \,\middle|\, \begin{array}{l} \textit{there exists an open cover } \{U_i \mid i = 1, \dots, k\} \textit{ of } M \textit{ such} \\ \textit{that } U_i \textit{ is homeomorphic to } \lambda^n = \mu^n \backslash \{pt\} \end{array} \right\}$$

and $\mathrm{Ker}_{n-1}(M)$ *is the* $(n-1)$*-homotopy kernel of* M*, introduced by Chigogidze.*

1. Introduction.

All spaces are assumed to be locally compact, separable and metrizable. *The Lusternik-Schnirelmann category* (*L-S category* for short) *of a space* X, denoted by $\mathrm{cat}(X)$, is defined as follows:

$$\mathrm{cat}(X) = \min\left\{ k \,\middle|\, \begin{array}{l} \text{there exists an open cover } \{U_i \mid i = 1, \dots, k\} \\ \text{such that each } U_i \text{ is contractible in } X \end{array} \right\}.$$

When X is a compact polyhedron, we may replace "an open cover" above with "a polyhedral cover", that is, a cover consisting of subpolyhedra of X. Throughout the present paper, the Hilbert cube (= the countable product of the closed interval [0, 1]) is denoted by Q. Also the n-dimensional universal Menger compactum (see [**1**], [**5**], [**11**] etc.) is denoted by μ^n. In [**17**], L. Montejano proved that for any compact connected Hilbert cube manifold (called a $\boldsymbol{Q}$-manifold in the sequel) X, a cover $\mathscr{U}$ of X such that $|\mathscr{U}| = \mathrm{cat}(X) + 1$ can be chosen so that each member of $\mathscr{U}$ is homeomorphic ($\approx$) to $\boldsymbol{Q} \times [0, 1)$. A similar

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result for noncompact $\boldsymbol{Q}$-manifolds was obtained by R. Wong [**21**] (see Theorem 1.1 below). The purpose of the present paper is to obtain analogous results on Menger manifolds (called μ^n-manifolds in the sequel). It is now widely recognized that there exists a strong similarity between $\boldsymbol{Q}$-manifold theory and μ^n-manifold theory, and the proofs of our results basically fit into the scheme similar to those of the previous results mentioned above, while details depend on finite dimensional feature and a special construction of Menger manifolds.

In order to state the results of Montejano and Wong precisely, we introduce the following notion. For a $\boldsymbol{Q}$-manifold M, let

$$\operatorname{gcat}_{\boldsymbol{Q}\times[0,1)}(M) = \min\left\{ k \;\middle|\; \begin{array}{l}\text{there exists an open cover}\\ \{U_i \mid i = 1, \dots, k\} \text{ such that each } U_i \text{ is}\\ \text{homeomorphic to } \boldsymbol{Q}\times[0,1)\end{array} \right\}.$$

Theorem 1.1. (a) ([**17**]). *For each compact connected $\boldsymbol{Q}$-manifold M, we have that* $\operatorname{cat}(M) + 1 = \operatorname{gcat}_{\boldsymbol{Q}\times[0,1)}(M)$.

(b) ([**21**]). *For each compact connected $\boldsymbol{Q}$-manifold M, we have that* $\operatorname{cat}(M) = \operatorname{gcat}_{\boldsymbol{Q}\times[0,1)}(M \times [0,1))$.

It is known that the $(n-1)$-homotopy theory in μ^n-manifold theory plays the same role as the one of the homotopy theory in $\boldsymbol{Q}$-manifold theory. So it is natural to introduce the n-homotopy theoretic counterpart of L-S category in order to obtain the corresponding results in Menger manifold theory (Theorems 3.2 and 3.4 below). An appropriate notion for this purpose has been introduced by R. H. Fox in the paper [**13**], where the notion is called the n-dimensional category.

Notation. For a subset A of a space X, $\operatorname{int}(A)$ and $\operatorname{bd}(A)$ denote the (topological) interior and the (topological) boundary of A in X respectively. On the other hand, when A is a manifold, the manifold interior and the manifold boundary of A are denoted by $\operatorname{Int} A$ and ∂A respectively. The q-dimensional ball $\{x \in \boldsymbol{R}^q \mid \|x\| \le 1\}$ ($\|\|$ denotes the standard norm on $\boldsymbol{R}^q$) is denoted by D^q and $S^{q-1} = \partial D^q$, the $(q-1)$-dimensional sphere. Also I denotes the unit interval $[0,1]$ and J denotes the half line $[0,\infty)$.

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2. Preliminary results on Menger manifolds and n-homotopy.

In this section, we recall some facts on Menger manifold theory and n-homotopy theory with emphasis on triangulated Menger manifolds. A geometric description of the $(n-1)$-homotopy kernels seems to be new in the literature. We begin with the definition of triangulated Menger manifolds.

Let M_0 be a PL manifold of dimension $\geq 2n+1$ (with or without boundary) and let $\mathscr{L}_0$ be a combinatorial triangulation of M_0. Take the second barycentric subdivision $\beta^2\mathscr{L}_0$ of $\mathscr{L}_0$ and let

$$M_1 = \mathrm{st}(|\mathscr{L}_0^{(n)}|, \beta^2\mathscr{L}_0) \quad \text{and} \quad \mathscr{L}_1 = \beta^2\mathscr{L}_0 \,|\, M_1.$$

Inductively, when we have defined M_i and its triangulation $\mathscr{L}_i$, take the second barycentric subdivision $\beta^2\mathscr{L}_i$ of $\mathscr{L}_i$ and define

$$M_{i+1} = \mathrm{st}(|\mathscr{L}_i^{(n)}|, \beta^2\mathscr{L}_i) \quad \text{and} \quad \mathscr{L}_{i+1} = \beta^2\mathscr{L}_i \,|\, M_{i+1}.$$

In this way, we have a decreasing sequence of PL manifolds (M_i). The intersection $M_\infty = \bigcap_{i=1}^\infty M_i$ is called a *triangulated μ^n-manifold* and the sequence $\{(M_i, \mathscr{L}_i) \mid i \geq 0\}$ is called the *defining sequence of M_∞*. Notice that this construction is different than that of M. Bestvina [**1**]. The present construction, called *the Lefschetz construction* in the sequel, is originally due to Lefschetz [**15**]. If we start with $M_0 = I^{2n+1}$ with the standard triagulation, then the resulting compactum is called the *n-dimensional universal Menger compactum*, and denoted by μ^n. A locally compact separable metrizable space is called a *μ^n-manifold* (or, *an n-dimensional Menger manifold*) if each of its points has a neighbourhood which is homeomorphic to μ^n. It is easily seen that any triangulated μ^n-manifold is a μ^n-manifold in the above sense. The outstanding theorem of Bestvina states that the converse holds.

Theorem 2.1 ([**1**]). (a) *A locally compact separable metrizable space X is a μ^n-manifold if and only if it is an n-dimensional LC^{n-1} space which satisfies the following condition, called the* disjoint n-cells property.

(DD^{n}P) *For each pair $\alpha, \beta : I^n \to X$ of maps and for each $\varepsilon > 0$, there exist two maps $\alpha', \beta' : I^n \to X$ which are ε-close to α and β respectively such that $\alpha'(I^n) \cap \beta'(I^n) = \emptyset$.*

(b) *Each μ^n-manifold is homeomorphic to a triangulated μ^n-manifold.*

It is known that the $(n-1)$-homotopy theory is a correct framework for the study of μ^n-manifolds. The definition of n-homotopy is originated by R. H. Fox [**13**] and later developed by A. Chigogidze in the context of Menger manifold theory [**5**].

Definition 2.2. Two maps $f, g : X \to Y$ between locally compact separable metrizable spaces are said to be *n-homotopic* (denoted by $f \overset{n}{\simeq} g$) if, for each map $\alpha : K \to X$ defined on a locally compact separable metrizable space K with $\dim K \leq n$, the compositions $f \circ \alpha$ and $g \circ \alpha$ are homotopic. If f and g are proper maps and $f \circ \alpha$ and $g \circ \alpha$ are *properly* homotopic for each *proper* map

$\alpha : K \to X$ with $\dim K \le n$, then we say that f and g are *properly n-homotopic* (denoted by $f \overset{n}{\simeq}_p g$).

The proof of Whitehead theorem and [**6**, Proposition 1.5] yield the following result which will be used repeatedly.

Theorem 2.3 (cf. [**9**, Proposition 1.1]). *Let $f : X \to Y$ be a map between locally compact LC^{n-1} (= locally $(n-1)$-connected) separable metrizable spaces with* $\dim X$, $\dim Y \le n$. *The map f is an $(n-1)$-homotopy equivalence if and only if the induced homomorphism $f_\sharp : \pi_i(X) \to \pi_i(Y)$ is an isomorphism for each $i = 0, 1, \dots, n-1$ and for every choice of base points.*

The proper $(n-1)$-homotopy types of μ^n-manifolds determine their topological types as is stated in the next theorem. We say that a proper map $f : X \to Y$ between locally compact separable metrizable space *induces an epimorphism* (*a monomorphism* resp.) *of the i-th homotopy groups of ends* if, every compactum K in Y is contained in some compactum L such that, for any choice of base point $* \in X - f^{-1}(L)$,

$$\begin{array}{ccc} \pi_i(X - f^{-1}(L), *) & \xrightarrow{f_\sharp} & \pi_i(Y - L, f(*)) \\ {\scriptstyle (j_X)_\sharp}\Big\downarrow & & \Big\downarrow{\scriptstyle (j_Y)_\sharp} \\ \pi_i(X - f^{-1}(K), *) & \xrightarrow{f_\sharp} & \pi_i(Y - K, f(*)) \end{array}$$

$\mathrm{im}(j_Y)_\sharp \subset \mathrm{im}\, f_\sharp$ ($\ker f_\sharp \subset \ker(j_X)_\sharp$ resp.), where j_X and j_Y are appropriate inclusions. An *isomorphism between homotopy groups of ends* is defined in the obvious way.

Theorem 2.4 ([**1**, 2.8.6, and Chap. 6]). *Let $f : M \to N$ be a proper map between μ^n-manifolds.*

(a) *The map f is properly $(n-1)$-homotopic to a homeomorphism if and only if f induces an isomorphism $f_\sharp : \pi_i(M) \to \pi_i(N)$ for each choice of base points, and induces an isomorphism between the i-th homotopy groups of the ends for each $i = 0, \dots, n-1$.*

(b) *In particular, if M and N are compact, the map f is $(n-1)$-homotopic to a homeomorphism if and only if f induces an isomorphism $f_\sharp : \pi_i(M) \to \pi_i(N)$ for each choice of base points and for each $i = 0, \dots, n-1$.*

Next we introduce the notion of Z-sets in Menger manifolds and state the Z-set Unknotting Theorem that will be used frequently.

Definition 2.5. A closed set A in a space X is called a Z_n-*set* if, for each map $\alpha : I^n \to X$ and for each $\varepsilon > 0$, there exists a map $\alpha' : I^n \to X - A$ which is

ε-close to α. A closed set in X which is a Z_n-set for each n is called a *Z-set*. It is known that every Z_n-set in each μ^n-manifold is always a Z-set.

The $\mathrm{DD^nP}$ and the Baire category argument imply the following:

THEOREM 2.6 ([**1**, 2.3.8 and Chap. 6]). *Let M be a μ^n-manifold and let K be a locally compact separable metrizable space with $\dim K \le n$. Then each map $f : K \to M$ is approximated by an embedding the image of which is a Z-set in M (such an embedding is called a Z-embedding).*

THEOREM 2.7 (the Z-set Unknotting Theorem, [**1**, 3.1.4 and Chap. 6]). *Let $f : A \to B$ be a homeomorphism between Z-sets A, B of a μ^n-manifold M and let $i_A : A \to M$ and $i_B : B \to M$ be the inclusions of A and B respectively. If $i_B \circ f \overset{n-1}{\simeq}_p i_A$, then there exists a homeomorphism $F : M \to M$ that is an extension of f.*

The following result will also be useful in the sequel.

THEOREM 2.8 ([**12**, Theorem 7] and [**5**, Theorem 8]). *For each μ^n-manifold M, there exists a locally compact polyhedron P with $\dim P \le n$ and a proper UV^{n-1}-map $\varphi : M \to P$.*

Here, a proper map $f : X \to Y$ between locally compact separable metrizable spaces is called a *UV^k-map* if each fibre $f^{-1}(y)$ satisfies the following condition:

(UV^k): for each embedding $e : f^{-1}(y) \to E$ into an ANR E and for each neighbourhood U of $e(f^{-1}(y))$ in E, there exists a neighbourhood V of $e(f^{-1}(y))$ contained in U such that $\pi_i(V) \to \pi_i(U)$ is trivial for each $i = 0, \dots, k$.

It is well known that proper UV^{n-1}-maps between LC^{n-1} spaces of dimension at most n are proper $(n-1)$-homotopy equivalences (cf. [**6**, Proposition 1.4]). Hence we have:

COROLLARY 2.9. *Each μ^n-manifold has the same proper $(n-1)$-homotopy type as a locally compact polyhedron of dimension at most n. In particular, each compact μ^n-manifold has the same $(n-1)$-homotopy type as a compact polyhedron of dimesion at most n.*

Before proceeding further, let us point out a property of triangulated μ^n-manifolds in the above consturction.

PROPOSITION 2.10. *Let $M_\infty = \bigcap_{i=1}^\infty M_i$ be a triangulated μ^n-manifold with the defining sequence $\{(M_i, \mathscr{L}_i) \mid i \ge 0\}$. Then*

(a) $|\mathscr{L}_i^{(n)}| \subset |\mathscr{L}_{i+1}^{(n)}| \subset \bigcup_{i=1}^\infty |\mathscr{L}_i^{(n)}| \subset M_\infty$ *for each i,*

(b) *for each map $f : K \to M_\infty$ of a compactum K with $\dim K \le n$ and for each $\varepsilon > 0$, there exists a map $g : K \to M_\infty$ which is ε-close to f such that $g(K) \subset \bigcup_{i=1}^\infty |\mathscr{L}_i^{(n)}|$, and*

(c) *if $\mathscr{N}_0$ is a subcomplex of $\mathscr{L}_0$ such that $|\mathscr{N}_0|$ is a PL manifold of dimension $\ge 2n+1$, then $|\mathscr{N}_0| \cap M_\infty$ is a μ^n-manifold.*

PROOF. The assertions (a) and (c) are immediate consequences of the construction. To verify the assertion (b), fix an index i and consider the map $f : K \to M_\infty \hookrightarrow M_i$. Factor the map through a PL map of an n-dimensional polyhedron and apply general position to obtain a map $f' : K \to M_i$ such that $f'(K)$ misses the dual $(\dim M_i - n - 1)$-skeleton of $\mathscr{L}_i$ (recall that $\dim M_i \geq 2n + 1$). Then the image of f' can be pushed into $|\mathscr{L}_i^{(n)}|$, thus we obtain a map $g : K \to |\mathscr{L}_i^{(n)}| \subset M_\infty$. If we take a sufficiently large i, then we may choose g as close to f as we wish. □

For a $\boldsymbol{Q}$-manifold M, the topological type of the $\boldsymbol{Q}$-manifold $M \times [0, 1)$ is determined by its homotopy type. The corresponding result has been obtained by Chigogidze for μ^n-manifolds [7]. He introduced the notion of the $(n-1)$-homotopy kernel of a μ^n-manifold which corresponds to $M \times [0, 1)$ for a $\boldsymbol{Q}$-manifold M.

DEFINITION 2.11. For a μ^n-manifold M, let $f : M \to M$ be a Z-embedding of M into itself which is properly $(n-1)$-homotopic to id. Then the topological type of $M \backslash f(M)$ does not depend on the choice of the embedding (by the Z-set Unknotting Theorem 2.7), and is called the $(n-1)$-*homotopy kernel* of M (denoted by $\mathrm{Ker}_{n-1}(M)$). It easily follows from Theorem 2.3 that the inclusion $\mathrm{Ker}_{n-1}(M) \hookrightarrow M$ is an $(n-1)$-homotopy equivalence.

We give a geometric construction of the homotopy kernel for the later use. For a triangulation $\mathscr{P}$ of a polyhedron P and a triangulation $\mathscr{I}$ of I, $\mathscr{P} \times \mathscr{I}$ denotes the cell complex structure of $P \times I$ whose cells are of the form:

$$\sigma \times \tau \quad \text{for } \sigma \in \mathscr{P} \text{ and } \tau \in \mathscr{I}.$$

PROPOSITION 2.12. *Let M be a μ^n-manifold with a defining sequence $\{(M_i, \mathscr{L}_i) \mid i \geq 0\}$. Take a triangulation $\mathscr{I}_0$ of the closed interval I and let $\mathscr{J}_0 = \mathscr{L}_0 \times \mathscr{I}_0$. Define $\beta^2 \mathscr{J}_0 = \beta^2 \mathscr{L}_0 \times \beta^2 \mathscr{I}_0$ and let $N_1 = \mathrm{st}(|\mathscr{J}_0^{(n)}|, \beta^2 \mathscr{J}_0)$, $\mathscr{J}_1 = \beta^2 \mathscr{J}_0 \mid N_1$. Proceed inductively as in the Lefschetz construction to define a sequence $\{(N_i, \mathscr{J}_i)\}$ by means of $\beta^2 \mathscr{J}_i = \beta^2 \mathscr{L}_i \times \beta^2 \mathscr{I}_i$, and let $N_\infty = \bigcap_{i=1}^\infty N_i$. Then N_∞ is homeomorphic to M and $\mathrm{Ker}_{n-1}(M)$ is homeomorphic to $N_\infty \cap M_0 \times [0, 1)$.*

PROOF. Notice first that $\mathscr{J}_i \mid N_i \times \{0\} = \mathscr{L}_i = \mathscr{J}_i \mid N_i \times \{1\}$. Let $r : M_0 \times I \to M_0 \times \{1\}$ be the projection and observe that $r(N_i) = M_i \times \{1\}$. Let $\rho = r \mid N_\infty : N_\infty \to M \times \{1\} = M \approx N_\infty \cap M_0 \times \{1\}$ be the restriction of r to N_∞. It is readily seen that ρ induces an isomorphism between the i-th homotopy groups for each $i = 0, \ldots, n-1$, thus M is homeomorphic to N_∞ by Theorem 2.4. Furthermore, the same argument shows that ρ is $(n-1)$-homotopic to id_{N_∞}. Also it is easy to see that $M \times \{1\}$ is a Z-set in N_∞. Thus $\mathrm{Ker}_{n-1}(M)$ is homeomorphic to $N_\infty \backslash M \times \{1\} = N_\infty \cap (M_0 \times [0, 1))$. □

By the Complement Theorem ([**4**]), we have that $\mathrm{Ker}_{n-1}(\mu^n)$ is homeomorphic to $\mu^n \backslash \{pt\}$.

NOTATION 2.13. Throughout this paper, $\mu^n \backslash \{pt\}$ is denoted by λ^n.

The space λ^n is topologically characterized as follows. A locally compact separable metrizable space X is said to be *k-tame at the infinity* if, for each compactum K in X, there exists a compactum $L \supset K$ such that the inclusion $i : X \backslash L \hookrightarrow X \backslash K$ factors through an at most $(k+1)$-dimensional compact polyhedron up to k-homotopy. That is, there exist a compact polyhedron P with $\dim P \leq k+1$ and two maps $\alpha : X \backslash L \to P$ and $\beta : P \to X \backslash K$ such that $\beta \circ \alpha$ is k-homotopic to i. A locally compact separable metrizable space Y is said to be *locally k-connected at the infinity* if, for each compactum K in Y, there exists a compactum $L \supset K$ such that each map $\alpha : S^i \to Y \backslash L$ extends to a map $\bar{\alpha} : D^{i+1} \to Y \backslash K$ for each $i = 0, \dots, k$.

THEOREM 2.14 [**9**, Corollary 2.8]. *A noncompact μ^n-manifold M is homeomorphic to λ^n if and only if M is $(n-1)$-connected, locally $(n-1)$-connected at infinity, and $(n-1)$-tame at infinity.*

DEFINITION 2.15. Let M be a μ^n-manifold. A closed μ^n-submanifold A of M is called a *clean μ^n-submanifold* if there is a μ^n-manifold δA in A such that

(a) $(M - A) \cup \delta A$ is a μ^n-manifold,

(b) δA is a Z-set in both of A and $(M - A) \cup \delta A$, and

(c) $A - \delta A$ is open.

If in addition, A and δA are homeomorphic to μ^n, then A and $(A, \delta A)$ are called a *clean μ^n* and a *clean μ^n-pair* respectively.

It is easy to see that there exists a sequence

$$W_1 \subset W_2 \backslash \delta W_2 \subset W_2 \subset W_3 \backslash \delta W_3 \subset \cdots$$

of clean μ^n-pairs $\{(W_i, \delta W_i)\}$ such that $\lambda^n = \bigcup_{i=1}^{\infty} W_i$.

For other aspects of Menger manifold theory, see for example, [**1**], [**4–12**].

3. Lusternik-Shnirelmann type invariant in n-homotopy theory and Menger manifolds.

The following definitions are natural counterparts to the $\mathrm{cat}(X)$ for a space X and $\mathrm{gcat}_{\boldsymbol{Q} \times [0,1)}(M)$ for a $\boldsymbol{Q}$-manifold M in the context of n-homotopy theory and μ^n-manifolds.

DEFINITION 3.1. (a) For a space X, let

$$\mathrm{cat}_n(X) = \min \left\{ k \;\middle|\; \begin{array}{l} \text{there exists an open cover } \{U_i \mid i = 1, \dots, k\} \\ \text{such that each inclusion } U_i \to X \text{ is} \\ n\text{-homotopic to a constant map} \end{array} \right\}.$$

(b) For a μ^n-manifold M, let

$$\mathrm{gcat}_{\lambda^n}(M) = \min\left\{ k \,\middle|\, \begin{array}{l} \text{there exists an open cover } \{U_i \mid i = 1, \dots, k\} \\ \text{such that each } U_i \text{ is homeomorphic to } \lambda^n \end{array} \right\}.$$

In [**13**], $\mathrm{cat}_n(X)$ is denoted by h_n-$\mathrm{cat}(X)$ and is called the n-dimensional category. Clearly, cat_n is an n-homotopy invariant.

Our first result is an analogue of Theorem 1.1 (a) (section 1).

Theorem 3.2. *For any compact connected μ^n-manifold M, we have that*

$$\mathrm{gcat}_{\lambda^n}(M) = \mathrm{cat}_{n-1}(M) + 1.$$

Corollary 3.3. *A compact connected μ^n-manifold is homeomorphic to μ^n if and only if it is the union of two open sets, each of which is homeomorphic to λ^n.*

Proof of Corollary 3.3. It is clear that the compactum μ^n is the union of the above open sets. To prove the reverse implication, take a compact connected μ^n-manifold M with $\mathrm{gcat}_{\lambda^n}(M) = 2$. Then we have that $\mathrm{cat}_{n-1}(M) = \mathrm{gcat}_{\lambda^n}(M) - 1 = 1$, that is, M is $(n-1)$-connected. Theorem 2.4 implies that μ^n is the unique $(n-1)$-connected compact μ^n-manifold. See also Appendix. □

Proof of Theorem 3.2. The idea of the proof is similar to the one of Theorem 1.1 (a). First we prove that $\mathrm{gcat}_{\lambda^n}(M) \geq \mathrm{cat}_{n-1}(M) + 1$. Let $\mathscr{U} = \{U_i \mid i = 1, \dots, k\}$ be an open cover such that each U_i is homeomorphic to λ^n. Take a closed shrinking $\{F_i\}$ of $\mathscr{U}$, that is, a closed (and hence compact) cover $\{F_i \mid i = 1, \dots, k\}$ of M such that $F_i \subset U_i$ for each i. Since U_K is homeomorphic to λ^n, there exists a clean μ^n-pair $(W, \delta W)$ in M such that $F_K \subset W \backslash \delta W \subset W \subset U_K$. The existence of such a pair easily follows from the construction of λ^n or from the proof of Lemma 2.3 of [**9**]. Then the space $N = (M \backslash W) \cup \delta W$ is a μ^n-manifold and it is easy to see (and as is proved in [**20**]) that the inclusion $N \hookrightarrow M$ is an $(n-1)$-homotopy equivalence, and hence $\mathrm{cat}_{n-1}(N) = \mathrm{cat}_{n-1}(M)$. It is easy to construct a retraction $W \to \delta W$ which extends to a retraction $r : M \to N$.

For $i = 1, \dots, k-1$, let $V_i = U_i \cap N$. Then $\{V_i \mid i = 1, \dots, k-1\}$ is an open cover of N and it is easily seen that the inclusion $V_i \to N$ is equal to the composition $V_i \hookrightarrow U_i \hookrightarrow M \xrightarrow{r} N$ which is $(n-1)$-homotopic to a constant map (because so is $U_i \hookrightarrow M$). Therefore $\mathrm{cat}_{n-1}(N) \leq k - 1$, which shows that $\mathrm{cat}_{n-1}(M) \leq \mathrm{gcat}_{\lambda^n}(M) - 1$.

In order to prove the reverse inequality, let P be a compact connected polyhedron which has the same $(n-1)$-homotopy type as M (Corollary 2.9). Embedding P into a Euclidean space and taking a regular neighbourhood, we may assume that P is a PL manifold. Let $m = \dim P$. By the invariance of cat_{n-1} under the $(n-1)$-homotopy equivalences, we have that $\mathrm{cat}_{n-1}(P) = \mathrm{cat}_{n-1}(M)$ (= denoted by k). There exists a polyhedral cover $\mathscr{P} = \{P_i \mid i = 1, \dots, k\}$ of P

such that each inclusion $P_i \hookrightarrow P$ is $(n-1)$-homotopic to a constant map. Throughout the proof, one fixes a triangulation $\mathscr{T}$ of P such that each P_i is supported by a subcomplex of $\mathscr{T}$. For simplicity of notation, $P_i^{(j)}$ denotes the j-skeleton of P_i with respect to $\mathscr{T}$. Also the point $(0,\dots,0) \in \operatorname{Int} D^{2n+1}$ is denoted by $\mathbf{0}$. By the assumption on P_i, the $(n-1)$-skeleton $P_i^{(n-1)}$ is contractible in P for each i.

Consider the product space $N = P \times D^{2n+1} \times [0, k+1]$ and let

$$S_i = P_i \times D^{2n+1} \times [0, i], \text{ and}$$

$$T_i = P_i \times D^{2n+1} \times [i, k+1], \quad i = 1, \dots, k.$$

Since $P_i^{(n-1)}$ is contractible in P, applying general position to $P \times D^{2n+1}$, the cone $c(P_i^{(n-1)})$ over $P_i^{(n-1)}$ can be embedded into $P \times D^{2n+1}$ so that $c(P_i^{(n-1)}) \cap (P \times \{\mathbf{0}\}) =$ the base of the cone $= P_i^{(n-1)}$. The embedding may be further modified to an embedding

$$s_i : c(P_i^{(n-1)}) \to P_i \times D^{2n+1} \times [i, i + 1/3]$$

such that

(1) $S_i \cap s_i(c(P_i^{(n-1)})) = s_i(P_i^{(n-1)}) = P_i^{(n-1)} \times \{\mathbf{0}\} \times \{i\}$, and

(2) $s_i(c(P_i^{(n-1)})) \cap \partial N = s_i(P_i^{(n-1)}) \cap \partial N$.

Let $S_i^* = S_i \cup s_i(c(P_i^{(n-1)}))$. It has the same homotopy type as $P_i \cup c(P_i^{(n-1)})$, and in particular is $(n-1)$-connected. Similarly we obtain an embedding

$$t_i : c(P_i^{(n-1)}) \to P \times D^{2n+1} \times [i - 1/3, i]$$

such that

(3) $T_i \cap t_i(c(P_i^{(n-1)})) = t_i(P_i^{(n-1)}) = P_i^{(n-1)} \times \{\mathbf{0}\} \times \{i\}$, and

(4) $t_i(c(P_i^{(n-1)})) \cap \partial N = t_i(P_i^{(n-1)}) \cap \partial N$.

Let $T_i^* = T_i \cup t_i(c(P_i^{(n-1)}))$ which is $(n-1)$-connected as well.

For each $i = 2, \dots, k$, take a polyhedral arc L_i in $\operatorname{Int} N$ connecting S_{i-1}^* with T_i^* such that

$$S_{i-1}^* \cap L_i \text{ is the vertex of the cone } s_{i-1}(c(P_{i-1}^{(n-1)})), \text{ and}$$

$$T_i^* \cap L_i \text{ is the vertex of the cone } t_i(c(P_i^{(n-1)}))$$

(Notice that the vertices of these cones are in $\operatorname{Int} N$). It is easy to see that

(5) $S_{i-1}^* \cup L_i \cup T_i^*$ is an $(n-1)$-connected subpolyhedron. Define subpolyhedra $N_1, \dots, N_{k+1}$ of N as follows.

$N_1 =$ a regular neighbourhood of T_1^*,

$N_i =$ a regular neighbourhood of $S_{i-1}^* \cup L_i \cup T_i^*$ for $i = 2, \dots, k$, and

$N_{k+1} =$ a regular neighbourhood of S_k^*.

Then $\{N_i \mid i = 1, \dots, k+1\}$ is a polyhedral cover of N and we prove that, for each $i = 1, \dots, k+1$,

(6) the polyhedra N_i and $\mathrm{bd}(N_i)$ (= the topological boundary of N_i in N) are $(n-1)$-connected.

Proof. Since N_i is a regular neighbourhood of an $(n-1)$-connected polyhedron, it is $(n-1)$-connected as well. To prove that $\mathrm{bd}(N_i)$ is $(n-1)$-connected, observe first that $\partial N_i = \mathrm{bd}(N_i) \cup (N_i \cap \partial N)$. Let

$$R_1 = P_1 \times \{\mathbf{0}\} \times [1, k+1] \cup t_1(c(P_1^{(n-1)})),$$

$$R_i = P_{i-1} \times \{\mathbf{0}\} \times [0, i-1] \cup s_{i-1}(c(P_{i-1}^{(n-1)})) \cup L_i \cup$$

$$\cup\, t_i(c(P_i^{(n-1)})) \cup P_i \times \{\mathbf{0}\} \times [i, k+1], \text{ for } i = 2, \dots, k, \text{ and}$$

$$R_{k+1} = P_{k+1} \times \{\mathbf{0}\} \times [k, k+1] \cup s_k(c(P_k^{(n-1)})).$$

The sets $T_1^*, S_{i-1}^* \cup L_i \cup T_i^*$ $(2 \le i \le k)$ and S_k^* collapse onto R_1, R_i $(2 \le i \le k)$ and R_{k+1} respectively and hence

(7) N_i is a regular neighbourhood of R_i $(i = 1, \dots, k+1)$. Recall that $\dim N_i = \dim N = m + 2n + 2$ and $\dim R_i \le \dim P + 1 = m + 1$. By general position and (7), the $(n-1)$-connectedness of ∂N_i easily follows from the $(n-1)$-connectedness of N_i. Also notice that

(8) $N_i \cap \partial N$ is a regular neighbourhood of $R_i \cap \partial N$ and

$$\dim(\partial N_i) = m + 2n + 1 \quad \text{and} \quad \dim(R_i \cap \partial N) \le m + 1.$$

Now given a map $\alpha : S^q \to \mathrm{bd}(N_i) \hookrightarrow \partial N_i$ $(q \le n-1)$, one can extend α to a map $\bar{\alpha} : D^{q+1} \to \partial N_i$ by the $(n-1)$-connectedness of ∂N_i. Applying the general position, one may push $\mathrm{im}\, \bar{\alpha}$ off $R_i \cap \partial N$ in ∂N_i and then push that image further into $\mathrm{cl}[\partial N_i \backslash (N_i \cap \partial N)] \subset \mathrm{bd}(N_i)$, due to (8). Thus α extends to a map $\hat{\alpha} : D^{q+1} \to \mathrm{bd}(N_i)$ and this proves that $\mathrm{bd}(N_i)$ is $(n-1)$-connected.

Since ∂N_i is a Z-set in N_i and $\mathrm{bd}(N_i) \subset \partial N_i$, we see that

(9) $\mathrm{bd}(N_i)$ is a Z_n-set in N_i.

Take a triangulation $\mathscr{L}_0$ of N such that each N_i and $\mathrm{bd}(N_i)$ are supported by subcomplexes of $\mathscr{L}_0$. Start the Lefschetz construction with $M_0 = N$ and the triangulation $\mathscr{L}_0$. This yields a μ^n-manifold P_∞ and a proof similar to the one of Proposition 2.12 works to prove that P_∞ is homeomorphic to M. Let U_i be an open set of P_∞ defined by

$$U_i = \mathrm{int}(N_i) \cap P_\infty = (N_i \backslash \mathrm{bd}(N_i)) \cap P_\infty \quad \text{for each } i = 1, \dots, k+1.$$

By the condition (6) and Theorem 2.4, we see that the μ^n-manifolds $N_i \cap P_\infty$ and $\mathrm{bd}(N_i) \cap P_\infty$ are homeomorphic to μ^n. By the condition (9) above, we see easily that $\mathrm{bd}(N_i) \cap P_\infty$ is a Z-set in $N_i \cap P_\infty$. Hence, $U_i = (N_i \backslash \mathrm{bd}(N_i)) \cap P_\infty$ is homeomorphic to λ^n by the Complement Theorem [**4**]. Thus we have that $\mathrm{gcat}_{\lambda^n}(M) \leq k+1$, completing the proof. □

Next we consider the $(n-1)$-homotopy kernel $\mathrm{Ker}_{n-1}(M)$ of a μ^n-manifold M and prove the following theorem that corresponds to Theorem 1.1 (b).

Theorem 3.4. *For any compact connected μ^n-manifold M,*

$$\mathrm{cat}_{n-1}(M) = \mathrm{gcat}_{\lambda^n}(\mathrm{Ker}_{n-1}(M)).$$

Proof. Since M and $\mathrm{Ker}_{n-1}(M)$ have the same $(n-1)$-homotopy type (see Definition 2.11), we see that

$$\mathrm{cat}_{n-1}(M) = \mathrm{cat}_{n-1}(\mathrm{Ker}_{n-1}(M)) \leq \mathrm{gcat}_{\lambda^n}(\mathrm{Ker}_{n-1}(M)).$$

We need the following lemma for the proof of the reverse inequality.

Lemma 3.5. *Let $(W, \delta W)$ be a clean μ^n-pair in μ^n and let Z be a closed set in μ^n such that $Z \cap W = \emptyset$. Let K be a closed set which is contained in $W \backslash \delta W$. Then there exists a homeomorphism $f : \mu^n \to \mu^n$ such that $f \,|\, Z = \mathrm{id}$ and $f(K) \cap W = \emptyset$.*

Proof of Lemma 3.5. A proof using the Z-set Unknotting theorem (Theorem 2.7) seems to be known to experts. We give a brief sketch here. One can easily take clean μ^n-pairs $(W_0, \delta W_0)$ and $(W_1, \delta W_1)$ such that $W_0 \subset (W \backslash \delta W) \backslash K$, $W \subset W_1 \backslash \delta W_1$ and $W_1 \cap Z = \emptyset$. By making use of Theorem 2.7, it is easy to construct a homeomorphism $f : W_1 \to W_1$ such that $f(W_0) = W$, $f(\delta W_0) = \delta W$, and $f \,|\, \delta W_1 = \mathrm{id}$. Extend f to a homeomorphism $f : \mu^n \to \mu^n$ by declaring that $f = \mathrm{id}$ outside W_1. Then it follows easily that $f \,|\, Z = \mathrm{id}$ and $f(K) \cap W = \emptyset$.

See [**14**, Lemma 2.1] for another construction. □

Now we proceed to the proof of the inequality:

$$\mathrm{cat}_{n-1}(M) \geq \mathrm{gcat}_{\lambda^n}(\mathrm{Ker}_{n-1}(M)).$$

Take a compact polyhedron S which has the same $(n-1)$-homotopy type as M (Corollary 2.9) and let $k = \mathrm{cat}_{n-1}(M) = \mathrm{cat}_{n-1}(S)$. As in Theorem 3.2, we may assume that S is a PL manifold. There exists a polyhedral cover $\{S_i \,|\, i = 1, \dots, k\}$ of S such that each inclusion $S_i \hookrightarrow S$ is $(n-1)$-homotopic to a constant map in S. Fix a triangulation of S such that each S_i is supported by a subcomplex of the triangulation. The $(n-1)$-skeleton of S_i (with respect to the triangulation) is contractible in S by the assumption, and hence the inclusion $S_i^{(n-1)} \hookrightarrow S$ extends to a map $c(S_i^{(n-1)}) \to S$ of the cone $c(S_i^{(n-1)})$ over $S_i^{(n-1)}$. As in the proof of Theorem 3.2, there exists a PL embedding $e_i : c(S_i^{(n-1)}) \to S \times D^{2n+1} \times [0,1]$ such that

$$e_i(c(S_i^{(n-1)})) \cap (S \times D^{2n+1} \times \{0\}) = e_i(S_i^{(n-1)}) = S_i^{(n-1)} \times \{\mathbf{0}\} \times \{0\}.$$

Let

$$T_i = S_i \times D^{2n+1} \times \{0\} \cup e_i(c(S_i^{(n-1)})),$$

which is an $(n-1)$-connected subpolyhedron of $S \times D^{2n+1} \times [0,1]$. A similar argument to the one in Theorem 3.2 works to prove that a regular neighbourhood N_i of T_i in $S \times D^{2n+1} \times [0,1]$ and its topological boundary $\mathrm{bd}(N_i)$ are $(n-1)$-connected (notice here that N_i is also a regular neighbourhood of $S_i \cup e_i(c(S_i^{(n-1)}))$ which has codimension $\geq 2n+1$ in N_i). For simplicity, $S \times D^{2n+1}$ and $S_i \times D^{2n+1}$ are denoted by P and P_i respectively. Take fine triangulations $\mathscr{P}_0$ of P and $\mathscr{I}_0$ of $[0,1]$ respectively and consider the cell complex structure $\mathscr{P}_0 \times \mathscr{I}_0$ of $P \times [0,1]$. Adjusting each N_i and taking a sufficiently fine triangulations $\mathscr{P}_0$ and $\mathscr{I}_0$, we may assume that each N_i is supported by a subcomplex of $\mathscr{P}_0 \times \mathscr{I}_0$.

Start the Lefschetz construction with $M_0 = P \times [0,1]$ and the cell complex $\mathscr{L}_0 = \mathscr{P}_0 \times \mathscr{I}_0$. The resulting compactum M_∞ is a μ^n-manifold which is homeomorphic to M, and the $(n-1)$-homotopy kernel $\mathrm{Ker}_{n-1}(M)$ is homeomorphic to $M_\infty \cap P \times [0,1)$ by Proposition 2.12. Recall that $|\mathscr{L}_i^{(n)}| \subset |\mathscr{L}_{i+1}^{(n)}|$ for $i \geq 1$ (Proposition 2.10). We define an increasing sequence (K_s) of compact polyhedra as follows.

$$K_s = |\mathscr{L}_s^{(n)}| \cap P \times [0, t_s] \quad \text{and}$$

$$K_s^i = |\mathscr{L}_s^{(n)}| \cap P_i \times [0, t_s], \text{ where } t_s \text{ is a vertex of } \mathscr{I}_s \text{ such that } t_s < 1 \text{ and } t_s \uparrow 1 \text{ as } s \to \infty \ (i = 1, \ldots, k).$$

Observe that

(1) $K_s \subset K_{s+1} \subset \bigcup_{s\geq 1} K_s = \bigcup_{s\geq 1} |\mathscr{L}_s^{(n)}| \cap P \times [0,1)$,
(2) $K_s = \bigcup_{i=1}^k K_s^i$, and
(3) each K_s is a Z-set in M_∞ (See the proof of Proposition 2.10 (b)).

Let $U_i = \mathrm{int}(N_i) \cap M_\infty$. Since N_i and $\mathrm{bd}(N_i)$ are supported by subcomplexes of $\mathscr{P}_0 \times \mathscr{I}_0$ and $\mathrm{bd}(N_i)$ is a Z-set in N_i, U_i is homeomorphic to λ^n (see the second part of the proof of Theorem 3.2).

The idea of the remaining part of the proof is similar to the one of [**21**]. We will construct an open subset V of $\mathrm{Ker}_{n-1}(M) \approx M_\infty \cap P \times [0,1)$ which is the union of k open sets homeomorphic to λ^n, and also which contains $\bigcup_{s\geq 1} |\mathscr{L}_s^{(n)}| \cap P \times [0,1)$. Then V is a μ^n-manifold and, by Theorem 2.3 and

Proposition 2.10, we see that the inclusion $V \hookrightarrow \mathrm{Ker}_{n-1}(M)$ is an $(n-1)$-homotopy equivalence. By [**7**, Theorem 2.2 and Proposition 2.2], we have that $\mathrm{Ker}_{n-1}(V) \approx \mathrm{Ker}_{n-1}(\mathrm{Ker}_{n-1}(M)) \approx \mathrm{Ker}_{n-1}(M)$. As a final step, we will construct an open cover of $\mathrm{Ker}_{n-1}(V)$, each member of which is homeomorphic to λ^n.

Construction of V: The open set V is defined as the union $V = \bigcup_{i=1}^{k} V_i$, where V_i is an open set which is homeomorphic to λ^n and contains $\bigcup_{s=1}^{\infty} K_s^i$. In what follows, we construct an open embedding $e_i : \lambda^n \to \mathrm{Ker}_{n-1}(M)$ such that $e_i(\lambda^n) \supset \bigcup_{s=1}^{\infty} K_s^i$. Such an embedding is obtained by modifying the argument of the one in [**21**]. Throughout the following construction, we fix an index i $(= 1, \ldots, k)$ and a homeomorphism $g_i : \lambda^n \to U_i$. Also fix a sequence (W_i) of clean (μ^n)'s in λ^n such that

$$g_i^{-1}(P_i \times \{0\}) \subset W_1 \backslash \delta W_1 \subset W_1 \subset W_2 \backslash \delta W_2 \subset W_2 \subset \cdots \subset \bigcup_{j=1}^{\infty} W_j = \lambda^n.$$

The embedding e_i is defined as the limit of a sequence of open embeddings $\{h_\ell : \lambda^n \to \mathrm{Ker}_{n-1}(M) \mid \ell = 1, 2, \ldots\}$.

Step 1. *There exists an open embedding* $h_1 : \lambda^n \to \mathrm{Ker}_{n-1}(M)$ *such that* $h_1 \mid W_1 = g_i \mid W_1$ *and* $h_1(W_2) \supset K_1^i$.

Proof. Suppose that $\hat{K}_1^i = K_1^i \backslash g_i(W_1 \backslash \delta W_1) \neq \emptyset$ (otherwise, just take g_i as h_1). First we show that

(1.1) there exists a map $f_1' : \hat{K}_1^i \to \mathrm{Ker}_{n-1}(M)$ which is $(n-1)$-homotopic to the inclusion $\hat{K}_1^i \to \mathrm{Ker}_{n-1}(M)$ in $\mathrm{Ker}_{n-1}(M)$ such that $f_1'(\hat{K}_1^i) \subset g_i(W_1) \backslash P_i \times \{0\}$.

There are a couple of ways to obtain such a map f_1' and one of them is described below. Take a fine triangulation $\mathscr{T}$ of K_1^i (that could be much finer than $\mathscr{L}_1$ of the defining sequence) and replace $\hat{K}_1^i$ by the star $\mathrm{st}(\hat{K}_1^i, \mathscr{T})$. It suffices to construct a map defined on $\mathrm{st}(\hat{K}_1^i, \mathscr{T})$ with the above property, so we assume at the outset that $\hat{K}_1^i$ is a compact polyhedron. The polyhedron $\hat{K}_1^i$ is a subset of $P_i \times [0,1]$. Using a homotopy along the $[0,1]$-direction in $P_i \times [0,1]$, $\hat{K}_1^i$ can be deformed to a subset of $g_i(W_1 \backslash \delta W_1)$ which does not intersect $P_i \times \{0\}$. Let $H : \hat{K}_1^i \times [0,1] \to P_i \times [0,1]$ be the homotopy such that $H_0 =$ the inclusion $\hat{K}_1^i \hookrightarrow P_i$ and $H_1(\hat{K}_1^i) \subset g_i(W_1 \backslash \delta W_1)$ (where $H_t(x) = H(x,t)$, $(x,t) \in \hat{K}_1^i \times [0,1]$). We may assume that $H(\hat{K}_1^i \times [0,1])$ is a subpolyhedron of $P_i \times [0,1]$ with $\dim H(\hat{K}_1^i \times [0,1]) \leq n+1$ and also that $\dim H_1(\hat{K}_1^i) \leq n$.

Consider the set $\hat{K}_1^i \cup H(\hat{K}_1^{i(n-1)} \times [0,1]) \cup H_1(\hat{K}_1^i)$ that is a polyhedron of dimension at most n. Using a similar trick to the one of [**1**, p. 34, Absorption Move] (cf. Proof of Proposition 2.10 (b)), the above set can be "pushed back" into $P \times [0,1] \cap M_\infty$ keeping $\hat{K}_1^i$ fixed. A careful control can be made so that the image of the set is contained in $P \times [0,1) \cap M_\infty = \mathrm{Ker}_{n-1}(M)$. Let u_1 :

$\hat{K}_1^i \cup H(\hat{K}_1^{i(n-1)} \times [0,1]) \cup H_1(\hat{K}_1^i) \to P \times [0,1) \cap M_\infty$ be the resulting map ($u_1 \mid \hat{K}_1^i =$ the inclusion). Then $f_1' = u_1 \circ H \mid : \hat{K}_1^i \times \{1\} \to \mathrm{Ker}_{n-1}(M)$ is the required map. Indeed, $f_1'(\hat{K}_1^i) \subset g_i(W_1) \backslash P_i \times \{0\}$ and also the map $u_1 \circ H \mid : \hat{K}_1^{i(n-1)} \times [0,1] \to \mathrm{Ker}_{n-1}(M)$ provides a homotopy between the inclusion $\hat{K}_1^{i(n-1)} \hookrightarrow \mathrm{Ker}_{n-1}(M)$ and $f_1' \mid \hat{K}_1^{i(n-1)}$ and this easily implies that f_1' is $(n-1)$-homotopic to the inclusion $\hat{K}_1^i \hookrightarrow \mathrm{Ker}_{n-1}(M)$ in $\mathrm{Ker}_{n-1}(M)$. This completes the proof of (1.1).

Let $L_1 = (\mathrm{Ker}_{n-1}(M) \backslash g_i(W_1)) \cup g_i(\delta W_1)$ which is a μ^n-manifold.

(1.2) There exists a map $f_1'' : f_1'(\hat{K}_1^i) \to L_1$ such that $f_1 = f_1'' \circ f_1'$ is $(n-1)$-homotopic to the inclusion $\hat{K}_1^i \hookrightarrow L_1$ in L_1.

Proof. Consider the sets

$$g_i^{-1}(f_1'(\hat{K}_1^i)) \subset W_1 \backslash \delta W_1 \subset W_1 \subset W_2 \backslash \delta W_2 \subset W_2.$$

Applying Lemma 3.5 to $\mu^n \approx W_2$, $Z = \delta W_2$, $(W, \delta W) = (W_1, \delta W_1)$, $K = g_i(f_1'(\hat{K}_1^i))$, we obtain a map $\tilde{f}_1 : W_2 \to W_2$ such that $\tilde{f}_1 \mid \delta W_2 = \mathrm{id}$ and $\tilde{f}_1(g_i^{-1}(f_1'(\hat{K}_1^i))) \cap W_1 = \varnothing$. Clearly $\tilde{f}_1$ is $(n-1)$-homotopic to id_{W_2} because $W_2 \approx \mu^n$. The map $g_i \circ \tilde{f}_1 \circ g_i^{-1} \mid g_i(W_2)$ naturally extends to a map $\hat{f_1} : \mathrm{Ker}_{n-1}(M) \to \mathrm{Ker}_{n-1}(M)$ (by defining $\hat{f_1} = \mathrm{id}$ outside $g_i(W_2)$) which is $(n-1)$-homotopic to id in $\mathrm{Ker}_{n-1}(M)$. Furthermore, there exists a retraction $g_i(W_1) \to g_i(\delta W_1)$ which extends to a retraction $r_1 : \mathrm{Ker}_{n-1}(M) \to L_1$. The required map f_1'' is defined by $f_1'' = r_1 \circ \hat{f_1}$.

Summarizing the situation, we have that

(1.3) there exists a map $f_1 : \hat{K}_1^i \to L_1$ which is $(n-1)$-homotopic to the inclusion $\hat{K}_1^i \to L_1$ in L_1 such that $f_1(\hat{K}_1^i) \subset g_i(W_2 \backslash \delta W_2) \cap L_1$.

Since $f_1 \mid \hat{K}_1^i \cap g_i(\delta W_1)$ is $(n-1)$-homotopic to the inclusion $\hat{K}_1^i \cap g_i(\delta W_1) \to g_i(\delta W_1) \to L_1$. By the Homotopy Extension Theorem [**4**, Proposition 2.2] or [**1**, Theorem 2.1.8], we may assume that $f_1 \mid \hat{K}_1^i \cap g_i(\delta W_1) = \mathrm{id}$. By the Z-set Approximation Theorem 2.6, we may further assume that f_1 is a Z-embedding such that $f_1(\hat{K}_1^i) \cap g_i(\delta W_1) = \hat{K}_1^i \cap g_i(\delta W_1)$. Note that $g_i(\delta W_1)$ is a Z-set in L_1. By the Z-set Unkotting Theorem 2.7, f_1 extends to a homeomorphism $\bar{f}_1 : L_1 \to L_1$ which satisfies:

(1.4) $\bar{f}_1 \mid g_i(\delta W_1) = \mathrm{id}$, $\bar{f}_1(\hat{K}_1^i) \subset g_i(W_2 \backslash (\delta W_2 \cup W_1))$, and

(1.5) $\bar{f}_1(\hat{K}_1^i) \cap g_i(\delta W_1) = \hat{K}_1^i \cap g_i(\delta W_1)$.

The desired open embedding $h_1 : \lambda^n \to \mathrm{Ker}_{n-1}(M)$ is defined as follows.

$$h_1 \mid W_1 = g_i \quad \text{and} \quad h_1 \mid \lambda^n \backslash W_1 = \bar{f}_1^{-1} \circ g_i.$$

This completes Step 1.

Step 2. *There exists an open embedding $h_2 : \lambda^n \to \mathrm{Ker}_{n-1}(M)$ such that $h_2 \mid W_2 = h_1$ and $h_2(W_3) \supset K_2^i$.*

Proof. Suppose that $\hat{K}_2^i = K_2^i \backslash h_1(W_2 \backslash \delta W_2) \neq \emptyset$ (otherwise, just take h_1 as h_2). As in Step 1 (1.1), one can show that

(2.1) there exist a map $f_2' : \hat{K}_2^i \to \mathrm{Ker}_{n-1}(M)$ which is $(n-1)$-homotopic to the inclusion $\hat{K}_2^i \to \mathrm{Ker}_{n-1}(M)$ in $\mathrm{Ker}_{n-1}(M)$ such that $f_2'(\hat{K}_2^i) \subset g_i(W_1 \backslash \delta W_1) \backslash P_i \times \{0\} = h_1(W_1 \backslash \delta W_1) \backslash P_i \times \{0\}$.

Let $L_2 = (\mathrm{Ker}_{n-1}(M) \backslash h_1(W_2)) \cup h_1(\delta W_2)$ which is a μ^n-manifold. Next we show that

(2.2) There exists a map $f_2'' : f_2'(\hat{K}_2^i) \to L_2$ such that $f_2 = f_2'' \circ f_2'$ is $(n-1)$-homotopic to the inclusion in L_2.

Here f_2'' is obtained as the composition of two maps that are similar to those described in (1.2). First we push $f_2'(\hat{K}_2^i)$ off $h_1(W_1)$ and next off $h_1(W_2)$, but staying in $h_1(W_3 \backslash \delta W_3)$. Let $\hat{f_2} : \mathrm{Ker}_{n-1}(M) \to \mathrm{Ker}_{n-1}(M)$ be the resulting map such that $\hat{f_2} = \mathrm{id}$ outside $h_1(W_3)$. A retraction $r_2 : \mathrm{Ker}_{n-1}(M) \to L_2$ is defined as the composition of two retractions

$$\mathrm{Ker}_{n-1}(M) \to (\mathrm{Ker}_{n-1}(M) \backslash h_1(W_1)) \cup h_1(\delta W_1) \to L_2.$$

As in (1.2), the composition $f_2'' = r_2 \circ f_2'$ is the desired map. Therefore,

(2.3) there exists a map $f_2 : \hat{K}_2^i \to L_2$ which is $(n-1)$-homotopic to the inclusion $K_1^i \hookrightarrow L_2$ in L_2 such that $f_2(\hat{K}_2^i) \subset h_1(W_3 \backslash \delta W_3) \backslash h_1(W_2)$.

By proceeding now the exactly in the same way as in Step 1, we have the desired open embedding h_2. This completes Step 2.

Continuing this process, we have a sequence $\{h_\ell : \lambda^n \to \mathrm{Ker}_{n-1}(M) \mid \ell = 1, 2, \ldots\}$ of open embeddings such that

(3) $h_\ell \mid W_\ell = h_{\ell-1}$ and $h_\ell(W_{\ell+1}) \supset K_\ell^i$ for each ℓ.

Using the same argument as in [**3**, 20.1], one can see that the limit $e_i = \lim_{\ell \to \infty} h_\ell$ is the desired open embedding.

Let $V_i = e_i(\lambda^n)$ and let $V = \bigcup_{i=1}^k V_i$. Clearly V is an open set in $\mathrm{Ker}_{n-1}(M)$ containing $\bigcup_{s=1}^\infty K_s$. As was previously mentioned, $\mathrm{Ker}_{n-1}(V)$ is homeomorphic to $\mathrm{Ker}_{n-1}(M)$.

This completes the construction of V.

It remains to construct an open cover of $\mathrm{Ker}_{n-1}(V)$, each member of which is homeomorphic to λ^n. This is carried out as follows. We consider the product space $P \times [0,1] \times [0,1]$ which contains the previous $P \times [0,1]$ as the subspace $P \times [0,1] \times \{0\}$. Take a triangulation $\mathscr{I}_0$ of $[0,1]$, and let $\mathscr{K} = \mathscr{P}_0 \times \mathscr{I}_0 \times \mathscr{I}_0$ be the cell complex structure of $P \times [0,1] \times [0,1]$. Begin the Lefshetz construction (as in Proposition 2.12) with $P \times [0,1] \times [0,1]$ and $\mathscr{K}$ to obtain another μ^n-manifold $\overline{\overline{M}}$ which is also homeomorphic to M. An important observation here is that

(4) $\overline{\overline{M}} \subset M \times [0,1]$.
We consider the previous open set V as a subset of $M \times \{0\}$ and let $\tilde{V} = V \times [0,1] \cap \overline{\overline{M}} \subset ((P \times [0,1]) \times [0,1]) \cap \overline{\overline{M}}$. It has clearly the same proper $(n-1)$-homotopy type as V. The above observation (4) implies that $\tilde{V}$ is open in $\overline{\overline{M}}$, and hence is a μ^n-manifold which is homeomorphic to V (Theorem 2.4). Furthermore, by the construction, we see that

$$\mathrm{Ker}_{n-1}(\tilde{V}) \approx \tilde{V} \cap (P \times [0,1] \times [0,1)) = \tilde{V} \backslash (V \times \{1\} \cap \overline{\overline{M}}).$$

Let $\tilde{V}_i = V_i \times [0,1) \cap \overline{\overline{M}} \subset \tilde{V}$. Then it is easy to verify the conditions of Theorem 2.14 (or to inspect the geometric situation directly) to see that $\tilde{V}_i \approx \lambda^n$.

Therefore $\{\tilde{V}_i \mid i = 1, \dots, k\}$ is the desired open cover of $\mathrm{Ker}_{n-1}(\tilde{V}) \approx \mathrm{Ker}_{n-1}(V)$. This completes the proof. □

APPENDIX. The monotone union of (λ^n)'s is λ^n.

In connection with Corollary 3.3, the following result would be worth mentioning. It is a Menger manifold analogue of Brown's Theorem [**2**].

THEOREM A. *Let $M = \bigcup_{i=1}^{\infty} U_i$ be a μ^n-manifold, where each U_i is an open set which is homeomorphic to λ^n such that $U_i \subset U_{i+1}$. Then M is homeomorphic to λ^n.*

As in the proof of Brown, the proof is based on the following engulfing lemma.

LEMMA B. *Let W_1 and W_2 be clean (μ^n)'s in λ^n such that $W_1 \subset W_2 \backslash \delta W_2$. For each compact set K of λ^n, there exists a homeomorphism $h : \lambda^n \to \lambda^n$ with compact support such that $h \mid W_1 = \mathrm{id}$ and $h(W_2) \supset K$.*

PROOF OF LEMMA. Let W_3 and W_4 be clean (μ^n)'s such that $W_2 \cup K \subset W_3 \backslash \delta W_3 \subset W_3 \subset W_4 \backslash \delta W_4$. For $i = 2, 3$, the subset $(W_i \backslash W_1) \cup \delta W_1$ is homeomorphic to μ^n in which $\delta W_1 \cup \delta W_i$ $(\approx \mu^n \oplus \mu^n)$ is contained as a Z-set. By the Z-set Unknotting Theorem, there exists a homeomorphism $h_1 : (W_2 \backslash W_1) \cup \delta W_1 \to (W_3 \backslash W_1) \cup \delta W_1$ such that $h_1 \mid \delta W_1 = \mathrm{id}$ and $h_1(\delta W_2) = \delta W_3$. Notice that $(W_4 \backslash W_i) \cup \delta W_i$ is homeomorphic to μ^n which contains $\delta W_4 \cup \delta W_1$ $(\approx \mu^n \oplus \mu^n)$ as a Z-set $(i = 2, 3)$. Applying the Z-set Unknotting Theorem again, we obtain a homeomorphism $h_2 : (W_4 \backslash W_2) \cup \delta W_2 \to (W_4 \backslash W_3) \cup \delta W_3$ such that $h_2 \mid \delta W_4 = \mathrm{id}$ and $h_2 \mid \delta W_2 = h_1 \mid \delta W_2$. Let $h : \lambda^n \to \lambda^n$ be the homeomorphism defined by $h \mid W_1 \cup (\lambda^n \backslash W_4) \cup \delta W_4 = \mathrm{id}$, $h \mid (W_2 \backslash W_1) \cup \delta W_1 = h_1$ and $h \mid (W_4 \backslash W_2) \cup \delta W_2 = \mathrm{id}$. Then $h(W_2) = W_3 \supset K$ and h is the desired homeomorphism. □

Having the above lemma at hand, the proof of Theorem proceeds exactly in the same way as the one of Brown's Theorem.

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Kazuhiro Kawamura
Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki 305-8571
Japan
E-mail: kawamura@math.tsukuba.ac.jp