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Asymptotic behavior of scattering amplitudes in magnetic fields at large separation

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1. Introduction.

In this paper we study the asymptotic behavior of scattering amplitudes by two magnetic fields at large separation in two dimensions. We denote by $x = (x_1, x_2)$ a generic point in $\boldsymbol{R}^2$, and we write

$$H(A) = (-i\nabla - A)^2 = \sum_{j=1}^{2} (-i\partial_j - a_j)^2, \quad \partial_j = \partial/\partial x_j,$$

for the Schrödinger operator with magnetic potential $A(x) = (a_1(x), a_2(x)) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$. The magnetic field $b(x)$ is defined as $b = \nabla \times A = \partial_1 a_2 - \partial_2 a_1$ and

$$\alpha = (2\pi)^{-1} \int b(x)\, dx$$

is called the total flux of field b, where the integration with no domain attached is taken over the whole space. This abbreviation is often used in the discussion below. We are now given two smooth magnetic fields $b_j \in C_0^\infty(\boldsymbol{R}^2)$, $j = 1, 2$, with compact support. Let $A_j(x)$, $\nabla \times A_j = b_j$, be the magnetic potential associated with b_j. For given magnetic field $b(x)$, the corresponding potential $A(x)$ is not uniquely determined, but the scattering amplitude is invariant under the gauge transformation $A \to A + \nabla g$. We fix one of such magnetic potentials. The precise form is specified in Section 2 (Proposition 2.1). We set

$$H_d = H(A_1 + A_{2d}) = (-i\nabla - A_1 - A_{2d})^2, \quad A_{2d}(x) = A_2(x - d), \tag{1.1}$$

for $d = (d_1, d_2) \in \boldsymbol{R}^2$ with $|d| \gg 1$, and we denote by $f_d(\omega \to \omega'; \lambda)$ the scattering amplitude for the pair (H_d, H_0), where $H_0 = -\Delta$ is the free Hamiltonian. The quantity $|f_d(\omega \to \omega'; \lambda)|^2$ is called the differential cross section for scattering from the initial direction $\omega \in S^1$ to the final direction ω' at energy $\lambda > 0$, S^1 being the unit circle. The precise representation for $f_d(\omega \to \omega'; \lambda)$ is also given in Section 2 (Proposition 2.2). The aim here is to study the asymptotic behavior of $f_d(\omega \to \omega'; \lambda)$ as the distance $|d|$ between the centers of fields $b_1(x)$ and $b_{2d}(x) =$

$b_2(x-d)$ goes to infinity. The two dimensional case is the most interesting. In fact, the magnetic effect is strongly reflected in this case for the topological reason that $\boldsymbol{R}^2\backslash\{0\}$ is not simply connected.

We shall formulate the obtained result. We set

$$H_j = H(A_j) = (-i\nabla - A_j)^2 \tag{1.2}$$

for $j = 1,2$, and denote by $f_j(\omega \to \omega'; \lambda)$ the scattering amplitude for the pair (H_j, H_0). We use the notation $\theta(x;\omega)$ to denote the azimuth angle from direction $\omega \in S^1$.

Theorem 1.1. *Let the notation be as above. Denote by α_j the total flux of field $b_j \in C_0^\infty(\boldsymbol{R}^2)$, $j = 1,2$, and define $\tau(d;\omega,\omega')$ by*

$$\tau(d;\omega,\omega') = \theta(d;\omega) - \theta(d;-\omega').$$

Assume that at least one of α_1 and α_2 is zero. If ω and ω' satisfy $\omega \neq \pm d/|d|$, $\omega' \neq \pm d/|d|$ and $\omega \neq \omega'$, then $f_d(\omega \to \omega';\lambda)$ behaves like

$$\begin{aligned} f_d(\omega \to \omega';\lambda) = {} & \exp(i\alpha_2\tau(-d;\omega,\omega'))f_1(\omega \to \omega';\lambda) \\ & + \exp(i\alpha_1\tau(d;\omega,\omega'))f_{2,d}(\omega \to \omega';\lambda) + o(1) \end{aligned} \tag{1.3}$$

as $|d| \to \infty$, where

$$f_{2,d}(\omega \to \omega';\lambda) = \exp(-i\sqrt{\lambda}d \cdot (\omega' - \omega))f_2(\omega \to \omega';\lambda)$$

is the scattering amplitude for the pair $(H_{2,d}, H_0)$ with $H_{2,d} = H(A_{2d})$.

We can derive the asymptotic formula even for the case $\omega = \pm\hat{d}$ or $\omega' = \pm\hat{d}$, where $\hat{d} = d/|d|$. For brevity, we assume that $\alpha_2 = 0$. Then $f_d(-\hat{d} \to \omega';\lambda)$ and $f_d(\omega \to \hat{d};\lambda)$ obey the same formula as (1.3). This can be seen from the proof of Theorem 1.1. However $f_d(\hat{d} \to \omega';\lambda)$ and $f_d(\omega \to -\hat{d};\lambda)$ take a different form. We can prove the following theorem.

Theorem 1.2. *Under the above situation, one has the following statements*:
(1) *If $\omega' \neq -\hat{d}$, then*

$$\begin{aligned} f_d(\hat{d} \to \omega';\lambda) = {} & f_1(\hat{d} \to \omega';\lambda) \\ & + (\cos\alpha_1\pi)\exp(i\alpha_1(\pi - \theta(d;-\omega')))f_{2,d}(\hat{d} \to \omega';\lambda) + o(1). \end{aligned}$$

(2) *If $\omega \neq \hat{d}$, then*

$$\begin{aligned} f_d(\omega \to -\hat{d};\lambda) = {} & f_1(\omega \to -\hat{d};\lambda) \\ & + (\cos\alpha_1\pi)\exp(i\alpha_1(\theta(d;\omega) - \pi))f_{2,d}(\omega \to -\hat{d};\lambda) + o(1). \end{aligned}$$

(3) *If* $\omega = \hat{d}$ *and* $\omega' = -\hat{d}$, *then*

$$f_d(\hat{d} \to -\hat{d}; \lambda) = f_1(\hat{d} \to -\hat{d}; \lambda) + (\cos \alpha_1 \pi)^2 f_{2,d}(\hat{d} \to -\hat{d}; \lambda) + o(1).$$

Magnetic potentials are not in general expected to fall off rapidly at infinity even if fields are assumed to be of compact support. Let $b \in C_0^\infty(\boldsymbol{R}^2)$. We define

$$A_b(x) = (a_{1b}(x), a_{2b}(x)) = (-\partial_2 \varphi(x), \partial_1 \varphi(x)) \tag{1.4}$$

with $\varphi = (2\pi)^{-1} \int \log|x - y| b(y)\, dy$. Then $\nabla \times A_b = \Delta\varphi = b$ and A_b becomes the potential associated with field b. However, if the flux α does not vanish, then $A_b(x)$ cannot decay faster than $O(|x|^{-1})$ at infinity. In fact, it behaves like

$$A_b(x) = A_\alpha(x) + O(|x|^{-2}), \quad |x| \to \infty, \tag{1.5}$$

where

$$A_\alpha(x) = \alpha(-x_2/|x|^2, x_1/|x|^2). \tag{1.6}$$

The motion of particles in quantum mechanical systems is subject to the influence of magnetic potentials as well as of magnetic fields. This fact can be found in the asymptotic formula (1.3). The phase factor $\exp(i\alpha_1 \tau(d; \omega, \omega'))$ depends on the flux α_1 of the field b_1. This means that b_1 has an influence upon the scattering by field $b_{2d} = \nabla \times A_{2d}$, although the support of b_1 is located in the long distance from that of b_{2d}. Such a quantum phenomenon is known as the Aharonov-Bohm effect ([**2**]). If, in particular, α_1 is a half-integer, then $f_d(\hat{d} \to \omega'; \lambda)$ obeys the asymptotic formula

$$f_d(\hat{d} \to \omega'; \lambda) = f_1(\hat{d} \to \omega'; \lambda) + o(1)$$

in Theorem 1.2. This means that the scattering by field b_{2d} does not make any contribution to the leading term in the asymptotic formula.

The present paper is motivated by the recent work [**7**], where the same problem has been studied in the case of scattering by potentials for the Schrödinger operator $-\Delta + V_1(x) + V_2(x - d)$ with potentials falling off rapidly at infinity. The case is quite different in the scattering by magnetic fields in two dimensions. Roughly speaking, the difference comes from the long-range property of magnetic potentials. Several new devices are required. We introduce

$$H_\alpha = H(A_\alpha) = (-i\nabla - A_\alpha)^2 \tag{1.7}$$

as an auxiliary operator, where $A_\alpha(x)$ is defined by (1.6). By (1.5), the difference $H(A_b) - H_\alpha$ becomes a perturbation of short-range class. The potential $A_\alpha(x)$ satisfies $\nabla \times A_\alpha = 2\pi\alpha\delta(x)$ in the distribution sense and the Hamiltonian H_α has a

δ-like magnetic field. The scattering amplitude $f_d(\omega \to \omega'; \lambda)$ is represented in terms of the eigenfunction of H_α. This Hamiltonian admits the partial wave expansion in polar coordinates and its outgoing eigenfunction takes the form

$$\varphi_+(x; \lambda, \omega) = \sum_{l \in Z} \exp(-i\nu\pi/2) \exp(il\theta(x; -\omega)) J_\nu(\sqrt{\lambda}|x|)$$

with $\nu = |l - \alpha|$, where $J_p(z)$ denotes the Bessel function of order p. One of important ingredients to prove the main theorems is the asymptotic behavior

$$\varphi_+(x; \lambda, \omega) = \exp(i\alpha(\theta(x; \omega) - \pi)) \exp(i\sqrt{\lambda}x \cdot \omega)(1 + o(1)), \quad \theta(x; \omega) \neq 0,$$

at infinity. This formula is known in the physical literatures (see [**2**], [**3**], [**9**] for example). If α is not an integer, the asymptotic form on the right side has the phase gap along direction ω. This makes the phase factors in the asymptotic formula (1.3).

We prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 6. The proof makes an essential use of the assumption that at least one of two flux α_1 and α_2 is zero. However the idea developed here, particularly in Section 4, seems to extend to the general case without such a restriction. We are going to discuss the detailed matter in another paper [**13**]. We end the section by making a comment on the extension to the case of several centers. If at most one field has a nonzero flux, then we can derive a similar asymptotic formula when all the distances between respective centers of fields go to infinity. We skip the details.

2. Scattering by magnetic field.

In this section we make a brief review on the scattering by magnetic fields with compact support. The aim here is to derive the representation formula for scattering amplitudes. The result is mentioned as Proposition 2.2 at the end of the section. The derivation is rather formal. The rigorous treatment can be found in [**10**], [**11**].

2.1. We begin by specifying the form of magnetic potential. Let $b \in C_0^\infty(\boldsymbol{R}^2)$ be given smooth magnetic field with compact support

$$\operatorname{supp} b \subset \{x \in \boldsymbol{R}^2 : |x| < M\} \tag{2.1}$$

for some $M > 0$, and we denote by α the total flux of b. We construct the magnetic potential $A(x)$, $\nabla \times A = b$, which has the property $A(x) = A_\alpha(x)$ for $|x| \gg 1$ large enough, where $A_\alpha(x)$ is defined by (1.6). Let $A_b(x) = (a_{1b}(x), a_{2b}(x))$ be defined by (1.4). By (1.5), we have $x_1 a_{1b}(x) + x_2 a_{2b}(x) = O(|x|^{-1})$ as $|x| \to \infty$, and hence we can define $a_b(x)$ as

$$a_b(x) = -\int_1^\infty (x_1 a_{1b}(sx) + x_2 a_{2b}(sx))\, ds$$

for $x \neq 0$.

Lemma 2.1. *One has the relation*

$$A_b(x) = A_\alpha(x) + \nabla a_b(x) + E(x)$$

for $x \neq 0$, *where* $E(x) = (e_1(x), e_2(x))$ *is given by*

$$e_1(x) = \int_1^\infty sx_2 b(sx)\, ds, \quad e_2(x) = -\int_1^\infty sx_1 b(sx)\, ds.$$

Proof. We set $b_{jk} = \partial_j a_{kb} - \partial_k a_{jb}$, $1 \leq j,\ k \leq 2$, for $A_b = (a_{1b}, a_{2b})$, so that $b(x) = b_{12}(x) = -b_{21}(x)$. A simple calculation yields

$$\partial_j a_b(x) = -\int_1^\infty (a_{jb}(sx) + s(d/ds)a_{jb}(sx) + sx_k b_{jk}(sx))\, ds$$

for $k \neq j$ and hence we obtain

$$\partial_j a_b(x) = a_{jb}(x) - \int_1^\infty sx_k b_{jk}(sx)\, ds - \lim_{R\to\infty} Ra_{jb}(Rx)$$

by partial integration. By (1.5), $RA_b(Rx) \to A_\alpha(x)$ as $R \to \infty$. This proves the lemma. □

We now introduce a cut-off function $\chi \in C_0^\infty[0, \infty)$ with the following properties: $\chi(s) \geq 0$ is nonnegative and

$$\chi(s) = 1 \quad \text{for } 0 \leq s \leq 1, \quad \chi(s) = 0 \quad \text{for } s > 2. \tag{2.2}$$

Let $M > 0$ be as in (2.1) and let $E(x)$ be as in Lemma 2.1. We set $\tilde{\chi}_M(x) = \chi(r/M)$ with $r = |x|$, and $\tilde{\chi}_{\infty M}(x) = 1 - \tilde{\chi}_M(x)$. Then $E(x)$ vanishes on the support of $\tilde{\chi}_{\infty M}$ and hence $A_b(x)$ is decomposed into

$$A_b = (\tilde{\chi}_{\infty M} + \tilde{\chi}_M)A_b = A(x) + \nabla(\tilde{\chi}_{\infty M} a_b)$$

by Lemma 2.1, where

$$A(x) = \tilde{\chi}_{\infty M}(x) A_\alpha(x) + B(x)$$

with $B = a_b \nabla \tilde{\chi}_M + \tilde{\chi}_M A_b$. The magnetic potential $A(x)$ still has $b(x)$ as a field, and it satisfies $A(x) = A_\alpha(x)$ for $|x| > 2M$. Thus we have proved the following proposition.

Proposition 2.1. *Let* $b \in C_0^\infty(\boldsymbol{R}^2)$ *be given smooth magnetic field with flux* α. *If* $b(x)$ *has support in* $\{|x| < M\}$ *for some* $M > 0$, *then there exists a smooth magnetic potential* $A(x)$ *associated with* b *such that*

$$A(x) = A_\alpha(x) = \alpha(-x_2/|x|^2, x_1/|x|^2), \quad |x| > 2M.$$

2.2. We fix the potential $A(x)$ as in the above proposition, and we write H for the operator $H(A) = (-i\nabla - A)^2$ throughout the section. This operator admits a unique self-adjoint realization in $L^2(\boldsymbol{R}^2)$. We denote by the same notation H this self-adjoint operator, which has the domain $\mathscr{D}(H) = H^2(\boldsymbol{R}^2)$, $H^s(\boldsymbol{R}^2)$ being the Sobolev space of order s. The operator H is known to have the following spectral properties ([**6**]): (1) H has no positive bound state energies; (2) The resolvents

$$R(\lambda \pm i\varepsilon; H) = (H - \lambda \mp i\varepsilon)^{-1}, \quad \varepsilon > 0,$$

have the boundary values to the positive axis

$$R(\lambda \pm i0; H) = \lim_{\varepsilon \to 0} R(\lambda \pm i\varepsilon; H), \quad \lambda > 0,$$

as an operator from the weighted space $L^2_s(\boldsymbol{R}^2) = L^2(\boldsymbol{R}^2; \langle x \rangle^{2s}\, dx)$ into $L^2_{-s}(\boldsymbol{R}^2)$ for $s > 1/2$, where $\langle x \rangle = (1 + |x|^2)^{1/2}$.

By Proposition 2.1, the difference $H - H_0$ between H and the free Hamiltonian $H_0 = -\Delta$ is a perturbation of long-range class. Nevertheless the ordinary wave operators

$$W_\pm(H, H_0) = s - \lim_{t \to \pm\infty} \exp(itH)\exp(-itH_0) : L^2(\boldsymbol{R}^2) \to L^2(\boldsymbol{R}^2)$$

are known to exist and to be asymptotically complete ([**8**]). Hence the scattering operator

$$S(H, H_0) = W_+^*(H, H_0) W_-(H, H_0) : L^2(\boldsymbol{R}^2) \to L^2(\boldsymbol{R}^2)$$

can be defined as a unitary operator.

Recall that $\theta(x; \omega)$ denotes the azimuth angle from direction $\omega \in S^1$. Let

$$\varphi_0(x; \lambda, \omega) = \exp(i\sqrt{\lambda} x \cdot \omega), \quad \lambda > 0, \quad \omega \in S^1,$$

be the generalized eigenfunction of H_0, $H_0 \varphi_0 = \lambda \varphi_0$, where the notation $\cdot$ denotes the scalar product in $\boldsymbol{R}^2$. As is well known, φ_0 is expanded as

$$\varphi_0(x; \lambda, \omega) = \sum_{l \in Z} \exp(i|l|\pi/2) \exp(il\theta(x; \omega)) J_{|l|}(\sqrt{\lambda}|x|) \tag{2.3}$$

in terms of the Bessel functions $J_p(z)$. If we define the unitary mapping $\mathscr{F}$ from $L^2(\boldsymbol{R}^2)$ to $L^2((0, \infty); d\lambda) \otimes L^2(S^1)$ by

$$(\mathscr{F}u)(\lambda, \omega) = 2^{-1/2}(2\pi)^{-1} \int \bar{\varphi}_0(x; \lambda, \omega) u(x)\, dx,$$

then H_0 is diagonalized as $\mathscr{F}^* H_0 \mathscr{F} = \lambda\times$ on the space $L^2((0,\infty); d\lambda) \otimes L^2(S^1)$, and the scattering operator $S(H, H_0)$ has the direct integral decomposition

$$S(H, H_0) \simeq \mathscr{F} S(H, H_0) \mathscr{F}^* = \int_0^\infty \oplus\, S(\lambda; H, H_0)\, d\lambda,$$

where the fibre $S(\lambda; H, H_0) : L^2(S^1) \to L^2(S^1)$ is called the scattering matrix at energy $\lambda > 0$ and it acts as

$$(\mathscr{F} S(H, H_0) u)(\lambda, \omega) = (S(\lambda; H, H_0)(\mathscr{F} u)(\lambda, \cdot))(\omega)$$

for $u \in L^2(\boldsymbol{R}^2)$.

2.3. We proceed to the representation for the integral kernel of $S(\lambda; H, H_0)$. Let $H_\alpha = H(A_\alpha)$ be as in (1.7). By Proposition 2.1, the perturbation $H - H_\alpha$ is of short-range class. We represent the kernel of $S(\lambda; H, H_0)$ as the sum of the kernels of two scattering matrices $S(\lambda; H_\alpha, H_0)$ and $S(\lambda; H, H_\alpha)$.

We first consider the kernel of $S(\lambda; H_\alpha, H_0)$. The operator H_α is rotationally invariant. We work in the polar coordinate system (r, θ) and write $L^2(\boldsymbol{R}_+)$ for $L^2((0,\infty); dr)$. Let Λ_l, $l \in \boldsymbol{Z}$, be the eigenspace associated with eigenvalue l of operator $-i\partial/\partial\theta$ acting on $L^2(S^1)$. Then we have the decomposition

$$L^2(\boldsymbol{R}_+) \otimes L^2(S^1) = \sum_{l \in Z} \oplus\, (L^2(\boldsymbol{R}_+) \otimes \Lambda_l).$$

If we define the unitary mapping

$$(Uu)(r, \theta) = r^{1/2} u(r\theta) : L^2(\boldsymbol{R}^2) \to L^2(\boldsymbol{R}_+) \otimes L^2(S^1),$$

then U yields the partial wave expansion for H_α. We formally write

$$H_\alpha \simeq U H_\alpha U^* = \sum_{l \in Z} \oplus\, (H_{l\alpha} \otimes Id),$$

where Id is the identity operator and $H_{l\alpha}$ is given by

$$H_{l\alpha} = -\partial_r^2 + (v^2 - 1/4) r^{-2}$$

with $v = |l - \alpha|$. The operator $H_{l\alpha}$ is self-adjoint in $L^2(\boldsymbol{R}_+)$ with domain

$$\mathscr{D}(H_{l\alpha}) = \{u \in L^2(\boldsymbol{R}_+) : H_{l\alpha} u \in L^2(\boldsymbol{R}_+),\ \lim_{r \to 0} r^{-1/2} u(r) < \infty\}$$

and hence H_α also becomes self-adjoint in $L^2(\boldsymbol{R}^2)$ with domain

$$\mathscr{D}(H_\alpha) = \{u \in L^2(\boldsymbol{R}^2) : H_\alpha u \in L^2(\boldsymbol{R}^2),\ \lim_{r = |x| \to 0} |u(x)| < \infty\}.$$

The potential $A_\alpha(x)$ has a strong singularity at the origin, so that the domain $\mathscr{D}(H_\alpha)$ does not necessarily coincide with the domain of H and H_0. Nevertheless it is known ([**10**]) that the wave operators $W_\pm(H_\alpha, H_0)$ exist and are asymptotically complete $\mathrm{Ran}\, W_\pm(H_\alpha, H_0) = L^2(\boldsymbol{R}^2)$.

We shall define the generalized eigenfunction $\varphi_\mp(x;\lambda,\omega)$ of H_α. To do this, we make use of expansion formula (2.3) for $\varphi_0(x;\lambda,\omega)$. We set

$$e_{\mp l}(r) = \exp(\pm i|l|\pi/2)J_{|l|}(r) - \exp(\pm i\nu\pi/2)J_\nu(r).$$

The Bessel function $J_p(r)$ obeys the asymptotic formula

$$J_p(r) = (2/\pi)^{1/2} r^{-1/2}\cos(r - (2p+1)\pi/4)(1 + g_m(r)) + O(r^{-m}), \quad r \to \infty,$$

for any $m \gg 1$ large enough, where $g_m(r)$ obeys $(d/dr)^k g_m(r) = O(r^{-1-k})$. Hence it follows that

$$e_{\mp l}(r) = \exp(\mp ir)(C_{\mp l} r^{-1/2} + O(r^{-3/2})) + \exp(\pm ir)O(r^{-3/2})$$

for some constant $C_{\mp l} \neq 0$. By definition, $e_{-l}(r)$ satisfies the incoming radiation condition $e'_{-l} + ie_{-l} = O(r^{-3/2})$ at infinity, while $e_{+l}(r)$ satisfies the outgoing radiation condition $e'_{+l} - ie_{+l} = O(r^{-3/2})$. Thus, if we take account of formula (2.3) and of the simple relation

$$\exp(il\theta(x;-\omega)) = \exp(i|l|\pi + il\theta(x;\omega))$$

between azimuth angles $\theta(x;\omega)$ and $\theta(x;-\omega)$, then $\varphi_\mp$ is defined by

$$\varphi_\mp(x;\lambda,\omega) = \sum_{l\in Z} \exp(\pm i\nu\pi/2)\exp(il\theta(x;\pm\omega))J_\nu(\sqrt{\lambda}|x|) \tag{2.4}$$

with $\nu = |l - \alpha|$ again. The series converges locally uniformly and $\varphi_\mp$ satisfies $H_\alpha\varphi_\mp = \lambda\varphi_\mp$. The eigenfunction $\varphi_\mp$ is formally represented as $\varphi_\mp = W_\pm(H_\alpha, H_0)\varphi_0$ by using the intertwining property of wave operators.

We often identify the coordinates over the unit circle S^1 with the azimuth angles from the positive x_1 axis. The scattering matrix $S(\lambda; H_\alpha, H_0) : L^2(S^1) \to L^2(S^1)$ has the property

$$S(\lambda; H_\alpha, H_0) : \bar{\varphi}_+(x;\lambda,\cdot) \to \bar{\varphi}_-(x;\lambda,\cdot).$$

Since

$$\exp(i\nu\pi/2)\exp(-il\theta(x;-\omega)) = \exp(i(\nu - l)\pi)\exp(-i\nu\pi/2)\exp(-il\theta(x;\omega))$$

by a simple computation, $S(\lambda; H_\alpha, H_0)$ has the kernel

$$S(\omega',\omega;\lambda,H_\alpha,H_0) = (2\pi)^{-1}\sum_{l\in Z}\exp(i(l-\nu)\pi)\exp(il(\omega'-\omega)).$$

According to [**10**], the sum on the right side equals

$$\sum_{l\in Z}\exp(i(l-\nu)\pi)\exp(il\theta)=2\pi(\cos\alpha\pi\delta(\theta)-(i\sin\alpha\pi/\pi)e^{i[\alpha]\theta}F_0(\theta)),$$

where $[\alpha]$ denotes the usual Gauss notation and $F_0(\theta)$ is defined by

$$F_0(\theta)=\text{v.p.}\, e^{i\theta}/(e^{i\theta}-1).$$

Thus we obtain

$$S(\omega',\omega;\lambda,H_\alpha,H_0)=\cos\alpha\pi\delta(\omega'-\omega)-(i\sin\alpha\pi/\pi)e^{i[\alpha](\omega'-\omega)}F_0(\omega'-\omega).$$

This kernel has been calculated by [**2**], [**10**]. We refer to [**1**], [**4**], [**12**] for the recent works related to the spectral theory for Hamiltonians with δ-like magnetic fields.

2.4. We shall derive the representation for the kernel of $S(\lambda;H,H_0)$. By the chain rule of wave operators, we have

$$W_\pm(H,H_0)=W_\pm(H,H_\alpha)W_\pm(H_\alpha,H_0)$$

and hence

$$S(H,H_0)=W_+^*(H_\alpha,H_0)S(H,H_\alpha)W_-(H_\alpha,H_0),$$

where $S(H,H_\alpha)=W_+^*(H,H_\alpha)W_-(H,H_\alpha)$. The existence and completeness of wave operators $W_\pm(H,H_\alpha)$ follow from those of $W_\pm(H_\alpha,H_0)$ and $W_\pm(H,H_0)$ at once. Thus $S(H,H_0)$ is decomposed into the sum

$$S(H,H_0)=S(H_\alpha,H_0)+W_+^*(H_\alpha,H_0)(S(H,H_\alpha)-Id)W_-(H_\alpha,H_0). \tag{2.5}$$

We have already calculated the kernel of $S(\lambda;H_\alpha,H_0)$. Let S_2 denote the second operator on the right side of (2.5) and we consider the operator

$$\mathscr{F}S_2\mathscr{F}^*:L^2((0,\infty);d\lambda)\otimes L^2(S^1)\to L^2((0,\infty);d\lambda)\otimes L^2(S^1).$$

If we make use of the formal relation $\varphi_\mp=W_\pm(H_\alpha,H_0)\varphi_0$, then this operator has the kernel

$$S_2(\omega',\omega;\lambda',\lambda)=2^{-1}(2\pi)^{-2}((S(H,H_\alpha)-Id)\varphi_+(\cdot\,;\lambda,\omega),\varphi_-(\cdot\,;\lambda',\omega')),$$

where the notation (,) stands for the L^2 scalar product in $L^2(\boldsymbol{R}^2)$. We set

$$\chi_{\infty M}(x)=1-\chi_M(x),\quad \chi_M(x)=\chi_M(r)=\chi(r/2M),$$

for the cut-off function $\chi(s)\in C_0^\infty[0,\infty)$ with property (2.2). The function $\chi_{\infty M}$ has support in $\{|x|>2M\}$. By Proposition 2.1, $A(x)=A_\alpha(x)$ on the support of $\chi_{\infty M}$ and hence $H=H_\alpha$ there. The wave operator $W_\pm(H,H_\alpha)$ is expressed through the limit

$$W_{\pm}(H, H_\alpha) = s - \lim_{t \to \pm\infty} \exp(itH)\chi_{\infty M} \exp(-itH_\alpha)$$

and hence we have

$$\begin{aligned} S(H, H_\alpha) - Id &= \exp(itH_\alpha)\chi_{\infty M} \exp(-itH) W_-(H, H_\alpha)\Big|_{t=-\infty}^{t=\infty} \\ &= i \int \exp(itH_\alpha) D_M W_-(H, H_\alpha) \exp(-itH_\alpha)\, dt \end{aligned}$$

by the intertwining property, where

$$D_M = H_\alpha \chi_{\infty M} - \chi_{\infty M} H = H\chi_{\infty M} - \chi_{\infty M} H = [H, \chi_{\infty M}] = [\chi_M, H]. \tag{2.6}$$

Since H_α and $\chi_{\infty M}$ are rotationally invariant, we may write the commutator D_M as

$$D_M = [H_\alpha, \chi_{\infty M}] = [H_0, \chi_{\infty M}] = [\chi_M, H_0]$$

and D_M fulfills the relation $D_M^* = -D_M$. We insert this integral representation. If we further make use of relation

$$\exp(-itH_\alpha)\varphi_{\mp} = \exp(-it\lambda)\varphi_{\mp}$$

and of formula

$$\int \exp(it(\lambda' - \lambda))\, dt = 2\pi\delta(\lambda' - \lambda),$$

then we obtain

$$S_2(\omega', \omega; \lambda', \lambda) = (i/4\pi) I(\omega', \omega; \lambda', \lambda)\delta(\lambda' - \lambda)$$

by a formal computation, where

$$I(\omega', \omega; \lambda', \lambda) = -(W_-(H, H_\alpha)\varphi_+(\cdot\,; \lambda, \omega), D_M\varphi_-(\cdot\,; \lambda', \omega')).$$

Since $W_-(H, H_\alpha)$ is represented in the integral form

$$W_-(H, H_\alpha) = \chi_{\infty M} - i \int_{-\infty}^{0} \exp(itH) D_M \exp(-itH_\alpha)\, dt,$$

we have

$$S_2(\omega', \omega; \lambda', \lambda) = (i/4\pi) I(\omega', \omega; \lambda)\delta(\lambda' - \lambda)$$

again by a formal computation, where

$$I(\omega', \omega; \lambda) = ((-\chi_{\infty M} + R(\lambda + i0; H) D_M)\varphi_+(\cdot\,; \lambda, \omega), D_M\varphi_-(\cdot\,; \lambda, \omega')).$$

Thus $S(\lambda; H, H_0) : L^2(S^1) \to L^2(S^1)$ has the integral kernel

$$S(\omega', \omega; \lambda, H, H_0) = S(\omega', \omega; \lambda, H_\alpha, H_0) + S_2(\omega', \omega; \lambda),$$

where

$$S_2(\omega', \omega; \lambda) = -(i/4\pi)(\psi_+(\cdot\,; \lambda, \omega), D_M \varphi_-(\cdot\,; \lambda, \omega'))$$

with

$$\psi_+(x; \lambda, \omega) = (\chi_{\infty M} - R(\lambda + i0; H) D_M)\varphi_+(x; \lambda, \omega). \tag{2.7}$$

As is easily seen, $\psi_+(x; \lambda, \omega)$ is a unique solution to equation $(H - \lambda)\psi_+ = 0$ such that $\psi_+ - \varphi_+(x; \lambda, \omega)$ satisfies the outgoing radiation condition at infinity.

We now define the scattering amplitude in question. The scattering amplitude $f(\omega \to \omega'; \lambda)$ for scattering from initial direction $\omega \in S^1$ to final one ω' at energy $\lambda > 0$ is defined by

$$f(\omega \to \omega'; \lambda) = c(\lambda)(S(\omega', \omega; \lambda, H, H_0) - \delta(\omega' - \omega))$$

with $c(\lambda) = (2\pi/i\sqrt{\lambda})^{1/2}$. We obtain the following proposition.

Proposition 2.2. *Assume that* $\omega \neq \omega'$. *Then the scattering amplitude* $f(\omega \to \omega'; \lambda)$ *is represented as*

$$f(\omega \to \omega'; \lambda) = c(\lambda)(f_\alpha(\omega' - \omega) - (i/4\pi) g_\alpha(\omega \to \omega'; \lambda))$$

with $c(\lambda) = (2\pi/i\sqrt{\lambda})^{1/2}$, *where*

$$f_\alpha(\omega' - \omega) = -(i \sin \alpha\pi/\pi) e^{i[\alpha](\omega' - \omega)} F_0(\omega' - \omega)$$

with $F_0(\theta) = e^{i\theta}/(e^{i\theta} - 1)$, *and*

$$g_\alpha(\omega \to \omega'; \lambda) = (\psi_+(\cdot\,; \lambda, \omega), D_M \varphi_-(\cdot\,; \lambda, \omega'))$$

with $\psi_+(x, ; \lambda, \omega)$ *defined by* (2.7).

3. Proof of Theorem 1.1: reduction to three lemmas.

In this section we prove Theorem 1.1. The proof is done by reduction to three lemmas. We recall the notation. Let $b_j \in C_0^\infty(\boldsymbol{R}^2)$, $j = 1, 2$, be two given magnetic fields with total flux α_j. We assume that b_j has support in the unit ball $\{|x| < 1\}$. By Proposition 2.1, we can construct the smooth magnetic potential $A_j(x)$ associated with b_j such that $A_j(x) = A_{\alpha_j}(x)$ for $|x| > 2$, where

$$A_{\alpha_j}(x) = \alpha_j(-x_2/|x|^2, x_1/|x|^2) \tag{3.1}$$

for $j = 1, 2$. We define $H_j = H(A_j)$ with this potential $A_j(x)$ and denote by $f_j(\omega \to \omega'; \lambda)$ the scattering amplitude for the pair (H_j, H_0).

Proof of Theorem 1.1. For brevity, we assume that $\alpha_2 = 0$. Then the field $b(x) = b_1(x) + b_2(x - d)$ has the flux $\alpha = \alpha_1$ and is supported in $\{|x| < M\}$ with $M = |d| + 1$. According to Proposition 2.1, there exists a magnetic potential $A(x)$ associated with b such that

$$A(x) = A_\alpha(x) = \alpha(-x_2/|x|^2, x_1/|x|^2)$$

for $|x| > 2M$. Note that $A_2(x) = 0$ for $|x| > 2$.

We set $H = H(A)$. By Proposition 2.2, the scattering amplitude $f_d(\omega \to \omega'; \lambda)$ for the pair (H, H_0) takes the form

$$f_d(\omega \to \omega'; \lambda) = c(\lambda)(f_\alpha(\omega' - \omega) - (i/4\pi)g_\alpha(\omega \to \omega'; \lambda)) \tag{3.2}$$

under natural modification of the notation in Proposition 2.2. Since $\alpha_2 = 0$, we have $f_{\alpha_2}(\theta) = 0$ and

$$f_\alpha(\theta) = f_{\alpha_1}(\theta) = -(i \sin \alpha_1\pi/\pi) \exp(i[\alpha_1]\theta) F_0(\theta)$$

for $\theta \neq 0$. Hence the first term $f_\alpha(\omega' - \omega)$ on the right side of (3.2) is decomposed into

$$f_\alpha(\omega' - \omega) = f_{\alpha_1}(\omega' - \omega) + \exp(i\alpha_1\tau(d; \omega, \omega')) f_{\alpha_2}(\omega' - \omega). \tag{3.3}$$

On the other hand, the second term $g_\alpha(\omega \to \omega'; \lambda)$ is represented as

$$g_\alpha(\omega \to \omega'; \lambda) = (\psi_+(\cdot\,; \lambda, \omega), D_M \varphi_-(\cdot\,; \lambda, \omega')),$$

where $\varphi_\mp(x; \lambda, \omega)$ is the generalized eigenfunction of the Hamiltonian $H_\alpha = H(A_\alpha) = H(A_{\alpha_1})$ with δ-like magnetic field $2\pi\alpha_1\delta(x)$ at the origin.

The two Hamiltonians H and H_d have the same magnetic field b and hence

$$H = e^{ig} H_d e^{-ig}$$

for some smooth real function $g(x)$. The function g satisfies

$$A(x) = A_1(x) + A_2(x - d) + \nabla g(x)$$

and we have $\nabla g = 0$ for $|x| \gg 1$ large enough. This function is uniquely determined up to a constant. If $g(x) \to 0$ at infinity, then

$$e^{ig(x)} = 1, \quad |x| \gg 1. \tag{3.4}$$

We turn back to $g_\alpha(\omega \to \omega'; \lambda)$. This term is rewritten as

$$g_\alpha(\omega \to \omega'; \lambda) = (\psi_{+g}(\cdot\,; \lambda, \omega), e^{-ig} D_M \varphi_-(\cdot\,; \lambda, \omega')) \tag{3.5}$$

with $\psi_{+g}(x; \lambda, \omega) = e^{-ig(x)}\psi_+(x; \lambda, \omega)$. Recall that $\psi_+(x; \lambda, \omega)$ satisfies $(H - \lambda)\psi_+ = 0$ and that $\psi_+ - \varphi_+$ obeys the outgoing radiation condition at infinity. Hence

$\psi_{+g}(x;\lambda,\omega)$ is a unique solution to $(H_d-\lambda)\psi_{+g}=0$ such that $\psi_{+g}-\varphi_+$ obeys the outgoing radiation condition. By construction, $A_1(x)=A_{\alpha_1}(x)$ for $|x|>2$, and $A_{2d}(x)=A_2(x-d)=0$ for $|x-d|>2$. Hence

$$H_d = H(A_1) = H_{\alpha_1} \tag{3.6}$$

on the region $\Pi_d=\{|x|>2\}\cap\{|x-d|>2\}$. This implies that $\varphi_+(x;\lambda,\omega)$ satisfies $(H_d-\lambda)\varphi_+=0$ in Π_d. We now set

$$\chi_0(x)=\chi_0(r)=\chi(r/2), \quad \chi_{0d}(x)=\chi_0(x-d) \tag{3.7}$$

and $\chi_{\infty d}(x)=1-\chi_0(x)-\chi_{0d}(x)$ for the cut-off function $\chi\in C_0^\infty[0,\infty)$ with property (2.2). The function $\chi_{\infty d}$ has support in Π_d and $\chi_0(x)\chi_{0d}(x)=0$ for $|d|\gg 1$. By uniqueness theorem, the solution $\psi_{+g}(x;\lambda,\omega)$ is represented as

$$\psi_{+g}=(\chi_{\infty d}-R(\lambda+i0;H_d)D_{1d}-R(\lambda+i0;H_d)D_{2d})\varphi_+, \tag{3.8}$$

where

$$D_{1d}=[\chi_0,H_d], \quad D_{2d}=[\chi_{0d},H_d]. \tag{3.9}$$

We consider the term $e^{-ig}D_M\varphi_-(x;\lambda,\omega')$ on the right side of (3.5). By (3.6), $\varphi_-(x;\lambda,\omega')$ obeys $\chi_{\infty d}(H_d-\lambda)\varphi_-=0$. Recall the notation $D_M=[H,\chi_{\infty M}]$ in (2.6) with $M=|d|+1$. Since

$$\chi_{\infty M}(H-\lambda)\varphi_-=\chi_{\infty M}(H_{\alpha_1}-\lambda)\varphi_-=0,$$

we may write the term $e^{-ig}D_M\varphi_-$ as

$$e^{-ig}D_M\varphi_-=e^{-ig}(H-\lambda)\chi_{\infty M}\varphi_-=(H_d-\lambda)\chi_{\infty M}e^{-ig}\varphi_-.$$

Thus we obtain

$$e^{-ig}D_M\varphi_-=(H_d-\lambda)(\chi_{\infty M}e^{-ig}\varphi_--\chi_{\infty d}\varphi_-)+(D_{1d}+D_{2d})\varphi_-, \tag{3.10}$$

where D_{1d} and D_{2d} are the commutators defined in (3.9). By (3.4), $e^{-ig}\varphi_--\varphi_-$ has compact support and ψ_{+g} obeys $(H_d-\lambda)\psi_{+g}=0$. We combine (3.10) with (3.8) to obtain that $g_\alpha(\omega\to\omega';\lambda)$ admits the decomposition

$$g_\alpha(\omega\to\omega';\lambda)=\gamma_1(d)+\gamma_2(d)+\gamma_{12}(d)+\gamma_{21}(d),$$

where

$$\gamma_1(d)=(((1-\chi_0)-R(\lambda+i0;H_d)D_{1d})\varphi_+(\cdot\,;\lambda,\omega),D_{1d}\varphi_-(\cdot\,;\lambda,\omega'))$$

$$\gamma_2(d)=(((1-\chi_{0d})-R(\lambda+i0;H_d)D_{2d})\varphi_+(\cdot\,;\lambda,\omega),D_{2d}\varphi_-(\cdot\,;\lambda,\omega'))$$

$$\gamma_{jk}(d)=-(R(\lambda+i0;H_d)D_{jd}\varphi_+(\cdot\,;\lambda,\omega),D_{kd}\varphi_-(\cdot\,;\lambda,\omega')).$$

This is the desired representation.

Lemma 3.1. *One has*

$$\gamma_{12}(d), \quad \gamma_{21}(d) \to 0$$

as $|d| \to \infty$.

Lemma 3.2. *Let* $D_0 = [\chi_0, H_0]$. *Then one has*

$$\gamma_1(d) = g_{\alpha_1}(\omega \to \omega'; \lambda) + o(1)$$

as $|d| \to \infty$, *where*

$$g_{\alpha_1}(\omega \to \omega'; \lambda) = (\psi_{+1}(\cdot\,; \lambda, \omega), D_0\varphi_-(\cdot\,; \lambda, \omega'))$$

and

$$\psi_{+1}(x; \lambda, \omega) = ((1 - \chi_0) - R(\lambda + i0; H_1)D_0)\varphi_+(x; \lambda, \omega)$$

is a unique solution to $(H_1 - \lambda)\psi_{+1} = 0$ *such that* $\psi_{+1} - \varphi_+$ *obeys the outgoing radiation condition at infinity.*

Lemma 3.3. *Let* $\varphi_0(x; \lambda, \omega) = \exp(i\sqrt{\lambda}x \cdot \omega)$ *and let* $D_0 = [\chi_0, H_0]$ *be as in Lemma* 3.2. *Then one has*

$$\gamma_2(d) = \exp(i\alpha_1\tau(d; \omega, \omega'))\exp(-i\sqrt{\lambda}d \cdot (\omega' - \omega))g_{\alpha_2}(\omega \to \omega'; \lambda) + o(1)$$

as $|d| \to \infty$, *where*

$$g_{\alpha_2}(\omega \to \omega'; \lambda) = (\psi_{+2}(\cdot\,; \lambda, \omega), D_0\varphi_0(\cdot\,; \lambda, \omega'))$$

and

$$\psi_{+2}(x; \lambda, \omega) = ((1 - \chi_0) - R(\lambda + i0; H_2)D_0)\varphi_0(x; \lambda, \omega)$$

is a unique solution to $(H_2 - \lambda)\psi_{+2} = 0$ *such that* $\psi_{+2} - \varphi_0$ *obeys the outgoing radiation condition at infinity.*

We shall complete the proof of the theorem, accepting the above lemmas as proved. These lemmas are proved in Section 5. If we recall the representation for $f_j(\omega \to \omega'; \lambda)$ in Proposition 2.2, then the theorem is obtained from (3.3) as an immediate consequence of the three lemmas. □

4. Resolvent estimates.

In this section we study the resolvent estimate for $R(\lambda + i0; H_d)$ as a preliminary step toward the proof of the three lemmas in the previous section. Throughout the section, the flux α_2 is still assumed to be zero.

4.1. We require several auxiliary operators. Let $\zeta_d \in C^\infty(\boldsymbol{R})$ be a real-valued periodic function with period 2π such that

$$\zeta_d(s) = \alpha_1 s, \quad |d|^{-\sigma} < s < 2\pi - |d|^{-\sigma}, \tag{4.1}$$

for $0 < \sigma \ll 1$ fixed small enough, and

$$|\zeta_d'(s)| = |(d/ds)\zeta_d(s)| \le c|d|^{\sigma}, \quad |\zeta_d''(s)| \le c|d|^{2\sigma}$$

for $c > 0$ independent of $|d| \gg 1$. We define $\eta_d(x) = \zeta_d(\theta(x; -\hat{d}))$ with $\hat{d} = d/|d| \in S^1$. By definition, we have

$$\nabla\eta_d(x) = \zeta_d'(\theta(x; -\hat{d}))\nabla\theta(x; -\hat{d}) = \zeta_d'(\theta(x; -\hat{d}))(-x_2/|x|^2, x_1/|x|^2)$$

and hence it follows from (4.1) that $\nabla\eta_d(x) = A_{\alpha_1}(x)$ on the cone region

$$\Gamma_d = \{x \in \boldsymbol{R}^2 : |d|^{-\sigma} < \theta(x; -\hat{d}) < 2\pi - |d|^{-\sigma}\}. \tag{4.2}$$

We can construct a smooth (not necessarily real-valued) function $p_{1d} \in C^\infty(\boldsymbol{R}^2)$ such that $|p_{1d}(x)| > c > 0$ and it satisfies

$$p_{1d}(x) = \exp(i\eta_d(x)) \tag{4.3}$$

for $|x| > R \gg 1$ fixed large enough. This function obeys the bound

$$|\nabla p_{1d}(x)| = |d|^{\sigma}O(|x|^{-1}), \quad |\nabla\nabla p_{1d}(x)| = |d|^{2\sigma}O(|x|^{-2}), \quad |x| \to \infty, \tag{4.4}$$

uniformly in $|d| \gg 1$. If we set $q_{1d}(x) = 1/p_{1d}(x)$, then q_{1d} also obeys the same bound as above and

$$q_{1d}(x) = \bar{p}_{1d}(x) = \exp(-i\eta_d(x)) \tag{4.5}$$

for $|x| > R$.

We now introduce the following two operators

$$K_{0d} = p_{1d}H_0 q_{1d}, \quad K_{2d} = p_{1d}H_{2,d}q_{1d} = p_{1d}H(A_{2d})q_{1d},$$

as an auxiliary operator, and we write $K_{1d} = H_1 = H(A_1)$. We set

$$\Sigma_{1d} = \{x \in \boldsymbol{R}^2 : |x| \ge R,\ x \in \Gamma_d^{\mathrm{c}}\}, \quad \Sigma_{2d} = \{x \in \boldsymbol{R}^2 : |x - d| < R\},$$

and denote by $\pi_{jd}(x)$ the characteristic function of Σ_{jd}, where Γ_d is defined by (4.2) and Γ_d^{c} is the complement of Γ_d. We consider the differences

$$W_{1d} = K_{1d} - K_{0d}, \quad W_{2d} = K_{2d} - K_{0d}.$$

The operator W_{1d} takes the form

$$W_{1d} = e_{1d}(x)\cdot\nabla + e_{0d}(x),$$

where the coefficients have support in $\Sigma_{1d} \cup \{|x| < R\}$. By (4.3) and (4.4), e_{0d} obeys $e_{0d}(x) = |d|^{2\sigma}O(|x|^{-2})$ at infinity, and e_{1d} satisfies

$$e_{1d}(x) = O(|d|^{\sigma})(-x_2/|x|^2, x_1/|x|^2) = O(|d|^{\sigma})\nabla\theta(x) \tag{4.6}$$

for $x \in \Sigma_{1d}$, where $\theta(x)$ is the azimuth angle from the positive x_1 axis. On the other hand, W_{2d} has uniformly bounded coefficients with support in Σ_{2d}. As is easily seen, $H_d = K_{0d}$ on

$$\{x \in \boldsymbol{R}^2 : |x-d| > R, |x| > R, x \in \Gamma_d\}.$$

We have $H_d = K_{1d} = K_{0d} + W_{1d}$ on $\Sigma_{1d} \cup \{|x| < R\}$ and $H_d = K_{2d} = K_{0d} + W_{2d}$ on Σ_{2d}. Since these two regions are disjoint with each other for $|d| \gg 1$, we obtain

$$H_d = K_{0d} + W_{1d} + W_{2d}. \tag{4.7}$$

We can write this relation as

$$H_d = K_{1d} + W_{2d}, \quad H_d = K_{2d} + W_{1d}. \tag{4.8}$$

4.2. We denote by $\| \ \|$ the norm of bounded operators on $L^2(\boldsymbol{R}^2)$. The remaining argument in this section is devoted to proving the following proposition.

Proposition 4.1. *Let ρ be fixed as*

$$1/2 < \rho < (1+\sigma)/2 < 1 \tag{4.9}$$

for $0 < \sigma \ll 1$ as in (4.1). *Then one has*

$$\|\langle x\rangle^{-\rho} R(\lambda + i0; H_d)\pi_{2d}\| = o(1), \quad |d| \to \infty.$$

The proof is done through a series of lemmas. Let $G_d(x, y; \lambda)$ be the Green kernel of $R(\lambda + i0; K_{0d})$. The resolvent $R(\lambda + i0; H_0)$ has the kernel

$$G_0(x, y : \lambda) = (i/4)H_0^{(1)}(\sqrt{\lambda}|x-y|),$$

where $H_0^{(1)}(z)$ is the Hankel function of first kind and order zero. As is well known, $H_0^{(1)}(z)$ behaves like

$$H_0^{(1)}(z) = (2/\pi)^{1/2}\exp(i(z-\pi/4))z^{-1/2}(1 + O(|z|^{-1}))$$

at infinity. Since $R(\lambda + i0; K_{0d}) = p_{1d}R(\lambda + i0; H_0)q_{1d}$, we have

$$G_d(x, y; \lambda) = c_0(\lambda)p_{1d}(x)\exp(i\sqrt{\lambda}|x-y|)|x-y|^{-1/2}q_{1d}(y)(1 + O(|x-y|^{-1}))$$

as $|x-y| \to \infty$, where $c_0(\lambda) = (1/8\pi)^{1/2}\exp(i\pi/4)\lambda^{-1/4}$.

Lemma 4.1. *Let ρ be as in Proposition* 4.1 *and let $f_0(x)$ be a bounded function with compact support. Then:*

(1) $\|f_0 R(\lambda + i0; K_{0d})\pi_{2d}\| = o(|d|^{-2\sigma})$.

(2) $\|\langle x\rangle^{-\rho} R(\lambda + i0; K_{0d})\pi_{2d}\| = o(1)$.

(3) $\|\pi_{1d}\langle x\rangle^{\rho-2} R(\lambda + i0; K_{0d})\pi_{2d}\| = o(|d|^{-2\sigma})$.

Proof. We prove (3) only. A similar argument applies to (1) and (2). To prove (3), we consider the integral

$$I_d = \iint_{x\in\Sigma_{1d},\, y\in\Sigma_{2d}} \langle x\rangle^{2\rho-4} |G_d(x, y; \lambda)|^2 \, dxdy.$$

If $x \in \Sigma_{1d}$ and $y \in \Sigma_{2d}$, then $|x - y| > c(|x| + |d|)$ for some $c > 0$. Thus the integral is evaluated as

$$I_d = O(1)\int_{\Sigma_{1d}} \langle x\rangle^{2\rho-4}(|x| + |d|)^{-1}\, dx = O(|d|^{-1-\sigma})\int_0^\infty (1 + r)^{2\rho-4} r\, dr$$

and hence $I_d = O(|d|^{-4\sigma}) \times O(|d|^{-(1-3\sigma)})$. We can take σ so small that $1 - 3\sigma > 0$. Then the desired estimate follows at once. □

Remark 4.1. By elliptic estimate, it follows from Lemma 4.1 (1) that

$$\|f_0 \nabla R(\lambda + i0; K_{0d})\pi_{2d}\| = o(|d|^{-\sigma}).$$

The differential operator K_{0d} does not necessarily have coefficients bounded uniformly in d, but this can be easily shown by use of (4.4).

Next we calculate

$$I(x, y) = (\nabla\theta \cdot \nabla)\exp(i\sqrt{\lambda}|x - y|), \quad \theta = \theta(x),$$

for $y \in \Sigma_{2d}$ fixed. A direct calculation yields

$$I(x, y) = i\sqrt{\lambda}|x|^{-1}|x - y|^{-1}|y|(\hat{x}_2\hat{y}_1 - \hat{x}_1\hat{y}_2)\exp(i\sqrt{\lambda}|x - y|),$$

where $\hat{x} = (\hat{x}_1, \hat{x}_2) = (x_1/|x|, x_2/|x|) \in S^1$. If $x \in \Sigma_{1d}$, $\hat{x} = -\hat{d} + O(|d|^{-\sigma})$, and if $y \in \Sigma_{2d}$, $\hat{y} = \hat{d} + O(|d|^{-1})$. Thus we have

$$\hat{x}_2\hat{y}_1 - \hat{x}_1\hat{y}_2 = O(|d|^{-\sigma})$$

and hence it follows that

$$I(x, y) = O(|d|^{1-\sigma})|x|^{-1}|x - y|^{-1}$$

uniformly in $x \in \Sigma_{1d}$ and $y \in \Sigma_{2d}$. If ρ fulfills (4.9), then we have

$$\iint_{x\in\Sigma_{1d},\, y\in\Sigma_{2d}} |x|^{2\rho}|I(x, y)|^2|x - y|^{-1}\, dxdy = O(|d|^{-2\sigma}) \times O(|d|^{-2\kappa})$$

with $\kappa = \sigma/2 - (\rho - 1/2) > 0$. Thus we have proved the following lemma.

Lemma 4.2. *Assume that* ρ *satisfies* (4.9). *Then*

$$\|\pi_{1d}\langle x\rangle^{\rho}(\nabla\theta \cdot \nabla)R(\lambda + i0; K_{0d})\pi_{2d}\| = o(|d|^{-\sigma}).$$

Lemma 4.3.

$$\|\langle x\rangle^{\rho} W_{1d} R(\lambda + i0; K_{0d})\pi_{2d}\| = o(1).$$

Proof. If we take account of (4.6), the lemma immediately follows from Lemmas 4.1 and 4.2 (see Remark 4.1 also). □

Remark 4.2. We can show in a similar way that

$$\|\langle x\rangle^{\rho} W_{1d} R(\lambda - i0; K_{0d})\pi_{2d}\| = o(1)$$

and hence we have

$$\|\pi_{2d} R(\lambda + i0; K_{0d}) W_{1d}\langle x\rangle^{\rho}\| = o(1)$$

by adjoint. Such immediate consequences are often used without further references in the discussion below.

Lemma 4.4.

$$\|\langle x\rangle^{-\rho} R(\lambda + i0; K_{1d})\pi_{2d}\| = o(1).$$

Proof. Since $K_{1d} = K_{0d} + W_{1d}$, we have

$$R(\lambda + i0; K_{1d}) = R(\lambda + i0; K_{0d}) - R(\lambda + i0; K_{1d}) W_{1d} R(\lambda + i0; K_{0d})$$

by the resolvent identity. By the principle of limiting absorption, we know ([**6**]) that

$$\langle x\rangle^{-\rho} R(\lambda + i0; K_{1d})\langle x\rangle^{-\rho} = \langle x\rangle^{-\rho} R(\lambda + i0; H_1)\langle x\rangle^{-\rho} : L^2(\boldsymbol{R}^2) \to L^2(\boldsymbol{R}^2)$$

is bounded uniformly in d. This, together with Lemmas 4.1 and 4.3, proves the lemma. □

Lemma 4.5.

$$\|\langle x\rangle^{\rho} W_{1d} R(\lambda + i0; K_{2d})\pi_{2d}\| = o(1).$$

Proof. The proof is almost the same as that of Lemma 4.4. Since $K_{2d} = K_{0d} + W_{2d}$, we have

$$R(\lambda + i0; K_{2d}) = R(\lambda + i0; K_{0d}) - R(\lambda + i0; K_{0d}) W_{2d} R(\lambda + i0; K_{2d})$$

by the resolvent identity. The operator

$$\pi_{2d}R(\lambda+i0;K_{2d})\pi_{2d}=\pi_{2d}p_{1d}R(\lambda+i0;H_{2,d})q_{1d}\pi_{2d}:L^2(\boldsymbol{R}^2)\to L^2(\boldsymbol{R}^2)$$

is bounded uniformly in d. The coefficients of $K_{2d}=p_{1d}H_{2,d}q_{1d}$ are bounded uniformly in d on Σ_{2d}. Hence

$$\|\pi_{2d}\nabla R(\lambda+i0;K_{2d})\pi_{2d}\|=O(1)$$

by elliptic estimate. Thus the lemma follows from Lemma 4.3. □

We are now in a position to prove the proposition in question.

Proof of Proposition 4.1. Since $H_d=K_{2d}+W_{1d}$ by (4.8), we have

$$R(\lambda+i0;H_d)=R(\lambda+i0;K_{2d})-R(\lambda+i0;H_d)W_{1d}R(\lambda+i0;K_{2d})$$

by the resolvent identity. Set

$$e_1=\|\pi_{2d}R(\lambda+i0;H_d)\langle x\rangle^{-\rho}\|,\quad e_2=\|\pi_{2d}R(\lambda+i0;H_d)\pi_{2d}\|.$$

Then it follows from Lemma 4.5 that

$$e_2=O(1)+o(1)e_1.$$

On the other hand, $H_d=K_{1d}+W_{2d}$ by (4.8) again. This yields

$$R(\lambda+i0;H_d)=R(\lambda+i0;K_{1d})-R(\lambda+i0;H_d)W_{2d}R(\lambda+i0;K_{1d}).$$

By Lemma 4.4, we have

$$\|\pi_{2d}\nabla R(\lambda+i0;K_{1d})\langle x\rangle^{-\rho}\|=o(1)$$

and hence it follows that

$$e_1=o(1)+o(1)e_2.$$

The proposition is obtained by combining the two relations above. □

5. Proof of Lemmas 3.1, 3.2 and 3.3.

In this section we prove Lemmas 3.1, 3.2 and 3.3, which remain unproved. To prove these lemmas, the asymptotic behavior at infinity of eigenfunction $\varphi_{\mp}(x;\lambda,\omega)$ plays an important role besides the resolvent estimates in the previous section.

Proposition 5.1. *Let* $\varphi_{\mp}(x;\lambda,\omega)$ *be defined by* (2.4). *If* $x/|x|\neq\omega$, *then* $\varphi_{+}(x;\lambda,\omega)$ *behaves like*

$$\varphi_{+}(x;\lambda,\omega)=\exp(i\alpha(\theta(x;\omega)-\pi))\exp(i\sqrt{\lambda}x\cdot\omega)(1+o(1))$$

as $|x| \to \infty$, *where the order estimate is uniform in* $x/|x| \in S^1$ *with* $|x/|x| - \omega| > \delta$, $\delta > 0$ *being fixed arbitrarily. Similarly the incoming eigenfunction* $\varphi_-(x; \lambda, \omega)$ *obeys*

$$\varphi_-(x; \lambda, \omega) = \exp(i\alpha(\theta(x; -\omega) - \pi)) \exp(i\sqrt{\lambda} x \cdot \omega)(1 + o(1))$$

for $|x/|x| + \omega| > \delta$.

We prove the three lemmas, accepting Proposition 5.1 as proved. The proposition is verified in the next section.

Proof of Lemma 3.1. Recall that $\varphi_\mp(x; \lambda, \omega)$ is the eigenfunction of $H_\alpha (= H_{\alpha_1})$. By Proposition 5.1,

$$\int |\pi_{2d}(x)\varphi_\mp(x; \lambda, \omega)|^2 \, dx = O(1) \tag{5.1}$$

is bounded uniformly in d. By elliptic estimate, it follows from Proposition 4.1 that

$$\|f_0 \nabla R(\lambda + i0; H_d) \nabla \pi_{2d}\| = o(1)$$

for bounded function f_0 with compact support. This, together with (5.1), completes the proof. □

Proof of Lemma 3.2. We represent $R(\lambda + i0; H_d)$ as

$$R(\lambda + i0; H_d) = R(\lambda + i0; K_{1d}) - R(\lambda + i0; H_d) W_{2d} R(\lambda + i0; K_{1d})$$

by use of the resolvent identity. Let $\chi_0(x) = \chi_0(r) = \chi(r/2)$ be as in (3.7) and let $D_{1d} = [\chi_0, H_d]$ be defined by (3.9). The coefficients of D_{1d} have support in $Q_1 = \{2 < |x| < 4\}$. Hence it follows from Proposition 4.1 and Lemma 4.4 that

$$\gamma_1(d) = (((1 - \chi_0) - R(\lambda + i0; K_{1d}) D_{1d})\varphi_+(\cdot\,; \lambda, \omega), D_{1d}\varphi_-(\cdot\,; \lambda, \omega')) + o(1)$$

as $|d| \to \infty$. The operator H_d coincides with $K_{1d} = H_1 = H(A_1)$ on Q_1, and H_1 is rotationally invariant there. Thus we have $D_{1d} = [\chi_0, H_1] = [\chi_0, H_0] = D_0$. This implies that

$$\gamma_1(d) = (((1 - \chi_0) - R(\lambda + i0; H_1) D_0)\varphi_+(\cdot\,; \lambda, \omega), D_0\varphi_-(\cdot\,; \lambda, \omega')) + o(1).$$

Thus the proof is completed. □

Proof of Lemma 3.3. We repeat the same argument as in the proof of Lemma 3.2. Let $D_{2d} = [\chi_{0d}, H_d]$ be as in (3.9). The coefficients of D_{2d} have support in $Q_{2d} = \{2 < |x - d| < 4\}$. Set $D_{0d} = [\chi_{0d}, H_0]$. Then we can calculate D_{2d} as

$$D_{2d} = [\chi_{0d}, K_{2d}] = p_{1d}[\chi_{0d}, H_{2,d}]q_{1d} = p_{1d}D_{0d}q_{1d}$$

and hence we have

$$R(\lambda + i0; K_{2d})D_{2d} = p_{1d}R(\lambda + i0; H_{2,d})D_{0d}q_{1d}.$$

By (4.5), $\bar{p}_{1d} = q_{1d}$ on Q_{2d}. Thus it follows from Proposition 4.1 and Lemma 4.5 that $\gamma_2(d)$ behaves like

$$\gamma_2(d) = (((1 - \chi_{0d}) - R(\lambda + i0; H_{2,d})D_{0d})q_{1d}\varphi_+(\cdot\,; \lambda, \omega), D_{0d}q_{1d}\varphi_-(\cdot\,; \lambda, \omega')) + o(1).$$

The term $q_{1d}(x)\varphi_\mp(x; \lambda, \omega)$ is calculated as

$$\begin{aligned} q_{1d}\varphi_\mp &= \exp(-i\eta_d(x))\varphi_\mp(x; \lambda, \omega) \\ &= \exp(-i\zeta_d(\theta(x; -\hat{d})))\varphi_\mp(x; \lambda, \omega) \\ &= \exp(-i\alpha_1\theta(x; -\hat{d}))\varphi_\mp(x; \lambda, \omega) \end{aligned}$$

on Q_{2d}. We apply Proposition 5.1 to $\varphi_\mp(x; \lambda, \omega)$. Then we obtain that

$$q_{1d}\varphi_\mp = \exp(i\alpha_1(\theta(x; \mp\omega) - \theta(x; -\hat{d}) - \pi))\varphi_0(x; \lambda, \omega) + o(1)$$

on Q_{2d}, where $\varphi_0(x; \lambda, \omega) = \exp(i\sqrt{\lambda}x \cdot \omega)$. If $x \in Q_{2d}$, then $\theta(x; \mp\omega) = \theta(d; \mp\omega) + O(|d|^{-1})$ and

$$\theta(x; -\hat{d}) = \theta(d; -\hat{d}) + O(|d|^{-1}) = \pi + O(|d|^{-1}).$$

Thus we see that $q_{1d}(x)\varphi_\mp(x; \lambda, \omega)$ behaves like

$$q_{1d}\varphi_\mp = \exp(i\alpha_1(\theta(d; \mp\omega) - 2\pi))\exp(i\sqrt{\lambda}d \cdot \omega)\varphi_0(x - d; \lambda, \omega) + o(1)$$

on Q_{2d}. This yields the desired result and the proof is complete. □

6. Asymptotic behavior at infinity of eigenfunction.

The Proposition 5.1 has played a basic role in proving the main theorem. We here prove this proposition. As stated in Section 1, the asymptotic behavior of eigenfunction $\varphi_\mp(x; \lambda, \omega)$ has been studied in the physical literatures [**2**], [**3**], [**9**]. We copy the proof from [**9**], the original idea of which is due to T. Takabayashi.

Proof of Proposition 5.1. If we write $\varphi_\mp(x; \lambda, \omega, \alpha)$ for $\varphi_\mp(x; \lambda, \omega)$, then

$$\varphi_-(x; \lambda, \omega, \alpha) = \bar{\varphi}_+(-x; \lambda, \omega, -\alpha).$$

Hence the asymptotic formula for $\varphi_-(x; \lambda, \omega)$ follows from that for $\varphi_+(x; \lambda, \omega)$ at once. We consider only the case that $\alpha \notin \boldsymbol{Z}$ is not an integer. The case $\alpha \in \boldsymbol{Z}$ is much simpler to deal with and we skip this case.

For brevity we assume that $0 < \alpha < 1$, and we write $\varphi(x;\omega)$ for $\varphi_+(x;\lambda,\omega)$ with $\lambda = 1$. The proposition is verified for $\varphi(x;\omega)$. The proof uses the integral representation

$$J_p(r) = \frac{i^p}{\pi}\left(\int_0^\pi e^{-ir\cos t}\cos pt\,dt - \sin p\pi\int_0^\infty e^{-pt+ir\cosh t}\,dt\right) \tag{6.1}$$

for the Bessel function $J_p(r)$, $r > 0$, with $p > 0$ ([**5**]). We denote by

$$\varphi_{\mathrm{inc}}(x;\omega) = \exp(i\alpha(\theta(x;\omega)-\pi))\exp(ix\cdot\omega)$$

the asymptotic form on the right side. If we make a change of variable

$$\gamma = \gamma(x;\omega) = \theta(x;\omega) - \pi, \quad -\pi \le \gamma < \pi,$$

then it follows from (2.4) that

$$\varphi_+(x;\omega) = \sum_{l\in Z}(-i)^\nu e^{il\gamma}J_\nu(|x|)$$

with $\nu = |l-\alpha|$, and also we have $\varphi_{\mathrm{inc}}(x;\omega) = e^{i\alpha\gamma - i|x|\cos\gamma}$. We expand $\varphi_{\mathrm{inc}}(x;\omega)$ as the Fourier series

$$\varphi_{\mathrm{inc}}(x;\omega) = \frac{1}{2\pi}\sum_{l\in Z}e^{il\gamma}\int_{-\pi}^\pi e^{i\alpha t - i|x|\cos t}e^{-ilt}\,dt = \frac{1}{\pi}\sum_{l\in Z}e^{il\gamma}\int_0^\pi e^{-i|x|\cos t}\cos\nu t\,dt.$$

On the other hand, we have

$$\varphi_+(x;\omega) = \frac{1}{\pi}\sum_{l\in Z}e^{il\gamma}\left(\int_0^\pi e^{-i|x|\cos t}\cos\nu t\,dt - \sin\nu\pi\int_0^\infty e^{-\nu t+i|x|\cosh t}\,dt\right)$$

by use of (6.1). Hence

$$\varphi_+(x;\omega) - \varphi_{\mathrm{inc}}(x;\omega) = -\frac{1}{\pi}\sum_{l\in Z}e^{il\gamma}\sin\nu\pi\int_0^\infty e^{-\nu t+i|x|\cosh t}\,dt.$$

We calculate the sum on the right side. By assumption, $\theta(x;\omega) \ne 0$ and hence $|\gamma| < \pi$. A simple computation shows that

$$\sum_{l\in Z}e^{il\gamma}e^{-\nu t}\sin\nu\pi = \sin\alpha\pi\left(\frac{e^{\alpha t}}{1+e^{-i\gamma}e^t} + \frac{e^{-\alpha t}}{1+e^{-i\gamma}e^{-t}}\right)$$

for $0 < \alpha < 1$. This yields

$$\varphi_+(x;\omega) - \varphi_{\mathrm{inc}}(x;\omega) = -\frac{\sin\alpha\pi}{\pi}\int_{-\infty}^\infty \frac{e^{-\alpha t}}{1+e^{-i\gamma}e^{-t}}e^{i|x|\cosh t}\,dt \tag{6.2}$$

for $|\gamma| < \pi$. By the stationary phase method, the integral obeys the bound $O(|x|^{-1/2})$ as $|x| \to \infty$. Thus the desired relation is obtained. □

Next we consider the asymptotic behavior of $\varphi_+(x;\lambda,\omega)$ along the forward direction ω. This has been already studied by [**3**], although the argument there is not rigorous.

Lemma 6.1. *Let* $G = \{x \in \boldsymbol{R}^2 : 0 < |x/|x| - \omega| < c|x|^{-1}\}$ *for some* $c > 0$. *Then*

$$\varphi_+(x;\lambda,\omega) = (\cos\alpha\pi)\exp(i\sqrt{\lambda}x\cdot\omega) + o(1), \quad |x| \to \infty,$$

for $x \in G$.

Proof. We use the same notation as in the proof of Proposition 5.1. The proof is rather sketchy but the justification is easy. We fix $0 < \delta \ll 1$ small enough and we write $\gamma = \gamma(x;\omega) = -\pi + \varepsilon$ or $\gamma = \pi - \varepsilon$ for $x \in G$, where $\varepsilon > 0$ and it obeys $\varepsilon = O(|x|^{-1})$. Let I be the integral on the right side of (6.2). If $\gamma = -\pi + \varepsilon$, then we have

$$I = e^{i|x|}\int_{-|x|^{-1/2+\delta}}^{|x|^{-1/2+\delta}} \frac{1}{t+i\varepsilon} e^{i|x|t^2/2}\,dt + o(1) = e^{i|x|}\int_{-|x|^{\delta}}^{|x|^{\delta}} \frac{1}{s+i\varepsilon|x|^{1/2}} e^{i|s|^2/2}\,ds + o(1)$$

by changing the variable $s = |x|^{1/2}t$. Note that $\varepsilon|x|^{1/2} = O(|x|^{-1/2})$ and hence it follows that

$$\int_{|x|^{-\delta}}^{|x|^{\delta}} \left(\frac{1}{s+i\varepsilon|x|^{1/2}} - \frac{1}{s}\right) e^{i|s|^2/2}\,ds = o(1).$$

This yields

$$I = e^{i|x|}\int_{-|x|^{-\delta}}^{|x|^{-\delta}} \frac{1}{s+i\varepsilon|x|^{1/2}}\,ds + o(1) = -i\pi e^{i|x|} + o(1)$$

for $\gamma = -\pi + \varepsilon$. Similarly we have $I = i\pi e^{i|x|} + o(1)$ for $\gamma = \pi - \varepsilon$. Thus the lemma immediately follows from (6.2). □

Theorem 1.2 follows from Lemma 6.1. We end the paper by proving this theorem.

Proof of Theorem 1.2. The theorem is proved in exactly the same way as in the proof of Theorem 1.1. We have only to replace

$$\exp(i\alpha_1\theta(d;\omega)) \to (1 + \exp(2i\alpha_1\pi))/2 = (\cos\alpha_1\pi)\exp(i\alpha_1\pi)$$

for $\omega = \hat{d}$. Then we can get the desired asymptotic formula for $f_d(\hat{d} \to \omega';\lambda)$ with $\omega' \neq -\hat{d}$. A similar argument applies to the other statements. □

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