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Lagrangian submanifolds of the three dimensional complex projective space

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Abstract. We investigate Lagrangian submanifolds of the 3-dimensional complex projective space. In case the second fundamental form takes a special form, we obtain several classification theorems. As a consequence we obtain several new examples of 3-dimensional Lagrangian submanifolds.

1. Introduction.

In this paper we investigate 3-dimensional Lagrangian submanifolds of the 3-dimensional complex projective space $\boldsymbol{C}P^3(4)$. A 3-dimensional submanifold of $\boldsymbol{C}P^3(4)$ is called Lagrangian if the complex structure J interchanges the tangent and normal spaces. Besides the complex submanifolds, the Lagrangian submanifolds form the most important class of submanifolds of complex projective spaces and have already been studied by many people.

In this paper, we particularly focus our attention on Lagrangian submanifolds which admit a special type of tangent frame. In particular we will consider two special cases. The paper is organised as follows. In Section 2 we recall the basic formulas for Lagrangian submanifolds of the complex projective space. This will include the basic existence and uniqueness theorem as well as the existence of a horizontal lift (see respectively [**CDVV1**] and [**R**]) in to the 7-dimensional sphere $S^7(1)$.

Next, we suppose that there exist one of two special type of orthonormal frame on our submanifold. We also show that such a frame always exists, if necessary by restricting to an open dense subset if M^3 admits one of the following geometric properties:

(i) M^3 is minimal, $\delta_M \neq 2$ and M^3 is quasi Einstein, where δ_M is the invariant introduced by Chen in [**C1**]. Minimal Lagrangian submanifolds with $\delta_M = 2$ were studied in [**CDVV1**] and [**CDVV2**]. A complete classification of the 3-dimensional ones was obtained in [**BSVW**]. Note that M is called quasi

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Einstein if the Ricci tensor has a double eigenvalue at each point p of M^3. Recall that a 3-dimensional manifold is Einstein if and only if it has constant sectional curvature and that the minimal Lagrangian submanifolds with constant sectional curvature were classified in [**E**],

(ii) M^3 is minimal and M is semi symmetric,

(iii) M^3 is Lagrangian H-umbilical in the sense of [**C3**], i.e. it is nowhere minimal, JH is an eigenvector of A_H and A_H restricted to $\{JH\}^\perp$ is a multiple of the identity. Here H denotes the mean curvature vector field. In particular this class is a generalization of the one studied in [**Ca**], [**C2**] and [**CV**],

(iv) M^3 is nowhere minimal and $JH/|H|$ is a Killing vector field whose integral curves lie in a complex vectorplane in $\boldsymbol{C}P^3$. Minimal Lagrangian submanifolds admitting a unit length Killing whose integral curves are geodesics were studied in [**CaV**].

In the next sections, we then express the Gauss and Codazzi equations for the two main cases. In order to solve this system of equations, we have to introduce several more subcases. We then, in the different subcases show how these equations can be solved explicitely and construct the corresponding Lagrangian immersions using the existence and uniqueness theorem.

In particular, we also obtain some new examples of Lagrangian submanifolds with constant sectional curvature. It becomes then straightforward to apply these results to obtain classification theorems for the different classes of Lagrangian submanifolds introduced above.

One of the main reasons for studying the above classes of Lagrangian submanifolds is the following problem, which can be seen as a Lagrangian analog of Chern's problem for minimal hypersurfaces in spheres:

PROBLEM 1. *Let M be a minimal Lagrangian submanifold of $\boldsymbol{C}P^n(4)$ with constant scalar curvature. Which are the possible values of the scalar curvature which can occur?*

2. Preliminaries.

First, we want to recall some basic definitions about distributions on Riemannian manifolds. For more details we refer to [**KN**]. Let E be a distribution. Denote by $E^\perp$ its orthogonal distribution. Then E is called *parallel* if $\nabla_X Y \in E$ for all vectorfields X tangent to M and $Y \in E$; it is called *autoparallel* if $\nabla_X Y \in E$ for all $X, Y \in E$; it is called *totally umbilical* if there exists a vector $H \in E^\perp$ such that $h(\nabla_X Y, Z) = h(X, Y)h(H, Z)$ for all $X, Y \in E$ and for all $Z \in E^\perp$, in this case H is called the *mean curvature normal* of the distribution E. We call E *spherical* if it is totally umbilical and its mean curvature normal H satisfies $h(\nabla_X H, Z) = 0$ for all $X \in E$ and $Z \in E^\perp$. If E is autoparallel, totally umbilical or spherical, then E is involutive and all the leaves of the foliation of M induced by E are totally geodesic, totally umbilical or spherical respectively.

Let M be a Lagrangian submanifold of $\boldsymbol{CP}^n(4)$. We denote the Levi-Civita connections of M and of $\boldsymbol{CP}^n(4)$ by ∇ and $\tilde{\nabla}$, respectively. The formulas of Gauss and Weingarten are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2}$$

for tangent vector fields X and Y and normal vector fields ξ, where D is the connection on the normal bundle. The second fundamental form h is related to the shape operator A_ξ by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle. \tag{3}$$

The mean curvature vector H of M is defined by $H = 1/n \operatorname{trace} h$.

For Lagrangian submanifolds of a Kaehler manifold, we have (cf. [**CO**])

$$D_X JY = J\nabla_X Y, \tag{4}$$

$$A_{JX} Y = -Jh(X, Y) = A_{JY} X. \tag{5}$$

The above formulas imply that $\langle h(X, Y), JZ \rangle$ is totally symmetric. If we denote the curvature tensors of ∇ and D by R and R^D, respectively, i.e.

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}, \quad R^D(X, Y) = [D_X, D_Y] - D_{[X, Y]},$$

then the equations of Gauss, Codazzi and Ricci are given respectively by

$$\begin{aligned} \langle R(X, Y)Z, W \rangle = {} & \langle A_{h(Y,Z)} X, W \rangle - \langle A_{h(X,Z)} Y, W \rangle \\ & + (\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle), \end{aligned} \tag{6}$$

$$(\nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z), \tag{7}$$

$$\begin{aligned} \langle R^D(X, Y)JZ, JW \rangle = {} & \langle [A_{JZ}, A_{JW}] X, Y \rangle \\ & + c(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle), \end{aligned} \tag{8}$$

where X, Y, Z, W (respectively, η and ξ) are vector fields tangent (respectively, normal) to M and ∇h is defined by

$$(\nabla h)(X, Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{9}$$

We recall the following Existence and Uniqueness Theorems for later use (cf. [**CDVV1**] and [**CDVV2**]).

THEOREM 1. *Let* $(M^n, \langle .,. \rangle)$ *be an n-dimensional simply connected Riemannian manifold. Let* σ *be a TM-valued symmetric bilinear form on M satisfying*

(i) $\langle \sigma(X, Y), Z \rangle$ *is totally symmetric,*

(ii) $(\nabla\sigma)(X, Y, Z) = \nabla_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)$ *is totally symmetric,*

(iii) $R(X, Y)Z = (\langle Y, Z \rangle X - \langle X, Z \rangle Y) + \sigma(\sigma(Y, Z), X) - \sigma(\sigma(X, Z), Y)$, *then there exists a Lagrangian isometric immersion* $L : (M, \langle .,. \rangle) \to \boldsymbol{CP}^n(4)$ *whose second fundamental form h is given by* $h(X, Y) = J\sigma(X, Y)$.

THEOREM 2. *Let* $L_1, L_2 : M \to \boldsymbol{CP}^n(4)$ *be two Lagrangian isometric immersions of a Riemannian manifold M with second fundamental forms* h^1 *and* h^2. *If*

$$\langle h^1(X, Y), JL_{1\star}Z \rangle = \langle h^2(X, Y), JL_{2\star}Z \rangle, \tag{10}$$

for all vector fields X, Y, Z *tangent to* M, *then there exists an isometry* ϕ *of* $\boldsymbol{CP}^n(4)$ *such that* $L_1 = \phi \circ L_2$.

In order to obtain the immersions more explicitly, it is often very convenient to consider the Hopf fibration $\pi : S^{2n+1}(1) \to \boldsymbol{CP}^n(4)$. On $S^{2n+1}(1) \subset \boldsymbol{C}^{n+1}$ we consider the Sasakian structure ϕ (the projection of the complex structure J of $\boldsymbol{C}^{n+1}$ on the tangent bundle of S^{2n+1}) and the structure vector field $\xi = Jx$, where x is the position vector. An isometric immersion $f : M \to S^{2n+1}$ is called C-totally real if ξ is normal to $f_*(TM)$. Note that for a C-totally real submanifold $\langle \phi(f_*(TM)), f_*(TM) \rangle = 0$. On $\boldsymbol{C}^{n+1}$ we consider the complex structure J. The main results of [**R**] can be specialized to our situation as follows.

First let $g : M \to \boldsymbol{CP}^n(4)$ be a totally real isometric immersion. Then there exist an isometric covering map $\tau : \hat{M} \to M$, and a C-totally real isometric immersion $f : M \to S^{2n+1}$ such that $g(\tau) = \pi(f)$. Hence every totally real immersion can be lifted locally (or globally if we assume the manifold is simply connected) to a C-totally real immersion of the same Riemannian manifold. Conversely, let $f : M \to S^{2n+1}$ be a C-totally real isometric immersion. Then $g = \pi(f) : M \to \boldsymbol{CP}^n(4)$ is again an isometric immersion, which is totally real. Under this correspondence, the second fundamental forms h^f and h^g of f and g satisfy $\pi_* h^f = h^g$. Moreover, h^f is horizontal w.r.t. π.

We now restrict ourselves to the case that our Lagrangian submanifold is 3-dimensional and admits a frame of a special type. Namely, we call M a **Lagrangian submanifold of Type 1** if and only if around each point p of an open dense subset of M there exists a local orthonormal basis $\{E_1, E_2, E_3\}$ such that

$$\nabla_{E_1}E_1 = 0, \qquad \nabla_{E_1}E_2 = a_1E_3, \qquad \nabla_{E_1}E_3 = -a_1E_2,$$
$$\nabla_{E_2}E_1 = b_1E_3, \qquad \nabla_{E_2}E_2 = b_2E_3, \qquad \nabla_{E_2}E_3 = -b_1E_1 - b_2E_2,$$
$$\nabla_{E_3}E_1 = -b_1E_2, \qquad \nabla_{E_3}E_2 = b_1E_1 + a_2E_3, \qquad \nabla_{E_3}E_3 = -a_2E_2,$$

$$h(E_1,E_1) = \lambda_1 JE_1, \qquad h(E_2,E_2) = \lambda_2 JE_1 + aJE_2 + bJE_3,$$
$$h(E_1,E_2) = \lambda_2 JE_2, \qquad h(E_2,E_3) = bJE_2 - aJE_3,$$
$$h(E_1,E_3) = \lambda_3 JE_3, \qquad h(E_3,E_3) = \lambda_3 JE_1 - aJE_2 - bJE_3,$$

and we call M a **Lagrangian submanifold of Type 2** if and only if around each point p of an open and dense subset of M there exists a local orthonormal basis $\{E_1, E_2, E_3\}$ such that

$$\nabla_{E_1}E_1 = a_1E_2 + a_2E_3, \qquad \nabla_{E_1}E_2 = -a_1E_1 + a_3E_3, \qquad \nabla_{E_1}E_3 = -a_2E_1 - a_3E_2,$$
$$\nabla_{E_2}E_1 = b_1E_2 + b_2E_3, \qquad \nabla_{E_2}E_2 = -b_1E_1 + b_3E_3, \qquad \nabla_{E_2}E_3 = -b_2E_1 - b_3E_2,$$
$$\nabla_{E_3}E_1 = c_1E_2 + c_2E_3, \qquad \nabla_{E_3}E_2 = -c_1E_1 + c_3E_3, \qquad \nabla_{E_3}E_3 = -c_2E_1 - c_3E_2,$$

$$h(E_1,E_1) = \lambda_1 JE_1, \qquad h(E_2,E_2) = \lambda_2 JE_1 + aJE_2,$$
$$h(E_1,E_2) = \lambda_2 JE_2, \qquad h(E_2,E_3) = -aJE_3,$$
$$h(E_1,E_3) = \lambda_2 JE_3, \qquad h(E_3,E_3) = \lambda_2 JE_1 - aJE_2.$$

We now give some examples of geometric conditions which imply that the Lagrangian submanifold M^3 is either of Type 1 or Type 2.

Lemma 1. *Let M^3 be a minimal Lagrangian submanifold of $\boldsymbol{C}P^3(4)$. Assume moreover that M is quasi Einstein and that $\delta_M \neq 2$. Then M is of Type* 2.

Proof. Denote by S the Ricci tensor of M^3 defined by

$$S(Y,Z) = \text{trace}\{X \mapsto R(X,Y)Z\},$$

and denote by ricci the associated 1-1 tensor field, i.e. $\langle \text{ricci}(Y), Z\rangle = S(Y,Z)$.

Let $p \in M$ and assume that p is not a totally geodesic point of M^3. Then, by choosing e_1 as the vector in which the function

$$f(v) = h(K(v,v),v),$$

defined on all unit length vectors at the point p, attains an absolute maximum it follows that e_1 is an eigenvector of A_{Je_1}. Next we choose e_2 and e_3 as eigenvectors of A_{Je_1} with respective eigenvalues λ_2 and λ_3. More details of this construction can be found in [**E**]. Using now that M^3 is minimal and that $\langle h(x,y), Jz\rangle$ is symmetric in x, y and z, it follows that

$$h(e_1,e_1) = \lambda_1 Je_1, \qquad h(e_2,e_2) = \lambda_2 Je_1 + aJe_2 + bJe_3, \tag{11}$$

$$h(e_1,e_2) = \lambda_2 Je_2, \qquad h(e_2,e_3) = bJe_2 - aJe_3, \tag{12}$$

$$h(e_1,e_3) = \lambda_3 Je_3, \qquad h(e_3,e_3) = \lambda_3 Je_1 - aJe_2 - bJe_3, \tag{13}$$

where $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Since f attains an absolute maximum in e_1, we must have that $\lambda_1 > 0$, $\lambda_1 - 2\lambda_2 \geq 0$, $\lambda_1 - 2\lambda_3 \geq 0$. If $\lambda_2 = \lambda_3$ it is also clear that by rotating e_2 and e_3, we can choose e_2 and e_3 such that $b = 0$.

A straightforward computation, using the Gauss equation now shows that

$$[S(e_i,e_j)] = \begin{pmatrix} 2-\lambda_1^2-\lambda_2^2-\lambda_3^2 & -(\lambda_2-\lambda_3)a & -(\lambda_2-\lambda_3)b \\ -(\lambda_2-\lambda_3)a & 2-2\lambda_2^2-2a^2-2b^2 & 0 \\ -(\lambda_2-\lambda_3)b & 0 & 2-2\lambda_3^2-2a^2-2b^2 \end{pmatrix} \tag{14}$$

It now follows that

$$\operatorname{ricci}(e_2) = -(\lambda_2-\lambda_3)ae_1 + 2(1-\lambda_2^2-a^2-b^2)e_2,$$

$$\langle \operatorname{ricci}(e_1), e_3\rangle = -(\lambda_2-\lambda_3)b.$$

Since M^3 is quasi-Einstein, we know that e_2, $\operatorname{ricci}(e_2)$ and $\operatorname{ricci}(\operatorname{ricci}(e_2))$ have to be linearly dependent. Hence the above formulas imply that

$$ab(\lambda_2-\lambda_3)^2 = 0.$$

So, if necessary by interchanging e_2 and e_3, we may assume that $b = 0$. If $\lambda_2 = \lambda_3$, we see that $\{e_1, e_2, e_3\}$ is a basis of Type 2 at the point p.

Therefore we may assume that $\lambda_2 \neq \lambda_3$. Suppose now that $a = 0$. Hence e_1, e_2 and e_3 are eigenvectors of ricci. Since we assumed that $\lambda_2 \neq \lambda_3$, we see that (if necessary after interchanging e_2 and e_3, which is allowed in this case since a and b both vanish) M^3 is quasi Einstein if and only if

$$2-\lambda_1^2-\lambda_2^2-\lambda_3^2 = 2-2\lambda_2^2,$$

which reduces to

$$-2\lambda_1^2 - 2\lambda_1\lambda_2 = 0.$$

Hence, since $\lambda_1 \neq 0$, we see that $\lambda_2 = -\lambda_1$ and $\lambda_3 = 0$. Thus e_3 is a vector such that $h(x, e_3) = 0$, for any vector x. It follows that $\delta_M(p) = 2$, which is a contradiction.

Finally, we consider the case that $\lambda_2 \neq \lambda_3$ and $a \neq 0$. Since $a \neq 0$, we see that M^3 is quasi-Einstein if and only if $2 - 2\lambda_3^2 - 2a^2$ is a double eigenvalue of S. This is the case if and only if

$$\det\begin{pmatrix} \lambda_3^2 - \lambda_1^2 - \lambda_2^2 + 2a^2 & (\lambda_3 - \lambda_2)a \\ (\lambda_3 - \lambda_2)a & 2(\lambda_3^2 - \lambda_2^2) \end{pmatrix} = 0.$$

Since $\lambda_2 \neq \lambda_3$ and $\lambda_3 = -\lambda_1 - \lambda_2$, this is the case if and only if

$$\det\begin{pmatrix} 2\lambda_1\lambda_2 + 2a^2 & -(\lambda_1 + 2\lambda_2)a \\ a & -2\lambda_1 \end{pmatrix} = 0,$$

i.e. if and only if

$$4\lambda_1^2\lambda_2 = -3\lambda_1 a^2 + 2\lambda_2 a^2. \tag{15}$$

Now, we consider the following change of basis

$$u_1 = \frac{1}{\sqrt{a^2 + 4\lambda_1^2}}(ae_1 - 2\lambda_1 e_2),$$

$$u_2 = \frac{1}{\sqrt{a^2 + 4\lambda_1^2}}(2\lambda_1 e_1 + ae_2),$$

$$u_3 = e_3.$$

Then, using (15), we have

$$\begin{aligned} h(ae_1 - 2\lambda_1 e_2, ae_1 - 2\lambda_1 e_2) &= (a^2\lambda_1 + 4\lambda_1^2\lambda_2)Je_1 + (-4a\lambda_1\lambda_2 + 4a\lambda_1^2)Je_2 \\ &= -2(\lambda_1 - \lambda_2)a(aJe_1 - 2\lambda_1 Je_2), \\ h(ae_1 - 2\lambda_1 e_2, e_3) &= a(\lambda_1 - \lambda_2)Je_3, \\ h(2\lambda_1 e_1 + ae_2, e_3) &= (2\lambda_1\lambda_3 - a^2)Je_3, \\ h(ae_1 - 2\lambda_1 e_2, 2\lambda_1 e_1 + ae_2) &= (2a\lambda_1^2 - 2a\lambda_1\lambda_2)Je_1 + (a^2 - 4\lambda_1^2)\lambda_2 Je_2 - 2a^2\lambda_1 Je_2 \\ &= a(\lambda_1 - \lambda_2)(2\lambda_1 Je_1 + aJe_2), \end{aligned}$$

from which it follows immediately that the basis $\{u_1, u_2, u_3\}$ is of Type 2 at the point p.

Since the Ricci tensor has two distinct eigenvalues, it is clear that such a basis can be extented differentiably on an open dense subset of M. $\square$

Lemma 2. *Let M^3 be a minimal semi-symmetric Lagrangian submanifold of $\boldsymbol{C}P^3(4)$ without totally geodesic points. Then, M^3 is of Type 2.*

Proof. It is well known, see for example [**Ver**] that a 3-dimensional semi-symmetric manifold is quasi Einstein. Therefore, applying the previous lemma, we see that the set of point with $\delta_M(p) \neq 2$ defines a Lagrangian submanifold of Type 2.

Therefore, in order to complete the proof, we have to show that the set of non-totally geodesic points with $\delta_M(p) = 2$ is empty. Assume there exists a point p such that $\delta_M(p) = 2$. Then, since M is minimal, we know from [**CDVV1**] and [**CDVV2**] that there exists an orthonormal basis $\{e_1, e_2, e_3\}$ at the point p such that

$$\begin{aligned} h(e_1, e_1) &= \lambda J e_1, & h(e_2, e_2) &= -\lambda J e_1, \\ h(e_1, e_2) &= -\lambda J e_2, & h(e_2, e_3) &= 0, \\ h(e_1, e_3) &= 0, & h(e_3, e_3) &= 0. \end{aligned}$$

Then, it follows from the Gauss equation that

$$\begin{aligned} &R(e_1, e_2)e_2 = (1 - 2\lambda^2)e_1 \\ &R(x, y)e_3 = \langle y, e_3 \rangle x - \langle x, e_3 \rangle y \\ &R(e_1, e_2)e_1 = -(1 - 2\lambda^2)e_2 \\ &R(x, e_3)y = \langle e_3, y \rangle x - \langle x, y \rangle e_3, \end{aligned}$$

where x and y are arbitrary tangent vectors. It now follows that

$$\begin{aligned} R.R(e_1, e_3, e_1, e_2, e_2) &= R(e_1, e_3)R(e_1, e_2)e_2 - R(R(e_1, e_3)e_1, e_2)e_2 \\ &\quad - R(e_1, R(e_1, e_3)e_2)e_2 - R(e_1, e_2)R(e_1, e_3)e_2 \\ &= (1 - 2\lambda^2)R(e_1, e_3)e_1 + R(e_3, e_2)e_2 \\ &= 2\lambda^2 e_3, \end{aligned}$$

which vanishes only if $\lambda = 0$, i.e. if p is a totally geodesic point. $\square$

Remark that the class of semi-symmetric submanifolds is a natural generalization of the one of locally symmetric manifolds. For a survey of further generalizations, we refer to [**Ver**].

Lemma 3. *Let M^3 be a Lagrangian H-umbilical submanifold of $\boldsymbol{C}P^3$. Then M^3 is of Type 2.*

Proof. Since H is a nowhere vanishing vector field, we can take E_1 as the unit length vector field in the direction of JH. Then, using the symmetries of the second fundamental form and the assumptions about H, it follows that we can write

$$h(E_1, E_1) = \lambda_1 JE_1, \qquad h(E_2, E_2) = \lambda_2 JE_1 + aJE_2 + bJE_3,$$
$$h(E_1, E_2) = \lambda_2 JE_2, \qquad h(E_2, E_3) = bJE_2 - aJE_3,$$
$$h(E_1, E_3) = \lambda_2 JE_3, \qquad h(E_3, E_3) = \lambda_2 JE_1 - aJE_2 - bJE_3.$$

It is now clear that, at least on an open dense subset, by rotating E_2 and E_3, we may assume that $b = 0$. □

Lemma 4. *Assume that M^3 is a minimal Lagrangian immersion which admits a unit length Killing vector field whose integral curves, when considered in $\boldsymbol{C}^4$, lie in a complex vector plane. Then M^3 is of Type* 1.

Proof. We denote the unit length Killing vector field by E_1. By the assumption that the integral curves lie in a complex vector plane, it follows that we can write

$$h(E_1, E_1) = \lambda_1 JE_1.$$

Hence $A_{JE_1}E_1 = \lambda_1 E_1$ and E_1 is an eigenvector of A_{JE_1}. Denote by E_2 and E_3 the other two eigenvectors of the symmetric operator A_{JE_1}. It follows that

$$h(E_1, E_3) = \lambda_2 JE_1,$$
$$h(E_2, E_3) = \lambda_3 JE_2,$$

where due to the minimality, we have that $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Using the symmetries of h, it now follows that we can introduce functions a and b such that

$$h(E_2, E_2) = \lambda_2 JE_1 + aJE_2 + bJE_3,$$
$$h(E_2, E_3) = bJE_2 - aJE_3,$$
$$h(E_3, E_3) = \lambda_3 JE_1 - aJE_2 - bJE_3.$$

Since E_1 is a unit length Killing vector field it follows that the connection coefficients are as for a Type 1 submanifold. □

Lemma 5. *Let M^3 be a Lagrangian submanifold of $\boldsymbol{CP}^3(4)$ with nowhere vanishing mean curvature vector H. Assume that $JH/|H|$ is a Killing vector field whose integrals, when considered in $\boldsymbol{C}^4$ lie in a complex vectorplane. Then M^3 is of Type* 1.

Proof. We define E_1 as the unit length vector field in the direction of JH. Proceeding now in the same way as in the previous lemma completes the proof. □

By a lengthy but straightforward computation we now compute that the Codazzi equations for a Lagrangian immersion of Type 1 are equivalent with

Lemma 6. *Assume that M is a 3-dimensional Lagrangian submanifold of Type 1. Denote by $\{E_1, E_2, E_3\}$ the corresponding orthonormal basis. Then, we have*

$$b_1(\lambda_2 + \lambda_3) = b_1\lambda_1 = 0, \tag{16}$$

$$a_1(\lambda_2 - \lambda_3) = 2b_1\lambda_2, \tag{17}$$

$$E_2(\lambda_1) = E_3(\lambda_1) = E_1(\lambda_2) = E_1(\lambda_3) = 0, \tag{18}$$

$$E_2(\lambda_2) = -E_2(\lambda_3) = a_2(\lambda_3 - \lambda_2), \tag{19}$$

$$E_3(\lambda_2) = -E_3(\lambda_3) = b_2(\lambda_2 - \lambda_3), \tag{20}$$

$$E_1(a) = b(3a_1 - b_1) + a_2(\lambda_3 - \lambda_2), \tag{21}$$

$$E_1(b) = a(b_1 - 3a_1) + b_2(\lambda_2 - \lambda_3), \tag{22}$$

$$E_3(b) + E_2(a) = 3(bb_2 - aa_2), \tag{23}$$

$$E_3(a) - E_2(b) = 4b_1\lambda_2 + 3(ab_2 + ba_2). \tag{24}$$

Proof. Assuming that M is of Type 1, and taking the corresponding orthonormal basis, we find that

$$\begin{aligned}(\nabla h)(E_2, E_3, E_2) &= \nabla^{\perp}_{E_2} h(E_2, E_3) - h(\nabla_{E_2}E_3, E_2) - h(\nabla_{E_2}E_2, E_3)\\ &= \nabla^{\perp}_{E_2}(bJE_2 - aJE_3) - h(-b_1E_1 - b_2E_2, E_2) - h(b_2E_3, E_3)\\ &= [ab_1 + b_2\lambda_2 - b_2\lambda_3]JE_1 + [E_2(b) + b_1\lambda_2 + 3b_2a]JE_2\\ &\quad + [-E_2(a) + 3b_2b]JE_3\end{aligned}$$

and

$$\begin{aligned}(\nabla h)(E_3, E_2, E_2) &= \nabla^{\perp}_{E_3} h(E_2, E_2) - 2h(\nabla_{E_3}E_2, E_2)\\ &= \nabla^{\perp}_{E_3}(\lambda_2 JE_1 + aJE_2 + bJE_3) - 2h(b_1E_1 + a_2E_3, E_2)\\ &= [E_3(\lambda_2) + ab_1]JE_1 + [E_3(a) - 3b_1\lambda_2 - 3a_2b]JE_2\\ &\quad + [E_3(b) + 3a_2a]JE_3.\end{aligned}$$

Consequently, it follows from the Codazzi equation that

$$ab_1 + b_2\lambda_2 - b_2\lambda_3 = E_3(\lambda_2) + ab_1$$

$$E_2(b) + b_1\lambda_2 + 3b_2a = E_3(a) - 3b_1\lambda_2 - 3a_2b$$

$$-E_2(a) + 3b_2b = E_3(b) + 3a_2a.$$

The other equations are then derived in a similar fashion using the other Codazzi equations. □

Using the definition of the curvature tensor and the Gauss equation, the following lemma follows by straightforward computations.

Lemma 7. *Assume that M is a 3-dimensional Lagrangian submanifold of Type 1. Denote by $\{E_1, E_2, E_3\}$ the corresponding orthonormal basis. Then, we have*

$$1 + \lambda_1\lambda_2 - \lambda_2^2 = b_1^2, \tag{25}$$

$$\lambda_1(\lambda_2 - \lambda_3) = (\lambda_2 + \lambda_3)(\lambda_2 - \lambda_3), \tag{26}$$

$$E_1(b_1) = 0, \tag{27}$$

$$E_2(b_1) = b(\lambda_2 - \lambda_3), \tag{28}$$

$$E_3(b_1) = a(\lambda_3 - \lambda_2), \tag{29}$$

$$E_1(b_2) - E_2(a_1) = b(\lambda_3 - \lambda_2) + a_2(a_1 - b_1), \tag{30}$$

$$E_3(a_1) - E_1(a_2) = a(\lambda_3 - \lambda_2) + b_2(a_1 - b_1), \tag{31}$$

$$E_3(b_2) - E_2(a_2) = 1 + \lambda_2\lambda_3 + b_1^2 + b_2^2 + a_2^2 - 2(a^2 + b^2) + 2a_1b_1. \tag{32}$$

Proof. By the definition of the curvature tensor, we obtain that

$$\begin{aligned} R(E_1, E_2)E_2 &= \nabla_{E_1}\nabla_{E_2}E_2 - \nabla_{E_2}\nabla_{E_1}E_2 - \nabla_{[E_1, E_2]}E_2 \\ &= \nabla_{E_1}b_2E_3 - \nabla_{E_2}a_1E_3 - \nabla_{a_1E_3 - b_1E_3}E_2 \\ &= b_1^2E_1 + [E_1(b_2) - E_2(a_1) - a_1a_2 + b_1a_2]E_3. \end{aligned}$$

On the other hand, from the Gauss equation it follows that

$$\begin{aligned} R(E_1, E_2)E_2 &= E_1 + Jh(Jh(E_2, E_2), E_1) - Jh(Jh(E_1, E_2)E_2) \\ &= E_1 + Jh(-\lambda_2E_1 - aE_2 - bE_3, E_1) - Jh(-\lambda_2E_2, E_2) \\ &= [1 + \lambda_1\lambda_2 - \lambda_2^2]E_1 + [b(\lambda_3 - \lambda_2)]E_3. \end{aligned}$$

So by comparing components of both expressions, we deduce that

$$b_1^2 = 1 + \lambda_1\lambda_2 - \lambda_2^2$$

$$E_1(b_2) - E_2(a_1) - a_1a_2 + b_1a_2 = b(\lambda_3 - \lambda_2).$$

The other equations can be derived similarly from the other Gauss equations. □

Similar, we obtain that if M is a Lagrangian of Type 2 that the Codazzi and Gauss equations reduce respectively to

LEMMA 8. *Assume that M is a 3-dimensional Lagrangian submanifold of Type 2. Denote by $\{E_1, E_2, E_3\}$ the corresponding orthonormal basis. Then, we have*

$$\lambda_2(c_1 - b_2) = \lambda_1(c_1 - b_2) = 0, \tag{33}$$

$$a(b_2 + c_1) = 2a_2\lambda_2 = -2a(3a_3 - b_2), \tag{34}$$

$$a(b_1 - c_2) = -2a_1\lambda_2, \tag{35}$$

$$b_2(\lambda_1 - 2\lambda_2) = aa_2, \tag{36}$$

$$(\lambda_1 - 2\lambda_2)(c_2 - b_1) = 2aa_1, \tag{37}$$

$$E_2(\lambda_1) = a_1(\lambda_1 - 2\lambda_2), \tag{38}$$

$$E_3(\lambda_1) = a_2(\lambda_1 - 2\lambda_2), \tag{39}$$

$$E_1(\lambda_2) = b_1(\lambda_1 - 2\lambda_2) + aa_1, \tag{40}$$

$$E_2(\lambda_2) = a(c_2 - b_1), \tag{41}$$

$$E_3(\lambda_2) = a(b_2 + c_1), \tag{42}$$

$$E_1(a) = \lambda_2 a_1 - c_2 a, \tag{43}$$

$$E_2(a) = \lambda_2(b_1 - c_2) - 3ac_3, \tag{44}$$

$$E_3(a) = \lambda_2(c_1 - 3b_2) + 3ab_3. \tag{45}$$

LEMMA 9. *Assume that M is a 3-dimensional Lagrangian submanifold of Type 2. Denote by $\{E_1, E_2, E_3\}$ the corresponding orthonormal basis. Then, we have*

$$E_2(a_1) - E_1(b_1) = 1 - \lambda_2^2 + \lambda_1\lambda_2 + a_1^2 + b_1^2 + b_2c_1 - b_2a_3 + a_2b_3 - a_3c_1, \tag{46}$$

$$E_3(a_2) - E_1(c_2) = 1 - \lambda_2^2 + \lambda_1\lambda_2 + a_2^2 + c_2^2 + b_2c_1 + b_2a_3 - a_1c_3 + a_3c_1, \tag{47}$$

$$E_2(a_2) - E_1(b_2) = b_1a_3 + b_1b_2 + a_1a_2 - a_1b_3 + c_2b_2 - c_2a_3, \tag{48}$$

$$E_3(a_1) - E_1(c_1) = a_2c_3 - a_3c_2 + a_1a_2 + b_1c_1 + c_1c_2 + b_1a_3, \tag{49}$$

$$E_2(c_2) - E_3(b_2) = b_1c_3 - b_3c_1 - a_2b_2 + a_2c_1 - b_2b_3 - c_2c_3, \tag{50}$$

$$E_2(c_1) - E_3(b_1) = b_3c_2 - c_3b_2 + a_1c_1 - a_1b_2 - c_1c_3 - b_1b_3, \tag{51}$$

$$E_3(a_3) - E_1(c_3) = a_1c_2 - a_2c_1 + a_2a_3 + a_3b_3 + b_3c_1 + c_2c_3, \tag{52}$$

$$E_1(b_3) - E_2(a_3) = b_1a_2 - a_1b_2 - a_1a_3 - b_1b_3 + a_3c_3 - c_3b_2, \tag{53}$$

$$E_3(b_3) - E_2(c_3) = 1 + \lambda_2^2 - 2a^2 + b_3^2 + c_3^2 + a_3b_2 - a_3c_1 + b_1c_2 - b_2c_1. \tag{54}$$

3. Lagrangian submanifolds of Type 1.

In this section, we will assume that M is a Lagrangian submanifold of $\boldsymbol{CP}^3(4)$ of Type 1, i.e. by restricting to an open and dense subset if necessary, we may assume that in a neighborhood of each point p, we have an orthonormal frame $\{E_1, E_2, E_3\}$ such that the equations Lemma 6 and 7 are satisfied. We denote the immersion of M^3 into $\boldsymbol{CP}^3$ by Φ.

Now we consider again different subcases. First, we assume that $b_1 \neq 0$ at p and hence in a neighborhood U of p. In this case it follows from (16) that $\lambda_1 = \lambda_2 + \lambda_3 = 0$. Hence $\Phi : U \to \boldsymbol{CP}^3(4)$ defines a minimal Lagrangian immersion. Notice that since $\lambda_1 = 0$, we also have that the integral curves of E_1 are geodesics in $\boldsymbol{CP}^3(4)$. These are precisely the Lagrangian immersions studied in [**CaV**]. We recall from there the following two examples:

Example 1. Let $\phi : N^2 \to \boldsymbol{CP}^3(4)$ be a horizontal holomorphic curve on a simply connected domain N^2. It is well known that using the Hopf fibration there exists an invariant (and hence minimal) immersion to the Sasakian space form $(S^7(1), I, \xi, \langle .,. \rangle)$ such that the following diagram commutes

$$\begin{array}{ccc} M^3 & \xrightarrow{\ \psi\ } & S^7(1) \subset \boldsymbol{C}^4 = \boldsymbol{H}^2 \\ \big\downarrow p & & \big\downarrow p \\ N^2 & \xrightarrow{\ \phi\ } & \boldsymbol{CP}^3(4), \end{array}$$

where p denotes the Hopf fibration. On $S^7 \subset \boldsymbol{H}^2$, there exist 3 orthogonal Sasakian structures I, J and K. Since ϕ is horizontal, the immersion ψ is horizontal with respect to J and K. Hence projecting using the Sasakian structure J produces a minimal Lagrangian immersion $\tilde{\psi}$ of M^3 into $\boldsymbol{CP}^3(4)$.

Example 2. Let $D \subset \boldsymbol{R}^2$ be a simply connected domain and let $f : D \to \boldsymbol{R}$ be a solution of the sinh-Gordon equation, i.e.

$$f_{xx} + f_{yy} + 8 \sinh f = 0.$$

Then, denoting by x and y the coordinates on $D \subset \boldsymbol{R}^2$ and by z the coordinate on $\boldsymbol{R}$, we define a metric on $D \times \boldsymbol{R}$ by

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right\rangle = \cosh f + \frac{1}{16} f_y^2, \qquad \left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial x} \right\rangle = -\frac{1}{16} f_x f_y, \tag{55a}$$

$$\left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right\rangle = \cosh f + \frac{1}{16} f_x^2, \qquad \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial z} \right\rangle = -\frac{1}{4} f_y, \tag{55b}$$

$$\left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle = \frac{1}{4} f_x, \qquad \left\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \right\rangle = 1, \tag{55c}$$

where f_x (resp. f_y) denotes the partial derivative of f with respect to x (resp. y) and a tensor T by

$$
\begin{aligned}
T\left(\frac{\partial}{\partial x},\frac{\partial}{\partial x}\right) &= -\frac{f_x}{2\cosh f}\frac{\partial}{\partial y}+\left(1+\frac{f_x^2}{8\cosh f)}\right)\frac{\partial}{\partial z},\\
T\left(\frac{\partial}{\partial y},\frac{\partial}{\partial y}\right) &= -\frac{f_y}{2\cosh f}\frac{\partial}{\partial x}-\left(1+\frac{f_y^2}{8\cosh f)}\right)\frac{\partial}{\partial z},
\end{aligned} \tag{56a}
$$

$$
\begin{aligned}
T\left(\frac{\partial}{\partial y},\frac{\partial}{\partial x}\right) &= -\frac{f_x}{4\cosh f}\frac{\partial}{\partial x}-\frac{f_y}{4\cosh f}\frac{\partial}{\partial y},\\
T\left(\frac{\partial}{\partial x},\frac{\partial}{\partial z}\right) &= \frac{1}{\cosh f}\left(\frac{\partial}{\partial x}+\frac{1}{4}f_y\frac{\partial}{\partial z}\right),
\end{aligned} \tag{56b}
$$

$$
T\left(\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right) = -\frac{1}{\cosh f}\left(\frac{\partial}{\partial y}-\frac{1}{4}f_x\frac{\partial}{\partial z}\right),\qquad T\left(\frac{\partial}{\partial z},\frac{\partial}{\partial z}\right) = 0. \tag{56c}
$$

A straightforward computation shows that, for more details see [**CaV**], the existence theorem of [**CDVV2**] can be applied and hence there exists a Lagrangian immersion $\psi_f : D\times \boldsymbol{R}\to \boldsymbol{C}P^3(4)$ with induced metric $\langle .,.\rangle$ and with second fundamental form $h = J(\psi_f)_*(T)$.

The Main Theorem of [**CaV**] then states that:

THEOREM 3. *The Lagrangian immersions defined in Example* 1 *and* 2 *are of Type* 1, *with* $b_1\neq 0$. *Conversely, every Lagrangian immersion of Type* 1 *with* $b_1\neq 0$ *can be locally obtained in this way.*

Therefore, restricting once more to an open and dense subset, we may now assume that $b_1 = 0$ in a neighborhood of the point p. In this case, it follows from (17) that $a_1(\lambda_2-\lambda_3) = 0$, so we have to consider again 2 subcases: $\lambda_2=\lambda_3$ and $\lambda_2\neq\lambda_3$. Restricting once more to an open and dense subset if necessary, we may assume that either $\lambda_2=\lambda_3$ in a neighborhood of the point p or that $\lambda_2\neq\lambda_3$ in a neighborhood of the point p.

3.1. $b_1 = 0$ **and** $\lambda_2=\lambda_3$**.**

In this case by rotating the vector fields E_2 and E_3, we may assume that $b = 0$. So from (22) we now find that

$$aa_1 = 0 \tag{57}$$

and from (25) we have

$$\lambda_1 = \frac{\lambda_2^2-1}{\lambda_2}. \tag{58}$$

On the other hand, we know from (18), (19) and (20) that λ_2 is constant, and therefore by (58) λ_1 is a constant too.

From (57) it follows that either $a = 0$ or $a_1 = 0$. Therefore by restricting once more to an open and dense subset, we may assume that either $a \neq 0$ (and hence $a_1 = 0$) in a neighborhood of p or $a = 0$ in a neighborhood of p. In the first case, it follows from (21), (23), (24), (30), (31) and (32) that we have the following system of differential equations for the other functions:

$$E_1(a) = 0 \qquad E_2(a) = -3aa_2 \qquad E_3(a) = 3ab_2$$

$$E_1(a_2) = 0 \qquad E_1(b_2) = 0 \qquad E_3(b_2) - E_2(a_2) = 1 + \lambda_2^2 - 2a^2 + b_2^2 + a_2^2.$$

And in the other case, i.e. when $a = 0$, we obtain the following system

$$E_1(b_2) - E_2(a_1) = a_2 a_1,$$

$$E_3(a_1) - E_1(a_2) = b_2 a_1,$$

$$E_3(b_2) - E_2(a_2) = 1 + \lambda_2^2 + b_2^2 + a_2^2.$$

We now can prove the following theorem.

THEOREM 4. *Let $G : N \to \boldsymbol{C}P^2(4)$ be a Lagrangian immersion. We denote by g a horizontal Hopf lift of G to $S^5(1)$. Then, for every constant number λ_2,*

$$\Phi(t,u,v) = \left[\left(1/\sqrt{1+\lambda_2^2}\right) g(u,v) e^{\lambda_2 it}, \left(\lambda_2/\sqrt{1+\lambda_2^2}\right) e^{-it/\lambda_2}\right]$$

defines a Lagrangian immersion of Type 1 with $b_1 = 0$ and $\lambda_2 = \lambda_3$. Conversely every Lagrangian immersion of Type 1, with $b_1 = 0$ and $\lambda_2 = \lambda_3$ can be locally obtained in this way.

PROOF. The fact that Φ as defined in the theorem is a Lagrangian submanifold of Type 1 with the desired properties can be straightforwardly verfied. In order to obtain the converse, we use the equations derived before together with the Lemma 7. We consider the disributions $T_1 = span\{E_1\}$ and $T_2 = span\{E_2, E_3\}$. It is clear that T_1 and T_2 are orthogonal distributions satisfying the following properties:

$$\nabla_{T_1} T_1 \subset T_1$$

$$\nabla_{T_2} T_1 \subset T_1$$

$$\nabla_{T_1} T_2 \subset T_2$$

$$\nabla_{T_2} T_2 \subset T_2.$$

From this, it is well known, see for instance [**KN**] that M can be locally written as a product manifold (with metric the product metric). So, we can write $M^3 = \boldsymbol{R} \times N^2$ and there exists coordinates

$$\frac{\partial}{\partial t} = E_1$$

$$span\left\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right\} = span\{E_2, E_3\}.$$

Since N^2 is a 2-dimensional we may also assume that $\partial/\partial u$ and $\partial/\partial v$ are isothermal coordinates on the surface N^2.

The immersion ϕ (which denotes the local horizontal lift of Φ to $S^7(1)$, and as usual we identify M with its image in $S^7(1)$) now satisfies the following system of differential equations:

$$\phi_{tt} = i\lambda_1\phi_t - \phi$$

$$\phi_{tu} = i\lambda_2\phi_u$$

$$\phi_{tv} = i\lambda_2\phi_v.$$

The first differential equation implies that we can write $\phi(t,u,v) = A_1(u,v)e^{\alpha_1 it} + A_2(u,v)e^{\alpha_2 it}$ where α_1 and α_2 are the solutions of the equation

$$\alpha^2 - \alpha\lambda_1 - 1 = 0,$$

from which it follows that $\alpha_1 = \lambda_2$ and $\alpha_2 = -1/\lambda_2$. On the other hand, the other two differential equation for ϕ now yield that $A_{2u} = A_{2v} = 0$. Therefore A_2 is a constant vector.

With respect to A_1 we know that

$$E_1 = \phi_t = A_1 i\lambda_2 e^{i\lambda_2 t} - \frac{iA_2}{\lambda_2}e^{-it/\lambda_2}$$

and so, we have

$$A_1 = \frac{-e^{-i\lambda_2 t}i\lambda_2}{\lambda_2^2 + 1}\left(E_1 + \frac{i\phi}{\lambda_2}\right)$$

a function of length $|A_1|^2 = 1/(1+\lambda_2^2)$. Similarly, we obtain that

$$A_2 = \frac{-e^{i(1/\lambda_2)t}i\lambda_2}{\lambda_2^2 + 1}(-E_1 + i\phi\lambda_2),$$

implying that A_1 and A_2 are orthogonal and that the length of the constant vector A_2 equals $|A_2|^2 = \lambda_2^2/(1+\lambda_2^2)$.

If we call $g = \sqrt{1+\lambda_2^2}A_1$, we can consider $g : N^2 \to S^7(1)$ and it follows that

$$D_{E_2}g = \sqrt{\lambda_2^2+1}e^{-i\lambda_2 t}E_2$$

$$D_{E_3}g = \sqrt{\lambda_2^2+1}e^{-i\lambda_2 t}E_3,$$

showing that $g_*(TN)$ is spanned by E_2 and E_3. Since

$$D_{E_2}E_2 = b_2E_3 + aJE_2 - e^{i\lambda_2 t}\sqrt{\lambda_2^2+1}g,$$

$$D_{E_2}E_3 = -b_2E_2 - aJE_3,$$

$$D_{E_3}E_3 = -a_2E_3 - aJE_2 - e^{i\lambda_2 t}\sqrt{\lambda_2^2+1}g,$$

it follows that $g(N)$ is obtained in a 3-dimensional complex linear subspace of $\boldsymbol{C}^4$ and thus g defines an immersion of N^2 into $S^5(1)$.

And as $g = -e^{-i\lambda_2 t}\left((i\lambda_2)/\sqrt{\lambda_2^2+1}(E_1 + (i\phi/\lambda_2))\right)$ we know that g is horizontal and moreover A_1 and A_2 are orthogonal, it follows that ϕ can be written as

$$\phi(t,u,v) = \left(\frac{1}{\sqrt{1+\lambda_2^2}}g(u,v)e^{\lambda_2 it}, \frac{\lambda_2}{\sqrt{1+\lambda_2^2}}e^{-it/\lambda_2}\right)$$

with λ_2 constant and the projection of g under the Hopf fibration defines a Lagrangian immersion of N^2 into CP^2. □

In the other case, $\lambda_2 \neq \lambda_3$, we have $a_1 = 0$, and from (25) and (26) it follows that $\lambda_1 = \lambda_2 + \lambda_3$ and $\lambda_2\lambda_3 = -1$. As $b_1 = 0$, for (28) and (29), we obtain $a = b = 0$. Thus, for (21) and (22) we have $a_2 = b_2 = 0$. Finally, from (19) and (20) we know that λ_2 is constant, and thus λ_1 and λ_3 are also constants. Also in this case, we can prove the corresponding theorem.

Theorem 5. *Let λ_2 and λ_3 be different constants satisfying $\lambda_2\lambda_3 = -1$. Then, we define $\Phi : \boldsymbol{R}^3 \to \boldsymbol{C}P^3(4)$ by*

$$\Phi(x,y,z) = \left(1/\left(\sqrt{2}\sqrt{1+\lambda_2^2}\right)e^{i\lambda_2 x}e^{\pm i\sqrt{1+\lambda_2^2}y}, 1/\left(\sqrt{2}\sqrt{1+\lambda_3^2}\right)e^{i\lambda_3 x}e^{\pm i\sqrt{1+\lambda_3^2}z}\right)$$

Then Φ defines a Lagrangian immersion of Type 1. *Conversely every Lagrangian immersion of Type* 1, *with $b_1 = 0$ and $\lambda_2 \neq \lambda_3$ can be obtained in this way.*

PROOF. It is straightforward to compute that Φ as defined above is a Lagrangian immersion of Type 1. In order to obtain the converse, we use the equations derived before. Since in this case $\nabla_{E_i}E_j = 0$, we can choose coordinates

$$\frac{\partial}{\partial x} = E_1, \quad \frac{\partial}{\partial y} = E_2, \quad \frac{\partial}{\partial z} = E_3.$$

So, the horizontal lift of the immersion satisfies

$$\phi_{xx} = i\lambda_1\phi_x - \phi, \tag{59}$$

$$\phi_{xy} = i\lambda_2\phi_y, \tag{60}$$

$$\phi_{xz} = i\lambda_3\phi_z, \tag{61}$$

$$\phi_{yy} = i\lambda_2\phi_x - \phi, \tag{62}$$

$$\phi_{yz} = 0, \tag{63}$$

$$\phi_{zz} = i\lambda_3\phi_x - \phi. \tag{64}$$

We now obtain from (59) that we can write our immersion as

$$\phi(x, y, z) = e^{i\alpha_1 x}A_1(y, z) + e^{i\alpha_2 x}A_2(y, z)$$

where α_1 and α_2 are the solutions of the equation

$$\alpha^2 - \lambda_1\alpha - 1 = 0,$$

thus, we have $\alpha_1 = \lambda_2$ and $\alpha_2 = \lambda_3$. Moreover, of (60), (61) and (63) we have

$$A_{1yz} = A_{2yz} = A_{2y} = A_{1z} = 0$$

and, of (62) and (64) we have

$$A_{1yy} = -A_1(1 + \lambda_2^2),$$
$$A_{2zz} = -A_2(1 + \lambda_3^2).$$

So, we can write our immersion as

$$\phi = B_1 e^{i\lambda_2 x} e^{i\sqrt{1+\lambda_2^2}y} + B_2 e^{i\lambda_2 x} e^{-i\sqrt{1+\lambda_2^2}y}$$
$$B_3 e^{i\lambda_3 x} e^{i\sqrt{1+\lambda_3^2}z} + B_4 e^{i\lambda_3 x} e^{-i\sqrt{1+\lambda_3^2}z}.$$

It follows that

$$\phi_x = i\lambda_2 B_1 e^{i\lambda_2 x} e^{i\sqrt{1+\lambda_2^2}y} + i\lambda_2 B_2 e^{i\lambda_2 x} e^{-i\sqrt{1+\lambda_2^2}y}$$

$$i\lambda_3 B_3 e^{i\lambda_3 x} e^{i\sqrt{1+\lambda_3^2}z} + i\lambda_3 B_4 e^{i\lambda_3 x} e^{-i\sqrt{1+\lambda_3^2}z},$$

$$\phi_y = i\sqrt{1+\lambda_2^2} B_1 e^{i\lambda_2 x} e^{i\sqrt{1+\lambda_2^2}y} - i\sqrt{1+\lambda_2^2} B_2 e^{i\lambda_2 x} e^{-i\sqrt{1+\lambda_2^2}y},$$

$$\phi_z = i\sqrt{1+\lambda_3^2} B_3 e^{i\lambda_3 x} e^{i\sqrt{1+\lambda_3^2}z} - i\sqrt{1+\lambda_3^2} B_4 e^{i\lambda_3 x} e^{-i\sqrt{1+\lambda_3^2}z}.$$

From which we deduce that

$$B_1 e^{i\lambda_2 x} e^{i\sqrt{1+\lambda_2^2}y} = \frac{1}{2(\lambda_3 - \lambda_2)\sqrt{(1+\lambda_2^2)}} \left(\sqrt{1+\lambda_2^2}\lambda_3 \phi + \sqrt{1+\lambda_2^2} i E_1 + i(\lambda_2 - \lambda_3) E_2 \right),$$

$$B_2 e^{i\lambda_2 x} e^{-i\sqrt{1+\lambda_2^2}y} = \frac{1}{2(\lambda_3 - \lambda_2)\sqrt{(1+\lambda_2^2)}} \left(\sqrt{1+\lambda_2^2}\lambda_3 \phi + \sqrt{1+\lambda_2^2} i E_1 - i(\lambda_2 - \lambda_3) E_2 \right),$$

$$B_3 e^{i\lambda_3 x} e^{i\sqrt{1+\lambda_3^2}z} = \frac{1}{2(\lambda_3 - \lambda_2)\sqrt{(1+\lambda_3^2)}} \left(-\sqrt{1+\lambda_3^2}\lambda_2 \phi - \sqrt{1+\lambda_3^2} i E_1 + i(\lambda_2 - \lambda_3) E_3 \right),$$

$$B_4 e^{i\lambda_3 x} e^{-i\sqrt{1+\lambda_3^2}z} = \frac{1}{2(\lambda_3 - \lambda_2)\sqrt{(1+\lambda_3^2)}} \left(-\sqrt{1+\lambda_3^2}\lambda_2 \phi - \sqrt{1+\lambda_3^2} i E_1 - i(\lambda_2 - \lambda_3) \right) E_3.$$

Since ϕ, E_1, E_2 and E_3 are mutually orthonormal and $\lambda_2\lambda_3 = -1$ it now follows that B_1, B_2, B_3 and B_4 are mutually orthogonal. Therefore, we can apply a isometry such that ϕ is given by

$$\phi(x,y,z) = \left(1/\left(\sqrt{2}\sqrt{1+\lambda_2^2}\right) e^{i\lambda_2 x} e^{\pm i\sqrt{1+\lambda_2^2}y}, 1/\left(\sqrt{2}\sqrt{1+\lambda_3^2}\right) e^{i\lambda_3 x} e^{\pm i\sqrt{1+\lambda_3^2}z} \right)$$

where λ_2 and λ_3 are constants satisfying $\lambda_2\lambda_3 = -1$. □

Summarizing the results of this section, we have

THEOREM 6. *Let $\Phi : M^3 \to \boldsymbol{CP}^3(4)$ be a Lagrangian immersion of Type* 1. *Then there exists an open and dense subset V of M^3 such that for each point p of V there exists a neighborhood U of p and $\Phi|_U$ is congruent to one of the immersions constructed in the previous* 3 *theorems.*

4. Lagrangian submanifolds of Type 2.

From (33) it follows that we can have two cases, namely either $c_1 \neq b_2$ or $c_1 = b_2$. In the first case it follows that $\lambda_1 = \lambda_2 = 0$, implying that M^3 is minimal and satisfies Chen's equality. Those submanifolds were classified in [**BSVW**].

In the second case, from (34) we obtain that $aa_3 = 0$, leading once more to two different subcases: namely $a = 0$ or $a \neq 0$. Restricting once more to an open and dense subset, and dividing M up into several pieces, we may assume that either $a = 0$ on M or $a \neq 0$ on M. In the first case, we call M of Type 2.1 whereas in the second case, we call M of Type 2.2.

4.1. Lagrangian submanifolds of Type 2.1.

If $a = 0$, we deduce from (34), (35), (44) and (45) that we have

$$a_2\lambda_2 = a_1\lambda_2 = (b_1 - c_2)\lambda_2 = b_2\lambda_2 = 0.$$

So, we consider again 2 cases: $\lambda_2 = 0$ or $\lambda_2 \neq 0$.

When $\lambda_2 \neq 0$, we get from the above equations that $a_2 = a_1 = b_2 = c_1 = 0$ and $b_1 = c_2$. And the other functions satisfy, using (40), (41), (42), (43), (44), (45), (46), (50), (51), (52), (53) and (54),

$$\begin{aligned}
E_2(\lambda_1) &= E_3(\lambda_1) = 0, \\
E_1(\lambda_2) &= b_1(\lambda_1 - 2\lambda_2), \\
E_2(\lambda_2) &= E_3(\lambda_2) = 0, \\
-E_1(b_1) &= 1 - \lambda_2^2 + \lambda_1\lambda_2 + b_1^2, \\
E_2(b_1) &= E_3(b_1) = 0, \\
E_3(a_3) - E_1(c_3) &= b_1c_3 + a_3b_3, \\
E_1(b_3) - E_2(a_3) &= a_3c_3 - b_1b_3, \\
E_3(b_3) - E_2(c_3) &= 1 + \lambda_2^2 + b_1^2 + b_3^2 + c_3^2.
\end{aligned}$$

In this case it is clear that we have 2 integrable distributions $T_1 = span\{E_2, E_3\}$ and $T_2 = span\{E_1\}$ with $T_1 \perp T_2$. We also see that T_1 is autoparallel, i.e.

$$\nabla_{T_1} T_1 \subset T_1.$$

The distribution T_2 is in general not autoparallel. However, since in this case

$$\nabla_{E_2} E_2 = -b_1 E_1 + b_3 E_3,$$
$$\nabla_{E_2} E_3 = -b_3 E_3,$$
$$\nabla_{E_3} E_2 = c_3 E_3,$$
$$\nabla_{E_3} E_3 = -b_1 E_1 - c_3 E_2,$$

we deduce that the distribution T_2 is umbilical with mean curvature normal $-b_1 E_1$. Since $E_2(b_1) = E_3(b_1) = 0$, it is also spherical. Therefore applying the result of Hiepko [**H**] we get that M can be written as a warped product with warping function f, $M = \boldsymbol{R} \times_{e^f} N^2$ with $f : M \to \boldsymbol{R}$, where $\partial/\partial t = E_3$ and such that

$$\frac{\partial f}{\partial t} = b_1, \tag{65}$$

$$\frac{\partial b_1}{\partial t} = -1 - b_1^2 + \lambda_2^2 - \lambda_1 \lambda_2, \tag{66}$$

$$\frac{\partial \lambda_2}{\partial t} = b_1(\lambda_1 - 2\lambda_2). \tag{67}$$

Using the standard formulas for warped product metrics, see for example [**ON**] it follows that the Gaussian curvature $\tilde{K}$ of the surface N is given by

$$\tilde{K} = e^{2f}(1 + \lambda_2^2 + b_1^2).$$

Since f, λ_2 and b_1 depend only on t, and $\tilde{K}$ has to be independent of t (which also can be varified by a straightforward computation). So, we deduce that N^2 has constant curvature $c^2 = e^{2f}(1 + \lambda_2^2 + b_1^2)$ and is therefore congruent with the sphere with radius $1/c$. Therefore, by applying the existence and uniqueness theorems, we have the following theorem:

Theorem 7. *Consider on an interval I a solution (f, b_1, λ_2) of the system of differential equations* (67) *for an arbitrary function λ_1. Introduce a constant c by*

$$c^2 = e^{2f}(1 + \lambda_2^2 + b_1^2)$$

and denote by $S^2(1/c)$ *the sphere with radius* $1/c$. *We then consider* $M = I \times_{e^f} S^2(1/c)$ *and define a* $(2,1)$*-tensorfield* σ *on* M *by*

$$\sigma\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \lambda_1 \frac{\partial}{\partial t},$$

$$\sigma\left(\frac{\partial}{\partial t}, X\right) = \lambda_2 X,$$

$$\sigma(X, Y) = \lambda_2 \langle X, Y \rangle \frac{\partial}{\partial t},$$

where $\partial/\partial t$ *is tangent to* I *and* X, Y *are tangent to* $S^2(1/c)$. *Then there exists a Lagrangian isometric immersion of Type* 2 *of* M *into* $\boldsymbol{CP}^3(4)$ *such that* $h = J\sigma$. *Conversely every Lagrangian immersion of Type* 2.1 *with* $\lambda_2 \neq 0$ *can be locally obtained in this way.*

In the case that $\lambda_2 = 0$, if follows from (40) that $b_1\lambda_1 = 0$. If $\lambda_1 = 0$ we again have that M is minimal and satisfies Chen's equality, and if $\lambda_1 \neq 0$ then, from (36), (37) and (40) we get

$$b_1 = c_2 = b_2 = c_1 = 0.$$

Since a is zero, we still are allowed to choose E_2 and E_3 appropriately. By applying a suitable rotation, we can choose $a_2 = 0$. So, (46) reduces to $E_2(a_1) = 1 + a_1^2$, which implies that a_1 is not a constant. Therefore, restricting once more to an open and dense subset and changing the sign of E_2 if necessary, we may assume that $a_1 > 0$. Now, from (48) it follows that we have $b_3 = 0$, and from (47) it follows that $c_3 = 1/a_1$. The other functions satisfy, using (46), (49), (52), (53), (38) and (39),

$$E_2(a_1) = 1 + a_1^2, \qquad E_3(a_1) = 0,$$

$$E_2(\lambda_1) = a_1\lambda_1, \qquad E_3(\lambda_1) = 0,$$

$$E_2(a_3) = a_1 a_3 - \frac{a_3}{a_1}, \qquad E_3(a_3) + \frac{E_1(a_1)}{a_1^2} = 0,$$

where λ_1 is not a constant. We now introduce a function f by $E_1(a_1) = f$. Computing $[E_1, E_3]a_1$ and $[E_1, E_2]a_1$ in two different ways it follows that $E_3(f) = a_3(1 + a_1^2)$ and $E_2(f) = 3a_1 f$. Next, we introduce functions α, β, γ and δ by

$$\alpha = \frac{a_1}{\sqrt{1+a_1^2}}, \tag{68}$$

$$\beta = -(1+a_1^2)\lambda_1^{-3}, \tag{69}$$

$$\gamma = f\lambda_1^{-3}, \tag{70}$$

$$\delta = a_3 a_1 \lambda_1^{-3}. \tag{71}$$

A straightforward computation then shows that

$$[E_2, \alpha E_3] = 0,$$
$$[\alpha E_3, \beta E_1 + \gamma E_2 + \delta E_3] = 0,$$
$$[E_2, \beta E_1 + \gamma E_2 + \delta E_3] = 0.$$

Hence there exist coordinates such that

$$\frac{\partial}{\partial x} = E_2,$$
$$\frac{\partial}{\partial y} = \alpha E_3,$$
$$\frac{\partial}{\partial z} = \beta E_1 + \gamma E_2 + \delta E_3$$

with

$$a_{1x} = 1 + a_1^2,$$
$$a_{1y} = 0,$$
$$\lambda_{1x} = a_1 \lambda_1,$$
$$\lambda_{1y} = 0,$$
$$f_x = 3a_1 f,$$
$$f_y = a_3\alpha(1+a_1^2),$$
$$a_{1z} = 0,$$
$$f = \frac{a_{1z} - \gamma(1+a_1^2)}{\beta},$$
$$\frac{a_{3y}}{\alpha} = -\frac{f}{a_1^2},$$

obtaining, after applying suitable translations of the coordinates, the following solutions:

$$f = \frac{A_1(z)\cos y + A_2(z)\sin y}{\cos^3 x} \tag{72}$$

$$a_1 = \tan x \tag{73}$$

$$\lambda_1 = \frac{A_3(z)}{\cos x} \tag{74}$$

$$a_3 = \frac{A_2(z)\cos y - A_1(z)\sin y}{\sin x \cos x} \tag{75}$$

where A_1, A_2 and A_3 are functions. Applying the existence and uniqueness theorem then gives the following result:

Theorem 8. *Let A_1, A_2 and A_3 be three arbitrary functions defined on an interval I and depending only on the variable z. Consider $M =]-\pi/2, \pi/2[\times \boldsymbol{R} \times I$. Let $\alpha, \dots, \delta, a_1, \dots, f$ be as defined in* (75) *and* (71). *We define a metric on M by assuming that E_1, E_2 and E_3 defined by*

$$\frac{\partial}{\partial x} = E_2,$$

$$\frac{\partial}{\partial y} = \alpha E_3,$$

$$\frac{\partial}{\partial z} = \beta E_1 + \gamma E_2 + \delta E_3$$

forms an orthonormal basis of M. We define a tensor field σ by

$$\sigma(E_i, E_j) = \lambda_1 \delta_{j1} \delta_{ij} E_1.$$

Then there exists a Lagrangian isometric immersion of Type 2 *of M into $\boldsymbol{CP}^3(4)$ such that $h = J\sigma$. Conversely every Lagrangian immersion of Type* 2.1 *with $\lambda_2 = 0$ and which is not minimal can be locally obtained in this way.*

4.2. Lagrangian submanifolds of Type 2.2.

If $a \neq 0$ then $a_3 = 0$ and of (34) we have $b_2 = a_2\lambda_2/a$. And from (35), (36) and (37) we have that

$$a_2(a^2 + \lambda_2(2\lambda_2 - \lambda_1)) = 0,$$

$$(b_1 - c_2)(a^2 + \lambda_2(2\lambda_2 - \lambda_1)) = 0.$$

Now, we again have to consider two subcases. First, we have that $a^2 + \lambda_2(2\lambda_2 - \lambda_1) \neq 0$. Then $a_2 = b_2 = c_1 = 0$ and $b_1 = c_2$. From (37) it follows that

$a_1 = 0$. The other functions satisfy the following equations, as can be deduced from (38), (39), (40), (41), (42), (43), (44), (45), (46), (50), (51), (52), (53) and (54),

$$\begin{aligned}
E_2(\lambda_1) &= E_3(\lambda_1) = 0,\\
E_1(\lambda_2) &= b_1(\lambda_1 - 2\lambda_2),\\
E_2(\lambda_2) &= E_3(\lambda_2) = 0,\\
E_1(a) &= -ab_1,\\
E_2(a) &= -3ac_3,\\
E_3(a) &= 3ab_3,\\
-E_1(b_1) &= 1 - \lambda_2^2 + \lambda_1\lambda_2 + b_1^2,\\
E_2(b_1) &= E_3(b_1) = 0,\\
E_1(c_3) &= -b_1c_3,\\
E_1(b_3) &= -b_1b_3,\\
E_3(b_3) - E_2(c_3) &= 1 + \lambda_2^2 - 2a^2 + b_1^2 + b_3^2 + c_3^2.
\end{aligned}$$

In this case we again have two integrable distributions $T_1 = span\{E_2, E_3\}$ and $T_2 = span\{E_1\}$ with $T_1 \perp T_2$. It follows again that T_2 is autoparallel and T_1 is spherical with mean curvature normal $-b_1E_1$. Therefore according to [**H**] we have that $M = \boldsymbol{R} \times_{e^f} N^2$ with $f : \boldsymbol{R} \to \boldsymbol{R}$ satisfying

$$\begin{aligned}
\frac{\partial f}{\partial t} &= b_1,\\
\frac{\partial b_1}{\partial t} &= -1 - b_1^2 + \lambda_2^2 - \lambda_1\lambda_2,\\
\frac{\partial \lambda_2}{\partial t} &= b_1(\lambda_1 - 2\lambda_2)
\end{aligned}$$

and the curvature of N^2 is given by

$$K(N^2) = e^{2f}(1 + \lambda_2^2 - 2a^2 + b_1^2),$$

which we verify by a straightforward computation is indeed independent of t. By translating f, i.e. by replacing a homothety N^2 with a homothetic copy of itself, we may assume that $e^{-2f}(1 + \lambda_2^2 + b_1^2) = 1$. It is also clear that $U_1 = e^f E_2$ and $U_2 = e^f E_3$ form an orthonormal basis on N^2. We denote by $\hat{\nabla}$ the Levi

Civita connection of the metric g on N^2. Using the formulas for warped product immersions, see [**ON**], we get that

$$\hat{\nabla}_{U_1} U_1 = e^f b_3 U_2, \quad \hat{\nabla}_{U_1} U_2 = -e^f b_3 U_1,$$

$$\hat{\nabla}_{U_2} U_1 = e^f c_3 U_2, \quad \hat{\nabla}_{U_2} U_2 = -e^f c_3 U_1.$$

We also define a tensor field $\hat{T}$ by

$$\hat{T}(U_1, U_1) = -\hat{T}(U_2, U_2) = e^f a U_1, \quad \hat{T}(U_1, U_2) = -e^f a U_2.$$

A straightforward computation now shows that $\hat{\nabla}\hat{T}$ is totally symmetric. Hence applying the existence and uniqueness theorem, we obtain that there exists an isometric horizontal minimal immersion $\psi : N^2 \to S^5(1)$. Therefore, we obtain the following theorem:

THEOREM 9. *Let $\psi : N^2 \to \boldsymbol{C}P^2(4)$ be a minimal isometric Lagrangian immersion. Denote by K its sectional curvature, by g its metric and by α its second fundamental form. Put $\hat{T} = -J\alpha$. We also consider a solution, defined on an interval I, of the system of differential equations*

$$\frac{\partial f}{\partial t} = b_1,$$

$$\frac{\partial b_1}{\partial t} = -1 - b_1^2 + \lambda_2^2 - \lambda_1 \lambda_2,$$

$$\frac{\partial \lambda_2}{\partial t} = b_1(\lambda_1 - 2\lambda_2),$$

where λ_1 is an arbitrary function, with initial conditions chosen such that $e^{2f}(1 + \lambda_2^2 + b_1^2) = 1$. We then define on the 3-dimensional warped product manifold $I \times_{e^f} N^2$ a tensor T by

$$T\left(\frac{\partial}{\partial t}, X\right) = \lambda_2 X,$$

$$T\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \lambda_1 \frac{\partial}{\partial t},$$

$$T(X, Y) = \langle X, Y \rangle \lambda_2 \frac{\partial}{\partial t} + \hat{T}(X, Y).$$

Then there exists a Lagrangian immersion $\Phi : I \times_{e^f} N^2 \to \boldsymbol{C}P^2(4)$ such that $JT = h$. Conversely every Lagrangian of Type 2.2 with $a^2 + \lambda_2(2\lambda_2 - \lambda_1) \neq 0$ can be locally obtained in this way.

Finally, we consider the case that $a^2 + \lambda_2(2\lambda_2 - \lambda_1) = 0$. Using (38), (39), (41), (42), (44) and (45) to compute the lefthandsides, we obtain that

$$E_2(a^2 + \lambda_2(2\lambda_2 - \lambda_1)) = E_3(a^2 + \lambda_2(2\lambda_2 - \lambda_1)) = 0$$

implies that $a_2 = 2b_3$ and $a_1 = -2c_3$. Now, after many tedious but straightforward computations, using the Mathematica computer program it follows that $a_1 = a_2 = 0$, thus $b_2 = c_1 = b_3 = c_3 = 0$ and in view of (35) also $b_1 = c_2$. From (54) we have

$$b_1^2 = 2a^2 - 1 - \lambda_2^2,$$

with $-E_1(b_1) = 1 + \lambda_2^2 + b_1^2 + a^2$, thus b_1 is not a constant. Therefore, restricting to an open dense subset and replacing E_1 by $-E_1$ if necessary, we may assume that $b_1 > 0$. Similarly, by choosing E_2 we may assume that $a > 0$. The differential equations determining the other functions are now obtained from (40), (41), (42), (43), (44) and (45) and reduce to

$$E_1(\lambda_2) = \frac{a^2\sqrt{|2a^2 - 1 - \lambda_2^2|}}{\lambda_2},$$

$$E_2(\lambda_2) = E_3(\lambda_2) = 0,$$

$$E_1(a) = -b_1 a,$$

$$E_2(a) = E_3(a) = 0,$$

where a, λ_1 and λ_2 are not constant. In this case a straightforward computation now shows that there exist coordinates x, y and z such that

$$\frac{\partial}{\partial x} = E_1,$$

$$\frac{\partial}{\partial y} = \frac{1}{a}E_2,$$

$$\frac{\partial}{\partial z} = \frac{1}{a}E_3.$$

Solving now the system of differential equations it follows that $\lambda_2^2 = -a^2 + d^2$, where d is a positive constant and $a'(x) = -a\sqrt{|3a^2 - 1 - d^2|}$ obtaining after a translation in the x coordinate that

$$a = \frac{\sqrt{1 + d^2}}{\sqrt{3}\sin(x\sqrt{1 + d^2})}.$$

Applying the existence and uniqueness theorem we then obtain the following

Theorem 10. *Consider* $]0, \pi/2[$ *and let* d *be a positive constant. Define* a *as above and denote by* I *the interval where* $d^2 - a^2 > 0$. *We define a metric on* $I \times \boldsymbol{R}^2$ *such that* E_1, E_2 *and* E_3 *defined by*

$$\frac{\partial}{\partial x} = E_1,$$
$$\frac{\partial}{\partial y} = \frac{1}{a}E_2,$$
$$\frac{\partial}{\partial z} = \frac{1}{a}E_3,$$

form an orthonormal basis and we define a tensor $T^{\pm}$ *by*

$$T^{\pm}(E_1,E_1) = \lambda_1 E_1, \qquad T^{\pm}(E_2,E_2) = \lambda_2 E_1 + aE_2,$$
$$T^{\pm}(E_1,E_2) = \lambda_2 E_2, \qquad T^{\pm}(E_2,E_3) = -aE_3,$$
$$T^{\pm}(E_1,E_3) = \lambda_2 E_3, \qquad T^{\pm}(E_3,E_3) = \lambda_2 E_1 - aJE_2,$$

where $\lambda_2 = \pm\sqrt{d^2 - a^2}$ *and* λ_1 *is determined by* $a^2 + \lambda_2(2\lambda_2 - \lambda_1) = 0$. *Then there exist Lagrangian immersions* $\phi^{\pm} : I \times \boldsymbol{R}^2 \to \boldsymbol{C}P^3(4)$ *with second fundamental forms respectively given by* $JT^{\pm}$. *Conversely every Lagrangian of Type* 2.2 *with* $a^2 + \lambda_2(2\lambda_2 - \lambda_1) = 0$ *can be locally obtained in this way.*

As seen before each of the geometric conditions described in Lemma 1 upto Lemma 5 leads upto the existence of a frame of Type 1 or Type 2. Combining the above formulas with those lemmas it is straightforward to compute which of the examples remain.

Corollary 1. *Let* M^3 *be a minimal Lagrangian submanifold of* $\boldsymbol{C}P^3(4)$. *Assume moreover that* M *is quasi Einstein and that* $\delta_M \neq 2$. *Then* M *is as obtained in Theorem* 7 *or* 9, *where* (f, b_1, λ_2) *is a solution of*

$$\frac{\partial f}{\partial t} = b_1,$$
$$\frac{\partial b_1}{\partial t} = -1 - b_1^2 + 3\lambda_2^2,$$
$$\frac{\partial \lambda_2}{\partial t} = -4 b_1 \lambda_2.$$

Corollary 2. *Let* M^3 *be a Lagrangian submanifold of* $\boldsymbol{C}P^3$ *with nowhere vanishing mean curvature vector* H. *Assume moreover that* JH *is an eigenvector of* A_H *and* A_H *restricted to* $\{JH\}^{\perp}$ *is a multiple of the identity. Then* M^3 *corresponds to the non-minimal examples in Theorems* 7, 8, 9 *and* 10.

Corollary 3. *Assume that* M^3 *is a minimal Lagrangian immersion which admits a unit length Killing vector field whose integral curves, when considered in* $\boldsymbol{C}^4$, *lie in a complex vector plane. Then* M^3 *is as in Theorem* 3, *or as in Theorem* 4 *with* $\lambda_2 = -1/\sqrt{3}$ *or as in Theorem* 5 *with* $\lambda_2 = 1$.

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