J. Math. Soc. Japan
Vol. 53, No. 3, 2001

Maillet type theorem for nonlinear partial differential equations and Newton polygons

By Akira Shirai

(Received Jul. 9, 1999)
(Revised Jan. 18, 2000)

Abstract. It is known that the formal solution to an equation of non-Kowalevski type is divergent in general. To this divergent solution it is important to evaluate the rate of divergence or the Gevrey order, and such a result is often called a Maillet type theorem. In this paper the Maillet type theorem is proved for divergent solutions to singular partial differential equations of non-Kowalevski type, and it is shown that the Gevrey order is characterized by a Newton polygon associated with an equation. In order to prove our results the majorant method is effectively employed.

1. Introduction and main result.

The purpose of this paper is to characterize the formal Gevrey order of divergent solutions to singular nonlinear partial differential equations of non Kowalevski type. Such characterization theorem is often called a Maillet type theorem.

In 1903, Maillet [**4**] proved that if an algebraic ordinary differential equation has a formal power series solution then this formal power series solution is in some formal Gevrey class. Later, by Gérard [**1**] and Malgrange [**5**] the result was extended to general analytic ordinary differential equations. This result was generalized to partial differential equations by Gérard-Tahara [**2**], and they got a critical value of Gevrey order for singular nonlinear partial differential equations [**3**, Chapter 6]. Moreover, they studied many other problems for singular nonlinear partial differential equations which are found in their book [**3**].

In linear partial differential equations, various kinds of Maillet type theorems were proved by Miyake [**7**], [**8**] and Miyake-Hashimoto [**9**].

In this paper we shall give a critical value of Gevrey order for more general equations than theirs.

Let $(t, x) = (t, x_1, \dots, x_d)$ denote $(1+d)$-dimensional complex variables. In this paper we use the following abbreviations. $\partial_t = \partial/\partial t, \partial_x = (\partial/\partial x_1, \dots, \partial/\partial x_d)$

2000 *Mathematics Subject Classification*. Primary 35A07, Secondary 35A20, 35C10.
Key Words and Phrases. singular PDE, divergent solution, Gevrey order, Newton polygon.

and for a multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \boldsymbol{N}^d$ ($\boldsymbol{N} = \{0, 1, 2, \dots\}$) we denote $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and $\partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_d)^{\alpha_d}$.

We denote by $\mathcal{O}_R$ the set of holomorphic functions in x variables in a neighborhood of the closed polydisk centered at the origin of radius R, by $\mathcal{O}_R[[t]]$ the ring of formal power series in t with coefficients in $\mathcal{O}_R$, and we set $\mathcal{O}[[t]] = \bigcup_{R>0} \mathcal{O}_R[[t]]$. We denote by $\mathcal{O}_R\{t\}$ the subring of $\mathcal{O}_R[[t]]$ of convergent power series in t near $t = 0$, and we set $\mathcal{O}\{t\} = \bigcup_{R>0} \mathcal{O}_R\{t\}$, which is the set of holomorphic functions in (t, x) variables in a neighborhood of the origin.

Let n, m, m_0, N, k be fixed nonnegative integers, which satisfy $m \le m_0 \le N$.

Let $\Delta = \{(j, \alpha) \in \boldsymbol{N} \times \boldsymbol{N}^d : 0 \le j \le m_0, 0 \le j + |\alpha| \le N\}$, and δ be the cardinal of Δ, i.e. $\delta = \sharp\Delta$. We denote by $X = (X_{j\alpha})_{(j,\alpha)\in\Delta} \in \boldsymbol{C}^\delta$ the complex variables.

Let $f(t, x, X)$ be a holomorphic function in a neighborhood of the origin of $\boldsymbol{C} \times \boldsymbol{C}^n \times \boldsymbol{C}^\delta$ with Taylor expansion

$$f(t, x, X) = \sum_{(p,q)\in\boldsymbol{N}\times\boldsymbol{N}^\delta} f_{pq}(x) t^p X^q, \quad X^q = \prod_{(j,\alpha)\in\Delta} X_{j\alpha}^{q_{j\alpha}}, (q = (q_{j\alpha}) \in \boldsymbol{N}^\delta).$$

We consider the following nonlinear partial differential equation,

$$\text{(1.1)} \qquad \begin{cases} t^n \partial_t^m u(t, x) = a(x) t^{k-m+n} + f(t, x, \{\partial_t^j \partial_x^\alpha u(t, x)\}_{(j,\alpha)\in\Delta}), \\ u(t, x) = O(t^k) \end{cases}$$

where $a(x)$ ($a(x) \not\equiv 0$) is holomorphic in a neighborhood of the origin.

In the equation (1.1) we assume the following relation

$$\text{(1.2)} \qquad m_0 < k$$

between the vanishing order k of $u(t, x)$ and m_0 in the set of multi-indices Δ. Moreover we assume the following condition.

$$\text{(1.3)} \qquad f_{pq}(x) \equiv 0, \quad \text{if} \quad V(p, q) := p + \sum_{(j,\alpha)\in\Delta} (k - j) q_{j\alpha} \le k - m + n.$$

Now we introduce some notations.

- $J = \{(p, q) : |q| \ge 1, f_{pq}(x) \not\equiv 0\}$,
- $N(p, q) = \max\{j + |\alpha| : q_{j\alpha} \ne 0\}$, for $(p, q) \in J$.

We define a non-negative constant σ by

$$\text{(1.4)} \qquad \sigma = \max_{(p,q)\in J} \left\{0, \frac{N(p, q) - m}{V(p, q) - (k - m + n)}\right\}.$$

Then the main result in this paper is stated as follows.

Theorem 1. *Under the assumptions* (1.2) *and* (1.3), *the equation* (1.1) *has a unique formal solution of the form*

$$u(t,x) = \sum_{i=k}^{\infty} u_i(x)t^i \in \mathcal{O}[[t]],$$

and it is in the formal Gevrey class $\mathcal{G}^{(1+\sigma)}$, *which is defined below. Especially if* $\sigma = 0$ *then the above formal solution is convergent.*

Definition 1 (s-Borel transformation and Gevrey order). Let $s \geq 1$. For a formal power series $u(t,x) = \sum_{i\geq 0} u_i(x)t^i \in \mathcal{O}[[t]]$, we define

$$v(t,x) = \sum_{i\geq 0} \frac{u_i(x)}{(i!)^{s-1}} t^i,$$

which is called the s-Borel transformation of $u(t,x)$.

If the s-Borel transformation $v(t,x)$ of $u(t,x)$ is in $\mathcal{O}\{t\}$, then we say that $u(t,x)$ is in the formal Gevrey class $\mathcal{G}^{(s)}$, and s is called the Gevrey order of $u(t,x)$.

Remark 1. It is obvious that $\mathcal{G}^{(1)} = \mathcal{O}\{t\}$, and by the definition we have $\mathcal{G}^{(s)} \subset \mathcal{G}^{(s')}$ if $s' \geq s$.

Remark 2. Here we give some remarks on the assumptions imposed in Theorem 1.

(i) The assumptions (1.2) and (1.3) in Theorem 1 are imposed to ensure the existence and uniqueness of the formal solution of (1.1). (See Examples 1 and 2.)

(ii) In general, $1+\sigma$ is the best constant for the Gevrey order of formal solution of (1.1) as we shall see in Example 3.

(iii) If an equation is not given in the normal form as (1.1), then there exist many formal solutions which belong to different Gevrey classes, in general, as we shall see in Example 4.

In linear ordinary differential equations it is convenient to draw a Newton polygon associated with the operator to characterize the Gevrey order of formal solutions. In fact, Ramis [**12**] characterized the Gevrey order by the Newton polygon for irregular singular ordinary differential operators, and Miyake [**7**], [**8**] and Miyake-Hashimoto [**9**] defined the Newton polygon for partial differential operators, and they proved the Maillet type theorem.

In Section 5, we shall define the Newton polygon for nonlinear partial differential equation of the form (1.1) to make it easy to understand our result. For the definition of the Newton polygon for nonlinear equation, see also S. Ōuchi [**11**].

2. Examples.

EXAMPLE 1. Let us consider the following nonlinear ordinary differential equation,

$$u(t) = t + tu(t)\sum_{r=0}^{\infty}\left(\frac{du}{dt}\right)^r, \quad u(t) = O(t). \tag{2.1}$$

This corresponds to the case $d = n = m = 0$, $m_0 = k = 1$ and $\sum_{r=0}^{\infty} T^r$ is holomorphic on $|T| < 1$. Therefore, (2.1) is a special case of (1.1) except the assumption (1.2). Let $u(t) = \sum_{i\geq 1} u_i t^i$ be a formal series, we substitute this series into (2.1), then we have $u_1 = 1$ and $u_2 = +\infty$. This shows that (2.1) does not have a formal power series solution.

EXAMPLE 2. Let us consider the following ordinary differential equation, which does not satisfy the assumption (1.3).

$$tu(t) = -t^4 + t^5 + u(t)\frac{d^2u}{dt^2}(t), \quad u(t) = O(t^3). \tag{2.2}$$

Then by an easy calculation we see that (2.2) has two formal solutions of the form

$$u(t) = \frac{1}{2}t^3 + \cdots, \quad \text{and} \quad u(t) = -\frac{1}{3}t^3 + \cdots.$$

EXAMPLE 3. Let us consider the following nonlinear partial differential equation,

$$\begin{cases} \partial_t u(t,x) = \dfrac{2}{1-x}t + (\partial_x^2 u)^2, \\ u(t,x) = O(t^2). \end{cases} \tag{2.3}$$

Our main theorem asserts that $u(t,x) \in \mathscr{G}^{(4/3)}$, which is the best class to which the formal solution $u(t,x)$ belongs as is proved as follows.

Let

$$u(t,x) = \sum_{n=1}^{\infty} u_{3n-1}(x)t^{3n-1}$$

be an expansion of $u(t,x)$. Then we have the following reccurence formula.

$$\begin{cases} u_2(x) = \dfrac{1}{1-x}, \\ u_{3n-1}(x) = \dfrac{1}{3n-1}\displaystyle\sum_{k+l=3n-2} \partial_x^2 u_k(x)\partial_x^2 u_l(x) \quad \text{for } n \geq 2. \end{cases} \tag{2.4}$$

By this formula we have

$$u_{3n-1}(x) = \frac{1}{(3n-1)(3n-4)\cdots 5}\partial_x^{2(n-1)}u_2(x)(\partial_x^2 u_2(x))^{n-1} + \cdots$$
$$= \frac{1}{(3n-1)(3n-4)\cdots 5}\cdot\frac{(2n-2)!}{(1-x)^{2n-1}}\cdot\frac{2^{n-1}}{(1-x)^{3(n-1)}} + \cdots .$$

By restricting at $x = 0$ in this formula, we have

$$u_{3n-1}(0) \geq C^n \frac{2^n(2n)!}{3^n n!}$$

by some positive constant C independent of n, because "$\cdots$" part is a linear sum of terms $\{(1-x)^{-p}\}$ with positive coefficients. This implies immediately that $u(t,x)$ just belongs to $\mathscr{G}^{(4/3)}$.

Example 4. Let us consider the following equation

$$\{u(t,x) - (1-x)t\}\frac{\partial u}{\partial t} = t^2, \tag{2.5}$$

where $(t,x) \in \boldsymbol{C}\times\boldsymbol{C}$.

Let $\sum_{n\geq 1} u_n(x)t^n \in \mathcal{O}[[t]]$ be a formal solution. Then we have the following relation for $u_1(x)$,

$$(u_1(x) - (1-x))u_1(x) = 0,$$

which implies $u_1(x) \equiv 0$ or $u_1(x) = 1-x$.

- Case of $u_1(x) \equiv 0$. The equation (2.5) is rewritten

$$\begin{cases} t\dfrac{\partial u}{\partial t} = \dfrac{-t^2}{1-x} + \dfrac{1}{1-x}\cdot u\cdot\dfrac{\partial u}{\partial t}, \\ u(t,x) = O(t^2). \end{cases} \tag{2.6}$$

In this case, a formal solution of (2.6) is holomorphic in the neighborhood of the origin of $\boldsymbol{C}\times\boldsymbol{C}$ by Theorem 1, because $\sigma = (1-1)/(3-2) = 0$.

- Case of $u_1(x) = 1-x$. We introduce a new unknown function $v(t,x)$ as follows.

$$v(t,x) = u(t,x) - (1-x)t.$$

Then $v(t,x)$ satisfies the following equation,

$$\begin{cases} v(t,x) = \dfrac{t^2}{1-x} - \dfrac{1}{1-x}\cdot v\cdot\dfrac{\partial v}{\partial t}, \\ v(t,x) = O(t^2). \end{cases} \tag{2.7}$$

In this case, a formal solution of (2.7) is in $\mathscr{G}^{(2)}$ by Theorem 1, because $\sigma = (1-0)/(3-2) = 1$. Here the Gevrey class $\mathscr{G}^{(2)}$ is the best possible class for $v(t,x)$ as can be shown by the same way as Example 3.

3. Reformulation of Theorem 1.

We shall reformulate our theorem for the sake of convenience to prove the theorem.

We decompose Δ into three disjoint subsets as follows:

- $\Delta_1 = \{(j,\alpha) \in \Delta : 0 \le j \le m-1, 0 \le j+|\alpha| \le m\}$,
- $\Delta_2 = \{(j,\alpha) \in \Delta : 0 \le j \le m-1, m+1 \le j+|\alpha| \le N\}$,
- $\Delta_3 = \{(j,\alpha) \in \Delta : m \le j \le m_0, m \le j+|\alpha| \le N\}$.

Under this notation, $f(t,x,X)$ is rewritten as follows:

$$f(t,x,\xi,\eta,\zeta) = \sum_{V(p,\beta,\gamma,\delta) \ge k-m+n+1} f_{p\beta\gamma\delta}(x) t^p \prod_{(j,\alpha)\in\Delta_1} \xi_{j\alpha}^{\beta_{j\alpha}} \prod_{(j,\alpha)\in\Delta_2} \eta_{j\alpha}^{\gamma_{j\alpha}} \prod_{(j,\alpha)\in\Delta_3} \zeta_{j\alpha}^{\delta_{j\alpha}},$$

where $X = (\xi,\eta,\zeta)$, $q = (\beta,\gamma,\delta)$ and β,γ,δ are multi-indices and

$$V(p,\beta,\gamma,\delta) = p + \sum_{(j,\alpha)\in\Delta_1} (k-j)\beta_{j\alpha} + \sum_{(j,\alpha)\in\Delta_2} (k-j)\gamma_{j\alpha} + \sum_{(j,\alpha)\in\Delta_3} (k-j)\delta_{j\alpha}. \tag{3.1}$$

Now the equation (1.1) is rewritten in the following form:

$$\begin{cases} t^n \partial_t^m u(t,x) = a(x) t^{k-m+n} \\ \qquad\qquad + f(t,x,\{\partial_t^j \partial_x^\alpha u(t,x)\}_{\Delta_1}, \{\partial_t^j \partial_x^\alpha u(t,x)\}_{\Delta_2}, \{\partial_t^j \partial_x^\alpha u(t,x)\}_{\Delta_3}), \\ u(t,x) = O(t^k), \end{cases} \tag{3.2}$$

where $\{\cdots\}_{\Delta_i}$ denotes $\{\cdots\}_{(j,\alpha)\in\Delta_i}$ for simplicity.

If $\Delta_2 = \varnothing$ and $\Delta_3 = \varnothing$, then it is trivial that $\sigma = 0$. We are not interested in this case so much. Indeed, in this case the proof of convergence of the formal solution is proved more easily (see Remark 3 in Section 4). Therefore in the following we assume

$$\Delta_2 \ne \varnothing \quad \text{or} \quad \Delta_3 \ne \varnothing. \tag{3.3}$$

The definition of J, $N(p,q)$ and σ can be rewritten as follows.

$$J = \{(p,\beta,\gamma,\delta) : |\gamma| + |\delta| \ge 1, f_{p\beta\gamma\delta}(x) \not\equiv 0\}, \tag{3.4}$$

$$N(p,\beta,\gamma,\delta) = \max\{j+|\alpha| : \gamma_{j\alpha} \ne 0 \text{ or } \delta_{j\alpha} \ne 0\}, \quad \text{for } (p,\beta,\gamma,\delta) \in J, \tag{3.5}$$

$$(3.6)\qquad \sigma = \max_{(p,\beta,\gamma,\delta)\in J}\left\{\frac{N(p,\beta,\gamma,\delta)-m}{V(p,\beta,\gamma,\delta)-(k-m+n)}\right\}.$$

By the assumption (3.3) we have $\sigma > 0$. Then theorem 1 is reduced to the following theorem.

Theorem 1′. *Under the assumptions* (1.2), (1.3) *and* (3.3), *the equation* (3.2) *has a unique formal solution of the form*

$$u(t,x) = \sum_{i=k}^{\infty} u_i(x)t^i \in \mathcal{O}[[t]],$$

and it is in the formal Gevrey class $\mathcal{G}^{(1+\sigma)}$.

4. Proof of Theorem 1′.

We shall give the proof of Theorem 1′ by the following four steps:

- Step 1: Construction of formal solution
- Step 2: Construction of majorant equation
- Step 3: Majorant estimate for integro-differential operator
- Step 4: Convergence of $(\sigma+1)$-Borel transformation

Now we begin the proof of Theorem 1′.

Step 1: Construction of formal solution.

In this step, we prove the following lemma.

Lemma 1. *The equation* (3.2) *has a unique formal solution of the form*

$$u(t,x) = \sum_{i=k}^{\infty} u_i(x)t^i \in \mathcal{O}[[t]].$$

Proof. We define an ideal $\mathcal{O}[[t]]_k$ of $\mathcal{O}[[t]]$ by $\mathcal{O}[[t]]_k = \{\sum_{i\ge k} u_i(x)t^i;\ u_i(x)\in\mathcal{O}\}$. Denote $P = t^n\partial_t^m$. We can easily see that the mapping

$$P : \mathcal{O}[[t]]_k \longrightarrow \mathcal{O}[[t]]_{k-m+n}$$

is invertible, and the inverse operator is given by $P^{-1} = \partial_t^{-m}t^{-n}$, where ∂_t^{-1} denotes integration in t variable from 0 to t. Let us introduce a new unknown function $U(t,x) = Pu(t,x)$, that is, $u(t,x) = P^{-1}U(t,x)$. Then the equation for $U(t,x)$ is given by

$$(4.1)\qquad U(t,x) = a(x)t^r + f(t,x,\{\partial_t^{j-m}\partial_x^{\alpha}(t^{-n}U(t,x))\}_{\Delta_1},$$
$$\{\partial_t^{j-m}\partial_x^{\alpha}(t^{-n}U(t,x))\}_{\Delta_2},\{\partial_t^{j-m}\partial_x^{\alpha}(t^{-n}U(t,x))\}_{\Delta_3}),$$

where we put $r = k - m + n$ for simplicity. Under this notation, (3.1) is rewritten

$$V(p,\beta,\gamma,\delta) = p + \sum_{(j,\alpha)\in\Delta_1} (r+m-n-j)\beta_{j\alpha} + \sum_{(j,\alpha)\in\Delta_2} (r+m-n-j)\gamma_{j\alpha} + \sum_{(j,\alpha)\in\Delta_3} (r+m-n-j)\delta_{j\alpha}.$$

It is trivial that if (4.1) has a unique formal solution then (3.2) has a unique formal solution.

Now we substitute the formal power series $U(t,x) = \sum_{i=r}^{\infty} U_i(x)t^i$ into (4.1). Then we have the following recurrence formula to determine the coefficients $\{U_i(x)\}$:

$$(4.2)\quad \begin{cases} U_r(x) = a(x), \\ U_i(x) = \displaystyle\sum_{V(p,\beta,\gamma,\delta)\geq r+1} f_{p\beta\gamma\delta}(x) \sum_{(*)} \prod_{\Delta_1,l} \frac{\partial_x^\alpha U_{n_{j\alpha l}}(x)}{\prod_{s=1}^{m-j}(n_{j\alpha l}-n+s)} \\ \qquad \times \displaystyle\prod_{\Delta_2,l} \frac{\partial_x^\alpha U_{m_{j\alpha l}}(x)}{\prod_{s=1}^{m-j}(m_{j\alpha l}-n+s)} \prod_{\Delta_3,l}\left(\prod_{s=0}^{j-m-1}(k_{j\alpha l}-n-s)\right)\partial_x^\alpha U_{k_{j\alpha l}}(x). \end{cases}$$

Here we used the following notations and abbreviations. The summation $\sum_{(*)}$ is taken over $(p,\{n_{j\alpha l}\},\{m_{j\alpha l}\},\{k_{j\alpha l}\})$ such that

$$(4.3)\quad i = p + \sum_{\Delta_1,l}(n_{j\alpha l}-n+m-j) + \sum_{\Delta_2,l}(m_{j\alpha l}-n+m-j) + \sum_{\Delta_3,l}(k_{j\alpha l}-n+m-j),$$

and

$$\prod_{\Delta_1,l} = \prod_{(j,\alpha)\in\Delta_1}\prod_{l=1}^{\beta_{j\alpha}}, \quad \prod_{\Delta_2,l} = \prod_{(j,\alpha)\in\Delta_2}\prod_{l=1}^{\gamma_{j\alpha}}, \quad \prod_{\Delta_3,l} = \prod_{(j,\alpha)\in\Delta_3}\prod_{l=1}^{\delta_{j\alpha}}$$

$$\sum_{\Delta_1,l} = \sum_{(j,\alpha)\in\Delta_1}\sum_{l=1}^{\beta_{j\alpha}}, \quad \sum_{\Delta_2,l} = \sum_{(j,\alpha)\in\Delta_2}\sum_{l=1}^{\gamma_{j\alpha}}, \quad \sum_{\Delta_3,l} = \sum_{(j,\alpha)\in\Delta_3}\sum_{l=1}^{\delta_{j\alpha}}.$$

The existence and uniqueness of the formal solution is proved as follows. Since all $n_{j\alpha l}$, $m_{j\alpha l}$, $k_{j\alpha l} \geq r$, we have

$$
\begin{aligned}
i &= p + \sum_{\Delta_1, l}(n_{j\alpha l} - n + m - j) + \sum_{\Delta_2, l}(m_{j\alpha l} - n + m - j) + \sum_{\Delta_3, l}(k_{j\alpha l} - n + m - j) \\
&\geq p + \sum_{(j,\alpha)\in\Delta_1}(r + m - n - j)\beta_{j\alpha} \\
&\quad + \sum_{(j,\alpha)\in\Delta_2}(r + m - n - j)\gamma_{j\alpha} + \sum_{(j,\alpha)\in\Delta_3}(r + m - n - j)\delta_{j\alpha} + n_{j\alpha l} - r \\
&= V(p,\beta,\gamma,\delta) + n_{j\alpha l} - r \\
&\geq n_{j\alpha l} + 1.
\end{aligned}
$$

Therefore, $n_{j\alpha l} \leq i - 1$ holds. Moreover, it is trivial that this relation does hold by replacing $n_{j\alpha l}$ by $m_{j\alpha l}$ or $k_{j\alpha l}$. Hence the right hand side of the reccurence formula of $U_i(x)$ is defined only by $\{U_r(x), U_{r+1}(x), \ldots, U_{i-1}(x)\}$ and their derivatives. This shows that the coefficients $U_i(x)$ $(i \geq r)$ are uniquely determined by induction on i.

Thus, Lemma 1 is proved. □

Step 2: Construction of majorant equation.

We shall define a majorant function of $U(t,x)$ obtained in Step 1.

When two formal power series $f(x) = \sum_{|\alpha|=0}^{\infty} f_\alpha x^\alpha$ and $g(x) = \sum_{|\alpha|=0}^{\infty} g_\alpha x^\alpha$ are given, a majorant relation

$$f(x) \ll g(x)$$

is defined by

$$|f_\alpha| \leq g_\alpha, \quad \text{for all } \alpha \in \boldsymbol{N}^n.$$

Moreover, when two formal power series $f(t,x) = \sum_{i=0}^{\infty} f_i(x)t^i$ and $g(t,x) = \sum_{i=0}^{\infty} g_i(x)t^i \in \mathcal{O}[[t]]$ are given, a majorant relation

$$f(t,x) \ll g(t,x)$$

is defined by

$$f_i(x) \ll g_i(x), \quad \text{for all } i = 0, 1, \ldots.$$

In the equation (1.1), functions $a(x)$ and $f(t,x,X)$ are assumed to be holomorphic in a neighborhood of the origin respectively. Therefore, we can take majorant functions for $a(x)$ and $f(t,x,\xi,\eta,\zeta)$ as follows:

$$a(x) \ll \frac{A}{(R - |x|)^r},$$

$$F(t,|x|,\xi,\eta,\zeta) := \sum_{V(p,\beta,\gamma,\delta)\geq r+1} \frac{F_{p\beta\gamma\delta}}{(R - |x|)^{p+|\beta|+|\gamma|+|\delta|}} t^p \xi^\beta \eta^\gamma \zeta^\delta \gg f(t,x,\xi,\eta,\zeta)$$

by taking some positive constants A, R and $F_{p\beta\gamma\delta}$, where $|x| = x_1 + \cdots + x_d$.

Let us consider the following equation for $W(t,|x|)$,

$$W(t,|x|)=\frac{A}{(R-|x|)^r}t^r+F(t,|x|,\{\partial_t^{j-m}\partial_x^\alpha(t^{-n}W(t,|x|))\}_{\Delta_1},\ \{\partial_t^{j-m}\partial_x^\alpha(t^{-n}W(t,|x|))\}_{\Delta_2},\{\partial_t^{j-m}\partial_x^\alpha(t^{-n}W(t,|x|))\}_{\Delta_3}) \tag{4.4}$$

with $W(t,|x|)=O(t^r)$.

We can prove the following lemma.

Lemma 2. *The formal solution of* (4.4) *is a majorant power series of the formal solution of* (4.1), *that is,* $U(t,x)\ll W(t,|x|)$.

Proof. We substitute the formal series $W(t,|x|)=\sum_{i=r}^\infty W_i(|x|)t^i$ into (4.4). Then we have

$$W_r(|x|)=\frac{A}{(R-|x|)^r}$$

and for $i\geq r+1$

$$W_i(|x|)=\sum_{V(p,\beta,\gamma,\delta)\geq r+1}\frac{F_{p\beta\gamma\delta}}{(R-|x|)^{p+|\beta|+|\gamma|+|\delta|}}\sum_{(*)}\prod_{\Delta_1,l}\frac{\partial_x^\alpha W_{n_{j\alpha l}}(|x|)}{\prod_{s=1}^{m-j}(n_{j\alpha l}-n+s)}\times\prod_{\Delta_2,l}\frac{\partial_x^\alpha W_{m_{j\alpha l}}(|x|)}{\prod_{s=1}^{m-j}(m_{j\alpha l}-n+s)}\prod_{\Delta_3,l}\left(\prod_{s=0}^{j-m-1}(k_{j\alpha l}-n-s)\right)\partial_x^\alpha W_{k_{j\alpha l}}(|x|), \tag{4.5}$$

where the summation $\sum_{(*)}$ is the same as (4.3). Since the recurrence formula (4.5) is similar to (4.2), we can easily show $U_i(x)\ll W_i(|x|)$ for $i=r,r+1,\dots$ by induction on i, that is,

$$U(t,x)\ll W(t,|x|). \tag{4.6}$$

□

Step 3: Majorant estimate for integro-differential operator.

Lemma 3. *The coefficients* $W_i(|x|)$ $(i\geq r)$ *can be written in the following form*:

$$W_i(|x|)=\sum_{h=\max\{r,Bi\}}^{Mi-(M-1)r}\frac{C_{hi}}{(R-|x|)^h},\quad i\geq r, \tag{4.7}$$

where C_{hi} *are non-negative constants and* M *and* B *are positive constants given by*

$$M=\max\left\{\left[\frac{(N-m+n+1)(r+1)}{k-m_0}\right]+2,\left[\frac{r+N-1}{k-m+1}\right]+1\right\},$$

$$B = \begin{cases} 1 & (n \geq m), \\ \dfrac{1}{m-n} & (n < m). \end{cases}$$

Here $[x]$ *denotes the integral part of* $x \in \boldsymbol{R}$.

Proof. We estimate the power of $1/(R - |x|)$. The upper bound estimation is calculated as follows:

$$\begin{aligned}
h &\leq p + |\beta| + |\gamma| + |\delta| + \sum_{\Delta_1, l}(Mn_{j\alpha l} - (M-1)r + |\alpha|) \\
&\quad + \sum_{\Delta_2, l}(Mm_{j\alpha l} - (M-1)r + |\alpha|) + \sum_{\Delta_3, l}(Mk_{j\alpha l} - (M-1)r + |\alpha|) \\
&= Mi - (M-1)p + |\beta| + |\gamma| + |\delta| \\
&\quad - M\sum_{\Delta_1}(-n + m - j)\beta_{j\alpha} - M\sum_{\Delta_2}(-n + m - j)\gamma_{j\alpha} \\
&\quad - M\sum_{\Delta_3}(-n + m - j)\delta_{j\alpha} - (M-1)\left(\sum_{\Delta_1} r\beta_{j\alpha} + \sum_{\Delta_2} r\gamma_{j\alpha} + \sum_{\Delta_3} r\delta_{j\alpha}\right) \\
&\quad + \sum_{\Delta_1} |\alpha|\beta_{j\alpha} + \sum_{\Delta_2} |\alpha|\gamma_{j\alpha} + \sum_{\Delta_3} |\alpha|\delta_{j\alpha} \\
&= Mi - (M-1)V(p, \beta, \gamma, \delta) + \sum_{\Delta_1}(1 + |\alpha| - m + j + n)\beta_{j\alpha} \\
&\quad + \sum_{\Delta_2}(1 + |\alpha| - m + j + n)\gamma_{j\alpha} + \sum_{\Delta_3}(1 + |\alpha| - m + j + n)\delta_{j\alpha} \\
&\leq Mi - (M-1)V(p, \beta, \gamma, \delta) \\
&\quad + (N - m + n + 1)\left(\sum_{\Delta_1} \beta_{j\alpha} + \sum_{\Delta_2} \gamma_{j\alpha} + \sum_{\Delta_3} \delta_{j\alpha}\right) \\
&\leq Mi - (M-1)V(p, \beta, \gamma, \delta) + \frac{N - m + n + 1}{k - m_0} V(p, \beta, \gamma, \delta) \\
&\leq Mi - \left(M - 1 - \frac{N - m + n + 1}{k - m_0}\right)(r + 1) \\
&\leq Mi - (M-1)r.
\end{aligned}$$

Next, the lower bound estimation is calculated by the definition of B as follows:

$$h \geq p + |\beta| + |\gamma| + |\delta| + \sum_{\Delta_1, l}(Bn_{j\alpha l} + |\alpha|) + \sum_{\Delta_2, l}(Bm_{j\alpha l} + |\alpha|) + \sum_{\Delta_3, l}(Bk_{j\alpha l} + |\alpha|)$$

$$= Bi - (B-1)p + \sum_{\Delta_1}(1 + |\alpha| + Bn - Bm + Bj)\beta_{j\alpha}$$

$$+ \sum_{\Delta_2}(1 + |\alpha| + Bn - Bm + Bj)\gamma_{j\alpha} + \sum_{\Delta_3}(1 + |\alpha| + Bn - Bm + Bj)\delta_{j\alpha}$$

$$\geq Bi - (B-1)p + \sum_{\Delta_1}(1 + |\alpha| + Bn - Bm + Bj)\beta_{j\alpha}$$

$$\geq Bi.$$

Thus, Lemma 3 is proved. □

The next lemma, which gives majorant relations between operators, plays a crucial role to construct a majorant equation for $W(t, |x|)$.

Lemma 4. *Let $W(t, |x|)$ be a formal solution of* (4.4). *Then we have the following majorant relations*:

(i) *If $(j, \alpha) \in \Delta_1$, then we have*

$$\partial_t^{j-m}\partial_x^\alpha(t^{-n}W(t, |x|)) \ll \left(\frac{M}{R - |x|}\right)^{|\alpha|} t^{m-j-n}W(t, |x|). \tag{4.8}$$

(ii) *If $(j, \alpha) \in \Delta_j$ $(j = 2, 3)$, then we have*

$$\partial_t^{j-m}\partial_x^\alpha(t^{-n}W(t, |x|)) \tag{4.9}$$

$$\ll \left(\frac{M}{R - |x|}\right)^{|\alpha|} t^{m-j-1}(t\partial_t)^{|\alpha|-m+j}t^{-n+1}W(t, |x|).$$

Proof. Let $(j, \alpha) \in \Delta_1$. Since $|\alpha| \leq m - j$, and by Lemma 3, we have

$$\partial_t^{j-m}\partial_x^\alpha(t^{-n}W(t, |x|))$$

$$= \sum_{i=r}^{\infty} \sum_{h=\max\{r, Bi\}}^{Mi-(M-1)r} \frac{\prod_{s=1}^{|\alpha|}(h + s - 1)}{\prod_{s=1}^{m-j}(i - n + s)} \frac{C_{hi}}{(R - |x|)^{h+|\alpha|}} t^{i-n+m-j}$$

$$\ll \sum_{i=r}^{\infty} \sum_{h=\max\{r, Bi\}}^{Mi-(M-1)r} M^{|\alpha|} \frac{C_{hi}}{(R - |x|)^{h+|\alpha|}} t^{i-n+m-j}$$

$$= \left(\frac{M}{R - |x|}\right)^{|\alpha|} t^{m-j-n}W(t, |x|).$$

Let $(j,\alpha)\in\Delta_2$, that is, $m+1\le j+|\alpha|\le N$. Then we have

$$\begin{aligned}
&\partial_t^{j-m}\partial_x^{\alpha}(t^{-n}W(t,|x|))\\
&\quad=\sum_{i=r}^{\infty}\sum_{h=\max\{r,Bi\}}^{Mi-(M-1)r}\frac{\prod_{s=1}^{|\alpha|}(h+s-1)}{\prod_{s=1}^{m-j}(i-n+s)}\frac{C_{hi}}{(R-|x|)^{h+|\alpha|}}t^{i-n+m-j}\\
&\quad\ll\sum_{i=r}^{\infty}\sum_{h=\max\{r,Bi\}}^{Mi-(M-1)r}M^{|\alpha|}(i-n+1)^{|\alpha|-m+j}\frac{C_{hi}}{(R-|x|)^{h+|\alpha|}}t^{i-n+m-j}\\
&\quad=\left(\frac{M}{R-|x|}\right)^{|\alpha|}t^{m-j-1}(t\partial_t)^{|\alpha|-m+j}(t^{-n+1}W(t,|x|)).
\end{aligned}$$

Next, let $(j,\alpha)\in\Delta_3$. Then we have

$$\begin{aligned}
&\partial_t^{j-m}\partial_x^{\alpha}(t^{-n}W(t,|x|))\\
&\quad=\sum_{i=r}^{\infty}\sum_{h=\max\{r,Bi\}}^{Mi-(M-1)r}\prod_{s=1}^{|\alpha|}(h+s-1)\prod_{s=1}^{j-m}(i-n-s+1)\frac{C_{hi}}{(R-|x|)^{h+|\alpha|}}t^{i-n+m-j}\\
&\quad\ll\sum_{i=r}^{\infty}\sum_{h=\max\{r,Bi\}}^{Mi-(M-1)r}M^{|\alpha|}(i-n+1)^{|\alpha|-m+j}\frac{C_{hi}}{(R-|x|)^{h+|\alpha|}}t^{i-n+m-j}\\
&\quad=\left(\frac{M}{R-|x|}\right)^{|\alpha|}t^{m-j-1}(t\partial_t)^{|\alpha|-m+j}(t^{-n+1}W(t,|x|)).
\end{aligned}$$

□

Step 4: Convergence of $(\sigma+1)$-Borel transformation.

We consider the following equation for $V(t,|x|)$.

$$(4.10)\quad \begin{cases} V(t,|x|)=\dfrac{A}{(R-|x|)^r}t^r\\ \qquad\qquad +F(t,x,\{G_1(j,\alpha)V\}_{\Delta_1},\{G_2(j,\alpha)V\}_{\Delta_2},\{G_2(j,\alpha)V\}_{\Delta_3}),\\ V(t,|x|)=O(t^r),\end{cases}$$

where $G_1(j,\alpha)$ and $G_2(j,\alpha)$ are the operators appeared in the right hand sides of (4.8) and (4.9), that is,

$$\begin{cases} G_1(j,\alpha)=\left(\dfrac{M}{R-|x|}\right)^{|\alpha|}t^{m-j-n},\\ G_2(j,\alpha)=\left(\dfrac{M}{R-|x|}\right)^{|\alpha|}t^{m-j-1}(t\partial_t)^{|\alpha|-m+j}t^{-n+1}.\end{cases}$$

We put the formal solution $V(t,|x|)=\sum_{i=r}^{\infty}V_i(|x|)t^i$. Then we have

$$V_r(|x|)=\frac{A}{(R-|x|)^r}$$

and for $i\geq r+1$

$$(4.11)\qquad V_i(|x|)=\sum_{V(p,\beta,\gamma,\delta)\geq r+1}\frac{F_{p\beta\gamma\delta}}{(R-|x|)^{p+|\beta|+|\gamma|+|\delta|}}\times\sum_{(*)}\prod_{\Delta_1,l}\left(\frac{M}{R-|x|}\right)^{|\alpha|}V_{n_{j\alpha l}}(|x|)$$
$$\times\prod_{\Delta_2,l}(m_{j\alpha l}-n+1)^{|\alpha|-m+j}\left(\frac{M}{R-|x|}\right)^{|\alpha|}V_{m_{j\alpha l}}(|x|)$$
$$\times\prod_{\Delta_3,l}(k_{j\alpha l}-n+1)^{|\alpha|-m+j}\left(\frac{M}{R-|x|}\right)^{|\alpha|}V_{k_{j\alpha l}}(|x|),$$

where the summation $\sum_{(*)}$ is the same as in (4.3). By the same argument as in the proof of Lemma 2 and from Lemma 4, we can easily show $W_i(|x|)\ll V_i(|x|)$ by induction on i, that is,

$$(4.12)\qquad U(t,x)\ll W(t,|x|)\ll V(t,|x|).$$

If we can prove

$$(4.13)\qquad V(t,|x|)\in\mathscr{G}^{(1+\sigma)},$$

then we obtain the consequence of Theorem 1′. Therefore, we will prove (4.13) in the remaining of this section.

Remark 3. If $\Delta_2=\emptyset$ and $\Delta_3=\emptyset$, then the equation for $V(t,|x|)$ is a functional equation. Therefore, we apply the implicit function theorem for this equation, we have a unique holomorphic solution $V(t,|x|)$, which is a majorant function of $U(t,x)$.

Now we put

$$X_i(|x|)=\frac{V_i(|x|)}{(i!)^{\sigma}}.$$

Then by dividing the formula (4.11) by $(i!)^{\sigma}$, we have

$$(4.14)\qquad X_i(|x|)=\sum_{V(p,\beta,\gamma,\delta)\geq r+1}\frac{F_{p\beta\gamma\delta}}{(R-|x|)^{p+|\beta|+|\gamma|+|\delta|}}\sum_{(*)}G(p,\beta,\gamma,\delta)$$
$$\times\prod_{\Delta_1,l}\left(\frac{M}{R-|x|}\right)^{|\alpha|}X_{n_{j\alpha l}}(|x|)$$
$$\times\prod_{\Delta_2,l}\left(\frac{M}{R-|x|}\right)^{|\alpha|}X_{m_{j\alpha 1}}(|x|)\prod_{\Delta_3,l}\left(\frac{M}{R-|x|}\right)^{|\alpha|}X_{k_{j\alpha l}}(|x|).$$

Here

$$G(p,\beta,\gamma,\delta) = \prod_{\Delta_2,l}(m_{j\alpha l}-n+1)^{|\alpha|-m+j}\prod_{\Delta_3,l}(k_{j\alpha l}-n+1)^{|\alpha|-m+j} \times \frac{\prod_{\Delta_1,l}(n_{j\alpha l}!)^{\sigma}\prod_{\Delta_2,l}(m_{j\alpha l}!)^{\sigma}\prod_{\Delta_3,l}(k_{j\alpha l}!)^{\sigma}}{(i!)^{\sigma}}. \tag{4.15}$$

Now we prove the following

LEMMA 5. *There exists a positive constant C independent of (p,β,γ,δ) such that*

$$G(p,\beta,\gamma,\delta) \le C^{|\beta|+|\gamma|+|\delta|}. \tag{4.16}$$

The most important tool in proving Lemma 5 is the following lemma.

LEMMA 6. *Let L and m_j be nonnegative integers such that $m_j \ge L$ for all $j=1,\dots,n$. Then we have*

$$m_1!\cdots m_n! \le (L!)^{n-1}(m_1+\cdots+m_n-(n-1)L)!. \tag{4.17}$$

PROOF OF LEMMA 6. Lemma 6 is proved by induction on n. The case $n=1$ is obvious. Assume that (4.17) is true up to $n-1$. For the case n, we have

$$\begin{aligned}
&\frac{m_1!\cdots m_n!}{(L!)^{n-1}(m_1+\cdots+m_n-(n-1)L)!}\\
&\quad\le \frac{(L!)^{n-2}(m_1+\cdots+m_{n-1}-(n-2)L)!m_n!}{(L!)^{n-1}(m_1+\cdots+m_n-(n-1)L)!}\\
&\quad= \frac{m_n!}{L!(m_1+\cdots+m_{n-1}-(n-2)L+1)\cdots(m_1+\cdots+m_n-(n-1)L)}\\
&\quad\le \frac{m_n!}{L!((n-1)L-(n-2)L+1)\cdots((n-1)L+m_n-(n-1)L)}\\
&\quad= \frac{m_n!}{L!(L+1)\cdots m_n}\\
&\quad= 1.
\end{aligned}$$

Thus, Lemma 6 is proved. □

PROOF OF LEMMA 5. For an arbitrary positive integer L, we have the following estimate by using Lemma 6.

$$
\begin{aligned}
&G(p,\beta,\gamma,\delta)\\
&\quad\le \prod_{\Delta_2,l}(m_{j\alpha l}-n+1)^{|\alpha|-m+j-L\sigma}\prod_{\Delta_3,l}(k_{j\alpha l}-n+1)^{|\alpha|-m+j-L\sigma}\\
&\qquad\times\frac{\left(\prod_{\Delta_1,l}(n_{j\alpha l}+L)!\prod_{\Delta_2,l}(m_{j\alpha l}+L)!\prod_{\Delta_3,l}(k_{j\alpha l}+L)!\right)^{\sigma}}{(i!)^{\sigma}}\\
&\quad\le \prod_{\Delta_2,l}(m_{j\alpha l}-n+1)^{|\alpha|-m+j-L\sigma}\\
&\qquad\times\prod_{\Delta_3,l}(k_{j\alpha l}-n+1)^{|\alpha|-m+j-L\sigma}(r+L)!^{(|\beta|+|\gamma|+|\delta|-1)\sigma}\\
&\qquad\times\left(\frac{\left(\sum_{\Delta_1,l}n_{j\alpha l}+\sum_{\Delta_2,l}m_{j\alpha l}+\sum_{\Delta_3,l}k_{j\alpha l}-r(|\beta|+|\gamma|+|\delta|)+r+L\right)!}{i!}\right)^{\sigma}.
\end{aligned}
$$

Now we decompose the set J into Ω_1 and Ω_2:

$$
\Omega_1=\left\{(p,\beta,\gamma,\delta)\in J:\frac{N(p,\beta,\gamma,\delta)-m}{V(p,\beta,\gamma,\delta)-r}=\sigma\right\},
$$

$$
\Omega_2=\left\{(p,\beta,\gamma,\delta)\in J:\frac{N(p,\beta,\gamma,\delta)-m}{V(p,\beta,\gamma,\delta)-r}<\sigma\right\}.
$$

When (p,β,γ,δ) is in Ω_1, we put $L=V(p,\beta,\gamma,\delta)-r=(N(p,\beta,\gamma,\delta)-m)/\sigma \in \boldsymbol{N}$. Recall i is given by (4.3). Then by an easy calculation we have

$$
\begin{aligned}
&\frac{\left(\sum_{\Delta_1,l}n_{j\alpha l}+\sum_{\Delta_2,l}m_{j\alpha l}+\sum_{\Delta_3,l}k_{j\alpha l}-r(|\beta|+|\gamma|+|\delta|)+r+L\right)!}{i!}\\
&\qquad=1.
\end{aligned}
$$

Moreover we get

$$
\prod_{\Delta_2,l}(m_{j\alpha l}-n+1)^{|\alpha|-m+j-L\sigma}\prod_{\Delta_3,l}(k_{j\alpha l}-n+1)^{|\alpha|-m+j-L\sigma}\le 1,
$$

since $|\alpha|-m+j-L\sigma\le 0$ for the exponents by the definition of $N(p,\beta,\gamma,\delta)$ and L (see (3.5)). Therefore, we have the following estimate in the case of $(p,\beta,\gamma,\delta)\in\Omega_1$:

$$G(p,\beta,\gamma,\delta) \le C_1^{|\beta|+|\gamma|+|\delta|-1}, \tag{4.18}$$

where $C_1 = \left\{ \Gamma\left(r + \frac{N-m}{\sigma} + 1 \right) \right\}^{\sigma} (\ge ((r+L)!)^{\sigma})$.

Next when (p,β,γ,δ) is in Ω_2, we put

$$L = \left[\frac{N(p,\beta,\gamma,\delta) - m}{\sigma} \right] + 1 \in \boldsymbol{N},$$

where $[x]$ denotes the integral part of x. By the definition of this symbol, it holds that

$$\frac{N(p,\beta,\gamma,\delta) - m}{\sigma} < L \le \frac{N(p,\beta,\gamma,\delta) - m}{\sigma} + 1,$$

and

$$\frac{N(p,\beta,\gamma,\delta) - m}{\sigma} < V(p,\beta,\gamma,\delta) - r$$

holds since $(p,\beta,\gamma,\delta) \in \Omega_2$. These imply

$$L \le \frac{N(p,\beta,\gamma,\delta) - m}{\sigma} + 1 < V(p,\beta,\gamma,\delta) - r + 1.$$

Since L and $V(p,\beta,\gamma,\delta) - r + 1$ are positive integers, we have

$$L \le V(p,\beta,\gamma,\delta) - r.$$

Now we can easily check

$$\frac{\left(\left(\sum_{\Delta_1, l} n_{j\alpha l} + \sum_{\Delta_2, l} m_{j\alpha l} + \sum_{\Delta_3, l} k_{j\alpha l} - r(|\beta| + |\gamma| + |\delta|) + r + L \right)! \right)^{\sigma}}{(i!)^{\sigma}} \le 1,$$

and

$$\prod_{\Delta_2, l} (m_{j\alpha l} - n + 1)^{|\alpha| - m + j - L\sigma} \prod_{\Delta_3, l} (k_{j\alpha l} - n + 1)^{|\alpha| - m + j - L\sigma} \le 1$$

does hold since the exponents $|\alpha| - m + j - L\sigma$ are negative in this case. Therefore, we have the following estimate in the case of $(p,\beta,\gamma,\delta) \in \Omega_2$:

$$G(p,\beta,\gamma,\delta) \le C_2^{|\beta|+|\gamma|+|\delta|-1}, \tag{4.19}$$

where $C_2 = \left\{ \Gamma\left(r + \frac{N-m}{\sigma} + 2 \right) \right\}^{\sigma} (\ge ((r+L)!)^{\sigma})$.

Since $C_1 < C_2$ and $1 \le C_2$, by (4.18) and (4.19) we have for all $(p,\beta,\gamma,\delta) \in J$

$$G(p,\beta,\gamma,\delta) \le C^{|\beta|+|\gamma|+|\delta|}, \tag{4.20}$$

by $C = C_2 = \left\{\Gamma\left(r+\dfrac{N-m}{\sigma}+2\right)\right\}^{\sigma} (\ge ((r+L)!)^{\sigma})$.

Thus, Lemma 5 is proved. □

By (4.14) and the above inequality (4.20), we have the following majorant relation for $X_i(|x|)$.

$$\begin{aligned} X_i(|x|) &\ll \sum_{V(p,\beta,\gamma,\delta)\ge r+1} \frac{F_{p\beta\gamma\delta}}{(R-|x|)^{p+|\beta|+|\gamma|+|\delta|}} \\ &\quad \times \sum_{(*)} \prod_{\Delta_1, l} C\left(\frac{M}{R-|x|}\right)^{|\alpha|} X_{n_{j\alpha l}}(|x|) \\ &\quad \times \prod_{\Delta_2, l} C\left(\frac{M}{R-|x|}\right)^{|\alpha|} X_{m_{j\alpha l}}(|x|) \prod_{\Delta_3, l} C\left(\frac{M}{R-|x|}\right)^{|\alpha|} X_{k_{j\alpha l}}(|x|). \end{aligned} \tag{4.21}$$

Now we consider the following analytic functional equation for $Y(t,|x|)$:

$$\begin{aligned} Y(t,|x|) &= \frac{At^r}{(r!)^{\sigma}(R-|x|)^r} \\ &\quad + F(t,|x|,\{H(j,\alpha)Y\}_{\Delta_1},\{H(j,\alpha)Y\}_{\Delta_2},\{H(j,\alpha)Y\}_{\Delta_3}) \end{aligned} \tag{4.22}$$

with $Y(t,|x|) = O(t^r)$, where

$$H(j,\alpha)Y := C\left(\frac{M}{R-|x|}\right)^{|\alpha|} t^{m-j-n} Y(t,|x|).$$

Then by the implicit function theorem we see that (4.22) has a unique solution

$$Y(t,|x|) = \sum_{i=r}^{\infty} Y_i(|x|)t^i$$

holomorphic in a neighborhood of the origin. We can easily examine that the coefficient $Y_i(|x|)$ is just obtained by the same formula as the right hand side of (4.21). This proves that

$$Y(t,|x|) \gg \sum_{i=r}^{\infty} X_i(|x|)t^i = \sum_{i=r}^{\infty} \frac{V_i(|x|)}{(i!)^{\sigma}} t^i \gg \sum_{i=r}^{\infty} \frac{U_i(x)}{(i!)^{\sigma}} t^i,$$

which we want to prove.

5. Newton polygon.

In this section, we define the Newton polygon for nonlinear partial differential equation.

First, let us consider the following linear ordinary differential operator,

$$P = \sum_{j,k=0}^{\text{finite}} a_{jk} t^j \left(\frac{d}{dt}\right)^k, \quad a_{jk} \in \boldsymbol{C}. \tag{5.1}$$

For an operator $a_{jk} t^j (d/dt)^k$ $(a_{jk} \neq 0)$ we associate a sector $S(j,k)$ such that

$$S(j,k) := \{(x,y) \in \boldsymbol{R}^2; x \leq k, y \geq j - k\}.$$

Then the Newton polygon $N(P)$ for the operator P is defined by

$$N(P) := \mathrm{Ch}\{S(j,k); (j,k) \text{ with } a_{jk} \neq 0\},$$

where $\mathrm{Ch}\{\cdots\}$ denotes the convex hull of $\{S(j,k)\}_{j,k}$.

Let $r_0, r_1, \dots, r_\tau$ be the slopes of sides of the Newton polygon $N(P)$ with τ-vertexes such that $0 = r_0 < r_1 < \cdots < r_\tau = +\infty$. Then Ramis proved the following Maillet type theorem.

Theorem 2 (Ramis [**11**]). *The formal power series solution of the following equation*

$$Pu(t) = f(t), \quad f(t) \in \mathcal{O}$$

belongs to $\mathscr{G}^{(1+\sigma)}$ *with* $\sigma = 1/r_1$.

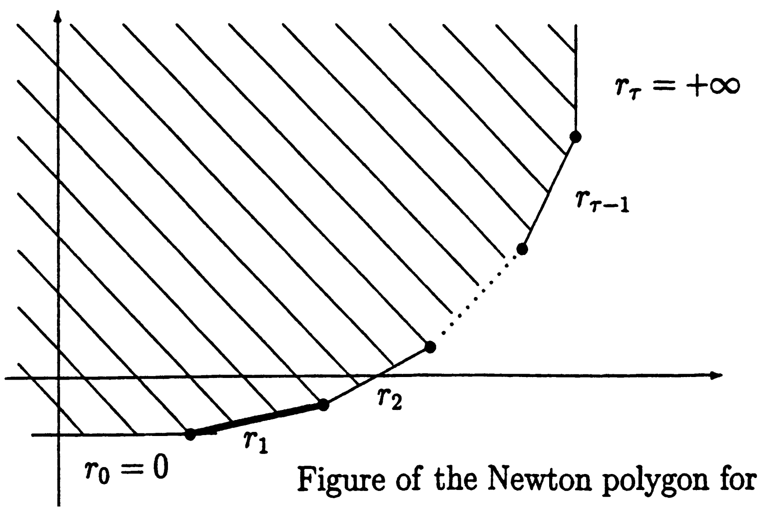


Figure of the Newton polygon for P

Miyake [7], [8] and Miyake-Hashimoto [9] showed that such an observation of Maillet type theorem from the Newton polygon is possible for generel singular partial differential equations, but for linear equations.

Here we shall introduce the Newton polygon for nonlinear partial differential equations and explain Theorem 1 by using the Newton polygon as Ramis' theorem.

Let us consider again the nonlinear partial differential equation considered in Theorem 1:

$$\begin{cases} t^n \partial_t^m u(t,x) = a(x)t^{k-m+n} + f(t,x,\{\partial_t^j \partial_x^\alpha u(t,x)\}_\Delta), \\ u(t,x) = O(t^k), \end{cases} \tag{5.2}$$

where

$$f(t,x,X) = \sum_{V(p,q) \ge k-m+n+1} f_{pq}(x) t^p \prod_{(j,\alpha)\in\Delta} X_{j\alpha}^{q_{j\alpha}},$$

$$V(p,q) = p + \sum_{(j,\alpha)\in\Delta} (k-j) q_{j\alpha}. \tag{5.3}$$

We introduced

- $J = \{(p,q) : |q| \ge 1, f_{pq}(x) \not\equiv 0\}$,
- $N(p,q) = \max\{j + |\alpha| : q_{j\alpha} \ne 0\}$, for $(p,q) \in J$.

By Theorem 1, we have already known that the formal solution in $\mathscr{O}[[t]]$ of (5.2) is in $\mathscr{G}^{(1+\sigma)}$ with

$$\sigma = \max_{(p,q)\in J} \left\{ 0, \frac{N(p,q) - m}{V(p,q) - (k-m+n)} \right\}.$$

Now the Newton polygon for nonlinear partial differential equation (5.2) is defined as follows. For each term

$$f_{pq}(x) t^p \prod_{(j,\alpha)\in\Delta} (\partial_t^j \partial_x^\alpha u)^{q_{j\alpha}}$$

we associate a point $(N(p,q), V(p,q))$ on (x,y)-plane, which means that the first component is the highest order of differentiations in the term, and the second component is the order of zeros of the term in t variable. For example,

$$t^n \partial_t^m u(t,x) \Leftrightarrow (m, k-m+n),$$

$$t^p (\partial_t^q \partial_x^\alpha u)^K (\partial_t^r u)^L \Leftrightarrow (\max\{q + |\alpha|, r\}, p + K(k-q) + L(k-r)).$$

Next we introduce a sector $S(p,q)$ for each point $(N(p,q), V(p,q))$ by

$$S(p,q) := \{(x,y) \in \boldsymbol{R}^2; x \le N(p,q), y \ge V(p,q)\}.$$

Then the Newton polygon $\mathcal{N}$ of the equation (5.2) is defined by

$$\mathcal{N} := \mathrm{Ch}\{S(p,q); (p,q) \in J\}.$$

Let $\gamma_0, \gamma_1, \ldots, \gamma_\tau$ be the slopes of sides of the Newton polygon $\mathcal{N}$ with τ-vertexes such that $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_\tau = +\infty$.

Then Theorem 1 can be read as follows.

THEOREM 3. *Under the assumptions* (1.2) *and* (1.3), *the formal solution of* (5.2) *is in* $\mathcal{G}^{(1+\sigma)}$ *with* $\sigma = 1/\gamma_1$ *which is the inverse of the least positive slope of the Newton polygon.*

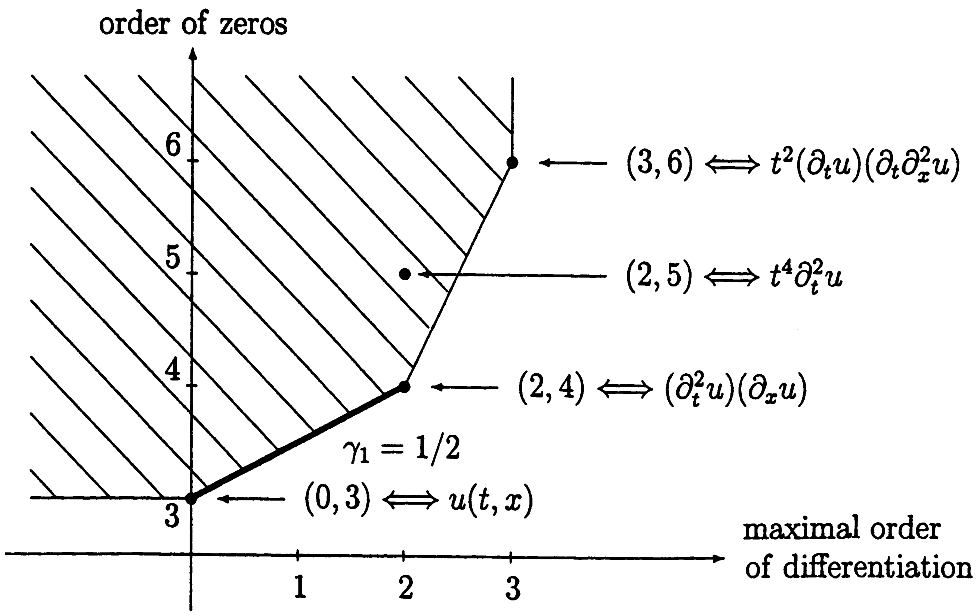
EXAMPLE 5. Let us consider the following equation

$$\left\{\begin{array}{l} u(t,x) = t^3 + (\partial_t^2 u)(\partial_x u) + t^2(\partial_t u)(\partial_t \partial_x^2 u) + t^2\partial_t^2 u, \\ u(t,x) = O(t^3). \end{array}\right. \tag{5.4}$$

The points corresponding to each terms are:

$$u(t,x) \Leftrightarrow (0,3), \quad (\partial_t^2 u)(\partial_x u) \Leftrightarrow (2,4)$$
$$t^2(\partial_t u)(\partial_t \partial_x^2 u) \Leftrightarrow (3,6), \quad t^4\partial_t^2 u \Leftrightarrow (2,5)$$

Therefore the Newton polygon of (5.4) is drawn as below.



Now Theorem 3 asserts that the formal solution of (5.4) is in $\mathcal{G}^{(3)}$.

EXPLANATION OF THEOREM 3. Why is the Gevrey order taken from the least positive slope of Newton polygon? The answer to this question is as follows.

By Theorem 1, the Gevrey order is given by the maximal value of

$$\frac{N(p,q)-m}{V(p,q)-(k-m+n)}$$

for $(p,q)\in J$.

Now the point associated with $t^n\partial_t^m u(u(t,x)=O(t^k))$ is given by $(m,k-m+n)$, and other points are given by $(N(p,q),V(p,q))$. Moreover, since $V(p,q)>k-m+n$, the point $(m,k-m+n)$ is the coordinate of the vertex on the horizontal side of the Newton polygon. Therefore, the minimal slope of segment from $(m,k-m+n)$ to $(N(p,q),V(p,q))$ is just the least positive slope of Newton polygon, that is,

$$0<\gamma_1=\min_{(p,q)\in J}\left\{\frac{V(p,q)-(k-m+n)}{N(p,q)-m}\right\}\leq+\infty.$$

By Theorem 1, we obtain a consequence of Theorem 3.

REMARK 4. We remark that for linear equations, the Newton polygon is defined for the operator and does not depend on individual solutions. In fact, even if we define the Newton polygon by taking care of the order of zeros of the formal solution as is being done in nonlinear equations, the Newton polygons so obtained will only be vertical shifts of the Newton polygon in the linear case, and the slopes of sides are still the same.

EXAMPLE 6. In Example 4, we considered the following equation

$$\{u(t,x)-(1-x)t\}\frac{\partial u}{\partial t}=t^2. \tag{5.5}$$

As in Example 4, in the case of $u_1(x)\equiv 0$, the equation (5.5) is rewritten into the form (2.6) and the Newton polygon of (2.6) is drawn as Figure 1; in the case of $u_1(x)=1-x$, the equation is reduced to (2.7) and the Newton polygon of (2.7) is drawn as Figure 2.

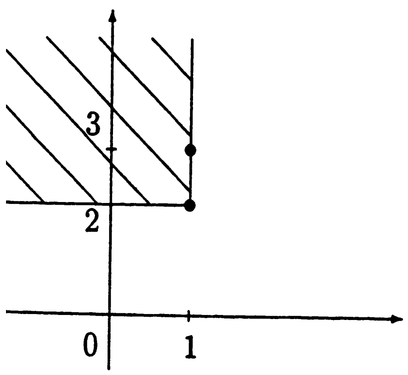


Figure 1

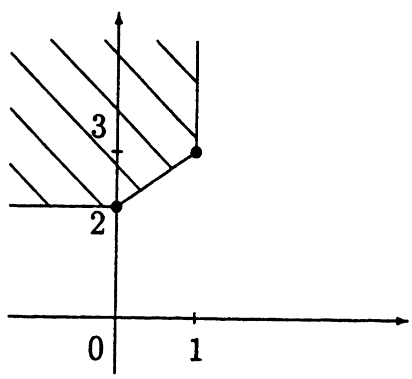


Figure 2

This shows that the order of zeros of formal solution changes the form of the Newton polygon for nonlinear partial differential equation.

References

[1] R. Gérard, Sur le théorème de Maillet, Funkcial. Ekvac., **34** (1991), 117–125.
[2] R. Gérard and H. Tahara, Formal power series solutions of nonlinear first order partial differential equations, Funkcial. Ekvac., **41** (1998), 133–166.
[3] R. Gérard and H. Tahara, Singular Nonlinear Partial Differential Equations. Vieweg, 1996.
[4] E. Maillet, Sur les séries divergentes et les équations différentielles, Ann. Ecole Normale, Ser. 3, **20** (1903), 487–518.
[5] B. Malgrange, Sur le théorème de Maillet, Asymptotic analysis, **2** (1989), 1–4.
[6] B. Malgrange, Sur les points singuliers des équations différentielles linéaires, Enseign. Math., **20** (1970), 147–176.
[7] M. Miyake, Global and local Goursat problems in a class of holomorphic or partially holomorphic functions, J. Differential Equations, **39** (1981), 445–463.
[8] M. Miyake, Newton polygons and formal Gevrey indices in the Cauchy-Gourat-Fuchs type equations, J. Math. Soc. Japan, **43** (1991), 305–330.
[9] M. Miyake and Y. Hashimoto, Newton polygons and Gevrey indices for linear partial differential operators, Nagoya Math. J., **128** (1992), 15–47.
[10] M. Miyake and A. Shirai, Convergence of formal solutions of first order singular nonlinear partial differential equations in complex domain, preprint (1999).
[11] S. Ōuchi, Genuine solutions and formal solutions with Gevrey type estimates of nonlinear partial differential equations, J. Math. Sci. Univ. Tokyo, **2** (1995), 375–417.
[12] J. P. Ramis, Théorèmes d'indices Gevrey pour les équations différentielles ordinaires, Mem. Amer. Math. Soc., No. 296, **48**, 1984.

Akira Shirai
Graduate School of Mathematics
Nagoya University
Chikusa-ku Nagoya 464-8602
Japan
E-mail: m96034q@math.nagoya-u.ac.jp