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Removable singularities for quasilinear degenerate elliptic equations with absorption term

Dedicated to Professor Kôzô Yabuta on the occasion of his sixtieth birthday

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Abstract. Let $N \geq 1$ and $p > 1$. Let F be a compact set and Ω be a bounded open set of $\boldsymbol{R}^N$ satisfying $F \subset \Omega \subset \boldsymbol{R}^N$. We define a generalized p-harmonic operator L_p which is elliptic in $\Omega \backslash F$ and degenerated on F. We shall study the genuinely degenerate elliptic equations with absorption term. In connection with these equations we shall treat two topics in the present paper. Namely, the one is concerned with removable singularities of solutions and the other is the unique existence property of bounded solutions for the Dirichlet boundary problem.

0. Introduction.

Let $N \geq 1$ and $p > 1$. Let F be a compact set and Ω be a bounded open set of $\boldsymbol{R}^N$ satisfying $F \subset \Omega \subset \boldsymbol{R}^N$. We also set $\Omega' = \Omega \backslash \partial F$, where $\partial F = F \backslash \mathring{F}$. Here by $\mathring{F}$ we denote the interior of F, which may be empty. We assume that the measure of ∂F is zero.

By $H^{1,p}(\Omega)$ we denote the space of all functions on Ω, whose generalized derivatives $\partial^\gamma u$ of order ≤ 1 satisfy

$$\|u\|_{1,p} = \sum_{|\gamma| \leq 1} \left(\int_\Omega |\partial^\gamma u(x)|^p \, dx \right)^{1/p} < +\infty, \tag{0-1}$$

and also, $H^{1,p}_{loc}(\Omega)$ is a local version of $H^{1,p}(\Omega)$. For the precise definition of function spaces, see §2. For $u \in H^{1,p}_{loc}(\Omega')$, we define a generalized p-harmonic operator by

$$L_p u = -\mathrm{div}(A(x)|\nabla u|^{p-2}\nabla u), \tag{0-2}$$

where $\nabla u = (\partial u/\partial x_1, \partial u/\partial x_2, \ldots, \partial u/\partial x_N)$, and $A(x) \in C^1(\Omega')$ is positive in $\Omega \backslash F$ and vanishes in $\mathring{F}$. Roughly speaking, the operators L_p considered here are not

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only permitted to vanish identically on a compact set $F \subset \Omega$, but also may be unbounded on ∂F. We shall consider the genuinely degenerate elliptic equations with absorption term $BQ(u)$, which are defined by

$$L_p u + B(x)Q(u) = f, \quad \text{in } \Omega' = \Omega \backslash \partial F. \tag{0-3}$$

Here $B(x)$ is a nonnegative function on Ω, and $Q(t)$ is continuous and strictly monotone increasing on $\boldsymbol{R}$ satisfying the growth condition (1-6). For instance we can adopt $|t|^{q-1}t$ with $q > 1$ and $(e^{|t|} - 1)\, sgn(t)$ for $Q(t)$. Since $A = 0$ in $\overset{\circ}{F}$, $L_p(u)$ is defined on $H^{1,p}_{loc}(\Omega \backslash F) \cap L^{\infty}_{loc}(\Omega')$ in a natural way by setting $L_p u = -\mathrm{div}(A(x)|\nabla u|^{p-2}\nabla u)$ in $\Omega \backslash F$ and $L_p u = 0$ in $\overset{\circ}{F}$.

In connection with these equations we shall treat two topics in the present paper. Namely, the one is concerned with removable singularities of solutions for (0-3) and the other is the unique existence property of bounded solutions for the Dirichlet boundary problem (0-7) below.

First we shall explain our results on removable singularities of solutions for (0-3). We assume that $u \in H^{1,p}_{loc}(\Omega \backslash F) \cap L^{\infty}(\Omega')$ satisfies (0-3) for $f \in L^1_{loc}(\Omega')$ in the weak sense. More precisely, we assume that u satisfies

$$\int_{\Omega'} (A|\nabla u|^{p-2}\nabla u \nabla \varphi + BQ(u)\varphi)\, dx = \int_{\Omega'} f\varphi\, dx, \quad \text{for all } \varphi \in C_0^{\infty}(\Omega'). \tag{0-4}$$

Then we shall show in Theorem 1 under some additional conditions that

$$\limsup_{x \to \partial F} |u(x)| < +\infty. \tag{0-5}$$

From this result we see in Theorem 2 that there is a bounded solution for (0-3) in Ω which coincides with u in $\Omega' = \Omega \backslash \partial F$. Namely, we shall show that every solution $u \in H^{1,p}_{loc}(\Omega \backslash F) \cap L^{\infty}(\Omega')$ of (0-3) can possess only removable singularities on ∂F. Our main assumption [**H-3**] is concerned with a variant of the so-called Minkowski content of tubular neighborhoods of ∂F. In order to see the geometrical meaning of this assumption we also introduce a relative capacity $C_K(F, \Omega)$ in §1 by the use of the conjugate function of the nonlinear term Q, assuming that Q is strictly convex. Then we shall show that F has a vanishing capacity if our conditions are satisfied. Roughly speaking, the boundary set ∂F is so small under our assumptions that the support of the distribution $L_p u$ and the set ∂F have no point in common. Here we remark that $u = Q^{-1}(f/B)$ in $\overset{\circ}{F}$ provided that u satisfies (0-3). Moreover we shall also see the sharpness of our results in the special case that F is either a discrete set or an m-dimensional compact smooth submanifolds $(0 < m \leq N - 1)$ of $\boldsymbol{R}^N$, and

$$\begin{cases} A(x) = d(x)^{p\alpha}, B(x) = d(x)^{p\beta}, C(x) = d(x)^{p\gamma}, \\ Q(t) = |t|^{q-1}t, \quad d(x) = dist(x, \partial F), \end{cases} \tag{0-6}$$

where $q > 1$ and α, β, γ are real numbers. (For the role of $C(x)$, see Theorem 1 in §2.) For example, there exists a characteristic number p_m^* defined by (4-2) such that if $q > p_m^*$, then every solution $u \in H_{loc}^{1,p}(\Omega') \cap L^\infty(\Omega')$ of (0-3) can possess only removable singularities on ∂F.

When F consists of finite points, $p = 2$, $Q(t) = |t|^{q-1}t$ and $A(x), B(x), C(x)$ are positive constants, H. Brezis and L. Veron initially studied in [**BV**] that if u satisfies (0-3) with some additional assumptions on p, u can possess only removable singularities on F. (See also [**VV1**], [**VV2**] and [**V**] for the quasilinear case.) In this paper we generalize their results for an arbitrary compact set F in place of finite set and for a wider class of (degenerate) elliptic operators L_p. We note that if $p = 2$, then this topics was already treated in the author's paper [**H2**] under the similar framework. By virtue of Kato's inequality and a maximum principle, the unique existence of bounded solutions was established. Since Kato's inequality does not work effectively in the quasilinear case, we shall employ a comparison principle, a priori estimates and a weak maximum principle instead. Since the operators L_p are quasilinear and rather general, we need to modify them suitably so that they are applicable to our problems.

Secondly we explain the existence and uniqueness result which is a direct application of the first part. We shall consider the Dirichlet boundary problem for the operators L_p with absorption term, that is

$$\begin{cases} L_p u + B(x)Q(u) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{0-7}$$

Then we shall establish the existence and uniqueness of bounded solutions u for this problem with $f/B \in L^\infty(\Omega)$. When the operator L_p is uniformly elliptic on $\bar{\Omega}$, this problem has been treated by many authors. When $p = 2$, H. Brezis and W. A. Strauss studied similar problems in [**BS**] for $f \in L^1(\Omega)$ with a monotone increasing non-linear term in u (possibly multi-valued). As for the degenerate case, the author proved in [**H2**] the existence and uniqueness of solutions of (0-7) for $f/B \in L^\infty(\Omega)$.

This paper is organized in the following way. In §1 we shall describe our precise framework and assumptions in this paper. We also introduce relative capacities by virtue of the conjugate function of the nonlinear term, and we study the meaning of our assumptions. In §2 and §3 we shall state our main results which concerns removable of singularities and the unique existence of solutions for Dirichlet boundary problem (0-7). In §4 we shall construct examples showing that in certain respects Theorem 1 gives best possible results. The §5 is devoted to prepare auxiliary lemmas which are needed to establish our theorems. In §6 we shall prove Theorem 1 by the use of a priori inequalities in §5 and the weak maximum principle. Theorem 3 will be finally established in §7 as an application of Theorem 1. In Appendix we shall prove Lemma 1-2 in §1 concerned with capacities.

1. Preliminaries.

In this section we prepare our basic framework and notations which are of importance through the present paper.

Let $N \geq 1$ and $p > 1$. Let F and Ω be a compact set and bounded open subset of $\boldsymbol{R}^N$ respectively, satisfying $F \subset \Omega$, and set

$$\Omega' = \Omega \backslash \partial F. \tag{1-1}$$

Here ∂F is defined as $\partial F = F \backslash \overset{\circ}{F}$. We assume that the measure of ∂F is zero. We define a distance to ∂F.

DEFINITION 1. By $d(x)$ we denote a distance function $d(x) = dist(x, \partial F)$.

REMARK 1. A distance function $d(x)$ is Lipschitz continuous and differentiable almost everywhere. Moreover one can approximate it by a smooth function. Namely there exists a nonnegative smooth function $D(x) \in C^\infty(\Omega')$ such that

$$C(0)^{-1} \leq \frac{D(x)}{d(x)} \leq C(0), \tag{1-2}$$

$$|\partial^\gamma D(x)| \leq C(|\gamma|) d(x)^{1-|\gamma|}, \quad x \in \Omega',$$

where γ is an arbitrary multi-index and $C(|\gamma|)$ is a positive number depending on $|\gamma|$. Therefore one can assume that $d(x)$ is smooth as well without a loss of generality. (For the construction of $D(x)$, see [**T**] for example.)

In this paper we treat the following degenerate nonlinear operator

$$\begin{aligned} L_p u &+ B(x) Q(u) \\ &= -\mathrm{div}(A(x)|\nabla u|^{p-2}\nabla u) + B(x) Q(u), \quad \text{in } \Omega. \end{aligned} \tag{1-3}$$

First we assume the following [**H-1**] on the nonnegative functions $A(x)$ and $B(x)$.

[**H-1**]

$$\begin{cases} A(x) \in C^1(\Omega') \cap L^1_{loc}(\Omega), & \\ A(x) = 0 & \text{in } \overset{\circ}{F} = F \backslash \partial F, \\ A(x) > 0 & \text{in } \Omega \backslash F, \end{cases} \tag{1-4}$$

and

$$\begin{cases} B(x) \in L^\infty_{loc}(\Omega') \cap L^1_{loc}(\Omega), & \\ B(x) > 0 & \text{in } \Omega' = \Omega \backslash \partial F. \end{cases} \tag{1-5}$$

Secondly we assume the following [**H-2**] on the nonlinear term $Q(t)$.

[**H-2**]

$Q(t)$ is a strictly monotone increasing and continuous function such that $Q(0) = 0$ and $t \cdot Q(t) > 0$ on $\boldsymbol{R}\backslash\{0\}$. Moreover we assume that there is a positive number δ_0 such that

$$\limsup_{|t|\to\infty} \frac{|t|^{p-1+\delta_0}}{|Q(t)|} < +\infty. \tag{1.6}$$

We need more notations.

Definition 2. Let δ_0 be a positive number. Let us set for any $t > 0$ and any $x \in \Omega' = \Omega\backslash\partial F$,

$$\begin{cases} \tilde{A}(x) = A(x) + d(x)|\nabla A(x)|, \\ \mathbf{M}(x) = \text{ess-sup}_{\{y\in\Omega:1/4<d(y)/d(x)<3\}} \dfrac{\tilde{A}(y)}{B(y)}, \\ K(x,t) = 2 + (\mathbf{M}(x)\cdot t^p)^{(p-1)/\delta_0}. \end{cases} \tag{1-7}$$

In this definition of $K(x,t)$ the constant term 2 can be replaced by any number strictly greater than 1. The following assumption is crucial in the present work.

[**H-3**]

For the same positive number $\delta_0 > 0$ as in [**H-2**], it holds that

$$\liminf_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^p}\int_{\varepsilon/2<d(x)<\varepsilon} A(x)K\left(x, \frac{1}{d(x)}\right)dx < +\infty. \tag{1-8}$$

We also assume that:

[**H-4**]

Let $B(x)$ and $C(x)$ satisfy

$$C(x) \in L^\infty_{loc}(\Omega')\cap L^1_{loc}(\Omega), \quad C(x)\geq 0 \quad \text{in } \Omega, \tag{1-9}$$

and

$$\sup_{x\in\Omega}\frac{C(x)}{B(x)} < +\infty. \tag{1-10}$$

In order to make clear in advance the role of the condition [**H-3**] as well as what it means, we shall trace the definition of the kernel $K(x,t)$ to its origin, and then we shall reconstruct it by using the conjugate function of the nonlinear term. After we have reconstructed the kernel $K(x,t)$, we shall interpret the condition [**H-3**] using the notion of the vanishing relative p-capacities of ∂F.

Originally the definition of the kernel $K(x,t)$ comes from the pointwise estimate of the supersolutions of the equation (0-3) under some additional assumptions (see Lemma 6-1). More precisely, we shall prove in §6 that every solution u of (0-3) in $H^{1,p}_{loc}(\Omega\backslash F)\cap L^{\infty}_{loc}(\Omega')$ is dominated by $K(x,1/d)^{1/(p-1)}$ up to constant times. Roughly speaking, the condition [**H-3**] guarantees the integrability of the term $B\cdot Q(u)$ near ∂F with u being the solution of (0-3). Then we can finally show the boundedness of the solution, which is one of the main purpose in the present paper.

It is very interesting that we can reconstruct the kernel without making use of the explicit supersolutions. To this end, we shall define the conjugate function of the nonlinear term in place of supersolutions. In the rest of this subsection we assume that Q is **strictly convex**. We need more notations.

Definition 3. For $x\in\Omega\backslash F$ and $t>0$, we set

$$(1\text{-}11)\qquad \begin{cases} \Phi_0(x,t)=\dfrac{B(x)}{A(x)}\cdot Q(t^{1/(p-1)}), \\ \Phi_1(x,t)=\dfrac{B(x)}{A(x)}\cdot t^{1+\delta_0/(p-1)}, \\ \Psi_j(x,t)=\sup_{s>0}[ts-\Phi_j(x,s)], \quad \text{for } j=0,1. \end{cases}$$

We also set

Definition 4. For any $t>0$ and $x\in\Omega\backslash F$

$$(1\text{-}12)\qquad \begin{cases} G_j(x,t)=c(p,\delta_0)^{-1}\Psi_j(x,t^p)\dfrac{1}{t^p}+2, \quad \text{for } j=0,1, \\ c(p,\delta_0)=\dfrac{\delta_0}{p-1+\delta_0}\left(\dfrac{p-1}{p-1+\delta_0}\right)^{(p-1)/\delta_0}. \end{cases}$$

Then by a direct calculation we have

$$(1\text{-}13)\qquad \begin{cases} \Psi_1(x,t)=c(p,\delta_0)\left(\dfrac{A(x)}{B(x)}\right)^{(p-1)/\delta_0}\cdot t^{1+(p-1)/\delta_0}, \\ G_1(x,t)=2+\left(\dfrac{A(x)}{B(x)}\cdot t^p\right)^{(p-1)/\delta_0}. \end{cases}$$

From the definition it immediately follows that

Lemma 1-1. (1) (*Young's inequality*) *For any positive numbers* s, t *and for almost all* $x \in \Omega \backslash F$, *it holds that*

$$st \leq \Phi_j(x,s) + \Psi_j(x,t) \quad \textit{for } j = 0, 1. \tag{1-14}$$

(2) *For any positive number* C_1, *there is a positive number* C_2 *such that for* $t \geq C_1$

$$\begin{cases} \Phi_1(x,t) \leq C_2 \cdot \Phi_0(x,t), \\ \Psi_0(x,t) \leq C_2 \cdot \Psi_1(x,t), \\ G_0(x,t) \leq C_2 \cdot G_1(x,t), \\ \Psi_1(x,t^p) \leq G_1(x,t) \cdot t^p \leq K(x,t) \cdot t^p. \end{cases} \tag{1-15}$$

(3) *For any* $x \in \Omega \backslash F$, *it holds that*

$$K(x,t) = \sup_{1/4 < d(y)/d(x) < 3} G_1(y,t). \tag{1-16}$$

Now we define the relative capacities of compactum $e \subset \Omega$ to Ω, and we shall explain [**H-3**] in terms of capacities.

Definition 5. For an arbitrary compactum $e \subset \Omega$ we define the weighted p-capacities relative to Ω by

(1-17)

$$\begin{cases} C(e,\Omega) = \inf\left[\int_\Omega A|\nabla\eta|^p\, dx; \eta \geq 1 \text{ on } e, \eta \in C_0^\infty(\Omega)\right] \\ C_K(e,\Omega) = \inf\left[\int_\Omega A \cdot K\left(x, \frac{1}{d}\right)|\nabla\eta|^p\, dx; \eta \geq 1 \text{ on } e, \eta \in C_0^\infty(\Omega)\right] \\ C_K^{\log}(e,\Omega) = \inf\left[\int_\Omega A \cdot K\left(x, \frac{1}{d}\right)\left(\log K\left(x, \frac{1}{d}\right)\right)^p |\nabla\eta|^p\, dx; \eta > 1 \text{ on } e, \eta \in C_0^\infty(\Omega)\right]. \end{cases}$$

Clearly it holds that $C(e,\Omega) \leq C_K(e,\Omega) \leq C_K^{\log}(e,\Omega)$. Moreover we show in §8 (Appendix) the following.

Lemma 1-2. *Assume that* [**H-1**]. *Then* [**H-3**] *implies that* $C_K(\partial F, \Omega) = 0$, *namely*, ∂F *has a vanishing capacity*.

If ∂F is sufficiently smooth, say, C^∞ compact submanifolds of $\boldsymbol{R}^n$ without boundary, then we can show that the opposite implication. Namely, the condition $C_K(\partial F, \Omega) = 0$ implies [**H-3**] under the assumption [**H-1**]. This lemma is not essentially new, but we shall show it in Appendix for the sake of self-containedness.

Lastly we define the following condition which is stronger than the condition $C_K(\partial F, \Omega) = 0$.

[**H-5**]

$$C_K^{\log}(\partial F, \Omega) = 0. \tag{1-17}$$

Remark 2. We shall see that [**H-3**] can be replaced by the condition [**H-5**] in many places. In Appendix we shall study more on capacities.

Remark 3. The condition (1-8) in [**H-3**] means that $B(x)$ does not vanish much faster than $A(x)$. If $1 \leq N \leq 2$, then either $\tilde{A}(x)$ or $\mathbf{M}(x)$ must vanish on ∂F in order to satisfy [**H-3**].

2. Removable singularities.

We use the following notations. Let D be an open subset of $\boldsymbol{R}^N$. Let $q \geq 1$ and let j be a positive integer. By $H^{j,q}(D)$ we denote the spaces of all functions on D, whose generalized derivatives $\partial^\gamma u$ of order $\leq j$ satisfy

$$\|u\|_{j,q} = \sum_{|\gamma| \leq j} \left(\int_D |\partial^\gamma u(x)|^q \, dx \right)^{1/q} < +\infty, \tag{2-1}$$

and also, $H_{loc}^{j,q}(D)$ is a local version of $H^{j,q}(D)$, and by $\|u\|_\infty$ we denote the essential supremum of u. By $H_0^{1,q}(D)$ we denote the completion of $C_0^\infty(D)$ with respect to the norm defined by (2-1). By $\mathscr{D}'(D)$ we denote the space of all distributions on D.

Now we are able to state our main results for removable singularities.

Theorem 1. *Assume that* [**H-1**], [**H-2**], [**H-3**] *and* [**H-4**]. *Assume that* $u \in H_{loc}^{1,p}(\Omega \backslash F) \cap L_{loc}^\infty(\Omega')$ *satisfies* $L_p u \in L_{loc}^1(\Omega')$ *in the distribution sense. Moreover we assume that for almost all* $x \in \{x \in \Omega'; u(x) \geq 0\}$,

$$L_p u + B(x) Q(u) \leq C(x). \tag{2-2}$$

Then we have $u_+ \in L_{loc}^\infty(\Omega)$, *where* $u_+ = \max(u, 0)$.

Moreover the condition [**H-3**] *can be replaced by* [**H-5**].

Remark 4. From Theorem 1 it follows that $u_+ \in L_{loc}^\infty(\Omega')$ can be extended as a locally bounded function on a whole Ω. Since the measure of ∂F is zero, this extension coincides with u_+ except on a set of measure zero. Theorem 1 will be established in §7.

Admitting this for the moment, we shall establish the result concerning removable singularities. To this end we give definitions of a weak solution of the equation

$$L_p u + BQ(u) = f \quad \text{in } D, \tag{2-3}$$

where D is an open subset of Ω. We recall that $L_p u$ is defined on $H_{loc}^{1,p}(\Omega \backslash F) \cap L_{loc}^\infty(\Omega')$ by setting $L_p u = 0$ in $\mathring{F}$.

Definition 6 (A weak solution in Ω'). For $f \in L^1_{loc}(\Omega')$ and $u \in H^{1,p}_{loc}(\Omega \backslash F) \cap L^\infty_{loc}(\Omega')$, u is said to be a weak solution of (2-3) in Ω', if it satisfies

$$\int_{\Omega'} (A|\nabla u|^{p-2}\nabla u \nabla\varphi + BQ(u)\varphi)\,dx = \int_{\Omega'} f\varphi\,dx, \quad \text{for all } \varphi \in C_0^\infty(\Omega'). \tag{2-4}$$

Since the assumption on the coefficient A is rather weak, we have to be careful to consider a weak solution of (2-3) in a whole Ω. Namely

Definition 7 (A weak solution in Ω). For $f \in L^1_{loc}(\Omega)$ and $u \in H^{1,p}_{loc}(\Omega \backslash F) \cap L^\infty_{loc}(\Omega)$, u is said to be a weak solution of (2-3) in Ω, if it satisfies that $A|\nabla u|^p \in L^1_{loc}(\Omega)$ and

$$\int_{\Omega} (A|\nabla u|^{p-2}\nabla u \nabla\varphi + BQ(u)\varphi)\,dx = \int_{\Omega} f\varphi\,dx, \quad \text{for all } \varphi \in C_0^\infty(\Omega). \tag{2-5}$$

Then it follows from Theorem 1 that we have the following result:

Theorem 2. *Assume that* [**H-1**], [**H-2**] *and either* [**H-3**] *or* [**H-5**]. *Instead of* [**H-4**] *assume that* $f(x) \in L^\infty_{loc}(\Omega') \cap L^1_{loc}(\Omega)$ *satisfies for some positive number* C

$$|f(x)| \le C \cdot B(x), \quad \text{for almost all } x \in \Omega. \tag{2-6}$$

Assume that $u \in H^{1,p}_{loc}(\Omega \backslash F) \cap L^\infty_{loc}(\Omega')$ *satisfies in the weak sense*

$$L_p u + B(x)Q(u) = f, \quad \text{in } \Omega'. \tag{2-7}$$

Then there exists a function $v \in H^{1,p}_{loc}(\Omega \backslash F) \cap L^\infty_{loc}(\Omega)$ *such that* v *satisfies in the weak sense*

$$\begin{cases} L_p v + B(x)Q(v) = f, & \text{in } \Omega \\ v|_{\Omega'} = u. \end{cases} \tag{2-8}$$

Proof of Theorem 2. This is a direct consequence of Theorem 1. In fact from Theorem 1 we have $u_+ \in L^\infty_{loc}(\Omega)$. The function $-u$ satisfies (2-7) with replacing f and $Q(t)$ by $-f$ and $-Q(-t)$ respectively. Since $-Q(-t)$ satisfies the same assumption as the one for $Q(t)$, we see in a similar way $u_- \in L^\infty_{loc}(\Omega)$, where $u_- = \max(-u, 0)$. According to Remark 4, u is extended as a locally bounded function on Ω. By v we denote this extension of u to a whole Ω. Thus $v \in L^\infty_{loc}(\Omega)$ and $v|_{\Omega'} = u$. From [**H-1**] and [**H-2**] we also see that $B \cdot Q(v) \in L^1_{loc}(\Omega)$. Here we note that since $A(x) = 0$ on $F \backslash \partial F$, $u(x) = v(x) = Q^{-1}(f(x)/B(x)))$ on $F \backslash \partial F$. Then it follows from Lemma 5-3 in §5 that v is extended as a solution of the same equation on a whole Ω. Here we note that the uniqueness of solutions in $L^\infty_{loc}(\Omega)$ follows from the same argument in the proof of Theorem 3 in §7. □

Remark 5. The monotonicity of the nonlinear term Q on $\boldsymbol{R}$ will be needed to establish the uniqueness of solutions in Theorem 2 and Theorem 3. For the proof of the existence of solutions it suffices to assume that there is a positive number C such that $Q(t)$ is monotone increasing for $t \in \boldsymbol{R} \backslash [-C, C]$.

3. Existence and uniqueness of solutions for Dirichlet problem.

As an application we consider the Dirichlet boundary value problem for degenerate quasi-linear elliptic equation:

$$\begin{cases} L_p u + B(x)Q(u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3-1}$$

Then we have the following result which will be established in §7.

THEOREM 3. *Assume that* [**H-1**], [**H-2**] *and* [**H-3**]. *Instead of* [**H-4**] *assume that* $f(x) \in L^\infty(\Omega)$ *satisfies for some positive number* C

$$|f(x)| \le C \cdot B(x), \quad \textit{for almost all } x \in \Omega. \tag{3-2}$$

Moreover we assume that $A(x), B(x) \in C^0(\bar{\Omega})$. *Then there exists a unique function*

$$u \in L^\infty(\Omega) \cap H^{1,p}_{loc}(\bar{\Omega} \backslash F) \tag{3-3}$$

which satisfies (3-1) *in the weak sense and satisfies*

$$\int_\Omega [A(x)|\nabla u|^p + B(x)Q(u)u]\, dx \le C[\|f/B\|_\infty^\lambda + \|f\|_\infty]. \tag{3-4}$$

Here $\lambda = (p + \delta_0)/(p - 1 + \delta_0)$ *and* C *is a positive number independent of each function* f.

REMARK 6. The condition [**H-3**] can be replace by [**H-5**] as before. For the proof of this Theorem 3 we shall regularize the problem. By virtue of Theorems 1 and 2, we shall prove that the unique solutions of this approximating nonlinear elliptic equations converge to the unique bounded solution of the original equation. Here we note that the operator L_p itself is not ε-regularizable, because it may be degenerate infinitely on ∂F.

REMARK 7. If we assume that ∂F is smooth, then we can also establish the Hölder continuity of the gradient $|\nabla u|$ of the solution u under some additional conditions. More precisely, in the coming paper we shall show $|\nabla u|$ belongs to the weighted Schauder space if $A(x)$ belongs to Muckenhoupt's A_p class and $A(x)$ is a power of the distance to ∂F.

4. Examples.

In this subsection we shall construct examples showing that in certain respects Theorem 1 gives best possible results. Let F be either the origin 0 or an

m-dimensional C^∞ compact submanifolds in $\boldsymbol{R}^N$ without boundary for $0 < m \le N-1$, and let $d(x)$ be a distance function. For example, if F consists of the origin 0, then we put $d(x) = |x|$ and $m = 0$. For $p > 1$ and $q > 0$, we set

$$Pu = -\operatorname{div}(d(x)^{p\alpha}|\nabla u|^{p-2}\nabla u) + b(x)\cdot d(x)^{p\beta}\cdot|u|^{q-1}u, \tag{4-1}$$

where $b(x)$ is a positive continuous function.

Assume that real numbers α, β and γ satisfy the following conditions. First we assume **(h-1)** and **(h-2)** which are equivalent to [**H-1**] and [**H-2**] respectively.

$$\min(\alpha, \beta, \gamma) > -\frac{N-m}{p}. \tag{h-1}$$

$$q > p-1. \tag{h-2}$$

From **(h-2)** we see that the condition [**H-2**] is satisfied for $\delta_0 = q - p + 1 > 0$. We need more notations. Let us set for $0 \le m \le N-1$ and $\alpha > -(N-m-p)/p$

$$p_m^* = \begin{cases} (p-1)\cdot\left(1 + p\dfrac{1-\alpha+\beta}{N+p\alpha-p-m}\right), & \text{if } \alpha < \beta+1, \\ p-1, & \text{if } \alpha \ge \beta+1. \end{cases} \tag{4-2}$$

Then we assume **(h-3)** which is equivalent to (1-8) in [**H-3**]. (See the proof of Theorem 4.)

$$\begin{cases} q \ge p_m^*, & \text{if } \alpha < \beta+1, \\ q > p_m^* = p-1, & \text{if } \alpha \ge \beta+1, \\ \alpha > -\dfrac{N-m-p}{p}. \end{cases} \tag{h-3}$$

Lastly we assume **(h-4)** which is equivalent to [**H-4**].

$$\beta \le \gamma. \tag{h-4}$$

Let us set $u_+ = \max[0, u]$ and $u_- = \max[0, -u]$. Then it follows from Theorem 1 that we have Theorem 4.

THEOREM 4. *Let F be either the origin or an m-dimensional C^∞ compact submanifolds in $\boldsymbol{R}^n$ without boundary for $0 < m \le N-1$. Assume that* **(h-1)**, **(h-2)**, **(h-3)** *and* **(h-4)**. *Assume that $u \in H_{loc}^{1,p}(\Omega') \cap L_{loc}^\infty(\Omega')$ satisfies $Pu \in L_{loc}^1(\Omega')$ in the distribution sense. Moreover we assume that for almost all $x \in \{x \in \Omega; u(x) \ge 0\}$*

$$Pu \le c(x) d(x)^{p\gamma}, \tag{4-3}$$

for some positive continuous function $c(x)$. Then we have $u_+ \in L_{loc}^\infty(\Omega)$.

Proof of Theorem 4. Since $Q(u) = |u|^{q-1}u$, we can put $\delta_0 = q - p + 1$ to obtain (1-6) in [**H-2**]. Putting $A(x) = d(x)^{p\alpha}$, $B(x) = b(x)d(x)^{p\beta}$ and $C(x) = c(x)d(x)^{p\gamma}$, we shall apply Theorem 1. Since the conditions [**H-1**], [**H-2**] and [**H-4**] are clearly satisfied, it suffices to examine the condition [**H-3**]. Note that $\mathbf{M}(x)$ is equivalent to $d(x)^{p(\alpha-\beta)}$ and so $K(x, 1/d(x))$ is equivalent to $1 + d(x)^{-p(p-1)((1-\alpha+\beta)/\delta_0)}$. Therefore we see

$$\begin{aligned}
(4\text{-}4)\qquad & \frac{1}{\varepsilon^p}\int_{\varepsilon/2<d(x)<\varepsilon} A(x)K\left(x, \frac{1}{d(x)}\right) dx \\
&= \frac{1}{\varepsilon^p}\int_{\varepsilon/2}^{\varepsilon} d\rho \int_{\{d(x)=\rho\}} (1 + d(x)^{-p(p-1)((1-\alpha+\beta)/\delta_0)}) d(x)^{p\alpha}\, dH^{N-1}(x) \\
&\leq C\, diam(F)^m \frac{1}{\varepsilon^p}\int_{\varepsilon/2}^{\varepsilon} (1 + \rho^{-p(p-1)((1-\alpha+\beta)/\delta_0)})\rho^{p\alpha+N-m-1}\, d\rho \\
&\leq C'\, diam(F)^m \cdot (\varepsilon^{((p\alpha-p+N-m)/(q-p+1))(q-p_m^*)} + \varepsilon^{p(\alpha+((N-m-p)/p))}) \\
&= O(1). \quad (\mathbf{h\text{-}1}) \text{ and } (\mathbf{h\text{-}3})
\end{aligned}$$

This proves the assertion. Here $H^{N-1}(\cdot)$ is the $(N-1)$-dimensional Hausdorff measure, and we used the fact:

Since F is compact and smooth, there is a positive number C such that we have

$$(4\text{-}5)\qquad |\{0 < d(x) < \varepsilon\}| \leq C\varepsilon^{N-m}\, diam(F)^m, \quad 0 < \varepsilon < 1.$$

Here by $|S|$ we denote the Lebesgue measure of the set $S \subset \boldsymbol{R}^N$. □

Counter-examples to Theorem 4. We shall see that Theorem 4 is best possible in certain respects. We note that $F = \partial F$ holds. Since it suffices to construct counter-examples in a sufficiently small neighborhood W of F, we may assume $d(x) = dist(x, F)$ is smooth so that we have $|\nabla d(x)| = 1$ in $W \backslash F$. Now we construct a null solution U for (4-7) in $W \backslash F$ of the form

$$(4\text{-}6)\qquad U(x) = d(x)^{-M}, \quad \text{for } M > 0.$$

Namely we want U to solve the following equation for some $M > 0$ and a suitable positive continuous function $b(x)$.

$$(4\text{-}7)\qquad PU(x) = 0, \quad \text{in } W \backslash F.$$

To do so, it suffices to put

$$(4\text{-}8)\qquad \begin{cases} b(x) = M^{p-1}E(x) \cdot d(x)^{M(p-1-q)-p(1-\alpha+\beta)} \\ E(x) = M(p-1) - p\alpha + p - 1 - d(x)\Delta d(x) \end{cases}$$

Since ∂F and $d(x) = dist(x, F)$ are smooth, we see that

$$\lim_{x \to F} d(x) \cdot \Delta d(x) = N - m - 1, \tag{4-9}$$

where $m = dim(F)$. In fact, if F is flat, then $d(x)\Delta d(x) \equiv N - m - 1$. Therefore, if the following conditions (4-10) are satisfied, we see that $M > 0$ and $b(x)$ is a positive continuous function, so that $U(x)$ becomes an unbounded null solution of (4.7).

$$q \le p - 1 + \frac{p(1 - \alpha + \beta)}{M}, \quad M > \max\left[\frac{N - m + p\alpha - p}{p - 1}, 0\right]. \tag{4-10}$$

After all we get

Proposition 4-1. *Assume that* **(h-1)** *and* $\alpha > -(N - m - p)/p$. *Then there exist unbounded null solutions for* (4-7), *if* (p, q, α, β) *satisfies one of the conditions listed below*:

$$\begin{cases} (1) & q < p_m^* \quad \text{and} \quad \alpha < \beta + 1, \\ (2) & q \le p - 1 \quad \text{and} \quad \alpha = \beta + 1, \\ (3) & q < p - 1 \quad \text{and} \quad \alpha > \beta + 1. \end{cases} \tag{4-11}$$

Remark 8. If $p = 2$ and $q = 1$, then the operator P is linear. Hence we can construct a local fundamental solution $E(x, y)$ of P in many cases. (If $\alpha = 0$, it is clear because P is elliptic.) Then $E(x, y)$ for $y \in F$ also becomes an unbounded solution of (4-7).

Remark 9. If $q = p - 1$ and $\beta + p(\alpha - \beta - 1) > 0$, then we can also construct a null solution of the form $e^{d(x)^{-M}}$ near F. In fact, if we put $U(x) = e^{d(x)^{-M}}$, then we have in a similar way

$$\begin{cases} b(x) = M^{p-1}\bar{E}(x) \cdot d(x)^{-Mp+\beta+p(\alpha-\beta-1)} \\ \bar{E}(x) = M(p - 1) + [(M + 1)(p - 1) - \alpha p - d(x)\Delta d(x)]d(x)^M. \end{cases} \tag{4-12}$$

Therefore if $\beta + p(\alpha - \beta - 1) > 0$ we see $b(x)$ is bounded and positive in a small neighborhood of F for sufficiently small $M > 0$.

Remark 10. Now we assume that $\alpha \le -(N - m - p)/p$ holds. Then we immediately see that there exist unbounded null solutions of (4-7) for an arbitrary q (respectively $q < p - 1$) provided $\alpha < 1 + \beta$ (respectively $\alpha \ge \beta + 1$). In fact we can choose any positive number for M in (4-10).

Lastly we consider **(h-4)**. We can show the following:

Proposition 4-2. *Assume that* **(h-1)** *and* $\dim(F) = m$ *for* $0 \le m \le N - 1$. *Then for the validity of Theorem* 4, *the assumption* $\beta \le \gamma$ (**(h-4)**) *is necessary if* $\alpha \ge \gamma + 1$.

PROOF OF PROPOSITION 4-2. Assume that $\beta > \gamma$. Let us set $U(x) = -\log d(x) \in L^1_{loc}(\Omega)$. Then it is easy to see that $U(x)$ becomes a counter-example, provided that $\alpha \geq \gamma + 1$. In fact we see that

$$-\mathrm{div}(d^{p\alpha}|\nabla U|^{p-2}\nabla U) = O(d(x)^{\alpha p - p}) \quad \text{in } \Omega\backslash F. \tag{4-13}$$

Since U is unbounded, **(h-4)** is necessary to avoid this candidate. □

5. Auxiliary lemmas.

In this section we shall establish a chain of auxiliary lemmas concerning basic estimates for weak solutions of the equation, which will be needed to establish Theorems stated in §2. Without loss of generality we assume that a fixed tubular neighborhood of F, say, $\{x : d(x) < 3\}$ is contained in Ω.

LEMMA 5-1 (A priori inequality 1). *Assume that* $\{x : d(x) < 3\} \subset \Omega$. *Assume that* [**H-1**], [**H-2**] *and* [**H-4**], *and assume that* $u \in H^{1,p}_{loc}(\Omega\backslash F) \cap L^\infty(\Omega')$ *satisfies* $L_p u \in L^1_{loc}(\Omega')$ *in the distribution sense. Moreover we assume that for almost all* $x \in \{x \in \Omega'; u(x) \geq 0\}$,

$$L_p u + B(x) \cdot Q(u) \leq C(x). \tag{5-1}$$

Then we have, for any number $q > 0$ *and any nonnegative function* $\eta \in C_0^\infty(\{x \in \Omega : 0 < d(x) < 3\})$,

$$\int_\Omega A(x)|\nabla(u-\mu)_+|^p (u-\mu)_+^{q-1}\eta^p\,dx \leq \left(\frac{p}{q}\right)^p \int_\Omega A(x)|\nabla\eta|^p (u-\mu)_+^{p+q-1}\,dx, \tag{5-2}$$

$$\int_\Omega A(x)|\nabla(u-\mu)_+|^p [1+(u-\mu)_+]^{-1}\eta^p\,dx \leq p^p \int_{\Omega'} A(x)|\nabla\eta|^p [1+(u-\mu)_+]^{p-1}[\log[1+(u-\mu)_+]]^p\,dx. \tag{5-3}$$

Here μ *is an arbitrary positive number satisfying*

$$Q(\mu) \geq \max\left[\sup_{x\in\Omega}\frac{C(x)}{B(x)}, \sup_{2<d(x)<3}|u|\right]. \tag{5-4}$$

PROOF OF LEMMA 5-1. We use the following test functions in this section:

$$\phi_j(x) = \eta(x)^p \rho_j((u(x) - \mu)_+), \quad (j = 1, 2), \tag{5-5}$$

where we set for positive number n

$$\rho_1(t) = \begin{cases} t_+^q, & \text{if } q > 0 \\ \log(1 + t_+), & \text{if } q = 0. \end{cases}$$

$$\rho_2(t) = \begin{cases} 1, & t \geq 1/n, \\ nt, & 0 \leq t \leq 1/n, \\ 0, & t \leq 0. \end{cases}$$

Since (5-3) can be treated in a similar way, we only prove (5-2) for $q > 0$.

Proof of (5-2). Using ϕ_1 as a test function, we have

$$\begin{aligned} q\int_\Omega A(x)|\nabla(u-\mu)_+|^p (u-\mu)_+^{q-1}\eta^p\,dx + \int_\Omega (B\cdot Q(u) - C(x))\phi_1(x)\,dx \\ + p\int_\Omega A(x)|\nabla u|^{p-2}(\nabla u\cdot\nabla\eta)\eta^{p-1}\rho_1((u-\mu)_+)\,dx \leq 0. \end{aligned} \tag{5-6}$$

Since the second term is nonnegative from the definition of μ, we have

$$\begin{aligned} q\int_\Omega & A(x)|\nabla(u-\mu)_+|^p (u-\mu)_+^{q-1}\eta^p\,dx \\ &\leq p\int_\Omega A(x)|\nabla u|^{p-1}|\nabla\eta|\eta^{p-1}\rho_1((u-\mu)_+)\,dx \\ &\leq p\left(\int_\Omega A(x)|\nabla u|^p (u-\mu)_+^{q-1}\eta^p\,dx\right)^{(p-1)/p} \\ &\qquad\cdot\left(\int_\Omega A(x)|\nabla\eta|^p (u-\mu)_+^{p+q-1}\,dx\right)^{1/p}. \end{aligned} \tag{5-7}$$

Here we used the equality $q = (q-1)(1-1/p) + (q-1+p)/p$. Hence the desired estimate holds. □

Using this we can show the following lemma which is of importance in the proof of Theorem 1.

Lemma 5-2 (A priori inequality 2). *Assume the same assumptions as in Lemma* 5-1.

Then we have, for any number $q > 0$ and any nonnegative function $\eta \in C_0^\infty(\{x \in \Omega : 0 < d(x) < 3\})$,

$$\int_\Omega [B(x)Q(u) - C(x)](u-\mu)_+^q \eta^p \, dx \le \frac{p^p}{q^{p-1}} \int_\Omega A(x)|\nabla\eta|^p (u-\mu)_+^{p+q-1} \, dx, \tag{5-8}$$

$$\int_\Omega [B(x)Q(u) - C(x)] \log[1+(u-\mu)_+]\eta^p \, dx \le p^p \int_{\Omega'} A(x)|\nabla\eta|^p [1+(u-\mu)_+]^{p-1} [\log[1+(u-\mu)_+]]^p \, dx, \tag{5-9}$$

$$\int_{\{x; u \ge \mu\}} [B(x)Q(u) - C(x)]\eta^p \, dx \le p^p \left(\int_\Omega A(x)|\nabla\eta|^p (u-\mu)_+^p \, dx \right)^{(p-1)/p} \cdot \left(\int_\Omega A(x)|\nabla\eta|^p \, dx \right)^{1/p}. \tag{5-10}$$

Here μ is an arbitrary positive number satisfying (5-4).

PROOF OF LEMMA 5-2. Since the arguments are quite similar, we concentrate on proving (5-8) and (5-10). Using ϕ_1 as the same test function as before, we have

$$\begin{aligned} \int_\Omega [B(x)Q(u) - C(x)]\phi_1(x) \, dx &\le p \int_\Omega A(x)|\nabla u|^{p-1} |\nabla\eta| \eta^{p-1} \rho_1((u-\mu)_+) \, dx \\ &\le p \left(\int_\Omega A(x)|\nabla u|^p (u-\mu)_+^{q-1} \eta^p \, dx \right)^{(p-1)/p} \\ &\qquad \cdot \left(\int_\Omega A(x)|\nabla\eta|^p (u-\mu)_+^{p+q-1} \, dx \right)^{1/p} \\ &\le \frac{p^p}{q^{p-1}} \int_\Omega A(x)|\nabla\eta|^p (u-\mu)_+^{p+q-1} \, dx. \quad \text{(Lemma 5-1)} \end{aligned} \tag{5-11}$$

This proves (5-8). In order to prove (5-10) we use ϕ_2 defined by (5-5) as a test function and the inequality (5-2) with $q = 1$. Then by letting $n \to +\infty$ the desired inequality follows in a similar way. □

LEMMA 5-3 (Extension). *Assume that* [**H-1**], [**H-2**] *and* [**H-3**]. *Moreover we assume that for $f \in L^1_{loc}(\Omega)$, $u \in H^{1,p}_{loc}(\Omega \backslash F) \cap L^\infty_{loc}(\Omega)$ satisfies in the weak sense*

$$L_p u + B(x) \cdot Q(u) = f, \qquad \text{in } \Omega'. \tag{5-12}$$

Then u can be extended as a weak solution of the same equation in whole Ω.

REMARK 11. In this lemma, we may relpace the condition [**H-3**] by a weaker condition $C(e, \Omega) = 0$. (See (1-17), Lemma 1-2 and Lemma A-1.)

PROOF. Since u is bounded, we see that $B \cdot Q(u) \in L^1_{loc}(\Omega)$. Let us set

$$\psi_\varepsilon = \begin{cases} 0, & \text{if } dist(x, \partial F) \le \varepsilon/2 \\ \dfrac{2}{\varepsilon}\, dist(x, \partial F_{\varepsilon/2}), & \text{if } \varepsilon/2 \le dist(x, \partial F) \le \varepsilon \\ 1, & \text{if } dist(x, \partial F) \ge \varepsilon \end{cases} \tag{5-13}$$

Here $F_\varepsilon = \{x \in \Omega; dist(x, F) \le \varepsilon\}$ and ε is sufficiently small. For any nonnegative $\phi \in C_0^\infty(\Omega)$ we put $\eta = u \cdot (\phi\psi_\varepsilon)^p$ for a test function. Then we see

$$\begin{aligned} (5\text{-}14) \qquad & -\int_\Omega (BQ(u) - f)u(\phi\psi_\varepsilon)^p\, dx \\ & = \int_\Omega A|\nabla u|^p |\phi\psi_\varepsilon|^p\, dx + p\int_\Omega A|\nabla u|^{p-2} u(\phi\psi_\varepsilon)^{p-1} \nabla u \cdot \nabla(\phi\psi_\varepsilon)\, dx, \end{aligned}$$

$$\begin{aligned} (5\text{-}15) \qquad & \left| \int A|\nabla u|^{p-2} \nabla u \cdot \nabla(\phi\psi_\varepsilon) u(\phi\psi_\varepsilon)^{p-1}\, dx \right| \\ & \le \left(\int A|\nabla u|^p |\phi\psi_\varepsilon|^p\, dx \right)^{1-(1/p)} \cdot \left(\int A|\nabla(\phi\psi_\varepsilon)|^p |u|^p\, dx \right)^{1/p}. \end{aligned}$$

Therefore we have for a fixed ϕ

$$\int A|\nabla u|^p |\phi\psi_\varepsilon|^p\, dx \le C + C' \int A|u|^p |\nabla(\phi\psi_\varepsilon)|^p\, dx. \tag{5-16}$$

From [**H-3**] we see that the right-hand side is bounded as $\varepsilon \to 0$. Since $\psi_\varepsilon \to 1$ on $\operatorname{supp}\phi$ as $\varepsilon \to 0$, it follows from Fatou's lemma that $A|\nabla u|^p \in L^1_{loc}(\Omega)$. Let $\phi \in C_0^\infty(\Omega)$. Now we put $\eta = \phi\psi_\varepsilon$ for a test function to obtain

$$\int_\Omega A|\nabla u|^{p-2} \nabla u \nabla(\phi\psi_\varepsilon)\, dx = -\int_\Omega (BQ(u) - f)\phi\psi_\varepsilon\, dx. \tag{5-17}$$

Note that

$$\begin{aligned} (5\text{-}18) \qquad & \left| \int_\Omega A|\nabla u|^{p-2} \phi \nabla u \cdot \nabla\psi_\varepsilon\, dx \right| \\ & \le \left(\int_{\operatorname{supp}|\nabla\psi_\varepsilon|} A|\nabla u|^p |\phi|\, dx \right)^{1-(1/p)} \cdot \left(\int_{\operatorname{supp}|\nabla\psi_\varepsilon|} A|\nabla\psi_\varepsilon|^p \phi\, dx \right)^{1/p}. \end{aligned}$$

Then from [**H-3**] and the local integrability of $A|\nabla u|^p$, we can show by letting

$\varepsilon \to 0$ ($\psi_\varepsilon \to 1$) that the right-hand side tends to zero, hence the desired equality follows. □

We recall the definition of $\mathbf{M}(x)$.

$$\mathbf{M}(x) = \underset{\{y \in \Omega : 1/4 < d(y)/d(x) < 3\}}{\text{ess-sup}} \frac{\tilde{A}(y)}{B(y)}, \quad \text{for } x \in \Omega \backslash F. \tag{5-19}$$

We also define

$$m(x) = \sup_{|y-x| \le d(x)/2} \frac{\tilde{A}(y)}{B(y)}. \tag{5-20}$$

Lastly we prepare the following.

Lemma 5-4 (The relation between m & $\mathbf{M}$). *Assume that for some $\varepsilon_0 > 0$, $\{x : d(x) \le \varepsilon_0\}$ is contained in Ω. Then it holds that*

(a)
$$\sup_{d(x)/2 \le d(y) \le d(x)} m(y) \le \mathbf{M}(x),$$

(b)
$$\sup_{\varepsilon/2 \le d(y) \le \varepsilon} m(y) \le \inf_{\varepsilon/2 \le d(y) \le \varepsilon} \mathbf{M}(y), \quad \textit{for any } \varepsilon \in (0, \varepsilon_0).$$

Proof. Since $\{y : |y - x| \le d(x)/2\} \subset \{y : 1/2 \le d(y)/d(x) \le 3/2\} \subset \{y : 1/4 \le d(y)/d(x) \le 3\}$, the inequality (a) is clear. We proceed to the proof of (b). From the definition of $m(x)$ we see that

$$\sup_{\varepsilon/2 \le d(y) \le \varepsilon} m(y) \le \sup_{(1/4)\varepsilon \le d(y) \le (3/2)\varepsilon} \frac{\tilde{A}(y)}{B(y)} \quad \text{for any } \varepsilon \in (0, \varepsilon_0). \tag{5-21}$$

Since it holds that $\{(1/4)\varepsilon \le d(y) \le (3/2)\varepsilon\} \subset \{(1/4)t \le d(y) \le 3t\}$ for any $t \in [\varepsilon/2, \varepsilon]$, we get

$$\sup_{(1/4)\varepsilon \le d(y) \le (3/2)\varepsilon} \frac{\tilde{A}(y)}{B(y)} \le \inf_{(1/2)\varepsilon \le t \le \varepsilon} \sup_{(1/4)t \le d(y) \le 3t} \frac{\tilde{A}(y)}{B(y)} = \inf_{(1/2)\varepsilon \le d(y) \le \varepsilon} \mathbf{M}(y). \tag{5-22}$$

Hence the desired estimate follows. □

6. Proof of Theorem 1.

In this section we shall establish Theorem 1 using the lemmas in the previous section. First we show an a priori bound for weak solutions of (0-3).

Lemma 6-1 (Supersolution). *Assume that $u \in H^{1,p}_{loc}(\Omega \backslash F) \cap L^\infty_{loc}(\Omega')$ satisfies $L_p u \in L^1_{loc}(\Omega')$ in the distribution sense. Assume that* [**H-1**], [**H-2**] *and* [**H-4**]. *Moreover we assume that for almost all $x \in \{x \in \Omega; u(x) \ge 0\}$*

(6-1) $$L_p u + B(x)Q(u) \le C(x).$$

Then we have, for some positive numbers C_1, C_2 *and* ε_0,

(6-2) $$u(x) \le C_1[m(x)^{1/\delta_0} d(x)^{-p/\delta_0} + 1] \le C_2 \cdot K\left(x, \frac{1}{d(x)}\right)^{1/(p-1)},$$

for any x *with* $0 < d(x) \le \varepsilon_0$. *Here* $m(x)$ *is defined by* (5-20).

Proof. First we note that the last inequality in (6-2) follows from the definition of the kernel $K(x,t)$ (See (1-7)). Let δ satisfy

(6-3) $$\delta \cdot \delta_0 = p.$$

Let $x_0 \in \Omega \backslash F$, with $0 < d(x_0) < 1/2$. For $R = d(x_0)/2$ and $r = |x - x_0|$, we set

(6-4) $$X = \{x \in \boldsymbol{R}^N; |x - x_0| < R\},$$

(6-5) $$\frac{\mu}{2} = Q^{-1}\left(3 \sup_{x \in \Omega} \frac{C(x)}{B(x)}\right),$$

and for $p' = p/(p-1)$,

(6-6) $$v(x) = \lambda w^{-\delta} + \mu, \quad w = R^{p'} - r^{p'}, \quad x \in X.$$

Now we determine constants λ so that v satisfies

(6-7) $$L_p v + B(x)Q(v) \ge C(x), \quad \text{in } X.$$

From a direct calculation, the monotonicity of $Q(\cdot)$ and the definition of μ it follows that

(6-8) $$\begin{cases} L_p v \ge -C_0 \lambda^{p-1} \tilde{A}(x) R^{p'} w^{-\delta(p-1)-p}, \\ Q(v) \ge C(x) + \dfrac{1}{3}\left(Q(\mu/2) + Q(\lambda w^{-\delta})\right) \end{cases}$$

where C_0 is a positive number independent of x_0, x, and R. Then we have

(6-9) $$\begin{aligned} &Pv + B(x)Q(v) \\ &\quad \ge C(x) - C_0 \lambda^{p-1} \tilde{A}(x) R^{p'} w^{-\delta(p-1)-p} + \frac{1}{3} B(x)(Q(\lambda w^{-\delta}) + Q(\mu/2)) \\ &\quad = C(x) + \frac{1}{3}\lambda^{-\delta_0} B(x) Q(\lambda w^{-\delta})\left(\lambda^{\delta_0} - 3C_0 \frac{\tilde{A}(x)}{B(x)} \frac{(\lambda w^{-\delta})^{\delta_0+p-1}}{Q(\lambda w^{-\delta})} \cdot R^{p'}\right) \\ &\qquad + \frac{1}{3} B(x) Q(\mu/2) \quad \text{in } X. \end{aligned}$$

Now we put

$$\lambda^{\delta_0} = 3C_0 \cdot m(x_0)R^{p'} \max\left[\sup_{|t|>1} \frac{t^{\delta_0+p-1}}{|Q(t)|}, \frac{1}{Q(\mu/2)}\right]. \tag{6-10}$$

If $\lambda w^{-\delta} \geq 1$, we immediately get the desired inequality (6-7) by the use of (1-6) in [**H-2**]. On the other hand, if $\lambda w^{-\delta} < 1$, we make use of the inequalities

$$\lambda^{\delta_0} \geq 3C_0 \cdot m(x_0)R^{p'} \frac{1}{Q(\mu/2)} \quad \text{and} \quad \lambda^{p-1} w^{-\delta(p-1)-p} \leq \lambda^{-\delta_0}. \tag{6-11}$$

Then we see

$$\frac{1}{3} B(x)Q(\mu/2) \geq C_0 \lambda^{p-1} \tilde{A}(x) R^{p'} w^{-\delta(p-1)-p}. \tag{6-12}$$

After all we get the desired conclusion. By the use of $\phi = (u-v)_+$ as a test function we get

$$\int_\Omega A(x)(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla\phi \, dx + \int_\Omega B(x)[Q(u) - Q(v)]\phi \, dx \leq 0. \tag{6-13}$$

Since $(u-v)_+ = 0$ near ∂X and Q is monotone, it follows from a weak maximum principle that

$$u(x_0) \leq v(x_0) = \lambda R^{-p'\delta} + \mu \leq C_1(m(x_0)^{1/\delta_0} d(x_0)^{-p/\delta_0} + 1), \tag{6-14}$$

and this proves the assertion. Here we used [**H-2**], $d(x_0) = R/2$ and $R \leq d(x) \leq 3R$ in X, and C_1 is a positive number independent of R. As for the weak maximum principle, see [**To**; Lemma 3.1] for example. $\square$

In this stage, the solution u of the inequality (2-2) in the distribution sense may still have singularities on ∂F. Combining this weak result with Lemma 5-2 we are able to show that u is bounded in Ω. First we assume [**H-1**], [**H-2**], [**H-3**] and [**H-4**]. Moreover we assume that $\{x : d(x) < 3\} \subset \Omega$ as before. Now we see from (5-10) in Lemma 5-2

$$\int_{\{x;u\geq\mu\}} [B(x)Q(u) - C(x)]\eta^p \, dx \leq p^p \left(\int_\Omega A(x)|\nabla\eta|^p (u-\mu)_+^p \, dx\right)^{(p-1)/p} \cdot \left(\int_\Omega A(x)|\nabla\eta|^p \, dx\right)^{1/p}, \tag{6-15}$$

Here $\operatorname{supp}\eta \subset \{x : 0 < d(x) < 3\}$ and μ is an arbitrary positive number satisfying

(6-16) $$Q(\mu/2) \geq \max\left[3 \sup_{x\in\Omega} \frac{C(x)}{B(x)}, \sup_{2<d(x)<3} |u|\right].$$

For η, we choose a Lipschitz continuous function η_ε for a sufficiently small $\varepsilon > 0$ such that

(6-17) $$\eta_\varepsilon(x) = \begin{cases} 0, & \text{if } dist(x, \partial F) \leq \varepsilon/2, \text{ or } d(x) \geq 3, \\ \dfrac{2}{\varepsilon}\, dist(x, \partial F_{\varepsilon/2}), & \text{if } \varepsilon/2 \leq dist(x, \partial F) \leq \varepsilon, \\ 1, & \text{if } \varepsilon \leq dist(x, \partial F) \text{ and } d(x) \leq 2. \end{cases}$$

Here F_ε is a tubular neighborhood of F defined by

(6-18) $$F_\varepsilon = \{x \in \Omega : dist(x, F) < \varepsilon\}.$$

By virtue of Lemma 6-1 we have, for some positive number C_1 independent of each x and $\mu > 0$,

(6-19) $$(u(x) - \mu)_+ \leq C_1 \cdot (1 + m(x)^{1/\delta_0} \cdot d(x)^{-p/\delta_0}).$$

From Lemma 5-4 we have

(6-20) $$\begin{aligned}\sup_{\varepsilon/2\leq d(x)\leq\varepsilon} (u-\mu)_+ &\leq C_1 \cdot \sup_{\varepsilon/2\leq d(x)\leq\varepsilon} (1 + m(x)^{1/\delta_0} \cdot d(x)^{-p/\delta_0}) \\ &\leq C_1' \cdot \inf_{\varepsilon/2\leq d(x)\leq\varepsilon} (1 + \mathbf{M}(x)^{1/\delta_0} \cdot d(x)^{-p/\delta_0}).\end{aligned}$$

By the definition of K, we have for some positive number C

(6-21) $$(1 + \mathbf{M}(x)^{1/\delta_0} \cdot d(x)^{-p/\delta_0})^{p-1} \leq C \cdot K\left(x, \frac{1}{d(x)}\right).$$

Then, it follows from (6-15) that for some positive number C',

(6-22) $$\begin{aligned}&\int_{\{x;u\geq\mu\}} [B(x)Q(u) - C(x)]\eta_\varepsilon^p\, dx \\ &\leq p^p \left(\int_\Omega A(x)|\nabla\eta_\varepsilon|^p (u-\mu)_+^{p-1}\, dx\right)^{(p-1)/p} \\ &\quad\times \left(\int_\Omega A(x)|\nabla\eta_\varepsilon|^p \sup_{\varepsilon/2\leq d(y)\leq\varepsilon} (u(y)-\mu)_+^{p-1}\, dx\right)^{1/p} \\ &\leq p^p \int_\Omega A(x)|\nabla\eta_\varepsilon|^p \sup_{\varepsilon/2\leq d(y)\leq\varepsilon} (u(y)-\mu)_+^{p-1}\, dx \\ &\leq C' p^p \cdot \frac{1}{\varepsilon^p} \int_{\{\varepsilon/2\leq d(x)\leq\varepsilon\}} A(x)K\left(x, \frac{1}{d(x)}\right) dx\end{aligned}$$

Since $\eta_\varepsilon \to 1$ as $\varepsilon \to 0$, it follows from [**H-3**] that $BQ(u) \in L^1_{loc}(\Omega)$. Again from (6-15) and [**H-3**] we have

$$\text{(6-23)} \quad \int_{\{x; u \geq \mu\}} [B(x)Q(u) - C(x)]\eta_\varepsilon^p \, dx \leq C\left(\int_\Omega A(x)|\nabla \eta_\varepsilon|^p (u-\mu)_+^{p-1} \, dx\right)^{(p-1)/p}.$$

From Lemma 1-1 (Young's inequality), (1-11), (1-13) and (1-15) we see that

$$\begin{aligned}\text{(6-24)} \quad & A|\nabla \eta_\varepsilon|^p t^{p-1} \\ & \leq A \cdot \Phi_0(x, L^{p-1} t^{p-1}) + A \cdot \Psi_0(x, |\nabla \eta_\varepsilon|^p / L^{p-1}) \\ & \leq B \cdot Q(Lt) + C \cdot A \cdot L^{-(p-1)(1+((p-1)/\delta_0))} \Psi_1(x, |\nabla \eta_\varepsilon|^p) \\ & \leq B \cdot Q(Lt) + C \cdot A \cdot L^{-(p-1)(1+((p-1)/\delta_0))} K(x, |\nabla \eta_\varepsilon|) |\nabla \eta_\varepsilon|^p,\end{aligned}$$

where L and t are arbitrary positive numbers. We used the following:

$$\text{(6-25)} \qquad \Psi_1(x, t^p) \leq G_1(x,t) \cdot t^p \leq K(x,t) \cdot t^p, \quad (x \in \Omega', t > 0).$$

We note that $d(x)$ is equivalent to ε on the support of η_ε and $|\nabla \eta_\varepsilon| = 2/\varepsilon$ holds there. Hence for any positive number κ, there is a large positive number L independent of each ε such that we have

$$\begin{aligned}\text{(6-26)} \quad \int_\Omega A|\nabla \eta_\varepsilon|^p (u-\mu)_+^{p-1} \, dx &\leq \int_{\operatorname{supp} \eta_\varepsilon} B \cdot Q(L(u-\mu)_+) \, dx \\ &\quad + \frac{C \cdot \varepsilon^{-p}}{L^{(p-1)(1+((p-1)/\delta_0))}} \int_{\operatorname{supp} \eta} A \cdot K\left(x, \frac{1}{d(x)}\right) dx \\ &\leq \int_{\operatorname{supp} \eta_\varepsilon} B \cdot Q(L(u-\mu)_+) \, dx + \kappa\end{aligned}$$

As $\varepsilon \to 0$, we easily see that

$$\text{(6-27)} \qquad \int_{\{x; u \geq \mu\}} [B(x)Q(u) - C(x)] \, dx = 0$$

Therefore we have showed that the positive part of u is bounded in Ω.

Lastly we assume [**H-5**] in stead of [**H-3**]. In this case the boundedness of u follows from (5-9) in Lemma 5-2, Lemma 6-1 and Lemma A1 in Appendix. Let ε and κ be arbitrary small positive numbers. By virtue of [**H-5**] and Lemma A1, we choose $\xi \in C_0^\infty(F_\varepsilon)$ such that $\xi \geq 1$ on F and

$$\text{(6-28)} \qquad \int_{F_\varepsilon} A \cdot K\left(x, \frac{1}{d}\right) \left(\log K\left(x, \frac{1}{d}\right)\right)^p |\nabla \xi(x)|^p \, dx < \kappa.$$

Here F_ε denotes a tubular neighborhood of F. We choose $\phi \in C_0^\infty(\Omega)$ such that $\phi = 1$ for $\{x : d(x) < 2\}$; 0 for $\{x : d(x) > 3\}$. Finally we set $\eta = (1 - \xi) \cdot \phi$. Since η is an admissible test function we have

$$\begin{aligned} &\int_\Omega [B(x)Q(u) - C(x)] \log[1 + (u - \mu)_+]\eta^p \, dx \\ &\qquad \le p^p \int_{\Omega'} A(x)|\nabla \eta|^p [1 + (u - \mu)_+]^{p-1} [\log[1 + (u - \mu)_+]]^p \, dx \\ &\qquad \le p^p \int_{\Omega'} A(x)|\nabla \xi|^p |\phi|^p [1 + (u - \mu)_+]^{p-1} [\log[1 + (u - \mu)_+]]^p \, dx \\ &\qquad \le C \int_{\Omega'} A(x)|\nabla \xi|^p K\left(x, \frac{1}{d(x)}\right) \left[\log K\left(x, \frac{1}{d(x)}\right)\right]^p dx < C\kappa. \end{aligned} \tag{6-29}$$

Therefore we see

$$\int_{\Omega \cap \{x : d(x) < 3\}} [B(x)Q(u) - C(x)] \log[1 + (u - \mu)_+] \, dx = 0. \tag{6-30}$$

This proves the assertion. □

7. Dirichlet boundary problem.

Uniqueness. First we prove the uniqueness of solutions in

$$T(\Omega) = \{u \in L^\infty(\Omega) \cap H^{1,p}_{loc}(\bar{\Omega} \backslash F); u = 0 \text{ on } \partial\Omega\}. \tag{7-1}$$

Assume that u and v are solutions to (3-1) in the space $T(\Omega)$. Note that $u = v$ in $F \backslash \partial F$ (Monotonicity of Q). From (5-16) in Lemma 5-3 we immediately see that $A(|\nabla u|^p + |\nabla v|^p) \in L^1(\Omega)$. By subtraction we get in the sense of the distribution

$$L_p(u - v) + B(x)(Q(u) - Q(v)) = 0, \quad \text{in } \Omega. \tag{7-2}$$

By the use of $\phi = (u - v)\eta^p$, where $\eta \in C^\infty(\Omega)$ will be specified later, we get

$$\begin{aligned} &\int_\Omega A(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla(u - v)\eta^p \, dx \\ &\qquad + p \int_\Omega A(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla \eta \eta^{p-1}(u - v) \, dx \\ &\qquad + \int_\Omega B[Q(u) - Q(v)](u - v)\eta^p \, dx = 0. \end{aligned} \tag{7-3}$$

Now we take a sequence of smooth functions η_j such that $\eta_j = 0$ near ∂F, $\lim_{j\to\infty} \eta_j = 1$ in Ω and

$$\limsup_{j\to\infty} \int_\Omega A(x)|\nabla \eta_j|^p \, dx < \infty. \tag{7-4}$$

This is possible from [**H-3**]. Then replacing η for η_j and letting $j \to \infty$, we get

$$\int_\Omega A(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla(u-v)\, dx + \int_\Omega B[Q(u) - Q(v)](u-v)\, dx = 0. \tag{7-5}$$

Since Q is monotone and $u = v$ in $F \backslash \partial F$, we see $u \equiv v$ in Ω. Thus the uniqueness holds.

EXISTENCE. We assume that $N > 1$. If $N = 1$, the proof below still works with obvious modifications. First we shall regularize the problem by approximating the operator L_p by uniformly elliptic operators $\{L_p^{(\varepsilon)}\}_{\varepsilon>0}$ in the following way. If L_p is uniformly elliptic, the existence of solutions to (3-1) in $H_0^{1,p}(\Omega)$ is well-known. Let us set for $\varepsilon > 0$

$$L_p^{(\varepsilon)} u = -\operatorname{div}[(\varepsilon + A(x))|\nabla u|^{p-2}\nabla u], \quad \text{for } u \in H_0^{1,p}(\Omega), \tag{7-6}$$

and consider the Dirichlet problem:

$$\begin{cases} L_p^{(\varepsilon)} u + B(x)Q(u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{7-7}$$

Then we prepare a lemma which concerns the existence and regularity of solutions of (7-7). We shall sketch the proof for convenience.

LEMMA 7-1. *Let $N > 1$. Assume that the same assumptions as those in Theorem 2. Then there is a unique $u_\varepsilon \in H_0^1(\Omega)$ which satisfies (7-7) in the weak sense. Moreover u_ε satisfies*

$$BQ(u_\varepsilon), BQ(u_\varepsilon)u_\varepsilon \in L^1(\Omega). \tag{7-8}$$

SKETCH OF PROOF. This lemma can be shown in the following way. We replace Q for $Q_n(u) = \min(|Q(u)|, n)\operatorname{sgn}(u)$ and consider the truncated equation below;

$$L_p^{(\varepsilon)} u_n + B(x)Q_n(u_n) = f, \quad \text{in } \Omega. \tag{7-9}$$

$$u = 0, \quad \text{on } \partial\Omega.$$

Since A, B and $f \in L^\infty(\bar{\Omega})$, we are able to prove the existence of bounded solutions in $H_0^{1,p}(\Omega)$ by the use of Schauder's fixed point theorem and the standard argument. It is easy to see that $\{u_n\}_{n=1}^\infty$ is bounded in $H_0^{1,p}(\Omega)$. As we make n tend to infinity, we show the weak convergence of solutions in $H_0^{1,p}(\Omega)$ using a priori estimates for a fixed $\varepsilon > 0$. Then by the compactness argument we see the limit u_ε satisfies (7-7) and (7-8). □

Remark 12. (1) For each compact set $K \subset \bar{\Omega} \backslash F$, it holds that $u_\varepsilon \in H^{1,p}(K)$ and $BQ(u_\varepsilon)u_\varepsilon \in L^1(\Omega)$. Since the operator L_p is uniformly elliptic on K and $A \in C^0(\bar{\Omega})$, there is a positive number $C(K)$ independent of each $\varepsilon > 0$ such that we have

$$\sup_{x \in K} |u_\varepsilon(x)| \le C(K). \tag{7-10}$$

Moreover if Q is uniformly Lipschitz continuous, then we see $u_\varepsilon \in H_{loc}^{2,p}(\Omega \backslash F)$ as well.

(2) Under a weaker assumption that $f \in L^1(\Omega)$, a similar existence result holds for the approximating problem (7-9). For the detailed, see [BS, Theorem 12 and its corollary].

End of the proof of Theorem 3. By u_ε we denote the solutions to (7-9) as before. From Lemma 7-1 and its remarks we see $u_\varepsilon \in H_0^{1,p}(\Omega)$ and $BQ(u_\varepsilon)u_\varepsilon \in H_0^{1,p}(\Omega)$. First we prove that u_ε satisfies (3-4) uniformly in $\varepsilon > 0$. We set

$$a = \frac{p + \delta_0}{p - 1 + \delta_0}, \quad \text{and} \quad b = p + \delta_0. \tag{7-11}$$

From (1-6) in [**H-2**], we see $|u_\varepsilon| \le C[(|u_\varepsilon|\,|Q(u_\varepsilon)|)^{1/(p+\delta_0)} + 1]$ for some positive number C. By young's inequality we have for any positive number h

$$\begin{aligned} \int_\Omega |f|\,|u_\varepsilon|\,dx &\le C \int_\Omega \frac{|f|}{B} [(|u_\varepsilon|\,|Q(u_\varepsilon)|)^{1/(p+\delta_0)} + 1] B\,dx \\ &\le Ca^{-1}h^{-a} \int_\Omega \left(\frac{|f|}{B}\right)^a B\,dx \\ &\quad + Cb^{-1}h^b \int_\Omega |u_\varepsilon||Q(u_\varepsilon)| B\,dx + C \int_\Omega |f|\,dx. \end{aligned} \tag{7-12}$$

Multilpying u_ε to the both side of (7-9) and integrating over Ω, we get

$$\begin{aligned} \int_\Omega (\varepsilon + A)|\nabla u_\varepsilon|^p\,dx &+ (1 - Cb^{-1}h^b) \int_\Omega B|u_\varepsilon|\,|Q(u_\varepsilon)|\,dx \\ &\le Ca^{-1}h^{-a} \int_\Omega \left(\frac{|f|}{B}\right)^a B\,dx + C \int_\Omega |f|\,dx. \end{aligned} \tag{7-13}$$

Now we put $h^b = b(2C)^{-1}$, then we have the desired inequality.

Secondly, by the method of a priori estimate and compactness, we derive a subsequence $\{u_{\varepsilon_j}\}_{j=1}^{\infty}$ from $\{u_\varepsilon\}_{\varepsilon>0}$ which converges weakly to some element $\bar{u} \in H^1_{loc}(\bar{\Omega}\backslash F)$ and u_{ε_j} converges $\bar{u}$ a.e. in $\Omega\backslash F$. Then by virtue of Fatou's lemma and a weakly lower semicontinuity of L^p-norm, we get

$$\int_{\Omega\backslash F} A|\nabla\bar{u}|^p\,dx + \int_{\Omega\backslash F} BQ(\bar{u})\bar{u}\,dx \le C[\|f/B\|_\infty^\lambda + \|f\|_\infty]. \tag{7-14}$$

Now we show that $BQ(u_{\varepsilon_j}) \to BQ(\bar{u})$ in the sense of distribution on $\Omega\backslash F$. From the definition of weak convergence of $\{u_{\varepsilon_j}\}_{j=1}^{\infty}$ and the estimates (7-13) and (7-14), we see that $f - L_p^{(\varepsilon_j)}u_{\varepsilon_j} \to f - L_p\bar{u}$ in the sense of distribution on $\Omega\backslash F$. Therefore the limit of $BQ(u_{\varepsilon_j})$ in $D'(\Omega\backslash F)$ as $j \to \infty$ exists. Hence it suffices to show that

$$\int_\Omega B(Q(u_{\varepsilon_j}) - Q(\bar{u}))\varphi\,dx \to 0, \quad \text{for all } \varphi \in C_0^\infty(\Omega\backslash F). \tag{7-15}$$

From Remark 12 just after the proof of Lemma 7-1, $\sup_{x\in\operatorname{supp}\varphi}|u_{\varepsilon_j}|$ is uniformly bounded on the support of φ, so that $BQ(u_{\varepsilon_j})$ is uniformly bounded with respect to ε_j. Since $u_{\varepsilon_j} \to \bar{u}$ a.e. in $\Omega\backslash F$, (7-15) follows from the dominated convergence theorem. After all we see that $\bar{u}$ satisfies (3-1) in $\Omega\backslash F$ in the weak sense. Now we define

$$u(x) = \begin{cases} \bar{u}(x), & \text{if } x \in \Omega\backslash F, \\ Q^{-1}(f(x)/B(x)), & \text{if } x \in F\backslash\partial F. \end{cases} \tag{7-16}$$

Then u clearly satisfies (3-1) in $\Omega\backslash\partial F$ in the sense of distribution. In $\Omega\backslash F$ the operator L_p is elliptic and the right-hand side of (3-1) belongs to $L^\infty(\Omega)$. Hence we see that $u \in L^\infty_{loc}(\Omega')$. Then it follows from Theorem 1 that u is bounded in Ω'. From Theorem 2 we see that there exists a unique function $v \in L^\infty(\Omega)$ which satisfies (2-8). Since $v = u$ in $\Omega\backslash\partial F$, we see that $v \in T(\Omega)$ is a unique weak solution to (3-1) in Ω and v satisfies (3-4) for some positive number C.

8. Appendix.

In this section we first study the relative capacity, and we shall prove Lemma 1-2. We assume that Q is strictly convex as before. Then we have

LEMMA 1-2. *Assume that* [**H-1**]. *Then* [**H-3**] *implies* $C_K(\partial F, \Omega) = 0$, *that is,* ∂F *has a vanishing capacity.*

PROOF. Without loss of generality we assume that $\{x : d(x) < 3\} \subset \Omega$. Let us set $\eta_j(x) = \eta_{\varepsilon_j}(x)$ for $\varepsilon_j = 2^{-j}$, $j = 1, 2, \dots$, where η_ε is defined by (6-17). Putting $\zeta_j = 1 - \eta_j$, $j = 1, 2, \dots$ we set

(8-1)
$$\zeta^N(x) = \frac{1}{N}\sum_{j=1}^{N}\zeta_j(x).$$

Clearly $\zeta^N \in C_0^{0,1}(\Omega)$ satisfies $\zeta^N \geq 1$ on ∂F. Then it follows from [**H-3**] that

(8-2)
$$\begin{aligned}
&\int_\Omega A(x)\cdot K\left(x,\frac{1}{d}\right)|\nabla\zeta^N|^p\,dx \\
&= \frac{1}{N^p}\sum_{j=1}^{N}\int_{\varepsilon_{j+1}\leq d(x)\leq\varepsilon_j} A(x)\cdot K\left(x,\frac{1}{d}\right)|\nabla\zeta_j|^p\,dx \\
&\leq C\frac{1}{N^p}\sum_{j=1}^{N}\frac{1}{\varepsilon_j^p}\int_{\varepsilon_{j+1}\leq d(x)\leq\varepsilon_j} A(x)\cdot K\left(x,\frac{1}{d}\right)dx \\
&= O(N^{1-p}).
\end{aligned}$$

Here C is a positive number independent of each j. Since $p > 1$ we see the capacity of ∂F must be zero. This proves the assertion. Lastly we state the following lemma without the proof.

Lemma A-1. *The followings are equivalent to each other.*
(1) $C_K(\partial F,\Omega) = 0$,
(2) $C_K(\partial F,F_\varepsilon) = 0$, *for some* $\varepsilon > 0$,
(3) $C_K(\partial F,F_\varepsilon) = 0$, *for any* $\varepsilon > 0$.
Here $F_\varepsilon = \{x \in \Omega; dist(x,F) \leq \varepsilon\}$.

From this we see that the removability of singularities does not depend on the shape of the boundary $\partial\Omega$. For the proof of this lemma it suffices to note that $A(x) = 0$ in the interior of F (c.f. [**H4**]).

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