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Combinatorial moves on ambient isotopic submanifolds in a manifold

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Abstract. In knot theory, it is well-known that two links in the Euclidean 3-space are ambient isotopic if and only if they are related by a finite number of combinatorial moves along 2-simplices. This fact is generalized for submanifolds in a manifold whose codimensions are positive.

1. Introduction.

In knot theory, it is well-known that two links in the Euclidean 3-space are ambient isotopic if and only if they are related by a finite number of combinatorial moves along 2-simplices (cf. [**7**], [**8**], [**2**], [**6**]). Such combinatorial moves are referred to Reidemeister's Δ-moves in [**2**]. In fact, the link isotopy type of a link is often defined to be the equivalence class of the link in this combinatorial sense, cf. [**3**], [**1**]. The purpose of this paper is to generalize this fact for submanifolds in a manifold whose codimensions are positive.

Let W^q be a PL q-manifold, which may be non-compact, non-orientable, or disconnected.

Theorem 1.1. *Let L and L' be compact proper locally flat n-manifolds in W^q with $n < q$, and Y a $(q-1)$-submanifold of the boundary ∂W^q of W^q. The following conditions are mutually equivalent.*

(1) *L is ambient isotopic to L' by an ambient isotopy keeping Y fixed.*
(2) *L is transformed into L' by proper moves of W^q relative to Y.*
(3) *L is transformed into L' by cellular moves relative to Y.*
(4) *L is transformed into L' by simplex moves relative to Y.*

It is well-known that (4) $\to$ (3) $\to$ (2) $\to$ (1). [(4) $\to$ (3) is obvious by definition. (3) $\to$ (2) is the cellular move lemma (see Proposition 4.15 of [**9**]). (2) $\to$ (1) is for example proved in Lemma 6.1 of [**4**].] The converse (1) $\to$ (2) is also well-known in a special case that W^q is compact and Y is empty (see Theorem 6.2 of [**4**]).

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As well as Reidemeister's $\varDelta$-moves for knots are generalized to moves for graphs in the 3-space (cf. [**5**], [**10**]), our moves are generalized to moves for polyhedra in a q-manifold such that the dimensions of the non-locally flat point sets are smaller than 2. (It remains open to treat such moves in a more general situation, since the argument might be complicated.)

Let G be an n-dimensional compact polyhedron in a q-manifold W^q, and $\varGamma$ the set of points of G at which G is not locally flat in W^q. (We say that G is locally flat at $x \in G$ in W^q if there exists a regular neighborhood $N(x)$ of x in W^q such that the triple $(N(x), G \cap N(x), x)$ is homeomorphic to (D^q, D^n, O) or (D^q_+, D^n_+, O), where (D^q, D^n) is the standard (q, n)-disk pair, (D^q_+, D^n_+) is the half disk pair of (D^q, D^n), and O is the origin.)

Theorem 1.2. *Suppose that* $\dim(\varGamma) \leq 1$ *and* $\dim(\varGamma \cap \partial W^q) \leq 0$. *If G is ambient isotopic to G' by an ambient isotopy of W^q keeping $\varGamma$ and a $(q-1)$-submanifold Y of ∂W^q fixed, then G is transformed into G' by a finite sequence of $(n+1)$-simplex moves relative to Y.*

When $\dim(\varGamma) \leq 0$ and W^q is simply connected, the assumption of Theorem 1.2 is more relaxed by the following proposition.

Proposition 1.3. *Let W^q be a simply connected q-manifold with $q \geq 3$, and $\varGamma$ a finite set of interior points of W^q. If there is an ambient isotopy $\{h_t\}$ $(t \in [0,1])$ of W^q keeping a $(q-1)$-submanifold Y of ∂W^q fixed such that $h_1|_\varGamma =$ id, then there is an ambient isotopy $\{h'_t\}$ $(t \in [0,1])$ of W^q such that $h'_1 = h_1$ and $\{h'_t\}$ keeps $\varGamma$ and Y fixed.*

For example, a *singular $\varSigma$-knot*, that is the image of a generic map of a closed surface $\varSigma$ into the 4-space R^4, is a 2-dimensional compact polyhedron such that $\varGamma$ is a finite set of points of R^4.

As a simple case, we formulate the 3-dimensional version of the above arguments.

Corollary 1.4. *Let W^3 be a 3-manifold, and L a compact proper 1-manifold in W^3. If L is ambient isotopic to L' by an ambient isotopy of W^3 keeping a 2-submanifold Y of ∂W^3 fixed, then L is transformed into L' by a finite sequence of 2-simplex moves relative to Y. For a finite graph G in W^3 such that $G \cap \partial W^3$ is the set of degree-one vertices of G, if G is ambient isotopic to G' by an ambient isotopy $\{h_t\}$ $(t \in [0,1])$ of W^3 keeping a 2-submanifold Y of ∂W^3 and the degree ≥ 3 vertices of G fixed, then G is transformed into G' by a finite sequence of 2-simplex moves relative to Y. Furthermore, when W^3 is simply connected, we can replace the assumption that $\{h_t\}$ keeps the degree ≥ 3 vertices of G fixed by that h_1 preserves the degree ≥ 3 vertices of G.*

Throughout this paper we work in the piecewise linear category.

2. Cellular moves and simplex moves.

An *ambient isotopy* of a manifold W means an isotopy $h_t : W \to W$ ($t \in [0,1]$) with h_0 the identity map. We say that two subsets of W are *ambient isotopic* (by an ambient isotopy keeping a subset Y of W fixed) if there exists an ambient isotopy $\{h_t\}$ of W whose terminal map h_1 carries one to the other (and each map h_t is identical on Y).

From now on, let W^q be a q-manifold and Y a $(q-1)$-submanifold of the boundary ∂W^q of W^q.

Let D be an $(n+1)$-disk. An i-disk in ∂D is called a *PL i-face* of D, where $i \in \{-1, 0, 1, 2, \ldots, n\}$. We assume that a PL (-1)-face is the empty set. In order to avoid confusion, for an $(n+1)$-simplex V, we shall call an i-face of V in the usual sense a *canonical i-face*. By a *combinatorial i-face* of V, we mean a PL i-face of V which is the union of some canonical i-faces of V.

For a homeomorphism $h : W^q \to W^q$, we denote by $\mathrm{supp}(h)$ the *support* of h, that is the closure of $\{x \in W^q \mid h(x) \neq x\}$ in W^q. If there exists a q-disk D in W^q such that $\mathrm{supp}(h) \subset D$, then the homeomorphism h is called a *move* supported by the q-disk D. Moreover, if $h|_{\partial W^q} = \mathrm{id}$ or if $D \cap \partial W^q$ is a PL $(q-1)$-face of D, then the move is called a *proper move* supported by D. A homeomorphism $h : W^q \to W^q$ is a proper move supported by D if and only if it is isotopic to the identity map by an ambient isotopy of W^q keeping $\mathrm{cl}(\partial W^q - D)$ fixed, see Lemma 6.1 of [**4**].

Let L be a compact proper locally flat n-manifold in W^q and D an $(n+1)$-disk in W^q, where we assume $n < q$. Put $D_0 = D \cap L$ and $D_1 = D \cap \partial W^q$. Suppose that D_0 is a PL n-face of D and one of the following conditions is satisfied:

(1) D_1 is a PL i-face of D_0 for some $i \in \{-1, 0, \ldots, n-1\}$.

(2) D_1 is a PL n-face of D such that $D_0 \cap D_1$ is a common PL $(n-1)$-face of D_0 and D_1.

Replacing D_0 with the PL n-face $\mathrm{cl}(\partial D - (D_0 \cup D_1))$ of D, we obtain another proper n-manifold in W^q from L. We call this replacement a *cellular move* for L along D. According as the case (1) or (2) occurs, we say that the cellular move is of *type* 1 or *type* 2. Let Y be a $(q-1)$-submanifold of ∂W^q. A cellular move for L along D is called a *cellular move relative to* Y if it is of type 1 or if it is of type 2 such that $D_1 \cap Y$ is a PL i-face of the $(n-1)$-disk $D_0 \cap D_1$ for some $i \in \{-1, 0, \ldots, n-2\}$.

The cellular move lemma (cf. Proposition 4.15 of [**9**]) states that if L' is obtained from L by a cellular move along an $(n+1)$-disk D relative to Y, then for any regular neighborhood U of D, there exists an ambient isotopy $\{h_t\}$ ($t \in [0,1]$) of W^q which carries L to L' and keeps $\mathrm{cl}(W^q - U)$ and Y fixed. In particular, a cellular move is a proper move.

Let L be a compact proper locally flat n-manifold in W^q and V an $(n+1)$-simplex in W^q. Put $V_0 = V \cap L$ and $V_1 = V \cap \partial W^q$. Suppose that V_0 is a combinatorial n-face of V and that one of the following conditions is satisfied:

(1) V_1 is a combinatorial i-face of V_0 for some $i \in \{-1, 0, \dots, n-1\}$.

(2) V_1 is a combinatorial n-face of V such that $V_0 \cap V_1$ is a common combinatorial $(n-1)$-face of V_0 and V_1.

We call the cellular move (relative to Y) along V an $(n+1)$-*simplex move* (*relative to* Y).

Notice that an $(n+1)$-simplex move of type 1 does not change ∂L. If L' is obtained from L by an $(n+1)$-simplex move along V of type 2, then ∂L is moved into $\partial L'$ by a cellular move in ∂W^q along the n-disk $V_1 = V \cap \partial W^q$.

Let G be an n-dimensional compact polyhedron in W^q such that $\dim(\Gamma) \le 1$ and $\dim(\Gamma \cap \partial W^q) \le 0$, where Γ is the set of non-locally flat points of G in W^q. Let $v_1, \dots, v_s$ be the vertices of Γ and $B_1, \dots, B_s$ regular neighborhoods of them in W^q. If v_i is an interior point (resp. a boundary point) of W^q, let $\partial_+ B_i$ be the $(q-1)$-sphere ∂B_i (resp. the $(q-1)$-disk $\mathrm{cl}(\partial B_i \cap \mathrm{int}(W^q))$). We assume that B_i is the cone over $\partial_+ B_i$ with v_i as the cone vertex. Put $\Lambda_i = G \cap \partial_+ B_i$ which is an $(n-1)$-dimensional compact polyhedron in $\partial_+ B_i$ such that the dimension of the non-locally flat point set is 0 or -1. Put $W_1 = \mathrm{cl}(W^q - \bigcup B_i)$ and $\Gamma_1 = \Gamma \cap W_1$. Then W_1 is a q-manifold and Γ_1 is the union of some proper simple arcs $e_1, \dots, e_t$ in W_1 (or the empty set). Let D_j $(j = 1, \dots, t)$ be a regular neighborhood of e_j in W_1. We regard the union $(\bigcup_{i=1}^{s} B_i) \cup (\bigcup_{j=1}^{t} D_j)$ as a regular neighborhood $N(\Gamma)$ of Γ in W^q. If necessary taking a subdivision of Γ, we may assume that for each j, the pair (D_j, e_j) is a cone of $(\partial D_j, \partial e_j)$ over a cone vertex w_j.

Let V be an $(n+1)$-simplex in W^q and put $V_0 = V \cap G$ and $V_1 = V \cap \partial W^q$. Suppose that V_0 and V_1 satisfy the condition of the definition of an $(n+1)$-simplex move as before. Moreover we suppose that one of the following conditions is satisfied:

(1) V is in $W_2 = \mathrm{cl}(W^q - N(\Gamma))$.

(2) V is in B_i for some i so that it is the join of v_i and an n-simplex in $\partial_+ B_i$ and $V \cap \Gamma$ is an i-face of V_0 for $i = 0$ or 1.

(3) V is in D_j for some j so that it is the join of a point of e_j and an n-simplex in $\partial_+ B_i$ or the join of an edge in e_j and an $(n-1)$-simplex in $\partial_+ B_i$ and $V \cap \Gamma$ is an i-face of V_0 for $i = 0$ or 1.

Then we define an $(n+1)$-*simplex move along* V (*with respect to* $N(\Gamma)$ *on* G) to be the replacement of V_0 by $\mathrm{cl}(\partial V - (V_0 \cup V_1))$.

3. Proof of Theorem 1.1.

Let V be an $(n+1)$-simplex in a Euclidean space R^ℓ and V_0 a canonical n-face of V. Suppose that $V = |v_0, v_1, \dots, v_{n+1}|$ and $V_0 = |v_0, v_1, \dots, v_n|$. Consider a linear map $p: V \to V_0$ such that $p(v_i) = v_i$ for $i = 0, 1, \dots, n$ and $p(v_{n+1})$ is a point of $\mathring{V}_0$ (= points of V_0 not contained in any face).

Let K be a simplicial complex with $|K| = V$, and K_0 its restriction to V_0. We say that K is *in rapport with* p if p is a simplicial map from K to K_0.

Lemma 3.1. *Let V, V_0 and p be as above. For any simplicial complex K with $|K| = V$, there exists a subdivision K' of K which is in rapport with p.*

Proof. For each $A \in K$, the image $p(A)$ is a convex linear cell. There exists an rth derived subdivision $K_0^{(r)}$ of K_0 such that for all $A \in K$, $p(A)$ are subdivided as subcomplexes of $K_0^{(r)}$. Let K^1 be the set $\{A \cap p^{-1}(B) \mid A \in K, B \in K_0^{(r)}\}$, which is a cellular subdivision of K such that the restriction over V_0 is $K_0^{(r)}$. Since p is a linear map carrying vertices of K^1 onto those of $K_0^{(r)}$, we obtain a desired subdivision K' by subdividing K^1 without introducing vertices. □

Let K be a simplicial complex with $|K| = V$ which is in rapport with $p: V \to V_0$. Each $(n+1)$-simplex $A = |a_0, a_1, \dots, a_{n+1}| \in K$ is mapped linearly onto an n-simplex $B = |b_0, b_1, \dots, b_n| \in K_0$. We may assume that

(1) $p(a_i) = b_i$ for $i = 0, 1, \dots, n$,
(2) $p(a_{n+1}) = b_0$, and
(3) $\mathrm{dist}(a_{n+1}, V_0) > \mathrm{dist}(a_0, V_0)$.

In this situation, we call the canonical n-face $|a_0, \dots, a_n|$ the *bottom face* of A, and the canonical n-face $|a_1, \dots, a_{n+1}|$ the *top face* of A. For each n-simplex $B \in K_0$, there exists a unique ordering $A_1, A_2, \dots, A_r$ of all the $(n+1)$-simplices in $K|p^{-1}(B)$ such that

(1) the bottom face of A_1 is B,
(2) the top face of A_i is the bottom face of A_{i+1} for $i = 1, \dots, r-1$.

Let D be an $(n+1)$-disk in a q-manifold W^q. Suppose that a (possibly non-proper) n-manifold L in W^q intersects with D such that $L \cap D = D_0$ for a PL n-face D_0 of D. Replacing D_0 with $\mathrm{cl}(\partial D - D_0)$, we obtain another (possibly non-proper) n-manifold in W^q from L. We call this replacement a *pseudo-cellular move* along D. (Notice that the result may be non-proper even if L is proper.)

Lemma 3.2. *Let V be an $(n+1)$-simplex and V_0 a canonical n-face of V. For any simplicial complex K with $|K| = V$, there exists a subdivision K' of K such that the canonical n-face V_0 is transformed into the combinatorial n-face $\mathrm{cl}(\partial V - V_0)$ by a finite sequence of pseudo-cellular moves along the $(n+1)$-simplices of K'.*

Proof. We use the induction on n. The case of $n = 0$ is obvious, for K itself is a desired one. Assume that $n > 0$. By Lemma 3.1, we may assume that

K is in rapport with a linear projection $p: V \to V_0$ being as in the lemma. By the induction hypothesis, there exists a subdivision K_0' of K_0 such that a canonical $(n-1)$-face V_{00} of V_0 is transformed into $\mathrm{cl}(\partial V_0 - V_{00})$ by a finite sequence of pseudo-cellular moves along the n-simplices of K_0'. Let $B_1, \dots, B_m$ be the n-simplices of K_0' and we assume that the sequence of pseudo-cellular moves is performed along $B_1, \dots, B_m$ in this order. Take a subdivision K' of K such that the projection $p: V \to V_0$ is a simplicial map from K' to K_0'. We claim that K' is a desired subdivision of K. For each n-simplex B_k $(k = 1, \dots, m)$ of K_0', let $A_1^k, \dots, A_{r_k}^k$ be the $(n+1)$-simplices mapped onto B_k such that the bottom face of A_1^k is B_k and, for each i $(i = 1, \dots, r_k - 1)$, the top face of A_i^k is the bottom face of A_{i+1}^k. Then V_0 is transformed into $\mathrm{cl}(\partial V - V_0)$ by a sequence of pseudo-cellular moves along the $(n+1)$-simplices $A_1^1, \dots, A_{r_1}^1, A_1^2, \dots, A_{r_2}^2, \dots, A_1^m, \dots, A_{r_m}^m$ in this order. □

Corollary 3.3. *Let (W, L) be homeomorphic to a standard disk pair (D^q, D^n). Then L is transformed into an n-disk L' contained in ∂W by a finite sequence of pseudo-cellular moves keeping ∂L fixed.*

Corollary 3.4. *If an n-manifold L in W^q is transformed into L' by a cellular move of type* 1, *then it is transformed into L' by a finite sequence of* $(n+1)$*-simplex moves of type* 1.

Proof. Let D be the $(n+1)$-disk in W^q along which L' is obtained from L by a cellular move of type 1, and put $D_0 = D \cap L$ and $D_1 = D \cap \partial W^q$. We notice that $\dim(D_1 \cap \partial W^q) \le n - 1$. Take a homeomorphism $f: (D, D_0) \to (V, V_0)$, where V and V_0 are as in Lemma 3.2. There exist simplicial complexes K_1 and K_2 such that $|K_1| = D$, $|K_2| = V$ and f is a simplicial map from K_1 to K_2. Let K_2' be a subdivision of K_2 as in Lemma 3.2 and K_1' be the corresponding subdivision of K_1. A sequence of pseudo-cellular moves as in Lemma 3.2 induces a sequence of $(n+1)$-simplex moves of type 1 transforming L into L'. □

Lemma 3.5. *Let (W, L) be homeomorphic to a standard disk pair (D^q, D^n) and $h: W \to W$ an orientation-preserving homeomorphism with $h|_{\partial L} = \mathrm{id}$. Then $h(L)$ is transformed into L by a finite sequence of* $(n+1)$*-simplex moves relative to ∂W.*

Proof. First we consider a special case that $h: W \to W$ is a homeomorphism with $h|_{\partial W} = \mathrm{id}$. Let $N(\partial W) \cong \partial W \times [0,1]$ be a collar neighborhood of ∂W in W with $\partial W \times \{0\} = \partial W$ and $N'(\partial W)$ the subset of $N(\partial W)$ corresponding to $\partial W \times [0, 1/2]$. Let $N(\partial L; \partial W) \cong \partial L \times D^{q-n}$ be a tubular neighborhood of ∂L in ∂W. For each point $y \in \partial L$, we denote by D_y^{q-n} the fiber of

$N(\partial L; \partial W) \cong \partial L \times D^{q-n}$ over y, which is a $(q-n)$-disk in ∂W. Let C_y and C'_y be the cones $(y \times \{0\}) * (D_y^{q-n} \times \{1\})$ in $\partial W \times [0,1]$ and $(y \times \{0\}) * (D_y^{q-n} \times \{1/2\})$ in $\partial W \times [0,1/2]$, respectively. Put $C = \bigcup_{y \in \partial L} C_y$ and $C' = \bigcup_{y \in \partial L} C'_y$. Let $M = \mathrm{cl}(N(\partial W) - C)$ and $M' = \mathrm{cl}(N'(\partial W) - C')$, which are collar neighborhoods of $\partial W - \partial L$ in W except ∂L. Since $\partial L = \partial h(L)$, taking $N(\partial W)$ to be sufficiently thin, we may assume that L and $h(L)$ restricted to $N(\partial W)$ are contained in C (and hence those restricted to $N'(\partial W)$ are in C'). Using the collar structure of M, one can isotope h so that $h|_{M'} = \mathrm{id}$. Put $B' = \mathrm{cl}(W - M')$, which is a q-disk such that $B' \cap \partial W = \partial L$ and the pair (B', L) is homeomorphic to a standard disk pair (D^q, D^n). By Corollary 3.3, the n-disk L is transformed into an n-disk L' contained in $\partial B'$ by a finite sequence of pseudo-cellular moves (in B') keeping ∂L fixed. This implies that L is transformed into L' in W by a finite sequence of cellular moves relative to ∂W and that $h(L)$ is transformed into $h(L')$ in W by a finite sequence of cellular moves relative to ∂W. Since $h(L') = L'$, L is transformed into $h(L)$ in W by a finite sequence of cellular moves relative to ∂W. By Corollary 3.4, we have the result in the case that $h|_{\partial W} = \mathrm{id}$.

Now we consider a general case that $h|_{\partial L} = \mathrm{id}$. We assert that $h|_{\partial W}$ is isotopic (in ∂W) to the identity map of ∂W keeping ∂L fixed. To see this, we use the following well-known fact due to Alexander (cf. [**1**, p. 161]).

Alexander's lemma: *If $f : B^m \to B^m$ is a homeomorphism from the conic m-disk to itself such that it keeps ∂B^m and a conic subset of B^m fixed, then f is isotopic to the identity map keeping ∂B^m and the conic subset fixed.* (A conic subset of B^m means a subset which is the cone from the origin over a subset of ∂B^m.)

Let $N(x)$ be a regular neighborhood of a point x of ∂L in the sphere ∂W. We may assume that $h|_{N(x)} = \mathrm{id}$. Identify the $(q-1)$-disk $B = \mathrm{cl}(\partial W - N(x))$ with the unit $(q-1)$-disk such that $B \cap \partial L$ is a conic subset of B. Since $h|_{\partial B \cup (B \cap \partial L)} = \mathrm{id}$, using Alexander's lemma, we see that $h|_{\partial W}$ is isotopic (in ∂W) to the identity map of ∂W keeping ∂L fixed. Let $\{g_t\}$ $(t \in [0,1])$ be an ambient isotopy of ∂W keeping ∂L fixed such that $g_1 = h|_{\partial W}$. Consider a collar neighborhood $N(\partial W) = \partial W \times [0,1]$ of ∂W in W with $\partial W = \partial W \times \{1\}$. We may assume that $L \cap N(\partial W) = \partial L \times [0,1] \subset \partial W \times [0,1] = N(\partial W)$. Define a homeomorphism $g : W \to W$ by

$$g(x) = \begin{cases} x & \text{for } x \in W - N(\partial W) \\ (g_t(x'), t) & \text{for } x = (x', t) \in \partial W \times [0,1] = N(\partial W). \end{cases}$$

Then $g(L) = L$ and $g|_{\partial W} = h|_{\partial W}$. Put $h' = h \circ g^{-1} : W \to W$, then $h'|_{\partial W} = \mathrm{id}$ and $h'(L) = h(L)$. From the previous case, we have the result. □

PROOF OF THEOREM 1.1. As stated before, it is sufficient to prove that (1) $\to$ (4). Suppose that L and L' are ambient isotopic by an ambient isotopy $\{h_t\}$ ($t \in [0,1]$) of W keeping Y fixed. There exists a finite sequence of compact proper locally flat n-manifolds $L = L_0, L_1, \dots, L_r = L'$ in W and q-disks $B_1^q, \dots, B_r^q$ in W^q such that for each i ($i = 1, \dots, r$), the pair $(B_i^q, L_{i-1} \cap B_i^q)$ is homeomorphic to the standard disk pair (D^q, D^n) or the half pair (D_+^q, D_+^n), and the n-manifold L_{i-1} is mapped to L_i by a proper move $f_i : W \to W$ supported by the q-disk B_i^q (see Theorem 6.2 and Remarks 6.2.1, 6.2.2, 6.2.4 of [**4**]). Moreover without loss of generality we may assume that if $B_i^q \cap \partial W^q$ is a PL $(q-1)$-face of B_i^q, say $B_i^{(q-1)}$, and $B_i^{(q-1)} \cap Y$ is not the empty set, then $\mathrm{cl}(B_i^{(q-1)} - (Y \cap B_i^{(q-1)}))$ is a PL $(q-1)$-face of B_i^q, say $B_i^{(q-1)'}$, and the pair $(B_i^{(q-1)'}, \partial L_{i-1} \cap B_i^{(q-1)'})$ is homeomorphic to the standard disk pair (D^{q-1}, D^{n-1}).

If $B_i^q \cap \partial W^q$ is empty or contained in Y, then by Lemma 3.5 we see that L_{i-1} is transformed into L_i by a finite sequence of $(n+1)$-simplex moves relative to ∂W^4. If $B_i^q \cap \partial W^q$ is not contained in Y, then $B_i^q \cap \partial W^q$ is a $(q-1)$-face $B_i^{(q-1)}$ of B_i^q, $\mathrm{cl}(B_i^{(q-1)} - (Y \cap B_i^{(q-1)}))$ is a $(q-1)$-face $B_i^{(q-1)'}$ of B_i^q, and the pair $(B_i^{(q-1)'}, \partial L_{i-1} \cap B_i^{(q-1)'})$ is homeomorphic to the standard disk pair (D^{q-1}, D^{n-1}). Notice that the restriction of the homeomorphism f_i to the $(q-1)$-disk $B_i^{(q-1)'}$ keeps $\partial B_i^{(q-1)'}$ fixed and maps the $(n-1)$-disk $\partial L_{i-1} \cap B_i^{(q-1)'}$ onto $\partial L_i \cap B_i^{(q-1)'}$. By Lemma 3.5, there exists a finite sequence of n-simplex moves in $B_i^{(q-1)'}$ transforming $\partial L_{i-1} \cap B_i^{(q-1)'}$ into $\partial L_i \cap B_i^{(q-1)'}$. Extending each n-simplex move to an $(n+1)$-simplex move in B_i^q, we have a finite sequence of $(n+1)$-simplex moves in W^q relative to Y which transforms $L_{i-1} \cap B_i^q$ into $L_i' \cap B_i^q$, where L_i' is a compact proper locally flat n-manifold in W^q with $L_i' \cap \mathrm{cl}(W^q - B_i^q) = L_{i-1} \cap \mathrm{cl}(W^q - B_i^q)$ and $\partial L_i' = \partial L_i$. Since a simplex move is a proper move, there exists an orientation-preserving homeomorphism $k_i : B_i^q \to B_i^q$ with $k_i(L_{i-1} \cap B_i^q) = L_i' \cap B_i^q$. Using this homeomorphism and f_i, we have an orientation-preserving homeomorphism $g_i : B_i^q \to B_i^q$ with $g_i(L_i' \cap B_i^q) = L_i \cap B_i^q$ and $g_i|_{\partial L_i'} = \mathrm{id}$. By Lemma 3.5 again, we see that L_i' is transformed into L_i by a finite sequence of $(n+1)$-simplex moves relative to ∂W^q. $\square$

4. Proof of Theorem 1.2.

Let G be an n-dimensional compact polyhedron in W^q such that $\dim(\Gamma) \le 1$ and $\dim(\Gamma \cap \partial W) \le 0$, where Γ is the non-locally flat point set of G. Let $K_0 \subset K$ be triangulations of $\Gamma \subset W^q$, $K_0' \subset K'$ first derived subdivisions and $K_0'' \subset K''$ second derived subdivisions. Let $N(\Gamma)$ be the derived neighborhood $|N(K_0''; K'')|$ of Γ. Let $v_1, \dots, v_s$ be vertices of K_0' which are vertices K_0, and $u_1, \dots, u_t$ the other vertices of K_0'. For each vertex v_i ($i = 1, \dots, s$), let $B_i = \overline{\mathrm{star}}(v_i; K'')$ and for each vertex u_j ($j = 1, \dots, t$), let D_j be $\overline{\mathrm{star}}(u_j; K'')$. Then

$N(\Gamma) = (\bigcup_{i=1}^{s} B_i) \cup (\bigcup_{j=1}^{t} D_j)$ and this is in a situation as in §2. In fact, put $W_1 = \mathrm{cl}(W^q - \bigcup B_i)$ and $\Gamma_1 = \Gamma \cap W_1$, which is the union of some proper simple arcs $e_1, \dots, e_t$ in W_1. Each vertex u_j $(j = 1, \dots, t)$ is on a unique edge e_j and D_j is a regular neigborhood of e_j in W_1. Put $W_2 = \mathrm{cl}(W^q - N(\Gamma))$ and $\partial_+ W_2 = \mathrm{cl}(\partial W_2 \cap \mathrm{int}(W^q))$.

Proof of Theorem 1.2. In the above situation, suppose that G is ambient isotopic to G' by an ambient isotopy $\{h_t\}$ $(t \in [0,1])$ of W^q keeping Γ and Y fixed. Without loss of generality, we may assume that $\{h_t\}$ preserves each B_i $(i = 1, \dots, s)$ and D_j $(j = 1, \dots, t)$ setwise and that h_1 restricted to $K''|_{N(\Gamma)}$ is a simplicial map. The intersection $G \cap W_2$ is a compact proper locally flat n-manifold in W_2 and it is ambient isotopic to $G' \cap W_2$ by the ambient isotopy $\{h_t\}$ restricted to W_2, which keeps $Y \cap W_2$ fixed. By Theorem 1.1, we have a finite sequence of $(n+1)$-simplex moves in W_2 relative to $Y \cap W_2$ carrying $G \cap W_2$ to $G' \cap W_2$. As stated in §2, the sequence of $(n+1)$-simplex moves induces a sequence of cellular moves in $\partial_+ W_2$ carrying $G \cap \partial_+ W_2$ to $G' \cap \partial_+ W_2$. By Theorem 1.1, each cellular move is replaced by a finite sequence of n-simplex moves in $\partial_+ W_2$.

Without loss of generality, we may assume that each n-simplex is contained in some $\partial_+ B_i$ or ∂D_j. Extending each n-simplex move to an $(n+1)$-simplex move conically with v_i in B_i or with w_j in D_j, we have a sequence of $(n+1)$-simplex moves which carries $G \cap N(\Gamma)$ to $G' \cap N(\Gamma)$. Thus we have a desired finite sequence of $(n+1)$-simplex moves with respect to $N(\Gamma)$.
□

5. Proof of Proposition 1.3.

Proof of Proposition 1.3. Let $\Gamma = \{x_1, \dots, x_s\}$. We claim that there exists a one-parameter family (parametrized by $u \in [0,1]$) of ambient isotopies $\{f_t^u\}$ $(t \in [0,1])$ of W^q keeping ∂W fixed such that $f_t^0(x_i) = h_t(x_i)$, $f_1^u(x_i) = x_i$ and $f_t^1 = \mathrm{id}$ for $i \in \{1, \dots, s\}$, $t \in [0,1]$ and $u \in [0,1]$. For each i $(i = 1, \dots, s)$, let $\beta_i : [0,1] \to W \times [0,1]$ be a path with $\beta_i(t) = (h_t(x_i), t)$ for $t \in [0,1]$, and let b_i^1 be the image of β_i. The images $b_1^1, \dots, b_s^1$ are mutually disjoint monotone arcs in $W \times [0,1]$ connecting points $(x_1, 0), \dots, (x_s, 0)$ of $W \times \{0\}$ to the corresponding points of $W \times \{1\}$, where a *monotone* arc means an arc intersecting $W \times \{t\}$ transversely for every $t \in [0,1]$. Let $b_1^0, \dots, b_s^0$ be the straight arcs in $W \times [0,1]$ connecting the same points as $b_1^1, \dots, b_s^1$. Using a level-preserving ambient isotopy of $W \times [0,1]$ keeping $\partial(W \times [0,1])$ fixed, we assume that b_j^1 is disjoint from b_i^0 for any distinct i and j. Let $\alpha_i : [0,1] \to W$ be a path determined from the trace of x_i by the ambient isotopy $\{h_t\}$ of W; i.e., $\alpha_i(t) = h_t(x_i)$

for $t \in [0,1]$. It is obtained from β_i by the projection $W \times [0,1] \to W$. Since W is simply connected, the path α_i is homotopic to the identity. Thus there is a one-parameter family $\{b_i^u\}$ $(u \in [0,1])$ of monotone arcs in $W \times [0,1]$ between b_i^0 and b_i^1 such that $\partial b_i^u = \partial b_i^1$ for $u \in [0,1]$. Let $\gamma_i : D^2 = [0,1] \times [0,1] \to W \times [0,1]$ be a map determined from $\{b_i^u\}$ so that the image $\gamma_i(u \times [0,1])$ is b_i^u for $u \in [0,1]$. Since $x_1, \dots, x_s$ are interior points of W, we may assume that the image of γ_i is disjoint from $\partial(W \times [0,1])$ except the end-points $(x_i, 0)$ and $(x_i, 1)$ of b_i^1. Since $q \geq 3$, the arcs $b_1^1, \dots, b_s^1$ (and $b_1^0, \dots, b_s^0$) are of codimension q submanifolds of $W \times [0,1]$ and we may assume that the image of γ_i is disjoint from b_j^1 (and b_j^0) for $j \in \{1, \dots, s\} - \{i\}$. Let K_i be a triangulation of D^2 such that γ_i is a simplicial map. Applying the cellular move lemma to the 2-simplices, we have a level-preserving ambient isotopy of $W \times [0,1]$ keeping $\partial(W \times [0,1])$ fixed such that b_i^1 is deformed into b_i^0 without moving the other arcs b_j^1 (and b_j^0), $j \in \{1, \dots, s\} - \{i\}$. Using this argument inductively, we have a level-preserving ambient isotopy of $W \times [0,1]$ keeping $\partial(W \times [0,1])$ fixed such that every b_i^1 $(i = 1, \dots, s)$ is deformed into b_i^0. Using this level-preserving ambient isotopy, we have a one-parameter family $\{f_t^u\}$ as in the claim.

Define an ambient isotopy $\{h_t'\}$ of W by

$$h_t' = \begin{cases} (f_{2t}^0)^{-1} h_{2t} & \text{for } t \in [0, 1/2] \\ (f_1^{2t-1})^{-1} h_1 & \text{for } t \in (1/2, 1], \end{cases}$$

then $h_1' = h_1$ and for each $t \in [0,1]$, $h_t'(x_i) = x_i$ $(i \in \{1, \dots, s\})$ and h_t' keeps Y fixed.

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