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Ahlfors functions on compact bordered Riemann surfaces

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Abstract. Let R be a compact bordered Riemann surface which is non-planar. We solve a conjecture posed by Gouma concerning the distribution of degrees of Ahlfors functions on R whose double is hyperelliptic. Besides we consider the problem when a linear transformation of an Ahlfors function on R is again an Ahlfors function. We give a necessary and sufficient condition for this problem when the degree of the Ahlfors function is maximal.

1. Introduction.

Let $\mathscr{S}$ be the set of Riemann surfaces R which is the interior of a compact bordered Riemann surface $\bar{R}$ of genus p with N (≥ 1) boundary components. For $R \in \mathscr{S}$ we shall denote by ϕ the canonical anti-conformal involution of $\hat{R}$ fixing the boundary ∂R, where $\hat{R}$ is the Schottky double [**2**] of R which is a compact Riemann surface of genus $g = 2p + N - 1$. Let $\boldsymbol{B}(R)$ be the set of bounded holomorphic functions $f(z)$ on R such that $|f(z)| \leq 1$ for all $z \in R$.

Given $a \in R$, a function $f_a \in \boldsymbol{B}(R)$ is called the Ahlfors function if $f_a'(a) = \sup\{\operatorname{Re} f'(a) \mid f \in \boldsymbol{B}(R)\}$. Here the derivative $f'(a)$ is evaluated with respect to a fixed holomorphic local coordinate z centered at a. By considering a linear transformation of $f_a(z)$, it is easy to see that $f_a(a) = 0$.

More generally, given $a, b \in R$ $(a \neq b)$ a function $f_{a,b} \in \boldsymbol{B}(R)$ is also called the Ahlfors function if $f_{a,b}(b) = \sup\{\operatorname{Re} f(b) \mid f \in \boldsymbol{B}(R), f(a) = 0\}$. For convenience sake we extend the definition of $f_{a,b}$ so that $f_{a,b} = f_a$ when $a = b$.

Ahlfors [**1**] showed that if $R \in \mathscr{S}$, then for any $a, b \in R$ the Ahlfors function $f_{a,b}$ is unique and that it gives an n-sheeted unlimited branched covering of R onto the unit disk Δ, where the integer n satisfies the inequality

$$(1) \qquad N \leq n \leq g + 1.$$

The number n is called the *degree* of the Ahlfors function f and is denoted by $\deg f$. The inequality (1) means that R is a disjoint union of the sets R_j

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$(j = N, \dots, g+1)$ where R_j is the subset $\{a \in R \mid \deg f_a = j\}$ of R. In view of this inequality there naturally arises the following question. What can be said about the distribution of the sets R_j in R when the surface R is of special type? When R is planar, i.e. $\rho = 0$, this problem becomes a trivial one, that is, the inequality (1) immediately gives $R = R_N$ or $\deg f_a = N$ for all $a \in R$. On the other hand, for non-planar surfaces, our knowledge about the above question is very incomplete.

Definition 1.1. Let $\mathscr{S}_H$ denote the set of Riemann surfaces $R \in \mathscr{S}$ whose double $\hat{R}$ is hyperelliptic of genus $g \geq 2$. Let $\mathscr{S}_L$ denote the set of Riemann surfaces $R \in \mathscr{S}$ with $g \geq 2$ which possesses an Ahlfors function f_a whose linear transformation is again an Ahlfors function f_b with some $b \neq a$ $(a, b \in R)$.

We summarize here some known facts about the degree of the Ahlfors function for non-planar surfaces with hyperelliptic double. In 1978 the author showed that a neighborhood of the set of the Weierstrass points of $\hat{R}$ in R is contained in the set R_{g+1} [**7**]. Also, in the same paper, the author constructed an example of a bordered surface $R \in \mathscr{S}_H$ of genus one with two boundary components such that the set R_2 has nonempty interior. This result easily implies the fact that the metric induced by the analytic capacity is not always real-analytic for non-planar surfaces [**7**]. Recently Gouma [**5**] showed that if $R \in \mathscr{S}_H$ is non-planar, then (i) $R = R_2 \cup R_{g+1}$ and (ii) R_{g+1} is a nonempty open subset in R.

Our first result concerns about the problem when a linear transformation of an Ahlfors function is also an Ahlfors function. Theorem 2.1 gives a necessary and sufficient condition for this problem in case the degree of the Ahlfors function is maximal, which extends our result [**7**] to non-planar surfaces. We next show by an example that $\mathscr{S}_H$ is a proper subset of $\mathscr{S}_L$ (Theorem 3.1).

By observing some examples of the case $(\rho, N) = (1, 1)$ with a help of computer graphics, Gouma stated a conjecture [**5**] that if $R \in \mathscr{S}_H$ is non-planar, then (i) R_2 is always nonempty, (ii) R_{g+1} consists of $g+1$ simply connected components and (iii) if the $g+1$ Weierstrass points of $\hat{R}$ contained in R are sufficiently close, then the region R_{g+1} is "very small". The main objective of this paper is to answer affirmatively to his conjectures (i) (Theorem 4.1) and (iii) (Main Theorem 5.1). In fact, our Theorem 5.1 is slightly stronger than the above conjecture (iii) in the sense that all Weierstrass points need not to be close together.

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2. Linear transformations of Ahlfors functions.

When $R \in \mathscr{S}$ is planar, Fay [3] gave a representation of Ahlfors functions by means of the Riemann theta function. Applying the trisecant formula to this representation, we obtained [7] a necessary and sufficient condition when a linear transformation of an Ahlfors function is also an Ahlfors function. In this section, we shall extend this result, by using duality argument, to the case where surfaces may have positive genus. Thus, we have eliminated the use of theta functions from the treatment of linear transformations of Ahlfors functions. For relevant properties of dual extremal problems on compact bordered Riemann surfaces, the reader is referred to [6].

Let Möb(Δ) denote the set of linear transformations mapping Δ onto itself and let $\mathrm{div}_K f$ denote the divisor of a meromorphic function or differential f on the set K.

First we recall the so-called dual extremal problem associated to Ahlfors functions: given $a, b \in R$, minimize the integral $\int_{\partial R} |\psi|$ among the differentials $\psi \in D_{a,b}(R)$ with residue 1 at b, where $D_{a,b}(R)$ is the set of meromorphic differentials $\psi \neq 0$ on $R \cup \partial R$ with $\mathrm{div}_{R \cup \partial R}\, \psi \geq -a - b$. Of course, when $a = b$, the condition on the singularity of the competing differential ψ must be modified. Indeed in this case we require that ψ is holomorphic except at a where it has a double pole of the form $\psi = z^{-2} + O(z^{-1})$ for the same local coordinate z used to evaluate the derivative at a.

Ahlfors [1] showed that if $R \in \mathscr{S}$, then for any $a, b \in R$ there exists a (not necessarily unique) differential $\psi_{a,b}$ extremal to the above problem. The differential $\psi_{a,b}$ is called a *Garabedian differential* associated to the Ahlfors function $f_{a,b}$. As in the case of Ahlfors function we write $\psi_a = \psi_{a,a}$ for short.

Definition 2.1. A function f on $R \cup \partial R$ is said to be *unitary* if $|f| = 1$ on ∂R. For $a \in R$ let $U_a(R)$ be the set of holomorphic unitary functions f on $R \cup \partial R$ vanishing at a. Given $a, b \in R$, a pair $(f, \psi) \in U_a(R) \times D_{a,b}(R)$ is said to be *positive* (respectively *strictly positive*) if $f\psi \geq 0$ (respectively $f\psi > 0$) on ∂R.

We summarize a result [1] which is crucial to our paper.

Lemma 2.1 (Ahlfors). *Let $R \in \mathscr{S}$. Then for any $a, b \in R$ the following hold*:
(a) $(f_{a,b}, \psi_{a,b}/\sqrt{-1}) \in U_a(R) \times D_{a,b}(R)$ *and it is positive.*
(b) *If $(f, \psi) \in U_a(R) \times D_{a,b}(R)$ is positive then $f = \varepsilon f_{a,b}$ for some constant ε of absolute value one.*

Lemma 2.2. *Let $R \in \mathscr{S}$. Then there exists a linear transformation $T \in$ Möb(Δ) such that $f_{c,d} = T \circ f_{a,b}$ if and only if there exists a meromorphic function $h \neq 0$ on $R \cup \partial R$ such that $\mathrm{div}_{R \cup \partial R}\, h\psi_{a,b} \geq -c - d$ and $h \geq 0$ on ∂R.*

Proof. Note that when $|f| = 1$, the identity

$$\frac{f-\alpha}{1-\bar{\alpha}f}\cdot(1-\bar{\alpha}f)^2\psi = |f-\alpha|^2 f\psi \tag{2}$$

holds.

If $f_{c,d} = \varepsilon(f_{a,b}-\alpha)/(1-\bar{\alpha}f_{a,b})$ for some $|\varepsilon| = 1$ and $|\alpha| < 1$, then setting $h = (\varepsilon\psi_{c,d})/((1-\bar{\alpha}f_{a,b})^2\psi_{a,b})$ we see that h is a function meromorphic on $R\cup\partial R$ with the divisor $\mathrm{div}_{R\cup\partial R}\, h\psi_{a,b} = \mathrm{div}_{R\cup\partial R}\,\psi_{c,d} \geq -c-d$. Using the identity (2) we have $h = (f_{c,d}\psi_{c,d})/(|f_{a,b}-\alpha|^2 f_{a,b}\psi_{a,b}) \geq 0$ on ∂R by Lemma 2.1 (a).

Conversely, assume that h is a function satisfying the properties stated in Lemma. Setting $\alpha = f_{a,b}(c)$, $T(z) = (z-\alpha)/(1-\bar{\alpha}z)$ and $\psi = h\cdot(1-\bar{\alpha}f_{a,b})^2\psi_{a,b}$ we see from (2) the pair $(T\circ f_{a,b}, \psi) \in U_c(R)\times D_{c,d}(R)$ is positive. Hence from Lemma 2.1 (b) we have $f_{c,d} = \varepsilon T\circ f_{a,b}$ for some constant ε of absolute value one. $\square$

Let $n(a,b)$ (≥ 0) be the degree of the positive divisor $a+b+\mathrm{div}_{R\cup\partial R}\,\psi_{a,b}$, where the zeros on ∂R are counted with a half of its multiplicity. Although $\psi_{a,b}$ is not unique in general, the number $n(a,b)$ is determined uniquely as follows.

Lemma 2.3. *For $R\in\mathscr{S}$ the following identity holds.*

$$\deg f_{a,b} + n(a,b) = g+1.$$

Proof. Lemma 2.1 shows that by reflection the Ahlfors function $f_{a,b}$ and a Garabedian differential $\psi_{a,b}$ are both extended meromorphically to the double $\hat{R}$. Thus, their divisors have the form

$$\begin{aligned}\mathrm{div}_{\hat{R}}\, f_{a,b} &= a + \mathscr{A} - \phi(a) - \phi(\mathscr{A}),\\ \mathrm{div}_{\hat{R}}\,\psi_{a,b} &= -a-b+\mathscr{B}+\phi(a)-\phi(b)+2\phi(\mathscr{A})+\phi(\mathscr{B}),\\ \mathrm{div}_{\hat{R}}\, f_{a,b}\psi_{a,b} &= -b+\mathscr{A}+\mathscr{B}-\phi(b)+\phi(\mathscr{A})+\phi(\mathscr{B}),\end{aligned}$$

where $\mathscr{A}$ and $\mathscr{B}$ are positive divisors on $R\cup\partial R$ defined by $\mathscr{A} = \mathrm{div}_{R\cup\partial R}\, f_{a,b} - a$ and $\mathscr{B} = a+b+\mathrm{div}_{R\cup\partial R}\,\psi_{a,b}$. Since the total degree of an Abelian differential is $2g-2$, we have $\deg\mathscr{A}+\deg\mathscr{B} = g$. Noting that $\deg f_{a,b} = \deg\mathscr{A}+1$ and $n(a,b) = \deg\mathscr{B}$, we obtain the desired identity. $\square$

Theorem 2.1. *Let a,b,c,d be points in $R\in\mathscr{S}$. Assume that the degree of the Ahlfors function $f_{a,b}$ is maximal i.e. $\deg f_{a,b} = g+1$. Then there exists a linear transformation $T\in \mathrm{Möb}(\Delta)$ such that $f_{c,d} = T\circ f_{a,b}$ if and only if there exists a meromorphic function $h\neq 0$ on $R\cup\partial R$ such that $\mathrm{div}_{R\cup\partial R}\, h = a+b-c-d$ and $h>0$ on ∂R.*

PROOF. Since $\deg f_{a,b} = g+1$, we have $\mathrm{div}_{R\cup\partial R}\,\psi_{a,b} = -a-b$ by Lemma 2.3. Thus $\mathrm{div}_{R\cup\partial R}\,h\psi_{a,b} \geq -c-d$ if and only if $\mathrm{div}_{R\cup\partial R}\,h \geq a+b-c-d$. If h satisfies an inequality $h \geq 0$ on ∂R, then by reflection h is extended to a meromorphic function on $\hat{R}$ whose divisor is symmetric about ∂R. Since the total degree of the divisor of h on $\hat{R}$ is zero, the degree of $\mathrm{div}_{R\cup\partial R}\,h$ is also zero. Thus $\mathrm{div}_{R\cup\partial R}\,h \geq a+b-c-d$ and $h \geq 0$ on ∂R if and only if $\mathrm{div}_{R\cup\partial R}\,h = a+b-c-d$ and $h > 0$ on ∂R. In view of Lemma 2.2 this completes the proof. □

REMARK 2.1. If $R \in \mathscr{S}$ is planar, then by (1) we see that the degree of the Ahlfors function is always maximal.

3. Example.

Theorem 2.1 shows that if there exist points $a, b \in R$ $(a \neq b)$ such that $f_b = \tau \circ f_a$ for some $\tau \in \mathrm{M\ddot{o}b}(\Delta)$ and if f_a is of maximal degree, then the double $\hat{R}$ possesses a nonconstant meromorphic function of degree four. In view of this it is of some interest to compare the sets $\mathscr{S}_L$ and $\mathscr{S}_H$. In fact we show that the set $\mathscr{S}_L$ is strictly larger than the set $\mathscr{S}_H$ by providing an example.

Consider the compact Riemann surface $\hat{R}$ defined by the algebraic equation

$$y^n = \prod_{j=1}^{m} \frac{x-\alpha_j}{1-\overline{\alpha_j}x} \tag{3}$$

where $\{\alpha_j\}_{j=1}^{m}$ is a set of distinct points in Δ and m and n are integers with

$$m > n \geq 2. \tag{4}$$

Let R be the open subset $\{p \in \hat{R} \,|\, |x(p)| < 1\}$ of $\hat{R}$ and define the mapping $\phi : \hat{R} \to \hat{R}$ by $\phi(x,y) = (1/\bar{x}, 1/\bar{y})$. Then it is clear that (i) $R \in \mathscr{S}$, (ii) $\hat{R}$ is the Schottky double of R and (iii) ϕ is the canonical anti-conformal involution of $\hat{R}$ fixing ∂R. Since the total branching number of the meromorphic function x is $2m(n-1)$, we see from the Riemann-Hurwitz relation [**4**] that $g = (m-1)(n-1) \geq 2$ and $N = (m,n)$ (gcd). The degree of the function x, y is given by $\deg x = n$, $\deg y = m$. Moreover, we find that a basis for the space of holomorphic differentials on $\hat{R}$ is given by

$$\frac{x^k\,dx}{y^j \prod_{l=1}^{m}(1-\overline{\alpha_l}x)} \quad (j = 1, \ldots, n-1; k = 0, \ldots, m-2). \tag{5}$$

We remark that $\hat{R}$ has an automorphism J defined by

$$J(x,y) = (x, \exp(2\pi i/n)y) \tag{6}$$

which is of order n and $\sharp\mathrm{Fix}(J) = 2m$, where $\mathrm{Fix}(J)$ denotes the set of fixed points of J.

The following Lemma is a slight extension of the fact well known for hyperelliptic surfaces.

Lemma 3.1. *If f is a nonconstant meromorphic function on $\hat{R}$ with $\deg f < m$, then f is a rational function of x and $\deg f$ is a multiple of n. If $\deg f = n$, then f is a linear transformation of x.*

Proof. From (4) we have an inequality $\sharp\mathrm{Fix}(J) > 2\deg f$. Then it follows from [**4**, Proposition V.1.4] that f is J-invariant, i.e. $f \circ J = f$. On the quotient surface $\hat{R}/\langle J\rangle$ the functions f and x project to well-defined functions $\tilde{f}$ and $\tilde{x}$. Since $\deg\tilde{x} = 1$, $\hat{R}/\langle J\rangle$ is of genus 0 and $\tilde{f}$ is a rational function of $\tilde{x}$. Thus there exists a rational function q with $f = q \circ x$. This implies $\deg f = \deg q \deg x$ so that $\deg f$ is a multiple of n. If $\deg f = n$, then $\deg q = 1$ so that q is a linear transformation. □

Corollary 3.1. *The surface $\hat{R}$ is hyperelliptic if and only if $n = 2$.*

Now we consider the special case where $\alpha_j = re^{(2j+1)\pi i/m}$ $(0 < r < 1)$ for $j = 1, \dots, m$, so that the equation of the double $\hat{R}$ is given by

$$y^n = \frac{x^m + r^m}{1 + r^m x^m}. \tag{7}$$

Let us write $x^{-1}(0) = \{O_1, \dots, O_n\}$ where O_j is the point $(0, r^s e^{2j\pi i/n}) \in R$ with $s = m/n$ $(j = 1, \dots, n)$.

Lemma 3.2. *Let ψ be a differential on $\hat{R}$ given by*

$$\psi = \left\{1 + \sum_{j=1}^{n-1} \frac{r^{js} + r^{(n-j)s}x^m}{y^j(1 + r^m x^m)}\right\} \frac{dx}{ix^2}.$$

Then, for r sufficiently small, the pair $(x, \psi) \in U_{O_n}(R) \times D_{O_n, O_n}(R)$ is strictly positive.

Proof. Clearly x is a unitary function vanishing at O_n. On ∂R, using $|x| = |y| = 1$ and (7), we have

$$\begin{aligned}\frac{x\psi}{dx/(ix)} &= 1 + \sum_{j=1}^{n-1} \frac{y^{n-j}(r^{js} + r^{(n-j)s}x^m)}{x^m + r^m} = 1 + \sum_{j=1}^{n-1} \frac{r^{(n-j)s} + r^{js}\bar{x}^m}{\bar{y}^{n-j}(1 + r^m\bar{x}^m)} \\ &= 1 + \mathrm{Re}\left\{\sum_{j=1}^{n-1} \frac{r^{js} + r^{(n-j)s}x^m}{y^j(1 + r^m x^m)}\right\} \geq 1 - \sum_{j=1}^{n-1} \frac{r^{js} + r^{(n-j)s}}{1 - r^m}.\end{aligned}$$

Since the last sum tends to 0 as $r \to 0$ and the differential $dx/(ix) > 0$ on ∂R, we see that $x\psi$ is strictly positive for r sufficiently small.

Next we show that $\psi \in D_{O_n, O_n}(R)$. However, it is easy to see that ψ is regular on $R \cup \partial R$ unless $x = 0$. In a neighborhood of $x^{-1}(0)$ we have

$$\psi = \left(1 + \sum_{j=1}^{n-1} r^{js}/y^j + O(x^m)\right) \frac{dx}{ix^2}.$$

Noting that (7) implies $y = r^s e^{2j\pi i/n} + O(x^m)$ near O_j $(j = 1, \dots, n)$ and using the identity $y^n - r^m = (y - r^s)y^{n-1}(1 + \sum_{j=1}^{n-1} r^{js}/y^j)$ we have

$$\operatorname{div}_{R \cup \partial R} \psi \geq -2O_n + (m-2)O_1 + \cdots + (m-2)O_{n-1}.$$

Thus $\psi \in D_{O_n, O_n}(R)$ because $m > 2$ by the inequality (4). □

THEOREM 3.1. *For every integer $n \geq 3$ there exist a non-planar Riemann surface $R \in \mathscr{S} \backslash \mathscr{S}_H$ and a nonempty open set $U \subset R_n$ such that f_b is a linear transformation of f_a for every $a, b \in U$. In particular, the set $\mathscr{S}_H$ is a proper subset of $\mathscr{S}_L$.*

PROOF. Choose an integer $n \geq 3$ and, for r sufficiently small, let R be the surface considered above whose double is given by (7). Then Corollary 3.1 implies that $R \notin \mathscr{S}_H$. Now we show that $R \in \mathscr{S}_L$. By using x itself as the local coordinate centered at O_n (with which the derivatives at O_n are evaluated) it follows from Lemmas 2.1 and 3.2 that x is the Ahlfors function at O_n. Moreover, since $x\psi$ is strictly positive by Lemma 3.2, [**7**, Lemma 4] implies that there exists a neighborhood U of O_n such that $\deg f_a = n$ for all $a \in U$. From Lemma 3.1 we conclude that f_a is a linear transformation of x for $a \in U$. In particular, $R \in \mathscr{S}_L$. That $\mathscr{S}_H$ is contained in $\mathscr{S}_L$ is clear from [**7**, Theorem 2]. □

REMARK 3.1. Let $\mathscr{S}_0$ be the set of surfaces $R \in \mathscr{S}$ which is planar, then we know that $\mathscr{S}_H \cap \mathscr{S}_0 = \mathscr{S}_L \cap \mathscr{S}_0$ [**7**].

4. Surfaces with hyperelliptic double.

From now on (to the end of the paper) we assume that all surfaces $R \in \mathscr{S}$ are non-planar with hyperelliptic double unless otherwise stated. Then it is known that the number N of components of ∂R is one or two [**7**] and that, by applying a conformal mapping, R may be regarded as the open subset $\{p \in \hat{R} \mid |x(p)| < 1\}$ of the compact Riemann surface $\hat{R}$ expressed by the algebraic equation of the form

$$(8) \qquad Y^2 = \prod_{j=1}^{g+1} (x - \alpha_j)(1 - \overline{\alpha_j} x), \quad |\alpha_j| < 1 \ (j = 1, \dots, g+1)$$

[5, Lemma 2]. We denote by $\mathscr{B}_k$ the set of subsets of Δ consisting of k distinct points. Then we have $\{\alpha_j\}_{j=1}^{g+1} \in \mathscr{B}_{g+1}$ which is called a *branch parameter* of R.

The equation (8) is birationally equivalent to (3) with $n = 2$ and $m = g+1$. Hence we may also assume that R is the subset $x^{-1}(\Delta)$ of the compact Riemann surface $\hat{R}$ given by an equation of the form

$$y^2 = \prod_{j=1}^{g+1} \frac{x - \alpha_j}{1 - \overline{\alpha_j} x}. \tag{9}$$

Note that $g \geq 2$ because R is non-planar. As in (5) a basis for the space of holomorphic differentials on $\hat{R}$ is given by $x^k\,dx/(y\prod_{j=1}^{g+1}(1-\overline{\alpha_j}x))$ $(k = 0, \dots, g-1)$.

By conformal invariance (up to a constant of absolute value one) of the Ahlfors functions, the set R_j is J-invariant where J denotes the hyperelliptic involution on $\hat{R}$ defined by (6) with $n = 2$. Thus we may identify the set R_j with the subset $x(R_j)$ of Δ, which we also denote by R_j. When we need to distinguish a branch parameter $\alpha \in \mathscr{B}_{g+1}$ of R, we use the notation $R(\alpha)$, $\hat{R}(\alpha)$ and $R_j(\alpha)$ in the obvious meaning.

For our application it is convenient to rewrite Lemma 2.1 as follows.

LEMMA 4.1. *Assume that f is a holomorphic function on $R \cup \partial R$ with $R \in \mathscr{S}$. Then f is the Ahlfors function f_a up to a multiplicative constant of absolute value one if and only if the following conditions* (i)–(iii) *are satisfied*:

(i) $f(a) = 0$,

(ii) $|f| = 1$ *on* ∂R,

(iii) *there exists a meromorphic differential* $\psi \neq 0$ *on* $R \cup \partial R$ *such that* $\operatorname{div}_{R\cup\partial R}\psi \geq -2a + \operatorname{div}_{R\cup\partial R} f$ *and* $\psi \geq 0$ *on* ∂R.

PROOF. First assume that $f = \varepsilon f_a$ $(|\varepsilon| = 1)$. Then by (a) in Lemma 2.1 the conditions (i)–(iii) are trivial if we choose $\psi = f_a\psi_a/\sqrt{-1}$. Conversely, assume that the conditions (i)–(iii) hold. Then the pair $(f, \psi/f) \in U_a \times D_{a,a}$ is positive. From (b) in Lemma 2.1 we have $f = \varepsilon f_a$ $(|\varepsilon| = 1)$. □

DEFINITION 4.1. For $n \in \boldsymbol{N} \cup \{0\}$ we denote by $\mathscr{P}_n$ the $\boldsymbol{R}$-linear space of polynomials p such that $L_n(p) = p$ where $L_n(p)(x) = x^n\overline{p(1/\bar{x})}(= x^n\overline{\phi^*}p(x))$.

Note that $p \in \mathscr{P}_n$ if and only if $p(x)$ is a polynomial of the form $p(x) = \sum_{k=0}^{n} c_k x^k$ with $\overline{c_k} = c_{n-k}$ for all $k = 0, \dots, n$. We list now some elementary properties about the set $\mathscr{P}_k$.

LEMMA 4.2. *Let p and q be polynomials. Then, for $m, n \in \boldsymbol{N} \cup \{0\}$,*

(i) $p \in \mathscr{P}_n$ *if and only if* $p(x)/x^{n/2} \in \boldsymbol{R}$ *for all* $|x| = 1$.

(ii) *If* $\deg p \leq n$ *then* $pL_n(p) \in \mathscr{P}_{2n}$.

(iii) *Assume* $p \in \mathscr{P}_m$ *and* $p \neq 0$. *Then* $pq \in \mathscr{P}_{m+n}$ *if and only if* $q \in \mathscr{P}_n$.

Proof. Easy. □

Lemma 4.3. *Let $a \in R$ and let ω be a meromorphic differential on $\hat{R}$. Then*
(i) *ω is holomorphic on R and real on ∂R if and only if ω has the form*

$$\frac{p(x)\,dx}{iy\prod_{j=1}^{g+1}(1-\overline{\alpha_j}x)}$$

with $p \in \mathscr{P}_{g-1}$.
(ii) *ω is real on ∂R and satisfies $\mathrm{div}_{R\cup\partial R}\,\omega \geq -a - J(a)$ if and only if ω has the form*

$$\frac{c\,dx}{i(x-x_a)(1-\overline{x_a}x)} + \frac{p(x)\,dx}{i(x-x_a)(1-\overline{x_a}x)y\prod_{j=1}^{g+1}(1-\overline{\alpha_j}x)} \tag{10}$$

where $c \in \boldsymbol{R}$, $p \in \mathscr{P}_{g+1}$ and $x_a = x(a)$.

Proof. A differential ω is holomorphic on $\hat{R}$ if and only if ω is of the form

$$\frac{p(x)\,dx}{iy\prod_{j=1}^{g+1}(1-\overline{\alpha_j}x)}$$

where $p(x)$ is a polynomial of degree $\leq g-1$. In view of the identities $\overline{\phi^*}(dx) = -x^{-2}\,dx$ and $\overline{\phi^*}(y\prod_{j=1}^{g+1}(1-\overline{\alpha_j}x)) = x^{-g-1}y\prod_{j=1}^{g+1}(1-\overline{\alpha_j}x)$, we have

$$\overline{\phi^*}\left(\frac{p(x)\,dx}{iy\prod_{j=1}^{g+1}(1-\overline{\alpha_j}x)}\right) = \frac{(L_{g-1}p)(x)\,dx}{iy\prod_{j=1}^{g+1}(1-\overline{\alpha_j}x)}.$$

Hence $\overline{\phi^*}\omega = \omega$ if and only if $p \in \mathscr{P}_{g-1}$. This proves (i).

Since the differential $dx/(i(x-x_a)(1-\overline{x_a}x))$ is positive, it is easily verified as above that the differential of the form (10) is real and its divisor satisfies $\mathrm{div}_{R\cup\partial R} \geq -a - J(a)$. From the Riemann-Roch theorem and the symmetry ϕ, it is seen that the real-dimension of the space of such differentials is $g+3$, which is the same as the real-dimension of the space of differentials of the form (10). This completes the proof of (ii). □

Lemma 4.4. *For any $a \in R$, the differential*

$$\eta_a = \frac{(y+y_a)(1+\overline{y_a}y)\,dx}{(x-x_a)(1-\overline{x_a}x)iy}$$

is strictly positive on ∂R and is a meromorphic differential of the third kind on $\hat{R}$ with simple poles at a and $\phi(a)$ where $x_a = x(a)$ and $y_a = y(a)$.

PROOF. The possible singularity of η_a occurs at the Weierstrass points, a, $J(a)$, $\phi(a)$ and $\phi(J(a))$. From the identity $y \circ J = -y$, it is easy to see that η_a is regular analytic on $\hat{R}$ except for simple poles only at a and $\phi(a)$. Since $|y| = |x| = 1$ on ∂R, for $p \in \partial R$ we have

$$\eta_a(p) = \frac{|y(p) + y_a|^2\, dx}{|x(p) - x_a|^2 ix}.$$

Thus η_a is strictly positive. □

THEOREM 4.1. *If $R \in \mathcal{S}_H$ is non-planar, then R_2 contains a neighborhood of the critical points of the function y. In particular $R_2 \neq \emptyset$.*

PROOF. Let E be the set of critical points of y in $R \cup \partial R$. Since y is unitary, E is a subset of R. We show next that $E \neq \emptyset$. Observe that the differential dy has zeros at $E \cup \phi(E)$ and has double poles at the $g+1$ Weierstrass points of $\hat{R}$ in $\phi(R)$. Since the divisor of dy on $\hat{R}$ has degree $2g-2$, we see easily that the function y has, counting multiplicities, $2g$ (≥ 4) critical points in R. Thus $E \neq \emptyset$.

For any $a \in E$ the function $y + y_a$ has at least double zero at $J(a)$. Hence, $\operatorname{div}_{R\cup\partial R} \eta_a \geq J(a) - a$. Putting $f = (x - x_a)/(1 - \overline{x_a}x)$ and applying Lemmas 4.1 and 4.4 we see that $f_a = \varepsilon f$ $(|\varepsilon| = 1)$, which implies that $a \in R_2$. Thus R_2 contains E. Since η_a is strictly positive for all $a \in E$, by [**7**, Lemma 4] R_2 contains some neighborhood of E. □

LEMMA 4.5. *For $a \in R$ the following conditions are equivalent*:

(i) $a \in R_2$.

(ii) *There exists a polynomial $p \in \mathcal{P}_{g-1}$ such that the differential $\omega = \eta_a + (p(x)\,dx)/(iy\prod_{j=1}^{g+1}(1 - \overline{\alpha_j}x))$ vanishes at $J(a)$ and ω is positive on ∂R.*

(iii) *There exists a polynomial $p \in \mathcal{P}_{g+1}$ such that $Y^2 = p(x)^2 + O((x - x_a)^2)$ near $x = x_a (= x(a)) \in \Delta$ and $|Y^2| \geq |p(x)|^2$ on $\partial\Delta$ where $Y^2 \in \mathcal{P}_{2g+2}$ is the polynomial of x given by* (8).

PROOF. By Lemma 3.1 every meromorphic function on $\hat{R}$ of degree two is a linear transformation of the function x. In view of Lemma 4.1, we see that $\deg f_a = 2$ if and only if there exists a meromorphic differential $\omega \neq 0$ such that ω is positive on ∂R and satisfies $\operatorname{div}_{R\cup\partial R} \omega \geq J(a) - a$. Since there exists no nontrivial holomorphic positive differential, we remark that $a \in R_2$ implies that $J(a) \neq a$, i.e. a is not a Weierstrass point.

Multiplying ω with a suitable positive constant we may assume that $\omega - \eta_a$ is regular on R. Since $\omega - \eta_a$ is real by Lemma 4.4, it is regarded as a holomorphic differential on $\hat{R}$. In view of Lemma 4.3 this established the equivalence of (i) and (ii).

To show that (i) and (iii) are equivalent, we make use of (ii) of Lemma 4.3. Now we know that the differential ω is of the form $\omega = \omega_1 + \omega_2$ where

$$\omega_1 = \frac{c\,dx}{i(x - x_a)(1 - \overline{x_a}x)}, \quad \omega_2 = \frac{p(x)\,dx}{i(x - x_a)(1 - \overline{x_a}x)\,Y}$$

with $c \in \boldsymbol{R}$ and $p \in \mathscr{P}_{g+1}$. Since $Y \circ J = -Y$ the sum of the residues of ω_2 at a and $J(a)$ vanishes. Thus by the residue theorem we have $c = (1 - |x_a|^2)/(4\pi) \int_{\partial R} \omega > 0$. Therefore we may assume that $c = 1$ and ω is of the form

$$\omega = \frac{(1 + p(x)/Y)\,dx}{i(x - x_a)(1 - \overline{x_a}x)}.$$

Since the differential $dx/(i(x - x_a)(1 - \overline{x_a}x))$ is strictly positive, ω is positive if and only if $1 + p(x)/Y \geq 0$ on ∂R. Since ∂R is J-invariant and $Y \circ J = -Y$, this is equivalent to the inequality $1 \geq |p(x)/Y|$ on ∂R or $|Y^2| \geq |p(x)|^2$ on $\partial \Delta$. On the other hand, we show next that $\operatorname{div}_{R \cup \partial R} \omega \geq J(a) - a$ with $J(a) \neq a$ if and only if $Y^2 - p(x)^2 = O((x - x_a)^2)$ near $x = x_a$. In fact, $\operatorname{div}_{R \cup \partial R} \omega \geq J(a) - a$ with $J(a) \neq a$ if and only if $Y + p(x) = O((x - x_a)^2)$ near $J(a)$ $(\neq a)$, which easily implies that $Y^2 - p(x)^2 = O((x - x_a)^2)$ near $x = x_a$. Conversely, assume that $Y^2 - p(x)^2 = O((x - x_a)^2)$. Then we claim that $Y(a) \neq 0$. If this were not the case, then we would have $p(x_a) = 0$. Thus $p(x)^2 = O((x - x_a)^2)$, which implies that $Y^2 = O((x - x_a)^2)$. However, this contradicts the definition of Y^2. Thus $Y(a) \neq 0$. Therefore $Y + p(x)$ and $Y - p(x)$ cannot vanish simultaneously at $J(a)$. Hence by taking $-p(x)$ for $p(x)$, if necessary, we conclude from the identity $Y^2 - p(x)^2 = (Y + p(x))(Y - p(x))$ that $Y + p(x) = O((x - x_a)^2)$ near $J(a)$ $(\neq a)$. This completes the proof of the equivalence of (i) and (iii). □

Remark 4.1. Lemma 4.5 (iii) shows that the problem on the degrees of Ahlfors functions on a surface with hyperelliptic double is reduced to one on polynomials in $\mathscr{P}_{g+1}$ on the closed unit disk.

To show that $R_2(\alpha)$ is "big" we will decompose a general δ-nonisolated set (c.f. Definition 5.1) into a union of such sets with simpler types. This is assured by the following

Lemma 4.6. *If $\alpha \in \mathscr{B}_{g+1}$ and $\alpha' \in \mathscr{B}_{g'+1}$ are branch parameters with $\alpha \cap \alpha' = \emptyset$, then we have $R_2(\alpha \cup \alpha') \supset R_2(\alpha) \cap R_2(\alpha')$ (as subsets of Δ).*

Proof. Lemma 4.5 (iii) implies that if $a \in R_2(\alpha) \cap R_2(\alpha')$, then there exist polynomials $p \in \mathscr{P}_{g+1}$ and $q \in \mathscr{P}_{g'+1}$ such that

$$Y^2 = p(x)^2 + O((x - x_a)^2), \quad |Y^2| \geq |p(x)|^2 \text{ on } \partial \Delta \tag{11}$$

$$Z^2 = q(x)^2 + O((x - x_a)^2), \quad |Z^2| \geq |q(x)|^2 \text{ on } \partial \Delta \tag{12}$$

where $Z^2 = \prod_{j=1}^{g'+1}(x - \alpha_j')(1 - \overline{\alpha_j'}x)$ with $\alpha' = \{\alpha_j'\}$. Multiplying (11) and (12) we see immediately that $a \in R_2(\alpha \cup \alpha')$, for $pq \in \mathscr{P}_{g+g'+2}$ by Lemma 4.2. $\square$

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \boldsymbol{C}^n$ we define $|\alpha| = \max\{|\alpha_1|, \dots, |\alpha_n|\}$ and write $0 = (0, \dots, 0) \in \boldsymbol{C}^n$ for short. Also for $a \in \boldsymbol{C}$ and $r > 0$ the open disk of radius r with center at a is denoted by $\Delta(a, r) = \{z \in \boldsymbol{C} \mid |z - a| < r\}$.

Lemma 4.7. *For $n \in \boldsymbol{N}$ let $\Phi : \Delta^n \times A \times X \to \boldsymbol{C}$ be a continuous function, where A is a compact subset of $\boldsymbol{C}$ such that $\{r_0 \le |z| \le 1\} \subset A \subset \bar{\Delta}$ with $0 < r_0 < 1$ and X is a neighborhood of $\partial\Delta$. Let E be either $\varnothing$ or $\partial\Delta$. Assume that, for some $k \in \boldsymbol{N}$, $\Phi_\alpha = \Phi(\alpha, \cdot, \cdot)$ satisfies the following four conditions*:

(i) *For any $(\alpha, a) \in \Delta^n \times A$, $\Phi_\alpha(a, \cdot)$ is a holomorphic function on X which is real-valued on $\partial\Delta$,*

(ii) *For any $(\alpha, a) \in \Delta^n \times E$ and r with $r_0 < r < 1$, the function $\Phi_\alpha(ra, \cdot)$ has at least k zeros, counting multiplicities, in $\Delta(a, 2(1 - r)) \backslash \partial\Delta$,*

(iii) *$\Phi_0 > 0$ on the set $A \times \partial\Delta \backslash \{(x, x) \mid x \in E\}$,*

(iv) *For any $a \in E$ there exist a neighborhood V_a of a in $\partial\Delta$ and a constant $\beta = \beta(a) > 0$ such that $\Phi_0(a, x) \ge \beta|x - a|^k$ for all x in V_a.*

Then there exists a constant $\delta > 0$ such that $\Phi_\alpha(a, x) > 0$ for all $|\alpha| < \delta$ and for all $(a, x) \in (A \backslash \partial\Delta) \times \partial\Delta$.

Proof. First, we claim that there exists a positive constant δ such that if $|\alpha| < \delta$, then $\Phi_\alpha(a, x) \ne 0$ for all $(a, x) \in (A \backslash \partial\Delta) \times \partial\Delta$. We show this by contradiction. If our claim were false, then there would exist sequences $\{\alpha_n\} \subset \Delta^n$, $\{a_n\} \subset A \backslash \partial\Delta$ and $\{x_n\} \subset \partial\Delta$ such that $\alpha_n \to 0$ $(n \to \infty)$ and $\Phi_{\alpha_n}(a_n, x_n) = 0$. Since both the sets A and $\partial\Delta$ are compact, we may assume without loss of generality that $a_n \to a$ and $x_n \to x$ $(n \to \infty)$ for some points $a \in A$ and $x \in \partial\Delta$. Then by continuity we conclude that $\Phi_0(a, x) = \lim_{n\to\infty} \Phi_{\alpha_n}(a_n, x_n) = 0$. If $E = \varnothing$, then (iii) implies that $\Phi_0(a, x) > 0$ which immediately gives a contradiction. On the other hand if $E = \partial\Delta$, then from (iii) we see that $a = x \in \partial\Delta$. Since continuity implies that Φ is uniformly continuous on compact sets we see by (i) that the sequence of holomorphic functions $\{\Phi_{\alpha_n}(a_n, \cdot)\}$ converges uniformly to $\Phi_0(a, \cdot)$ on a compact neighborhood of $x = a$. The condition (iv) implies that the order of $\Phi_0(a, \cdot)$ at $x = a$ is at most k. Thus by Rouché's theorem the function $\Phi_{\alpha_n}(a_n, \cdot)$, for sufficiently large n, has at most k zeros, counting multiplicities, in a neighborhood V of $x = a$. The condition (ii), however, implies that $\Phi_{\alpha_n}(a_n, \cdot)$, for sufficiently large n, has k zeros in $V \backslash \partial\Delta$. Thus we conclude that $\Phi_{\alpha_n}(a_n, \cdot)$ has no zeros on $V \cap \partial\Delta$. This, however, contradicts the fact that $\Phi_{\alpha_n}(a_n, x_n) = 0$ and $x_n \in V \cap \partial\Delta$ for sufficiently large n. Hence our claim is proved.

For any fixed $(\alpha, a, x) \in \Delta^n \times (A \backslash \partial\Delta) \times \partial\Delta$ with $|\alpha| < \delta$, consider a function $f : [0, 1] \to \boldsymbol{C}$ defined by $f(t) = \Phi_{t\alpha}(a, x)$. By (i) and (iii) the function f is real-valued, continuous and satisfies $f(0) = \Phi_0(a, x) > 0$. Moreover, we know from our claim that $f(t) \neq 0$ for all $t \in [0,1]$. The intermediate value theorem then implies that $f(1) > 0$. Thus $\Phi_\alpha(a, x) > 0$ which completes the proof. $\square$

5. Main theorem and a reduction of the problem.

Definition 5.1. Let $\beta = \{\beta_j\}$ be a set consisting of m points in the complex plane and assume that $\delta > 0$ is so small that the disks $\Delta(\beta_j, \delta)$ $(j = 1, \dots, m)$ are disjoint. A set $\alpha = \{\alpha_1, \dots, \alpha_n\}$ $(\alpha_i \neq \alpha_j$ if $i \neq j)$ is said to be *δ-nonisolated* with respect to β if $\alpha \subset \Delta(\beta_1, \delta) \cup \cdots \cup \Delta(\beta_m, \delta)$ and if $n_j = \sharp(\alpha \cap \Delta(\beta_j, \delta)) \geq 2$ for every $j = 1, \dots, m$. The vector $(n_1, \dots, n_m)$ is called the *type* of δ-nonisolated set α where the numbers $\{n_j\}$ satisfy $\sum_{j=1}^m n_j = n$. There exists a unique sequence $\tilde{\alpha} = \{\tilde{\alpha}_j\}_{j=1}^n$ of points in $\boldsymbol{C}$ such that $\tilde{\alpha}_j \in \beta$ and $|\alpha_j - \tilde{\alpha}_j| < \delta$ $(j = 1, \dots, n)$, which is called the *center* of the set α.

With this definition our Main Theorem can be stated as follows.

Theorem 5.1. *For every $\beta = \{\beta_j\} \in \mathscr{B}_s$ and every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\alpha \in \mathscr{B}_{g+1}$ $(g \geq 2)$ is δ-nonisolated with respect to β, then $R_2(\alpha) \supset \Delta \backslash \bigcup_{j=1}^s \Delta(\beta_j, \varepsilon)$.*

In order to prove Theorem 5.1, we first study the condition $a \in R_2$ more explicitly. From Lemma 4.5 (ii) we know that $a \in R_2$ if and only if there exists a polynomial $q \in \mathscr{P}_{g-1}$ such that the meromorphic function

$$\Psi_\alpha(a, p) = (\eta_a + \omega_a) / \frac{dx}{ix} = \frac{(y + y(a))(y^{-1} + \overline{y(a)})}{(x - x(a))(x^{-1} - \overline{x(a)})} + \frac{xq(x)}{y \prod_{j=1}^{g+1} (1 - \overline{\alpha_j} x)}$$

of p on $\hat{R}$ vanishes at $p = J(a)$ and is positive on ∂R, where $\omega_a(x)$ denotes the differential $q(x)\,dx/(iy \prod_{j=1}^{g+1} (1 - \overline{\alpha_j} x))$. Letting $p \to J(a)$ we evaluate $\Psi_\alpha(a, J(a))$. Thus the condition for the function $\Psi_\alpha(a, \cdot)$ to vanish at $J(a)$, which we call the *vanishing condition*, is given by

$$q(x(a)) = \frac{1 - |y(a)|^2}{1 - |x(a)|^2} y'(a) \prod_{j=1}^{g+1} (1 - \overline{\alpha_j} x(a)) \tag{13}$$

where $y'(a)$ denotes the derivative of y at a with respect to the local coordinate x. Also, the positivity of $\Psi_\alpha(a, \cdot)$ on ∂R is given by the inequality

$$\left| \frac{y + y(a)}{x - x(a)} \right|^2 + \frac{q(x)}{x^{(g-1)/2} \prod_{j=1}^{g+1} |x - \alpha_j|} \geq 0 \quad \text{on } \partial R, \tag{14}$$

which we call the *boundary condition*. Thus to prove our Main Theorem we have to find a polynomial $q \in \mathscr{P}_{g-1}$ satisfying both the vanishing and the boundary conditions.

To avoid considering general δ-nonisolated sets, we introduce the following reduction by decomposition of these sets. In fact, one verifies easily that every δ-nonisolated set in $\mathscr{B}_n$ $(n \geq 3)$ is a disjoint union of finitely many δ-nonisolated sets in $\mathscr{B}_{n_j}$ $(n_j \geq 3)$ of the following special types:

(I) $(v_1, \dots, v_t)$ where $v_1 \geq 3$ is odd and $v_j \geq 2$ $(j = 2, \dots, t;\ t \geq 1)$ is even,

(II) $(v_1, \dots, v_t)$ where all $v_j \geq 2$ is even.

In the next two sections we shall prove special cases of Main Theorem which treats δ-nonisolated set of the form (I) and (II) respectively. By noting that the number of types of δ-nonisolated set in $\mathscr{B}_{g+1}$ is finite, a repeated application of Lemma 4.6 will prove Main Theorem.

REMARK 5.1. Theorem 5.1 disproves the assertion in [**7**, p. 168] that "$N(p) = 4$ also in a neighborhood of ∂S" which was stated without proof. Indeed this erroneous statement was the starting point of the study in [**5**] and the present paper.

6. Proof of Case (I).

In this section we assume that $g = m + 2n - 1$ is even and $\alpha = \{\alpha_1, \dots, \alpha_{g+1}\}$ is ε-nonisolated with respect to $\{\beta_0, \beta_1, \dots, \beta_n\} \subset \varDelta$. Here $m \geq 3$ is an odd integer, $n \geq 0$ and the center $\tilde{\alpha} = \{\tilde{\alpha}_j\}_{j=1}^{g+1}$ of α is such that $\tilde{\alpha}_1 = \cdots = \tilde{\alpha}_m = \beta_0$ and $\tilde{\alpha}_{m+2j-1} = \tilde{\alpha}_{m+2j} = \beta_j \neq \beta_0$ $(j = 1, \dots, n)$. For simplicity's sake we do not require that the points β_j $(j = 1, \dots, n)$ are distinct. By choosing ε sufficiently small we may assume that $\bar{\varDelta}(\beta_j, \varepsilon) \subset \varDelta$ $(j = 0, \dots, n)$ and if $\beta_i \neq \beta_j$, then $\bar{\varDelta}(\beta_i, \varepsilon) \cap \bar{\varDelta}(\beta_j, \varepsilon) = \varnothing$ $(i, j = 0, \dots, n)$. Since a linear transformation of the branch parameter induces a conformal equivalence of surfaces in $\mathscr{S}_H$ [**5**, Lemma 5], we may assume without loss of generality that $\beta_0 = 0$. In order to overcome the difficulty in treating multi-valuedness of the function y it is convenient to introduce a uniformizing variable $x = t^2$. Thus substituting $x \to x^2$ in (9), we see that the function $y = y(x)$ given by

$$\sqrt{\prod_{j=1}^{g+1} \frac{x^2 - \alpha_j}{1 - \overline{\alpha_j} x^2}} = x^m \prod_{j=1}^{n} \frac{x^2 - \beta_j}{1 - \overline{\beta}_j x^2} \sqrt{\prod_{j=1}^{m} \frac{1 - \alpha_j x^{-2}}{1 - \overline{\alpha_j} x^2} \frac{\prod_{j=m+1}^{g+1} (1 - (\alpha_j - \tilde{\alpha}_j)/(x^2 - \tilde{\alpha}_j))}{\prod_{j=m+1}^{g+1} (1 - (\overline{\alpha_j} - \overline{\tilde{\alpha}_j})/(x^{-2} - \overline{\tilde{\alpha}_j}))}}$$

is single-valued, odd, unitary, nowhere-vanishing and holomorphic on the region X which is the interior of the set $A \cup A'$, where A is a compact set $\bar{\varDelta} \setminus \bigcup_{j=0}^{n} \{x \mid x^2 \in \varDelta(\beta_j, \varepsilon)\}$ and $A' = \{1/\bar{x} \mid x \in A\}$. The sign of the above square root is so chosen that $\sqrt{1} = 1$. Also it is important to observe that the function y is real-analytic in the parameter $(\alpha, x) \in A \times X$. Hence substituting $x \to x^2$ and $a \to a^2$ we may consider, instead of $\Psi_\alpha(a, p)$, the function

$$\tilde{\Psi}_\alpha(a, x) = \frac{(y + y(a))(y^{-1} + \overline{y(a)})}{(x^2 - a^2)(x^{-2} - \bar{a}^2)} + \frac{x^2 q(x^2)}{y \prod_{j=1}^{g+1} (1 - \overline{\alpha_j} x^2)} \tag{15}$$

defined for $(\alpha, a, x) \in \{|\alpha - \tilde{\alpha}| < \varepsilon\} \times A \times X$, where $q(x) \in \mathscr{P}_{g-1}$ is a suitable polynomial to be determined later. Note that, as a meromorphic function of x, $\tilde{\Psi}_\alpha(a, x)$ is regular except for simple poles, counting multiplicities, at a and $1/\bar{a}$. The hyperelliptic involution J now corresponds to the mapping $J(x) = -x$, so the vanishing condition (13) is given by

$$q(a^2) = B_1(\alpha, a) \tag{16}$$

where

$$B_1(\alpha, a) = \frac{1 - |y(a)|^2}{1 - |a|^4} \frac{y'(a)}{2a} \prod_{j=1}^{g+1} (1 - \overline{\alpha_j} a^2).$$

We remark that, for fixed α, $B_1(\alpha, a)$ is an odd function of a.

Lemma 6.1. *If $g(z)$ is a function real-analytic in a neighborhood Ω of $\partial\Delta$ which is identically zero on $\partial\Delta$, then the function $g(z)/(1 - |z|^2)$ is extended to be real-analytic on Ω.*

Proof. Note that $(1 - |z|^2, \operatorname{Re} z)$ or $(1 - |z|^2, \operatorname{Im} z)$ is a real-analytic local coordinate on a neighborhood of a point in $\boldsymbol{C} \setminus \{0\}$. Lemma is easily proved by expanding $g(z)$ in a power series using these coordinates. □

Lemma 6.2. *If $f(z)$ is holomorphic on a region containing $D = \{r < |z| \le 1\}$ such that $|f(z)| = 1$ on $\partial\Delta$ and $|f(z)| \le 1$ on D, then*

$$\lim_{z \to \zeta \in \partial\Delta} \frac{1 - |f(z)|^2}{1 - |z|^2} = \frac{\zeta f'(\zeta)}{f(\zeta)} \ge 0.$$

Equality holds if and only if $f(z)$ is constant.

Proof. From Lemma 6.1, the function $(1-|f(z)|^2)/(1-|z|^2)$ is real-analytic near $\partial\Delta$. Thus for $\zeta \in \partial\Delta$ we have

$$\lim_{z \to \zeta} \frac{1 - |f(z)|^2}{1 - |z|^2} = \frac{1}{2} \frac{\partial}{\partial r} |f(r\zeta)|^2 \bigg|_{r=1} = \operatorname{Re} [\zeta f'(\zeta) \overline{f(\zeta)}] = \operatorname{Re} \left[\frac{\zeta f'(\zeta)}{f(\zeta)} \right].$$

Also, by $|f(\zeta)| = 1$ on $\partial\Delta$ we have

$$0 = \frac{\partial}{\partial \theta} |f(e^{i\theta}\zeta)|^2 \bigg|_{\theta=0} = -2 \operatorname{Im} \left[\frac{\zeta f'(\zeta)}{f(\zeta)} \right].$$

Thus $\zeta f'(\zeta)/f(\zeta) \in \boldsymbol{R}$. Since $|f(z)| \le 1$, this implies that $\zeta f'(\zeta)/f(\zeta) \ge 0$. Equality holds if and only if $f'(\zeta) = 0$. Since $f(z)$ is unitary on $\partial\Delta$, this implies that $f(z)$ must be a constant. □

LEMMA 6.3. *$B_1(\alpha, a)$ is real-analytic for $(\alpha, a) \in \{|\alpha - \tilde{\alpha}| < \varepsilon/2\} \times A$. If $a \in \partial\Delta$, then*

$$B_1(\alpha, a) = \left(\frac{ay'(a)}{2y(a)}\right)^2 a^{g-1} \prod_{j=1}^{g+1} |a^2 - \alpha_j|.$$

In particular, $a^{1-g} B_1(\alpha, a) > 0$ for $a \in \partial\Delta$.

PROOF. Since y is unitary on $\partial\Delta$ Lemma 6.1 easily implies that $B_1(\alpha, a)$ is real-analytic on $\{|\alpha - \tilde{\alpha}| < \varepsilon/2\} \times A$. The remained assertions of Lemma follow from Lemma 6.2. □

On the other hand, the boundary condition (14) is rewritten as

$$\left|\frac{y + y(a)}{x^2 - a^2}\right|^2 + \frac{q(x^2)}{x^{g-1} \prod_{j=1}^{g+1} |x^2 - \alpha_j|} \geq 0 \quad \text{for all } |x| = 1. \tag{17}$$

Our task is to find a polynomial $q \in \mathscr{P}_{g-1}$ satisfying (16) and (17) under the condition $|\alpha - \tilde{\alpha}| < \delta$ for sufficiently small δ.

In order to guess the form of the polynomial $q(x)$ we first consider the limiting case $\alpha = \tilde{\alpha}$. By putting $F(x) = x^m \prod_{j=1}^n (x^2 - \beta_j)/(1 - \overline{\beta_j} x^2)$, the conditions (16) and (17) are given respectively by

$$q(a^2) = a^{g-1} \frac{1 - |F(a)|^2}{1 - |a|^4} \frac{aF'(a)}{2F(a)} \prod_{j=1}^n (a^2 - \beta_j)(a^{-2} - \overline{\beta_j}) \tag{18}$$

and

$$\left|\frac{F(x) + F(a)}{x^2 - a^2}\right|^2 + \frac{q(x^2)}{x^{g-1} \prod_{j=1}^n |x^2 - \beta_j|^2} \geq 0 \quad \text{for all } |x| = 1.$$

Observe that $F(x)$ is of the form $xp(x^2)/(L_k p)(x^2)$ where $k = g/2 = (m-1)/2 + n$ and $p(x)$ is the polynomial $x^{(m-1)/2} \prod_{j=1}^n (x - \beta_j)$. In this situation the following lemma gives us a polynomial $Q_a(x) \in \mathscr{P}_{g-1}$ which satisfies similar but slightly different conditions as above.

LEMMA 6.4. *For $k \in \boldsymbol{N}$, let $h(x)$ be a rational function $xp(x^2)/(L_k p)(x^2)$ where $p(x)$ is a polynomial of degree $\leq k$. Then for every $a \in \boldsymbol{C} \backslash h^{-1}(\infty)$ there exists a unique polynomial $Q_a(x) \in \mathscr{P}_{2k-1}$ satisfying*

$$\left|\frac{h(x) + h(a)}{x^2 - a^2}\right|^2 + \frac{Q_a(x^2)}{x^{2k-1} |p(x^2)|^2} = \frac{1 + |h(a)|^2}{1 + |a|^2} \frac{1}{|x - a|^2} \qquad \textit{for all } |x| = 1$$

and

$$Q_a(a^2) = a^{2k-1} p(a^2)\overline{p(1/\bar{a}^2)}\left\{\frac{1-|h(a)|^2}{1-|a|^4}\frac{ah'(a)}{2h(a)} - \frac{1+|h(a)|^2}{2(1+|a|^2)^2}\right\}. \tag{19}$$

Every coefficient of the polynomial $Q_a(x)$ is an odd and real-analytic function of a on the region $\boldsymbol{C}\backslash h^{-1}(\infty)$.

Proof. By definition the rational function $h(x)$ is unitary on $\partial\Delta$. Thus from an elementary identity

$$|h(x)+h(a)|^2 + |h(x)-h(a)|^2 = c(|x+a|^2 + |x-a|^2) \quad \text{for all } |x| = 1$$

with $c = (1+|h(a)|^2)/(1+|a|^2)$, we obtain for $x \in \partial\Delta$

$$\frac{c}{|x-a|^2} - \left|\frac{h(x)+h(a)}{x^2-a^2}\right|^2 = \left|\frac{h(x)-h(a)}{x^2-a^2}\right|^2 - \frac{c}{|x+a|^2}.$$

Then by analytic continuation both sides of the above identity are extended to a rational function $f(x)$ satisfying

$$\begin{aligned} f(x) &= \frac{c}{(x-a)(x^{-1}-\bar{a})} - \frac{(h(x)+h(a))(h(x)^{-1}+\overline{h(a)})}{(x^2-a^2)(x^{-2}-\bar{a}^2)} \\ &= \frac{(h(x)-h(a))(h(x)^{-1}-\overline{h(a)})}{(x^2-a^2)(x^{-2}-\bar{a}^2)} - \frac{c}{(x+a)(x^{-1}+\bar{a})}. \end{aligned} \tag{20}$$

For fixed $a \in \boldsymbol{C}\backslash h^{-1}(\infty)$, put $g(x) = f(x)p(x^2)(L_k p)(x^2)$. Now we show that $g(x)$ is a polynomial. If $a = 0$, then 0 is not a pole of $h(x)$. Thus $h(0) = 0$ because $h(x)$ is odd. Hence it is clear that $f(x) \equiv 0$ and so $g(x) \equiv 0$. If $a \neq 0$, then we see easily from (20) that the rational function $g(x)$ has possible poles only at $x = \infty, \pm a$ and $\pm 1/\bar{a}$. Indeed, comparing the two expressions of $f(x)$ in (20) and noting that the function $h(x)$ is odd and unitary, one verifies easily that $g(x)$ has poles only at ∞ of order $\leq 4k$. Thus $g(x)$ is a polynomial of degree $\leq 4k$. The identity (20) implies that $f(x)$ is odd. Thus we find a polynomial $Q_a(x)$ such that

$$f(x)p(x^2)(L_k p)(x^2) = xQ_a(x^2). \tag{21}$$

Since $f(x)$ is by definition real-valued on $\partial\Delta$, Lemma 4.2 implies that $g(x) \in \mathscr{P}_{4k}$. Again, by $x \in \mathscr{P}_2$, Lemma 4.2 implies that $Q_a(x) \in \mathscr{P}_{2k-1}$. The uniqueness of $Q_a(x)$ is obvious. The identity (19) is obtained by substituting $x = a$ to the right end of (20).

Writing $Q_a(x) = \sum_{j=0}^{2k-1} c_j(a)x^j$ we next show that the coefficients $c_j(a)$ $(j = 0, \dots, 2k-1)$ depend real-analytically in a. For any fixed $a_0 \in \boldsymbol{C}\backslash h^{-1}(\infty)$, let E be the finite set $h^{-1}(\{0,\infty\}) \cup \{\pm a_0, \pm 1/\overline{a_0}\}$. From the identities (20) and (21) we see that, for fixed $x \in \boldsymbol{C} \backslash E$, $Q_a(x^2)$ is a real-analytic function of a on a neighborhood of a_0. Thus choosing $2k$ points $x_1, \dots, x_{2k}$ in $\boldsymbol{C} \backslash E$ such that $x_1^2, \dots, x_{2k}^2$ are mutually distinct, the coefficients $\{c_j(a)\}_{j=0}^{2k-1}$ satisfy a system of linear equations $\sum_{j=0}^{2k-1} c_j(a) x_k^{2j} = Q_a(x_k^2)$ $(k = 1, \dots, 2k)$. By noting Vandermonde's determinant $\det(x_k^{2j}) \neq 0$, Cramer's formula now yields that every $c_j(a)$ is real-analytic on a neighborhood of a_0. Since $a_0 \in \boldsymbol{C} \backslash h^{-1}(\infty)$ is arbitrary, $c_j(a)$ is real-analytic on $\boldsymbol{C} \backslash h^{-1}(\infty)$. The identity $Q_{-a}(x) = -Q_a(x)$ follows from the fact that $f(x)$ is an odd function of a. Thus every $c_j(a)$ is odd. $\square$

REMARK 6.1. For $k = 1$ and $p(x) = \alpha x + \beta$, a simple calculation gives $Q_a(x) = \bar{c}x + c$ with $c = (|\alpha|^2 - |\beta|^2)\bar{\alpha}(\alpha a + \beta \bar{a})/|\alpha + \beta \bar{a}^2|^2$.

Now we shall modify $Q_a(x)$ to obtain the desired polynomial.

DEFINITION 6.1. For $k = g/2$ and $p(x) = x^{(m-1)/2} \prod_{j=1}^n (x - \beta_j)$, let $Q_a(x) \in \mathscr{P}_{g-1}$ be the polynomial given in Lemma 6.4. Then we define a polynomial $q_a(x)$ by

$$q_a(x) = Q_a(x) + x^{(g-2)/2}(\gamma_0 + \gamma_1 + (\overline{\gamma_0} + \overline{\gamma_1})x) \prod_{j=1}^n (x - \beta_j)(x^{-1} - \overline{\beta_j}) \tag{22}$$

where $\gamma_0 = a(1 + |F(a)|^2)/(2(1 + |a|^2)^3)$ and $\gamma_1 = a(b_1 - |a|^2 \overline{b_1})/(1 - |a|^4)$ with $b_1 = a^{1-g}(B_1(\alpha, a) - B_1(\tilde{\alpha}, a)) / \prod_{j=1}^n (a^2 - \beta_j)(a^{-2} - \overline{\beta_j})$.

We remark that the identity $q_{-a}(x) = -q_a(x)$ holds. Now we show that $q_a(x)$ is indeed a polynomial in $\mathscr{P}_{g-1}$ satisfying (16) and (17).

LEMMA 6.5. *$q_a(x)$ is a polynomial in $\mathscr{P}_{g-1}$ which satisfies the vanishing condition* (16) *and is real-analytic for the variables* $(\alpha, a, x) \in \{|\alpha - \tilde{\alpha}| < \varepsilon/2\} \times A \times X$.

PROOF. Rewrite the function $q_a(x) - Q_a(x)$ as $(\gamma_0 + \gamma_1 + (\overline{\gamma_0} + \overline{\gamma_1})x) x^{(m-3)/2} \cdot \prod_{j=1}^n (x - \beta_j)(1 - \overline{\beta_j} x)$. From our assumption that $m \geq 3$ is odd, we have immediately $\gamma_0 + \gamma_1 + (\overline{\gamma_0} + \overline{\gamma_1})x \in \mathscr{P}_1$, $x^{(m-3)/2} \in \mathscr{P}_{m-3}$ and $(x - \beta_j)(1 - \overline{\beta_j} x) \in \mathscr{P}_2$. Lemma 4.2 (iii) implies that $q_a(x) - Q_a(x) \in \mathscr{P}_{m-2+2n} = \mathscr{P}_{g-1}$. Thus $q_a(x) \in \mathscr{P}_{g-1}$.

Simple calculation using (19) shows that $q_a(x)$ satisfies (16). By Lemmas 6.1 and 6.3 we see that the constant γ_1 is real-analytic in the parameter $(\alpha, a) \in \{|\alpha - \tilde{\alpha}| < \varepsilon/2\} \times A$. This implies $q_a(x)$ is also real-analytic, as desired. $\square$

LEMMA 6.6. *If $\alpha = \tilde{\alpha}$ and $a \in A$, then for all $|x| = 1$ we have an inequality*

$$\left| \frac{F(x) + F(a)}{x^2 - a^2} \right|^2 + \frac{q_a(x^2)}{x^{g-1} \prod_{j=1}^n |x^2 - \beta_j|^2} \geq \frac{1 + |F(a)|^2}{1 + |a|^2} \max\left\{ \frac{1}{|x - a|^2} - 1, \frac{|x + a|^2}{2(1 + |a|^2)^2} \right\}.$$

Proof. Since $\gamma_1 = 0$, in view of Lemma 6.4 we have, for $|x| = 1$,

$$\left|\frac{F(x)+F(a)}{x^2-a^2}\right|^2 + \frac{q_a(x^2)}{x^{g-1}\prod_{j=1}^{n}|x^2-\beta_j|^2} = \frac{1+|F(a)|^2}{1+|a|^2}\left\{\frac{1}{|x-a|^2} + \frac{\mathrm{Re}(\bar{a}x)}{(1+|a|^2)^2}\right\}$$

$$\geq \frac{1+|F(a)|^2}{1+|a|^2}\left\{\frac{1}{(1+|a|)^2} + \frac{|x+a|^2-(1+|a|^2)}{2(1+|a|^2)^2}\right\}$$

$$= \frac{1+|F(a)|^2}{1+|a|^2}\left\{\frac{(1-|a|)^2}{2(1+|a|)^2(1+|a|^2)} + \frac{|x+a|^2}{2(1+|a|^2)^2}\right\}.$$

This immediately gives the inequality of Lemma. □

Now we return to the proof of the special case of Theorem 5.1. For $(\alpha, a, x) \in \Delta^n \times A \times X$, define a function $\Phi_\alpha(a,x)$ by

$$\Phi_\alpha(a,x) = (x+a)(x^{-1}+\bar{a})\tilde{\Psi}_{\varepsilon\alpha/2+\tilde{\alpha}}(-a,x),$$

where $\tilde{\Psi}$ is the function (15) with the polynomial $q(x)$ substituted by $q_a(x)$.

We must show that the function $\Phi_\alpha(a,x)$ satisfies all the assumptions of Lemma 4.7. This is proved as follows. Putting $\alpha = \varepsilon\alpha'/2 + \tilde{\alpha}$ and using $q_{-a} = -q_a$, we have

$$\Phi_{\alpha'}(a,x) = \frac{(y-y(a))(y^{-1}-\overline{y(a)})}{(x-a)(x^{-1}-\bar{a})} - \frac{(x+a)(1+\bar{a}x)xq_a(x^2)}{y\prod_{j=1}^{g+1}(1-\overline{\alpha_j}x^2)}. \tag{23}$$

One verifies easily that by expanding in a power series the functions $(y-y(a))/(x-a)$ and $(y^{-1}-\overline{y(a)})/(x^{-1}-\bar{a})$ are real-analytic in the variables (α, a, x). Since $y \neq 0, \infty$ on X, the second term of (23) is also real-analytic. Thus $\Phi_\alpha(a,x)$ is real-analytic in $(\alpha, a, x) \in \Delta^n \times A \times X$ which suffices to prove the continuity of Φ. Clearly, the function $\Phi_\alpha(a,\cdot)$ is holomorphic on X, and for $x \in \partial\Delta$ we have

$$\Phi_{\alpha'}(a,x) = \left|\frac{y-y(a)}{x-a}\right|^2 - \frac{|x+a|^2 q_a(x^2)}{x^{g-1}\prod_{j=1}^{g+1}|x^2-\alpha_j|},$$

so that Lemma 4.2 implies the real-valuedness of $\Phi_\alpha(a,\cdot)$ on $\partial\Delta$. Thus the condition (i) of Lemma 4.7 is proved. The condition (ii) with $k=2$ and $E = \partial\Delta$ is clear from Lemma 6.5 and the symmetry of the function $\Phi_\alpha(a,\cdot)$ with respect to $\partial\Delta$. Also, the conditions (iii) and (iv) are proved easily from Lemma 6.6.

We can now apply Lemma 4.7 to the function $\Phi_\alpha(a,x)$. By noting the identity $\Phi_\alpha(a,x) = |x+a|^2\tilde{\Psi}_{\varepsilon\alpha/2+\tilde{\alpha}}(-a,x)$ for $x \in \partial\Delta$, Lemma 4.7 implies that there exists a $\delta > 0$ such that if $|\alpha - \tilde{\alpha}| < \delta$ and if $a^2 \in \Delta\backslash\bigcup_{j=0}^{n}\Delta(\beta_j, \varepsilon)$, then $\tilde{\Psi}_\alpha(a,\cdot)$ is positive on $\partial\Delta$. Thus both the vanishing and the boundary conditions are satisfied for all $a^2 \in \Delta\setminus\bigcup_{j=0}^{n}\Delta(\beta_j,\varepsilon)$. Therefore we have established the inclusion $R_2(\alpha) \supset \Delta\setminus\bigcup_{j=0}^{n}\Delta(\beta_j,\varepsilon)$. This completes the proof of Case (I) of Main Theorem.

7. Proof of Case (II).

In this section we assume that $g = 2n-1$ $(n \geq 2)$ is an odd integer and that $\alpha = \{\alpha_1,\dots,\alpha_{g+1}\}$ is ε-nonisolated with center at $\tilde{\alpha} = \{\tilde{\alpha}_1,\dots,\tilde{\alpha}_{g+1}\}$ where $\tilde{\alpha}_{2j-1} = \tilde{\alpha}_{2j} = \beta_j \in \Delta$ for $j = 1,\dots,n$. We do not require that the points β_j $(j = 1,\dots,n)$ are distinct. By choosing ε sufficiently small we may assume that $\bar{\Delta}(\beta_j,\varepsilon) \subset \Delta$ $(j = 1,\dots,n)$ and if $\beta_i \neq \beta_j$, then $\bar{\Delta}(\beta_i,\varepsilon) \cap \bar{\Delta}(\beta_j,\varepsilon) = \emptyset$ $(i,j = 1,\dots,n)$. Since our argument here is similar as in the previous section, we merely outline the proof.

Let X be the interior of $A \cup A'$ where $A = \bar{\Delta}\setminus\bigcup_{j=1}^{n}\Delta(\beta_j,\varepsilon)$ and $A' = \{1/\bar{x} \mid x \in A\}$. Since the genus g is odd, the function $y = y(x) = \sqrt{\prod_{j=1}^{g+1}(x-\alpha_j)/(1-\bar{\alpha}_j x)}$ has a single-valued branch on X

$$G(x)\cdot\sqrt{\frac{\prod_{j=1}^{g+1}(1-(\alpha_j-\tilde{\alpha}_j)/(x-\tilde{\alpha}_j))}{\prod_{j=1}^{g+1}(1-(\bar{\alpha}_j-\bar{\tilde{\alpha}}_j)/(x^{-1}-\bar{\tilde{\alpha}}_j))}}$$

which is holomorphic and nowhere-vanishing, where $G(x)$ is a finite Blaschke product $\prod_{j=1}^{n}(x-\beta_j)/(1-\bar{\beta}_j x)$. Clearly, the function y is real-analytic in the variable $(\alpha,x) \in \{|\alpha-\tilde{\alpha}| < \varepsilon\} \times X$. According to the sign of $\sqrt{y^2} = \pm y$, we define two functions, for $(\alpha,a,x) \in \{|\alpha-\tilde{\alpha}| < \varepsilon\} \times A \times X$,

$$\Psi_\alpha^+(a,x) = \frac{(y+y(a))(y^{-1}+\overline{y(a)})}{(x-a)(x^{-1}-\bar{a})} + \frac{xq(x)}{y\prod_{j=1}^{g+1}(1-\bar{\alpha}_j x)}$$

and

$$\Psi_\alpha^-(a,x) = \frac{(y-y(a))(y^{-1}-\overline{y(a)})}{(x-a)(x^{-1}-\bar{a})} - \frac{xq(x)}{y\prod_{j=1}^{g+1}(1-\bar{\alpha}_j x)}.$$

Note that, on the region X, $\Psi_\alpha^-(a,\cdot)$ is holomorphic, while $\Psi_\alpha^+(a,\cdot)$ is meromorphic with poles only at a and $1/\bar{a}$. Since y is unitary on $\partial\Delta$, we have $|y+y(a)|^2 + |y-y(a)|^2 = 2(1+|y(a)|^2)$ for all $x \in \partial\Delta$. Then by analytic continuation we have the identity

$$\Psi_\alpha^+(a,x) + \Psi_\alpha^-(a,x) = \frac{2(1+|y(a)|^2)}{(x-a)(x^{-1}-\bar{a})}. \tag{24}$$

Our task is now to find a polynomial $q(x) \in \mathscr{P}_{g-1}$ such that $\Psi_\alpha^-(a,a) = 0$ and that the functions $\Psi_\alpha^+(a,x)$ and $\Psi_\alpha^-(a,x)$ satisfy the boundary condition (14). The vanishing condition $\Psi_\alpha^-(a,a) = 0$ is equivalent to $q(a) = B_2(\alpha,a)$ where

$$B_2(\alpha,a) = \frac{1-|y(a)|^2}{1-|a|^2} y'(a) \prod_{j=1}^{g+1}(1-\overline{\alpha_j}a).$$

Lemma 7.1. *For every $a \in \boldsymbol{C}\backslash G^{-1}(\infty)$ there exists a unique polynomial $P_a(x) \in \mathscr{P}_{g-1}$ such that the function $\Psi_{\tilde{\alpha}}^-(a,x)$ defined with $q(x) = P_a(x)$ satisfies $\Psi_{\tilde{\alpha}}^-(a,\cdot) \equiv 0$ and $P_a(a) = B_2(\tilde{\alpha},a)$. Every coefficient of the polynomial $P_a(x)$ is a real-analytic function of a on the region $\boldsymbol{C}\backslash G^{-1}(\infty)$.*

Proof. The uniqueness is clear. Consider the function

$$h_a(x) = \frac{G(x)-G(a)}{x-a}\prod_{j=1}^{n}(1-\overline{\beta_j}x)$$

which is easily seen to be a polynomial of degree $\leq n-1$. Since $G(x)$ is unitary, we have

$$L_{n-1}h_a(x) = \frac{G(x)^{-1}-\overline{G(a)}}{x(x^{-1}-\bar{a})}\prod_{j=1}^{n}(x-\beta_j).$$

Put $P_a(x) = h_a(x)L_{n-1}h_a(x)$. Then applying Lemma 4.2 we see that $P_a(x) \in \mathscr{P}_{g-1}$ which satisfies

$$\frac{(G(x)-G(a))(G(x)^{-1}-\overline{G(a)})}{(x-a)(x^{-1}-\bar{a})} = \frac{xP_a(x)}{G(x)\prod_{j=1}^{n}(1-\overline{\beta_j}x)^2}.$$

Thus $\Psi_{\tilde{\alpha}}^-(a,\cdot) \equiv 0$, and the identity $P_a(a) = B_2(\tilde{\alpha},a)$ is obvious.

The real-analyticity of the coefficients of $P_a(x)$ is proved similarly as in Lemma 6.4. □

Definition 7.1. We define a polynomial $q_a(x)$ by

$$q_a(x) = P_a(x) - \gamma_0 x^{(g-1)/2}(x-a)(x^{-1}-\bar{a}) + x^{(g-1)/2}(\gamma_1 x^{-1} + \overline{\gamma_1}x),$$

where $\gamma_0 = 1/8\prod_{j=1}^{n}(1-|\beta_j|)^2$ and $\gamma_1 = a(b_2 - |a|^2\overline{b_2})/(1-|a|^4)$ with $b_2 = a^{(1-g)/2}(B_2(\alpha,a) - B_2(\tilde{\alpha},a))$.

In view of our assumption $g \geq 3$ is odd, we note that $q_a(x)$ is indeed a polynomial. The following Lemmas show that the functions $\Psi_\alpha^+(a,x)$ and $\Psi_\alpha^-(a,x)$ defined with $q_a(x)$ satisfy both the vanishing and the boundary conditions.

LEMMA 7.2. *$B_2(\alpha, a)$ is real-analytic for $(\alpha, a) \in \{|\alpha - \tilde{\alpha}| < \varepsilon/2\} \times A$. If $a \in \partial\Delta$, then*

$$B_2(\alpha, a) = \left(\frac{ay'(a)}{y(a)}\right)^2 a^{(g-1)/2} \prod_{j=1}^{g+1} |a - \alpha_j|.$$

In particular, $a^{(1-g)/2} B_2(\alpha, a) > 0$ for $a \in \partial\Delta$.

PROOF. See the proof of Lemma 6.3. □

LEMMA 7.3. *The polynomial $q_a(x) \in \mathscr{P}_{g-1}$ satisfies $q_a(a) = B_2(\alpha, a)$. Moreover, $q_a(x)$ is real-analytic for the variables $(\alpha, a, x) \in \{|\alpha - \tilde{\alpha}| < \varepsilon/2\} \times A \times X$.*

PROOF. See the proof of Lemma 6.5. □

LEMMA 7.4. *There exists a constant $\gamma_2 > 0$ such that for $(a, x) \in A \times \partial\Delta$ the inequalities $\Psi_{\tilde{\alpha}}^-(a, x) \geq \gamma_2 |x - a|^2$ and $|x - a|^2 \Psi_{\tilde{\alpha}}^+(a, x) > 0$ hold.*

PROOF. Since $\gamma_1 = 0$, from Lemma 7.1 we have

$$\Psi_{\tilde{\alpha}}^-(a, x) = \frac{\gamma_0 x^n (x - a)(x^{-1} - \bar{a})}{G(x) \prod_{j=1}^n (1 - \overline{\beta_j} x)^2}.$$

Consequently, for $x \in \partial\Delta$

$$\Psi_{\tilde{\alpha}}^-(a, x) = \frac{\gamma_0 |x - a|^2}{\prod_{j=1}^n |x - \beta_j|^2} \geq \gamma_2 |x - a|^2$$

where $\gamma_2 = 1/8 \prod_{j=1}^n ((1 - |\beta_j|)/(1 + |\beta_j|))^2 > 0$. On the other hand, for $x \in \partial\Delta$ the identity (24) implies

$$|x - a|^2 \Psi_{\tilde{\alpha}}^+(a, x) = 2(1 + |G(a)|^2) - \frac{\gamma_0 |x - a|^4}{\prod_{j=1}^n |x - \beta_j|^2} > 2 - \frac{16\gamma_0}{\prod_{j=1}^n (1 - |\beta_j|)^2} = 0.$$

Here we have used the fact that $G(a) \neq 0$ for $a \in A$. □

Now we are ready to apply Lemma 4.7. Define the functions $\Phi_\alpha^+(a, x)$ and $\Phi_\alpha^-(a, x)$ by $\Phi_\alpha^+(a, x) = (x - a)(x^{-1} - \bar{a}) \Psi_{\varepsilon\alpha/2 + \tilde{\alpha}}^+(a, x)$ and $\Phi_\alpha^-(a, x) = \Psi_{\varepsilon\alpha/2 + \tilde{\alpha}}^-(a, x)$. Then the hypothesis of Lemma 4.7 is satisfied for $\Phi_\alpha^+(a, x)$ if $E = \varnothing$ and for $\Phi_\alpha^-(a, x)$ if $k = 2$ and $E = \partial\Delta$. This completes the proof of the special case (II) of Theorem 5.1 and thus we have finally established Main Theorem.

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