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The behavior of the principal distributions around an isolated umbilical point

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Abstract. Let f be a homogeneous polynomial in two variables such that on its graph G_f, the origin $o = (0,0,0)$ of $\boldsymbol{R}^3$ is an isolated umbilical point. In this paper, the behavior of the principal distributions around o is studied in relation to the existence of other umbilical points than o and the behavior of the gradient vector field of f.

1. Introduction.

Let S be a smooth surface in the 3-dimensional Euclidean space $\boldsymbol{R}^3$ and $\mathrm{Umb}(S)$ the set of the umbilical points of S. If $S \backslash \mathrm{Umb}(S) \neq \emptyset$, then there exists *a principal distribution* D_S *on* S, which is a one-dimensional continuous distribution on $S \backslash \mathrm{Umb}(S)$ such that $D_S(p)$ is one of the principal directions at $p \in S \backslash \mathrm{Umb}(S)$. If D_S has an isolated singularity p_0, i.e., if p_0 is an isolated umbilical point, then as a quantity in relation to the behavior of the principal distributions around p_0, *the index* $\mathrm{ind}_{p_0}(S)$ *of* p_0 is defined ([**2**, p. 137]).

Let P_o^k be the set of the homogeneous polynomials of degree $k \geqq 3$ such that on their graphs, the origin $o = (0,0,0)$ of $\boldsymbol{R}^3$ is an isolated umbilical point, and f an element of P_o^k and $\tilde{f}$ the function on $\boldsymbol{R}$ defined by $\tilde{f}(\theta) = f(\cos\theta, \sin\theta)$. A real number at which $d\tilde{f}/d\theta = 0$ is called *a root of* f and the set of the roots of f is represented by R_f. Each root $\theta_0 \in R_f$ determines a straight line

$$L(\theta_0) := \{(x,y) \in \boldsymbol{R}^2; x\sin\theta_0 - y\cos\theta_0 = 0\}$$

on $\boldsymbol{R}^2$ through o. The straight line determined by a root is called *a root line of* f. The natural coordinates (x,y) on the xy-plane may be considered as coordinates on the graph G_f of f. Then a root line is considered not only as a subset of $\boldsymbol{R}^2$ but also as a subset of G_f. The set of the root lines of f is represented by $\tilde{R}_f$. Let r be a positive number such that on $0 < x^2 + y^2 \leqq r^2$, there exists no umbilical point, and r_0 the supremum of such numbers as r. A continuous function ϕ_{r,θ_0,ϕ_0} is called *the argument function on* $x^2 + y^2 = r^2$ *with initial values*

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(θ_0, ϕ_0) if ϕ_{r,θ_0,ϕ_0} satisfies $\phi_{r,\theta_0,\phi_0}(\theta_0) = \phi_0$ and if for any $\theta \in \boldsymbol{R}$, $\cos\phi_{r,\theta_0,\phi_0}(\theta)\cdot(\partial/\partial x) + \sin\phi_{r,\theta_0,\phi_0}(\theta)(\partial/\partial y)$ is in the principal directions at $(r\cos\theta, r\sin\theta)$. For $r \in (0, r_0)$ and for $\theta_0 \in R_f$, there exists the argument function $\phi_{r,\theta_0,\theta_0}$ (see [**1**]). It is said that *the sign of* $\theta_0 \in R_f$ *is positive* (*resp. negative*) if there exists a positive number $\varepsilon > 0$ such that for any $r \in (0, r_0)$ and for any $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)\backslash\{\theta_0\}$,

$$\{\theta - \phi_{r,\theta_0,\theta_0}(\theta)\}(\theta - \theta_0) > 0 \ \ (\text{resp.} < 0).$$

A root $\theta_0 \in R_f$ is said to be *related* (*resp. non-related*) *to the origin* if the sign of θ_0 is either positive or negative (resp. neither positive nor negative). Let N_+ (resp. N_-) be the number of the root lines determined by the positive (resp. negative) roots. Then the index of o is represented as

$$\mathrm{ind}_o(G_f) = 1 - \frac{N_+ - N_-}{2},$$

and moreover the following holds ([**1**]):

$$N_+ - N_- \in \{k - 2i\}_{i=0}^{[k/2]},$$

where $[k/2]$ is the Gauss' symbol for $k/2$.

One of the purposes of this paper is to describe the relation between the sign of a root θ_0 related to the origin and the set $\mathrm{Umb}(G_f; L(\theta_0))$ of the umbilical points on $L(\theta_0)\backslash\{o\}$. If $d\tilde{f}/d\theta \equiv 0$, then any $\theta_0 \in \boldsymbol{R}$ is a root non-related to the origin. Suppose that $d\tilde{f}/d\theta \not\equiv 0$. Then for any $\theta_0 \in R_f$, there exists a positive integer m such that $(d^{m+1}\tilde{f}/d\theta^{m+1})(\theta_0) \neq 0$. The minimum of such integers as m is called *the multiplicity of* θ_0 and denoted by $\mu(\theta_0)$. The multiplicity $\mu(\theta_0)$ is odd (resp. even) if and only if θ_0 is related (resp. non-related) to the origin (see [**1**]).

Proposition 1.1. *If* θ_0 *is a root of* f *non-related to the origin, then the following holds*:

$$\mathrm{Umb}(G_f; L(\theta_0)) \neq \emptyset.$$

Suppose that $\theta_0 \in R_f$ is related to the origin. It is said that *the critical sign of* θ_0 *is positive* (*resp. negative*) if $\tilde{f}(\theta_0)(d^{\mu(\theta_0)+1}\tilde{f}/d\theta^{\mu(\theta_0)+1})(\theta_0) \leqq 0$ (resp. > 0). The sign and the critical sign of θ_0 are represented by $\mathrm{sign}(\theta_0)$ and $\text{c-sign}(\theta_0)$, respectively. Let $\tilde{K}_f(\theta_0)$ be the Gaussian curvature of G_f at $(\cos\theta_0, \sin\theta_0)$. If $\tilde{K}_f(\theta_0) \neq 0$, then the sign of $\tilde{K}_f(\theta_0)$ is represented by $\mathrm{sign}[\tilde{K}_f(\theta_0)]$. Let $\{+,-\}$ be the set of symbols $+, -$. In the natural way, $\mathrm{sign}(\theta_0)$, $\text{c-sign}(\theta_0)$ and $\mathrm{sign}[\tilde{K}_f(\theta_0)]$ may be considered as elements of the set $\{+,-\}$. Let $\cdot$ be the law of composition of the set $\{+,-\}$ such that

$$+\cdot+ = -\cdot- = +, \quad +\cdot- = -\cdot+ = -.$$

Then one of the main results in this paper is stated as follows.

Theorem 1.2. *Let θ_0 be a root of f.*
(1) *If $\tilde{K}_f(\theta_0) = 0$, then θ_0 is related to the origin, and the following hold*:

$$(\text{sign}(\theta_0), \text{c-sign}(\theta_0)) = (+,+),$$
$$\text{Umb}(G_f; L(\theta_0)) = \varnothing;$$

(2) *If θ_0 is related to the origin and satisfies $\tilde{K}_f(\theta_0) \neq 0$, then*

$$\text{sign}(\theta_0) \cdot \text{c-sign}(\theta_0) \cdot \text{sign}[\tilde{K}_f(\theta_0)] = -\,(\text{resp.} = +)$$

if and only if

$$\text{Umb}(G_f; L(\theta_0)) = \varnothing \ (\text{resp.} \neq \varnothing).$$

The other of the purposes of this paper is to describe the relation between the sign of a root θ_0 related to the origin and the behavior of the gradient vector field of f near a root line $L(\theta_0)$. A number θ_0 is called *a gradient root of f* if for any $\rho \in \boldsymbol{R}$, the gradient

$$\frac{\partial f}{\partial x}(\rho\cos\theta_0, \rho\sin\theta_0)\frac{\partial}{\partial x} + \frac{\partial f}{\partial y}(\rho\cos\theta_0, \rho\sin\theta_0)\frac{\partial}{\partial y}$$

of f at $(\rho\cos\theta_0, \rho\sin\theta_0)$ is in the principal directions. The set of the gradient roots of f is represented by R_f^G.

Proposition 1.3. *A number θ_0 is an element of R_f^G if and only if θ_0 is an element of R_f or satisfies $\tilde{K}_f(\theta_0) = 0$.*

There exists a continuous function ψ such that for any $\theta \in \boldsymbol{R}$, the gradient at $(\cos\theta, \sin\theta) \in G_f$ is represented by a tangent vector $\cos\psi(\theta)(\partial/\partial x) + \sin\psi(\theta) \cdot (\partial/\partial y)$ with constant multiplication. Such a function ψ is called *an argument function of the gradient*. It is said that *the gradient sign of $\theta_0 \in R_f^G$ is positive* (*resp. negative*) if there exists a positive number $\varepsilon > 0$ such that for any $r \in (0, r_0)$ and for any $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon) \backslash \{\theta_0\}$,

$$\{\phi_{r,\theta_0,\psi(\theta_0)}(\theta) - \psi(\theta)\}(\theta - \theta_0) > 0 \ (\text{resp.} < 0).$$

An element $\theta_0 \in R_f^G$ is said to be *related* (*resp. non-related*) *to the gradient* if the gradient sign of θ_0 is positive or negative (resp. neither positive nor negative). If $\theta_0 \in R_f^G$ is related to the gradient, then the gradient sign of θ_0 is represented by g-sign(θ_0).

It is said that *the curvature sign of $\theta_0 \in R_f^G$ is positive* (*resp. negative*) if there exists a positive number $\varepsilon > 0$ such that for any $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon) \backslash \{\theta_0\}$,

$$\tilde{K}_f(\theta) > 0 \ (\text{resp.} < 0).$$

An element $\theta_0 \in R_f^G$ is said to be *related* (*resp. non-related*) *to the curvature* if the curvature sign of θ_0 is neither positive nor negative (resp. either positive or negative). If $\theta_0 \in R_f^G$ is non-related to the curvature, then the curvature sign of θ_0 is also represented by $\mathrm{sign}[\tilde{K}_f(\theta_0)]$.

Proposition 1.4. *Let θ_0 be an element of $R_f^G \backslash R_f$. Then θ_0 is related or non-related to both of the gradient and the curvature.*

Proposition 1.5. *Let θ_0 be an element of R_f. Then exactly one of the following happens*:

(1) *A root θ_0 is non-related to just one of the origin, the gradient and the curvature*;

(2) *A root θ_0 is non-related to each of the origin, the gradient and the curvature.*

The other of the main results in this paper is the following.

Theorem 1.6. *Suppose that $\theta_0 \in R_f$ is related to the origin and the gradient.*

(1) *If $\tilde{f}(\theta_0) = 0$, then the following holds*:

$$(\mathrm{sign}(\theta_0), \text{g-sign}(\theta_0), \mathrm{sign}[\tilde{K}_f(\theta_0)]) = (+, -, -);$$

(2) *If $\tilde{f}(\theta_0) \neq 0$, then the following holds*:

$$\mathrm{sign}(\theta_0) \cdot \text{g-sign}(\theta_0) \cdot \mathrm{sign}[\tilde{K}_f(\theta_0)] = -.$$

If $\theta_0 \in R_f$ satisfies $\tilde{f}(\theta_0) = 0$, then it is seen that $\tilde{K}_f(\theta_0) = 0$. Therefore from Theorem 1.2 and from Theorem 1.6, the following is obtained.

Theorem 1.7. *Suppose that $\theta_0 \in R_f$ satisfies* (1) *in Proposition* 1.5 *and $\tilde{K}_f(\theta_0) \neq 0$. Then* $\text{c-sign}(\theta_0) \cdot \text{g-sign}(\theta_0) = +$ (*resp.* $= -$) *if and only if*

$$\mathrm{Umb}(G_f; L(\theta_0)) = \emptyset \ \ (\text{resp.} \neq \emptyset).$$

Remark 1.8. A condition $\tilde{K}_f(\theta_0) \neq 0$ in Theorem 1.7 may not be omitted. For example, we give an element $f(x, y) = x^4 + y^4 \in P_o^4$. We see that

(1) 0 is a root of f related to the origin and the gradient;

(2) $\mathrm{sign}(0) = \text{c-sign}(0) = \mathrm{sign}[\tilde{K}_f(0)] = +$;

(3) $\text{g-sign}(0) = -$.

However we also see that $\mathrm{Umb}(G_f; L(\theta_0)) = \emptyset$, because of $\tilde{K}_f(0) = 0$.

This paper is organized as follows. In Section 2, notations and fundamental results are prepared. In Section 3, the set of the umbilical points on each root line of $f \in P_o^k$ is studied. Particularly, Proposition 1.1 and Theorem 1.2 are proved. In Section 4, the behavior of a principal distribution is compared with the behavior of the gradient vector field near a point $(r\cos\theta_0, r\sin\theta_0)$ where $\theta_0 \in R_f^G$. Particularly, Proposition 1.3, Proposition 1.4, Proposition 1.5 and Theorem 1.6 are proved.

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2. Preliminaries.

Let $f(x,y)$ be a homogeneous polynomial in two real variables x, y of degree $k \geqq 3$ and G_f the graph of f. We set

$$p_f := \frac{\partial f}{\partial x}, \quad q_f := \frac{\partial f}{\partial y}, \quad r_f := \frac{\partial^2 f}{\partial x^2}, \quad s_f := \frac{\partial^2 f}{\partial x \partial y}, \quad t_f := \frac{\partial^2 f}{\partial y^2}.$$

Moreover we set

$$\begin{aligned}
&\tilde{p}_f(\theta) := p_f(\cos\theta, \sin\theta), \quad \tilde{q}_f(\theta) := q_f(\cos\theta, \sin\theta), \\
&\tilde{r}_f(\theta) := r_f(\cos\theta, \sin\theta), \quad \tilde{s}_f(\theta) := s_f(\cos\theta, \sin\theta), \\
&\tilde{t}_f(\theta) := t_f(\cos\theta, \sin\theta),
\end{aligned}$$

and

$$\begin{aligned}
d_f(\theta,\phi) &:= \tilde{s}_f(\theta)\cos^2\phi + \{\tilde{t}_f(\theta) - \tilde{r}_f(\theta)\}\cos\phi\sin\phi - \tilde{s}_f(\theta)\sin^2\phi, \\
n_f(\theta,\phi) &:= \{\tilde{s}_f(\theta)\tilde{p}_f(\theta)^2 - \tilde{p}_f(\theta)\tilde{q}_f(\theta)\tilde{r}_f(\theta)\}\cos^2\phi \\
&\qquad + \{\tilde{t}_f(\theta)\tilde{p}_f(\theta)^2 - \tilde{r}_f(\theta)\tilde{q}_f(\theta)^2\}\cos\phi\sin\phi \\
&\qquad + \{\tilde{p}_f(\theta)\tilde{q}_f(\theta)\tilde{t}_f(\theta) - \tilde{s}_f(\theta)\tilde{q}_f(\theta)^2\}\sin^2\phi.
\end{aligned}$$

Then (r, θ_0, ϕ_0) satisfies the equation

$$r^{k-2} d_f(\theta_0, \phi_0) + r^{3k-4} n_f(\theta_0, \phi_0) = 0 \tag{2.1}$$

if and only if a tangent vector $\cos\phi_0(\partial/\partial x) + \sin\phi_0(\partial/\partial y)$ at $(r\cos\theta_0, r\sin\theta_0)$ is in the principal directions. We set

$$\mathrm{grad}_f(\theta) := \begin{pmatrix} \tilde{p}_f(\theta) \\ \tilde{q}_f(\theta) \end{pmatrix}, \quad \mathrm{Hess}_f(\theta) := \begin{pmatrix} \tilde{r}_f(\theta) & \tilde{s}_f(\theta) \\ \tilde{s}_f(\theta) & \tilde{t}_f(\theta) \end{pmatrix}.$$

We denote by $\langle\,,\rangle$ the scalar product in $\boldsymbol{R}^2$, and for a vector $v \in \boldsymbol{R}^2$, we set $\|v\| := \sqrt{\langle v, v\rangle}$.

Lemma 2.1. *For real numbers θ_0, ϕ_0, the following hold*:
(1)

$$d_f(\theta_0,\phi_0) = \frac{1}{2}\frac{\partial}{\partial\phi}\left\langle \mathrm{Hess}_f(\theta_0)\begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}, \begin{pmatrix}\cos\phi\\ \sin\phi\end{pmatrix}\right\rangle\Big|_{\phi=\phi_0}$$

$$= \left\langle \mathrm{Hess}_f(\theta_0)\begin{pmatrix}\cos\phi_0\\ \sin\phi_0\end{pmatrix}, \begin{pmatrix}-\sin\phi_0\\ \cos\phi_0\end{pmatrix}\right\rangle,$$

(2)

$$n_f(\theta_0,\phi_0) = \frac{(1+\|\mathrm{grad}_f(\theta_0)\|^2)^2}{k-1}\tilde{K}_f(\theta_0)\left\langle \mathrm{grad}_f(\theta_0), \begin{pmatrix}\cos\phi_0\\ \sin\phi_0\end{pmatrix}\right\rangle \sin(\phi_0-\theta_0).$$

Proof. We immediately obtain (1). We rewrite $n_f(\theta_0,\phi_0)$ by

$$(k-1)\,\mathrm{grad}_f(\theta) = \mathrm{Hess}_f(\theta)\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \tag{2.2}$$

as follows.

$$\begin{aligned} n_f(\theta_0,\phi_0) = \frac{1}{(k-1)^2}&\{[\tilde{s}_f(\theta_0)(\tilde{r}_f(\theta_0)\cos\theta_0+\tilde{s}_f(\theta_0)\sin\theta_0)^2\\ &-\tilde{r}_f(\theta_0)(\tilde{r}_f(\theta_0)\cos\theta_0+\tilde{s}_f(\theta_0)\sin\theta_0)\\ &\quad\times(\tilde{s}_f(\theta_0)\cos\theta_0+\tilde{t}_f(\theta_0)\sin\theta_0)]\cos^2\phi_0\\ &+[\tilde{t}_f(\theta_0)(\tilde{r}_f(\theta_0)\cos\theta_0+\tilde{s}_f(\theta_0)\sin\theta_0)^2\\ &-\tilde{r}_f(\theta_0)(\tilde{s}_f(\theta_0)\cos\theta_0+\tilde{t}_f(\theta_0)\sin\theta_0)^2]\cos\phi_0\sin\phi_0\\ &+[-\tilde{s}_f(\theta_0)(\tilde{s}_f(\theta_0)\cos\theta_0+\tilde{t}_f(\theta_0)\sin\theta_0)^2\\ &+\tilde{t}_f(\theta_0)(\tilde{r}_f(\theta_0)\cos\theta_0+\tilde{s}_f(\theta_0)\sin\theta_0)\\ &\quad\times(\tilde{s}_f(\theta_0)\cos\theta_0+\tilde{t}_f(\theta_0)\sin\theta_0)]\sin^2\phi_0\}\\ =\{-[\tilde{r}_f(\theta_0)&\cos\theta_0+\tilde{s}_f(\theta_0)\sin\theta_0]\sin\theta_0\cos^2\phi_0\\ &+[\tilde{r}_f(\theta_0)\cos^2\theta_0-\tilde{t}_f(\theta_0)\sin^2\theta_0]\cos\phi_0\sin\phi_0\\ &+[\tilde{s}_f(\theta_0)\cos\theta_0+\tilde{t}_f(\theta_0)\sin\theta_0]\cos\theta_0\sin^2\phi_0\}\\ \times\frac{\tilde{r}_f(\theta_0)\tilde{t}_f(\theta_0)-\tilde{s}_f(\theta_0)^2}{(k-1)^2}& \end{aligned}$$

$$= \{-\tilde{p}_f(\theta_0)\sin\theta_0\cos^2\phi_0$$

$$+ [\tilde{p}_f(\theta_0)\cos\theta_0 - \tilde{q}_f(\theta_0)\sin\theta_0]\cos\phi_0\sin\phi_0$$

$$+ \tilde{q}_f(\theta_0)\cos\theta_0\sin^2\phi_0\}\frac{\tilde{r}_f(\theta_0)\tilde{t}_f(\theta_0) - \tilde{s}_f(\theta_0)^2}{k-1}$$

$$= \frac{\tilde{r}_f(\theta_0)\tilde{t}_f(\theta_0) - \tilde{s}_f(\theta_0)^2}{k-1}\left\langle \mathrm{grad}_f(\theta_0), \begin{pmatrix}\cos\phi_0\\ \sin\phi_0\end{pmatrix}\right\rangle \sin(\phi_0 - \theta_0).$$

Gaussian curvature $K_f(x,y)$ of G_f at (x,y) is represented as

$$K_f(x,y) = \frac{r_f(x,y)t_f(x,y) - s_f(x,y)^2}{\{1 + p_f(x,y)^2 + q_f(x,y)^2\}^2}. \tag{2.3}$$

Therefore we obtain (2) of Lemma 2.1. □

Proposition 2.2. *For a number θ_0, the following are mutually equivalent*:

(1) *A number θ_0 is a root of f*;

(2) *Two vectors $(\cos\theta_0, \sin\theta_0)$ and $(-\sin\theta_0, \cos\theta_0)$ are eigenvectors of* $\mathrm{Hess}_f(\theta_0)$;

(3) *A vector $\mathrm{grad}_f(\theta_0)$ is represented as*

$$\mathrm{grad}_f(\theta_0) = k\tilde{f}(\theta_0)\begin{pmatrix}\cos\theta_0\\ \sin\theta_0\end{pmatrix};$$

(4) *A tangent vector $\cos\theta_0(\partial/\partial x) + \sin\theta_0(\partial/\partial y)$ is in the principal directions at $(r\cos\theta_0, r\sin\theta_0)$.*

Proof. We set

$$\tilde{d}_f(\theta) := d_f(\theta,\theta), \quad \tilde{n}_f(\theta) := n_f(\theta,\theta).$$

From (2) of Lemma 2.1, we obtain $\tilde{n}_f \equiv 0$. By (2.2) and by (1) of Lemma 2.1, we obtain

$$\tilde{d}_f(\theta) = (k-1)\frac{d\tilde{f}}{d\theta}(\theta) \tag{2.4}$$

for any $\theta \in \boldsymbol{R}$. Therefore it is seen that (1) and (4) are equivalent. By

$$k\tilde{f}(\theta) = \left\langle \mathrm{grad}_f(\theta_0), \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right\rangle, \tag{2.5}$$

we see that (1) and (3) are equivalent. By (2.2) and by (2.5), we see that (2) and (3) are equivalent. □

Corollary 2.3. *If* $(r\cos\theta_0, r\sin\theta_0)$ *is an umbilical point of* G_f, *then* θ_0 *is an element of* R_f.

From now on, suppose that $f \in P_o^k$, and let r be a positive constant such that on $0 < x^2 + y^2 \leqq r^2$, there exists no umbilical point, and r_0 the supremum of such numbers as r. The argument function ϕ_{r,θ_0,ϕ_0} on $x^2 + y^2 = r^2$ with initial values (θ_0, ϕ_0) is characterized as the function satisfying $\phi_{r,\theta_0,\phi_0}(\theta_0) = \phi_0$ and

$$r^{k-2} d_f(\theta, \phi_{r,\theta_0,\phi_0}(\theta)) + r^{3k-4} n_f(\theta, \phi_{r,\theta_0,\phi_0}(\theta)) = 0 \tag{2.6}$$

for any $\theta \in \boldsymbol{R}$. For any $r \in (0, r_0)$ and for (θ_0, ϕ_0) satisfying (2.1), the argument function ϕ_{r,θ_0,ϕ_0} is smooth ([**1**]). For an integer $n \in \{0, 1, \ldots, \mu(\theta_0)\}$, the following holds ([**1**]):

$$\frac{d^n}{d\theta^n}(\theta - \phi_{r,\theta_0,\theta_0})\bigg|_{\theta=\theta_0} = \frac{\dfrac{d^n[\tilde{d}_f]}{d\theta^n}(\theta_0)}{\dfrac{\partial(d_f + r^{2k-2} n_f)}{\partial\phi}\bigg|_{(\theta,\phi)=(\theta_0,\theta_0)}}. \tag{2.7}$$

Therefore by (2.4) and by (2.7), we see that for a root θ_0 related to the origin, $\mathrm{sign}(\theta_0) = +$ (resp. $= -$) if and only if

$$\frac{d^{\mu(\theta_0)}}{d\theta^{\mu(\theta_0)}}(\theta - \phi_{r,\theta_0,\theta_0})\bigg|_{\theta=\theta_0} > 0 \ \text{(resp.} < 0).$$

3. The set of the umbilical points on a root line.

Let f be an element of P_o^k with $k \geqq 3$ and θ_0 an element of R_f. Let $\lambda_{\theta_0}^{(1)}$ be the eigenvalue of $\mathrm{Hess}_f(\theta_0)$ corresponding to an eigenvector $(\cos\theta_0, \sin\theta_0)$, and $\lambda_{\theta_0}^{(2)}$ the other eigenvalue of $\mathrm{Hess}_f(\theta_0)$.

Proposition 3.1. *There exists an umbilical point on* $L(\theta_0)\backslash\{o\}$ *if and only if*

$$(\lambda_{\theta_0}^{(1)} - \lambda_{\theta_0}^{(2)})(\lambda_{\theta_0}^{(1)})^2 \lambda_{\theta_0}^{(2)} > 0. \tag{3.1}$$

In addition, if (3.1) *holds, then the following holds*:

$$\mathrm{Umb}(G_f; L(\theta_0)) = \{\pm(r_{\theta_0}\cos\theta_0, r_{\theta_0}\sin\theta_0)\}, \tag{3.2}$$

where

$$r_{\theta_0} = \left\{\frac{(\lambda_{\theta_0}^{(1)} - \lambda_{\theta_0}^{(2)})(k-1)^2}{(\lambda_{\theta_0}^{(1)})^2 \lambda_{\theta_0}^{(2)}}\right\}^{1/(2k-2)}.$$

Proof. For any $\phi_0 \in \boldsymbol{R}$, the following hold:

$$\begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \end{pmatrix} = \cos(\phi_0 - \theta_0)\begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \end{pmatrix} + \sin(\phi_0 - \theta_0)\begin{pmatrix} -\sin\theta_0 \\ \cos\theta_0 \end{pmatrix}, \tag{3.3}$$

$$\begin{pmatrix} -\sin\phi_0 \\ \cos\phi_0 \end{pmatrix} = -\sin(\phi_0 - \theta_0)\begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \end{pmatrix} + \cos(\phi_0 - \theta_0)\begin{pmatrix} -\sin\theta_0 \\ \cos\theta_0 \end{pmatrix}. \tag{3.4}$$

Applying (3.3) and (3.4) to (1) of Lemma 2.1, we obtain

$$d_f(\theta_0, \phi_0) = (\lambda_{\theta_0}^{(2)} - \lambda_{\theta_0}^{(1)})\cos(\phi_0 - \theta_0)\sin(\phi_0 - \theta_0).$$

Applying (2.2) to (2) of Lemma 2.1, we obtain

$$n_f(\theta_0, \phi_0) = \frac{(\lambda_{\theta_0}^{(1)})^2 \lambda_{\theta_0}^{(2)}}{(k-1)^2}\cos(\phi_0 - \theta_0)\sin(\phi_0 - \theta_0).$$

Therefore noticing (2.1), we see that $\mathrm{Umb}(G_f; L(\theta_0)) \neq \emptyset$ holds if and only if (3.1) holds. We immediately obtain (3.2). □

If $\theta_0 \in R_f$, then from (2.2) and from (2.5), we obtain

$$\lambda_{\theta_0}^{(1)} = k(k-1)\tilde{f}(\theta_0). \tag{3.5}$$

Then by Proposition 2.2, we obtain

$$\|\mathrm{grad}_f(\theta_0)\| = \frac{|\lambda_{\theta_0}^{(1)}|}{k-1}. \tag{3.6}$$

Applying (3.6) to (2.3), we may represent Gaussian curvature K_f at a point $(r\cos\theta_0, r\sin\theta_0)$ as

$$K_f(r\cos\theta_0, r\sin\theta_0) = \frac{\lambda_{\theta_0}^{(1)}\lambda_{\theta_0}^{(2)} r^{2k-4}}{\left\{1 + \left(\frac{\lambda_{\theta_0}^{(1)}}{k-1}\right)^2 r^{2k-2}\right\}^2}.$$

Particularly the following holds:

$$\tilde{K}_f(\theta_0) = \frac{\lambda_{\theta_0}^{(1)}\lambda_{\theta_0}^{(2)}}{\left\{1 + \left(\frac{\lambda_{\theta_0}^{(1)}}{k-1}\right)^2\right\}^2}. \tag{3.7}$$

We shall prove

Proposition 3.2. *Let θ_0 be a number such that $\tilde{K}_f(\theta_0) = 0$. Then*
(1) *just one of $\lambda^{(1)}_{\theta_0}$ and $\lambda^{(2)}_{\theta_0}$ is equal to* 0;
(2) *The following holds*:

$$\mathrm{Umb}(G_f; L(\theta_0)) = \emptyset.$$

Proof. Since $\tilde{K}_f(\theta_0) = 0$, it follows from (3.7) that $\lambda^{(1)}_{\theta_0} = 0$ or $\lambda^{(2)}_{\theta_0} = 0$. However from $f \in P^k_o$, we see that just one of $\lambda^{(1)}_{\theta_0}$ and $\lambda^{(2)}_{\theta_0}$ is nonzero. If $\theta_0 \in R_f$, then from Proposition 3.1, we see that $\mathrm{Umb}(G_f; L(\theta_0)) = \emptyset$. If $\theta_0 \notin R_f$, then from Corollary 2.3, we see that $\mathrm{Umb}(G_f; L(\theta_0)) = \emptyset$. □

Corollary 3.3. *For any $\theta_0 \in R_f$, either $\tilde{f}(\theta_0)$ or $(d^2\tilde{f}/d\theta^2)(\theta_0)$ is not equal to* 0.

Proof. By (2.2), we obtain

$$\frac{d^2\tilde{f}}{d\theta^2}(\theta_0) = -\frac{1}{k-1}\left\langle \begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \end{pmatrix}, \mathrm{Hess}_f(\theta_0)\begin{pmatrix} \cos\theta_0 \\ \sin\theta_0 \end{pmatrix}\right\rangle + \left\langle \mathrm{Hess}_f(\theta_0)\begin{pmatrix} -\sin\theta_0 \\ \cos\theta_0 \end{pmatrix}, \begin{pmatrix} -\sin\theta_0 \\ \cos\theta_0 \end{pmatrix}\right\rangle. \tag{3.8}$$

Therefore we obtain

$$\frac{d^2\tilde{f}}{d\theta^2}(\theta_0) = -\frac{1}{k-1}\lambda^{(1)}_{\theta_0} + \lambda^{(2)}_{\theta_0}. \tag{3.9}$$

Therefore from (3.5), (3.9) and from (1) of Proposition 3.2, we obtain Corollary 3.3. □

We want to construct a map $\mathrm{D.Q}_f$ from R_f to $\tilde{\mathbf{R}} := \boldsymbol{R} \cup \{\infty\}$. Suppose that θ_0 is a root of f at which $\tilde{f}(\theta_0) = 0$. Then we set $\mathrm{D.Q}_f(\theta_0) := \infty$. Suppose that θ_0 is a root of f at which $\tilde{f}(\theta_0) \neq 0$. Then we set

$$\mathrm{D.Q}_f(\theta_0) := \frac{d^2\tilde{f}}{d\theta^2}(\theta_0)\Big/\tilde{f}(\theta_0).$$

The value $\mathrm{D.Q}_f(\theta_0)$ is called *the determinant quotient at θ_0*. By (3.5) and by (3.9), we obtain

$$\frac{d^2\tilde{f}}{d\theta^2}(0)\Big/k(k-1)\tilde{f}(\theta_0) = \frac{\lambda^{(2)}_{\theta_0}}{\lambda^{(1)}_{\theta_0}} - \frac{1}{k-1}. \tag{3.10}$$

From (3.10), we obtain

LEMMA 3.4. *Let θ_0 be a root at which $\tilde{f}(\theta_0) \neq 0$. Then the following holds*:

$$\frac{\lambda_{\theta_0}^{(2)}}{\lambda_{\theta_0}^{(1)}} = \frac{1}{k-1}\left\{1 + \frac{\mathrm{D.Q}_f(\theta_0)}{k}\right\}.$$

For any $a \in \boldsymbol{R}$, we define the subset $[a, \infty]$ (resp. $[\infty, a]$) of $\tilde{\mathbf{R}}$ by

$$\{x \in \mathbf{R}; a \leqq x \ \ (\text{resp. } \ x \leqq a)\} \cup \{\infty\}.$$

Similarly we define the subsets

$$[a, \infty), (a, \infty], (a, \infty), (\infty, a], [\infty, a), (\infty, a)$$

of $\tilde{\mathbf{R}}$.

Noticing (3.7) and Lemma 3.4, we obtain

PROPOSITION 3.5. *Let θ_0 be a root. Then*
(1) $\tilde{K}_f(\theta_0) = 0$ *if and only if* $\mathrm{D.Q}_f(\theta_0) = -k$ *or* ∞;
(2) $\tilde{K}_f(\theta_0) > 0$ *if and only if* $\mathrm{D.Q}_f(\theta_0) \in (-k, \infty)$;
(3) $\tilde{K}_f(\theta_0) < 0$ *if and only if* $\mathrm{D.Q}_f(\theta_0) \in (\infty, -k)$.

In this section, an element of R_f related to the origin is merely called *a related root*.

PROPOSITION 3.6. *If θ_0 is a root such that $\tilde{K}_f(\theta_0) = 0$, then θ_0 is a related root with* c-sign$(\theta_0) = +$.

PROOF. We see from Proposition 3.5 that $\mathrm{D.Q}_f(\theta_0) = -k$ or ∞. If $\mathrm{D.Q}_f(\theta_0) = -k$, then we see that $\mu(\theta_0) = 1$ and that θ_0 is a related root with c-sign$(\theta_0) = +$. If $\mathrm{D.Q}_f(\theta_0) = \infty$, then it follows that $\tilde{f}(\theta_0) = 0$. Then from Corollary 3.3, we obtain $(d^2\tilde{f}/d\theta)(\theta_0) \neq 0$. Therefore we see that $\mu(\theta_0) = 1$, which implies that θ_0 is a related root. Since $\tilde{f}(\theta_0) = 0$, it follows that c-sign$(\theta_0) = +$. □

From Proposition 3.1 and from Lemma 3.4, we see that $\mathrm{Umb}(G_f; L(\theta_0)) \neq \emptyset$ if and only if

$$\frac{1}{k-1}\left\{1 + \frac{\mathrm{D.Q}_f(\theta_0)}{k}\right\} \in (0, 1).$$

Therefore we obtain

PROPOSITION 3.7. *Let θ_0 be a root and $L(\theta_0)$ the root line determined by θ_0. Then* $\mathrm{Umb}(G_f; L(\theta_0)) \neq \emptyset$ *if and only if the determinant quotient* $\mathrm{D.Q}_f(\theta_0)$ *satisfies*

$$\mathrm{D.Q}_f(\theta_0) \in (-k, k(k-2)). \tag{3.11}$$

In addition, if (3.11) *holds, then the following holds*:

$$\sharp \mathrm{Umb}(G_f; L(\theta_0)) = 2.$$

Corollary 3.8. *Let θ_0 be a root such that $\mu(\theta_0) \geqq 2$. Then the following holds*:

$$\sharp \mathrm{Umb}(G_f; L(\theta_0)) = 2.$$

Particularly Corollary 3.8 implies Proposition 1.1.

Proposition 3.9. *Let f be a homogeneous polynomial of degree $k \geqq 3$ satisfying $\tilde{f} \not\equiv 0$ and $d\tilde{f}/d\theta \equiv 0$, and G_f the graph of f. Then the set* $\mathrm{Umb}(G_f)$ *of the umbilical points on G_f is represented as follows*:

$$\mathrm{Umb}(G_f) = \{o\} \sqcup \{x^2 + y^2 = c_0^2\}, \tag{3.12}$$

where c_0 is a nonzero number.

Proof. For a homogeneous polynomial f satisfying $\tilde{f} \not\equiv 0$ and $d\tilde{f}/d\theta \equiv 0$, it is obvious that $\mathrm{D.Q}_f(\theta_0) = 0$ for any $\theta_0 \in \boldsymbol{R}$. Therefore if the degree k of f is not less than 3, then we see from Proposition 3.7 that $\sharp\mathrm{Umb}(G_f; L) = 2$ for any straight line L on $\boldsymbol{R}^2$ through o. From Lemma 3.4, we obtain

$$\frac{\lambda^{(2)}_{\theta_0}}{\lambda^{(1)}_{\theta_0}} = \frac{1}{k-1} \tag{3.13}$$

for any $\theta_0 \in \boldsymbol{R}$. By Proposition 3.1, (3.5) and by (3.13), we see that for any $\theta_0 \in \boldsymbol{R}$, the set of the umbilical points on $L(\theta_0)$ is represented as

$$\left\{ o, \pm\left(\frac{k-2}{k^2\tilde{f}(\theta_0)^2}\right)^{1/2(k-1)} (\cos\theta_0, \sin\theta_0) \right\}.$$

Since $\tilde{f}$ is a constant function, we see that the set $\mathrm{Umb}(G_f)$ is represented as

$$\mathrm{Umb}(G_f) = \{o\} \sqcup \left\{ x^2 + y^2 = \left(\frac{k-2}{k^2c^2}\right)^{1/(k-1)} \right\},$$

where c is a nonzero number. Hence we have proved Proposition 3.9. □

Remark 3.10. If f satisfies $d\tilde{f}/d\theta \equiv 0$, then k is even and f is represented by $(x^2 + y^2)^{k/2}$ with constant multiplication ([**1**]).

From now on, we suppose that $d\tilde{f}/d\theta \not\equiv 0$.

Proposition 3.11. *Let f be an element of P_o^k with $k \geqq 3$. Then the following holds*:

$$\sharp \mathrm{Umb}(G_f) \in \{2i+1\}_{i=0}^k.$$

Proof. If we set

$$D_f(x,y) := x^2 s_f(x,y) + xy\{t_f(x,y) - r_f(x,y)\} - y^2 s_f(x,y),$$

then we see that $D_f(x,y)$ is a homogeneous polynomial of degree k and that $\tilde{d}_f(\theta) = D_f(\cos\theta, \sin\theta)$. Therefore we see that the number $\sharp \tilde{R}_f$ is less than or equal to k. Noticing Corollary 2.3 and Proposition 3.7, we obtain $\sharp \mathrm{Umb}(G_f) \in \{2i+1\}_{i=0}^k$. □

Proposition 3.12 ([**1**]). *Let θ_0 be a related root with* c-sign$(\theta_0) = +$. *Then* sign$(\theta_0) = +$ *holds.*

Proof. We shall show

$$\frac{\partial d_f}{\partial \phi}(\theta_0, \theta_0) \frac{d^{\mu(\theta_0)+1} \tilde{f}}{d\theta^{\mu(\theta_0)+1}}(\theta_0) > 0. \tag{3.14}$$

From (2.7) and from (3.14), we obtain sign$(\theta_0) = +$.

By (1) of Lemma 2.1, we obtain

$$\frac{\partial d_f}{\partial \phi}(\theta_0, \theta_0) = \lambda_{\theta_0}^{(2)} - \lambda_{\theta_0}^{(1)}. \tag{3.15}$$

Therefore by (3.9) and by (3.15), we obtain

$$\frac{d^2 \tilde{f}}{d\theta^2}(\theta_0) = \frac{\partial d_f}{\partial \phi}(\theta_0, \theta_0) + \frac{k-2}{k-1} \lambda_{\theta_0}^{(1)}. \tag{3.16}$$

If $\tilde{f}(\theta_0) = 0$, then from (3.5) we obtain $(d^2 \tilde{f}/d\theta^2)(\theta_0) = (\partial d_f/\partial \phi)(\theta_0, \theta_0)$. By Corollary 3.3, we obtain (3.14). Suppose that $\tilde{f}(\theta_0) \neq 0$. Then we see from (3.5) and from (3.16) that

$$\mathrm{D.Q}_f(\theta_0) = \frac{1}{\tilde{f}(\theta_0)} \frac{\partial d_f}{\partial \phi}(\theta_0, \theta_0) + k(k-2). \tag{3.17}$$

Since c-sign$(\theta_0) = +$, we obtain

$$\frac{1}{\tilde{f}(\theta_0)} \frac{\partial d_f}{\partial \phi}(\theta_0, \theta_0) \leqq -k(k-2). \tag{3.18}$$

Therefore we see that

$$\frac{\partial d_f}{\partial \phi}(\theta_0,\theta_0)\frac{d^{\mu(\theta_0)+1}\tilde{f}}{d\theta^{\mu(\theta_0)+1}}(\theta_0) = \left[\frac{1}{\tilde{f}(\theta_0)}\frac{\partial d_f}{\partial \phi}(\theta_0,\theta_0)\right]\left[\tilde{f}(\theta_0)\frac{d^{\mu(\theta_0)+1}\tilde{f}}{d\theta^{\mu(\theta_0)+1}}(\theta_0)\right].$$

By c-sign$(\theta_0) = +$ and by (3.18), we obtain (3.14). □

Remark 3.13. We obtain (1) of Theorem 1.2, from Proposition 3.2, Proposition 3.6 and from Proposition 3.12.

We want to study the number $\sharp\mathrm{Umb}(G_f; L(\theta_0))$ determined by a related root θ_0 with c-sign$(\theta_0) = +$.

Proposition 3.14. *Let θ_0 be a related root with* c-sign$(\theta_0) = +$ *and $L(\theta_0)$ the root line determined by θ_0. Then $\sharp\mathrm{Umb}(G_f; L(\theta_0)) = 0$ (resp. $= 2$) if and only if $\tilde{K}_f(\theta_0) \leqq 0$ (resp. > 0).*

Proof. Since c-sign$(\theta_0) = +$, we see that $\mathrm{D.Q}_f(\theta_0) \in [\infty, 0]$. Therefore by Proposition 3.5 and by Proposition 3.7, we obtain Proposition 3.14. □

We want to study the number $\sharp\mathrm{Umb}(G_f; L(\theta_0))$ determined by a related root θ_0 with c-sign$(\theta_0) = -$. It is seen that $\mathrm{D.Q}_f(\theta_0) \in [0, \infty)$. Therefore from Proposition 3.5, we obtain

Proposition 3.15. *Let θ_0 be a related root with* c-sign$(\theta_0) = -$. *Then $\tilde{K}_f(\theta_0)$ is a positive number.*

Next, we shall prove

Lemma 3.16. *Let θ_0 be a related root satisfying $(\partial d_f/\partial\phi)(\theta_0, \theta_0) = 0$. Then the following holds*:

$$(\text{c-sign}(\theta_0), \text{sign}(\theta_0)) = (-, +).$$

Proof. Noticing (3.14), we see that c-sign$(\theta_0) = -$, and by (3.15) we obtain $\lambda^{(1)}_{\theta_0} = \lambda^{(2)}_{\theta_0}$. By (2) of Lemma 2.1 and by Proposition 3.15, we obtain $\lambda^{(1)}_{\theta_0}(\partial n_f/\partial\phi)(\theta_0, \theta_0) > 0$. Since c-sign$(\theta_0) = -$, we see from (2.7) that sign$(\theta_0) = +$. □

Proposition 3.17. *Let θ_0 be a related root with* c-sign$(\theta_0) = -$. *Then* sign$(\theta_0) = +$ *(resp. $= -$) if and only if $\sharp\mathrm{Umb}(G_f; L(\theta_0)) = 0$ (resp. $= 2$).*

Proof. If sign$(\theta_0) = +$, then by (2.7) we obtain $(1/\tilde{f}(\theta_0))(\partial d_f/\partial\phi)(\theta_0, \theta_0) \geqq 0$. By (3.17), we see that $\mathrm{D.Q}_f(\theta_0) \in [k(k-2), \infty)$. Therefore it follows from Proposition 3.7 that $\sharp\mathrm{Umb}(G_f; L(\theta_0)) = 0$.

Conversely, if $\sharp\mathrm{Umb}(G_f; L(\theta_0)) = 0$, then from Proposition 3.5, Proposition 3.7 and from Proposition 3.15, we see that $\mathrm{D.Q}_f(\theta_0) \in [k(k-2), \infty)$. If $\mathrm{D.Q}_f(\theta_0) \in (k(k-2), \infty)$, then it is seen that sign$(\theta_0) = +$. If $\mathrm{D.Q}_f(\theta_0) =$

$k(k-2)$, i.e., if $(\partial d_f/\partial\phi)(\theta_0,\theta_0)=0$, then it follows from Lemma 3.16 that $\mathrm{sign}(\theta_0)=+$.

Therefore considering Proposition 3.7, we obtain Proposition 3.17. □

From Proposition 3.12, we see that the critical sign of a negative root is negative. Therefore noticing Proposition 3.17, we obtain

Corollary 3.18. *Let θ_0 be a related root with* $\mathrm{sign}(\theta_0)=-$. *Then the following holds*:

$$\sharp\mathrm{Umb}(G_f;L(\theta_0))=2.$$

Noticing Corollary 3.8 and Proposition 3.17, we obtain

Corollary 3.19. *Let θ_0 be a related root such that* $(\mathrm{sign}(\theta_0),\text{c-sign}(\theta_0))=(+,-)$. *Then $\mu(\theta_0)=1$ holds.*

Proof of Theorem 1.2. Noticing Remark 3.13, suppose that θ_0 is a related root satisfying $\tilde{K}_f(\theta_0)\neq 0$. Then from Proposition 3.12, Proposition 3.14, Proposition 3.15 and from Proposition 3.17, we obtain Theorem 1.2. □

4. The behaviors of the principal distributions and of the gradient vector field.

Proposition 4.1. *Let f be an element of P_o^k. Then for a real number θ_0, the following are mutually equivalent*:

(1) *A real number θ_0 is an element of R_f^G*;

(2) *A real number θ_0 is an element of R_f or satisfies $\tilde{K}_f(\theta_0)=0$*;

(3) *Let ψ be an argument function of the gradient. Then a vector*

$$(\cos\psi(\theta_0),\sin\psi(\theta_0))$$

is an eigenvector of $\mathrm{Hess}_f(\theta_0)$ *corresponding to a nonzero eigenvalue*;

(4) *Let ϕ_0 be a number such that for a nonzero number r,* $\cos\phi_0(\partial/\partial x)+\sin\phi_0(\partial/\partial y)$ *is in the principal directions at $(r\cos\theta_0,r\sin\theta_0)$. Then for any $\rho\in\boldsymbol{R}$,*

$$\cos\phi_0\frac{\partial}{\partial x}+\sin\phi_0\frac{\partial}{\partial y},\quad -\sin\phi_0\frac{\partial}{\partial x}+\cos\phi_0\frac{\partial}{\partial y}$$

are in the principal directions at $(\rho\cos\theta_0,\rho\sin\theta_0)$, and

$$(\cos\phi_0,\sin\phi_0),\quad(-\sin\phi_0,\cos\phi_0).$$

are eigenvectors of $\mathrm{Hess}_f(\theta_0)$.

To prove Proposition 4.1, we need the following.

Lemma 4.2. *A number* θ_0 *is an element of* R_f *satisfying* $\tilde{f}(\theta_0) = 0$ *if and only if* θ_0 *satisfies* $\mathrm{grad}_f(\theta_0) = (0,0)$. *In addition, if* $\mathrm{grad}_f(\theta_0) = (0,0)$, *then the following hold*:

(1) $\tilde{K}_f(\theta_0) = 0$;

(2) *There exists an integer* n *such that*

$$\psi(\theta_0) = \theta_0 + \pi/2 + n\pi;$$

(3) *A vector* $(\cos\psi(\theta_0), \sin\psi(\theta_0))$ *is an eigenvector of* $\mathrm{Hess}_f(\theta_0)$ *corresponding to a nonzero eigenvalue.*

Proof. By Proposition 2.2, we see that a number θ_0 is an element of R_f satisfying $\tilde{f}(\theta_0) = 0$ if and only if θ_0 satisfies $\mathrm{grad}_f(\theta_0) = (0,0)$.

Suppose that $\mathrm{grad}_f(\theta_0) = (0,0)$. Then by (3.5), (3.7) and by $\tilde{f}(\theta_0) = 0$, we obtain $\tilde{K}_f(\theta_0) = 0$. A homogeneous polynomial $f(x,y)$ is represented as

$$f(x,y) = \{-(\sin\theta_0)x + (\cos\theta_0)y\}^2 g(x,y), \tag{4.1}$$

where $g(x,y)$ is a homogeneous polynomial such that $\tilde{g}(\theta_0) \neq 0$. Then we see that

$$\mathrm{grad}_f(\theta) = \sin(\theta-\theta_0)\left\{\begin{pmatrix} -2\sin\theta_0 \\ 2\cos\theta_0 \end{pmatrix}\tilde{g}(\theta) + \sin(\theta-\theta_0)\,\mathrm{grad}_g(\theta)\right\}. \tag{4.2}$$

Therefore we see that there exists an integer n satisfying (2) of Lemma 4.2. From (2.2), we see that $(\cos\theta_0, \sin\theta_0)$ is an eigenvector of $\mathrm{Hess}_f(\theta_0)$ corresponding to an eigenvalue 0. Therefore we see from Proposition 3.2 and from (2) of Lemma 4.2 that $(\cos\psi(\theta_0), \sin\psi(\theta_0))$ is an eigenvector of $\mathrm{Hess}_f(\theta_0)$ corresponding to the nonzero eigenvalue. □

We shall prove Proposition 4.1.

Proof of (4) from (2). If θ_0 is a root of f, then we see from Proposition 2.2 that a vector $\cos\theta_0(\partial/\partial x) + \sin\theta_0(\partial/\partial y)$ is in the principal directions at $(\rho\cos\theta_0, \rho\sin\theta_0)$ and that two vectors $(\cos\theta_0, \sin\theta_0)$ and $(-\sin\theta_0, \cos\theta_0)$ are eigenvectors of $\mathrm{Hess}_f(\theta_0)$. By Lemma 2.1 and by Proposition 2.2, we see that $-\sin\theta_0(\partial/\partial x) + \cos\theta_0(\partial/\partial y)$ is in the principal directions at $(\rho\cos\theta_0, \rho\sin\theta_0)$.

If a number θ_0 satisfies $\tilde{K}_f(\theta_0) = 0$, then from (2) of Lemma 2.1, we see that $n_f(\theta_0, \phi) = 0$ for any $\phi \in \boldsymbol{R}$. Let $\cos\phi_0(\partial/\partial x) + \sin\phi_0(\partial/\partial y)$ be in the principal directions at $(r\cos\theta_0, r\sin\theta_0)$. Then from $n_f(\theta_0, \phi_0) = 0$, we obtain $d_f(\theta_0, \phi_0) = 0$. Then we also obtain $d_f(\theta_0, \phi_0 + \pi/2) = 0$. Therefore we see that

$$\cos\phi_0\frac{\partial}{\partial x} + \sin\phi_0\frac{\partial}{\partial y}, \quad -\sin\phi_0\frac{\partial}{\partial x} + \cos\phi_0\frac{\partial}{\partial y}$$

are in the principal directions at $(\rho\cos\theta_0, \rho\sin\theta_0)$, and that $(\cos\phi_0, \sin\phi_0)$ and $(-\sin\phi_0, \cos\phi_0)$ are eigenvectors of $\mathrm{Hess}_f(\theta_0)$. Hence we have proved (4) from (2).

Proof of (2) from (4). If (4) in Proposition 4.1 holds, then we obtain $n_f(\theta_0, \phi_0) = 0$. Noticing Lemma 4.2, we suppose that $\mathrm{grad}_f(\theta_0) \neq (0,0)$. Then we may suppose that ϕ_0 satisfies

$$\left\langle \mathrm{grad}_f(\theta_0), \begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \end{pmatrix} \right\rangle \neq 0.$$

Therefore by (2) of Lemma 2.1 and by Proposition 2.2, we see that $\theta_0 \in R_f$ or that $\tilde{K}_f(\theta_0) = 0$. Hence we have proved (2) from (4).

Proof of (3) from (2). Let θ_0 be an element of R_f with $\tilde{f}(\theta_0) \neq 0$. Then by (2.2) and by Proposition 2.2, we see that $(\cos\psi(\theta_0), \sin\psi(\theta_0))$ is an eigenvector of $\mathrm{Hess}_f(\theta_0)$ corresponding to a nonzero eigenvalue.

By (2.2), we see that for $\phi_0 \in \boldsymbol{R}$, the following holds:

$$(4.3)\quad (k-1)\,\mathrm{grad}_f(\theta_0) = \mathrm{Hess}_f(\theta_0)\left\{\cos(\theta_0-\phi_0)\begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \end{pmatrix} + \sin(\theta_0-\phi_0)\begin{pmatrix} -\sin\phi_0 \\ \cos\phi_0 \end{pmatrix}\right\}.$$

We choose as ϕ_0 a number such that $(\cos\phi_0, \sin\phi_0)$ is an eigenvector of $\mathrm{Hess}_f(\theta_0)$. If $\theta_0 \notin R_f$, then it follows that $\tilde{K}_f(\theta_0) = 0$. Then just one of the eigenvalues of $\mathrm{Hess}_f(\theta_0)$ is zero. By Lemma 4.2, we see that $\mathrm{grad}_f(\theta_0) \neq (0,0)$. Therefore we see from (4.3) that a vector $(\cos\psi(\theta_0), \sin\psi(\theta_0))$ is an eigenvector of $\mathrm{Hess}_f(\theta_0)$ corresponding to the nonzero eigenvalue. Hence we have proved (3) from (2).

Proof of (2) from (3). We suppose that a vector $(\cos\psi(\theta_0), \sin\psi(\theta_0))$ is an eigenvector of $\mathrm{Hess}_f(\theta_0)$. Moreover noticing Lemma 4.2, we suppose that $\mathrm{grad}_f(\theta_0) \neq (0,0)$. Then by Proposition 2.2 and by (4.3), we see that $\theta_0 \in R_f$ or that just one of the eigenvalues of $\mathrm{Hess}_f(\theta_0)$ is zero. Hence we have proved (2) from (3).

Proof of (3) from (1). We suppose that $\tilde{p}_f(\theta_0)(\partial/\partial x) + \tilde{q}_f(\theta_0)(\partial/\partial y)$ is in the principal directions at $(\cos\theta_0, \sin\theta_0)$, and that $\mathrm{grad}_f(\theta_0) \neq (0,0)$. Then we may suppose that

$$(4.4)\qquad \frac{1}{\|\mathrm{grad}_f(\theta_0)\|}\mathrm{grad}_f(\theta_0) = \begin{pmatrix} \cos\psi(\theta_0) \\ \sin\psi(\theta_0) \end{pmatrix}.$$

A number $\psi(\theta_0)$ satisfies the equation

$$d_f(\theta_0, \psi(\theta_0)) + \rho^{2k-2} n_f(\theta_0, \psi(\theta_0)) = 0.$$

By direct computations, we obtain

$$n_f(\theta_0, \psi(\theta_0)) = \|\mathrm{grad}_f(\theta_0)\|^2 d_f(\theta_0, \psi(\theta_0)). \tag{4.5}$$

Therefore we obtain

$$\{1 + \rho^{2k-2}\|\mathrm{grad}_f(\theta_0)\|^2\} d_f(\theta_0, \psi(\theta_0)) = 0,$$

which implies that $d_f(\theta_0, \psi(\theta_0)) = 0$. Therefore noticing (1) of Lemma 2.1 and (4.3), we obtain (3) from (1).

Proof of (1) from (3). Suppose that (3) in Proposition 4.1 holds. Then the number $\psi(\theta_0)$ satisfies $d_f(\theta_0, \psi(\theta_0)) = 0$. We may suppose that $\mathrm{grad}_f(\theta_0) \neq 0$ and that $\psi(\theta_0)$ satisfies (4.4). Then by (4.5), we see that θ_0 is an element of R_f^G.

Hence we have proved Proposition 4.1.

Corollary 4.3. *If $\theta_0 \in R_f^G$ satisfies $\tilde{f}(\theta_0) = 0$, then $\theta_0 \in R_f$ holds.*

Proof. Noticing (2) of Propostion 4.1, we may suppose that $\tilde{K}_f(\theta_0) = 0$. Then just one of the eigenvalues of $\mathrm{Hess}_f(\theta_0)$ is zero. By (2.2), (2.5) and by (1) of Lemma 2.1, we see that $(\cos\theta_0, \sin\theta_0)$ is an eigenvector of $\mathrm{Hess}_f(\theta_0)$ corresponding to the zero eigenvalue and that $(-\sin\theta_0, \cos\theta_0)$ is also an eigenvector of $\mathrm{Hess}_f(\theta_0)$. Then Proposition 2.2 says that $\theta_0 \in R_f$. □

For $\theta_0 \in R_f^G$, there exists a positive number $\varepsilon_0 > 0$ such that each element of $(\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0) \backslash \{\theta_0\}$ is not an element of R_f^G. Let $\eta_{\theta_0}(\theta)$ be a continuous function on $(\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0)$ such that

$$\boldsymbol{e}_{\theta_0}^{(1)}(\theta) := \begin{pmatrix} \cos\eta_{\theta_0}(\theta) \\ \sin\eta_{\theta_0}(\theta) \end{pmatrix}, \quad \boldsymbol{e}_{\theta_0}^{(2)}(\theta) := \begin{pmatrix} -\sin\eta_{\theta_0}(\theta) \\ \cos\eta_{\theta_0}(\theta) \end{pmatrix}$$

are eigenvectors of $\mathrm{Hess}_f(\theta)$, and $\lambda_{\theta_0}^{(1)}(\theta)$, $\lambda_{\theta_0}^{(2)}(\theta)$ the eigenvalues of $\mathrm{Hess}_f(\theta)$ corresponding to $\boldsymbol{e}_{\theta_0}^{(1)}(\theta)$, $\boldsymbol{e}_{\theta_0}^{(2)}(\theta)$, respectively.

Lemma 4.4. *An argument function ψ of the gradient satisfies $\psi(\theta_0) \in \{\eta_{\theta_0}(\theta_0) + n\pi/2; n \in \mathbf{Z}\}$ if and only if $\lambda_{\theta_0}^{(1)}(\theta_0) \neq \lambda_{\theta_0}^{(2)}(\theta_0)$.*

Proof. If $\lambda_{\theta_0}^{(1)}(\theta_0) \neq \lambda_{\theta_0}^{(2)}(\theta_0)$, then by Proposition 4.1, we obtain $\psi(\theta_0) \in \{\eta_{\theta_0}(\theta_0) + n\pi/2; n \in \mathbf{Z}\}$.

Suppose that $\lambda_{\theta_0}^{(1)}(\theta_0) = \lambda_{\theta_0}^{(2)}(\theta_0) = 1$ and that $\theta_0 = 0$. Then f is represented as

$$f(x, y) = \frac{1}{k(k-1)} x^k + \frac{1}{2} x^{k-2} y^2 + g(x, y) y^3,$$

where g is a homogeneous polynomial of degree $k-3$. We obtain

$$\mathrm{Hess}_f(\theta) = (\cos^{k-2}\theta)E + (k-2)(\cos^{k-3}\theta\sin\theta)\begin{pmatrix} 0 & 1 \\ 1 & -2c \end{pmatrix} + (\sin^2\theta)M(\theta),$$

where $c \in \boldsymbol{R}$ and $M(\theta)$ is a continuous, matrix-valued function. Then we see that $\cot 2\eta_0(0) = c$, which implies that $\eta_0(0) \notin \{n\pi/2; n \in \boldsymbol{Z}\}$. On the other hand, by (2.2), we see that ψ satisfies $\psi(0) \in \{n\pi; n \in \mathbf{Z}\}$. Therefore we obtain $\psi(0) \notin \{\eta_0(0) + n\pi/2; n \in \mathbf{Z}\}$.

Hence we have proved Lemma 4.4. $\square$

Suppose that $\lambda_{\theta_0}^{(1)}(\theta_0) = \lambda_{\theta_0}^{(2)}(\theta_0)$. Then noticing (2.2) and Lemma 4.4, we suppose that there exists the argument function ψ_{θ_0} of the gradient satisfying

$$\psi_{\theta_0}(\theta_0) = \theta_0 \in (\eta_{\theta_0}(\theta_0) - \pi/2, \eta_{\theta_0}(\theta_0) + \pi/2)\backslash\{\eta_{\theta_0}(\theta_0)\}.$$

Suppose that $\lambda_{\theta_0}^{(1)}(\theta_0) \neq \lambda_{\theta_0}^{(2)}(\theta_0)$. Then noticing Lemma 4.4, we suppose that there exists the argument function ψ_{θ_0} of the gradient such that $\psi_{\theta_0}(\theta_0) = \eta_{\theta_0}(\theta_0)$. Then by Proposition 4.1, we see that $\lambda_{\theta_0}^{(1)}(\theta) \neq 0$ for $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0)$. In addition, noticing Proposition 2.2, Proposition 4.1, Lemma 4.2 and Corollary 4.3, we suppose that

$$\begin{cases} \psi_{\theta_0}(\theta_0) = \theta_0, & \text{if } \lambda_{\theta_0}^{(1)}(\theta_0)\lambda_{\theta_0}^{(2)}(\theta_0) \neq 0, \\ \psi_{\theta_0}(\theta_0) = \theta_0 + \pi/2, & \text{if } \tilde{f}(\theta_0) = 0, \\ |\psi_{\theta_0}(\theta_0) - \theta_0| \in (0, \pi/2), & \text{if } \theta_0 \in R_f^G \backslash R_f. \end{cases}$$

We set

$$\Lambda_{\theta_0}(\theta) := \frac{\lambda_{\theta_0}^{(2)}(\theta)}{\lambda_{\theta_0}^{(1)}(\theta)}.$$

Then by Proposition 4.1, we see that $\Lambda_{\theta_0}(\theta) \neq 0, 1$ for any $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0)\backslash \{\theta_0\}$.

Lemma 4.5. *Let θ_0 be an element of R_f^G.*

(1) *If $\tilde{f}(\theta_0) \neq 0$, then for any $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0)\backslash\{\theta_0\}$, the following hold*:

$$\{\theta - \psi_{\theta_0}(\theta)\}\{\psi_{\theta_0}(\theta) - \eta_{\theta_0}(\theta)\}\{1 - \Lambda_{\theta_0}(\theta)\}\Lambda_{\theta_0}(\theta) > 0, \tag{4.6}$$

$$\{\theta - \eta_{\theta_0}(\theta)\}\{\psi_{\theta_0}(\theta) - \eta_{\theta_0}(\theta)\}\Lambda_{\theta_0}(\theta) > 0, \tag{4.7}$$

(2) *If $\tilde{f}(\theta_0) = 0$, then for any $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0)\backslash\{\theta_0\}$, the following holds*:

$$(\theta - \theta_0)\{\psi_{\theta_0}(\theta) - \eta_{\theta_0}(\theta)\}\{1 - \Lambda_{\theta_0}(\theta)\}\Lambda_{\theta_0}(\theta) < 0.$$

Proof. For $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0)$, the following holds:

$$(4.8)\qquad \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \cos(\theta - \eta_{\theta_0}(\theta))\boldsymbol{e}^{(1)}_{\theta_0}(\theta) + \sin(\theta - \eta_{\theta_0}(\theta))\boldsymbol{e}^{(2)}_{\theta_0}(\theta).$$

By (2.2) and by (4.8), we obtain

$$(4.9)\qquad \mathrm{grad}_f(\theta) = \frac{1}{k-1}\{\cos(\theta - \eta_{\theta_0}(\theta))\lambda^{(1)}_{\theta_0}(\theta)\boldsymbol{e}^{(1)}_{\theta_0}(\theta) + \sin(\theta - \eta_{\theta_0}(\theta))\lambda^{(2)}_{\theta_0}(\theta)\boldsymbol{e}^{(2)}_{\theta_0}(\theta)\}.$$

Therefore we see that for $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0)\backslash\{\theta_0\}$, there exists the nonzero number $c(\theta)$ satisfying

$$(4.10)\qquad \begin{pmatrix} \cos\psi_{\theta_0}(\theta) \\ \sin\psi_{\theta_0}(\theta) \end{pmatrix} = \frac{c(\theta)}{k-1}\{\cos(\theta - \eta_{\theta_0}(\theta))\lambda^{(1)}_{\theta_0}(\theta)\boldsymbol{e}^{(1)}_{\theta_0}(\theta) + \sin(\theta - \eta_{\theta_0}(\theta))\lambda^{(2)}_{\theta_0}(\theta)\boldsymbol{e}^{(2)}_{\theta_0}(\theta)\}.$$

Suppose that $\tilde{f}(\theta_0) \neq 0$. Then we see that $|\theta_0 - \eta_{\theta_0}(\theta_0)| < \pi/2$. From (4.8) and from (4.10), we see that for $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0)\backslash\{\theta_0\}$,

$$\begin{cases} \psi_{\theta_0}(\theta) < \theta < \eta_{\theta_0}(\theta) \quad \text{or} \quad \eta_{\theta_0}(\theta) < \theta < \psi_{\theta_0}(\theta), & \text{if } \Lambda_{\theta_0}(\theta) > 1, \\ \theta < \psi_{\theta_0}(\theta) < \eta_{\theta_0}(\theta) \quad \text{or} \quad \eta_{\theta_0}(\theta) < \psi_{\theta_0}(\theta) < \theta, & \text{if } \Lambda_{\theta_0}(\theta) \in (0,1), \\ \psi_{\theta_0}(\theta) < \eta_{\theta_0}(\theta) < \theta \quad \text{or} \quad \theta < \eta_{\theta_0}(\theta) < \psi_{\theta_0}(\theta), & \text{if } \Lambda_{\theta_0}(\theta) < 0. \end{cases}$$

Hence we obtain (4.6).

We set

$$c_1(\theta) := \frac{c(\theta)\cos(\theta - \eta_{\theta_0}(\theta))\lambda^{(1)}_{\theta_0}(\theta)}{k-1}, \quad c_2(\theta) := \frac{c(\theta)\sin(\theta - \eta_{\theta_0}(\theta))\lambda^{(2)}_{\theta_0}(\theta)}{k-1}.$$

Then for any $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0)\backslash\{\theta_0\}$, we see that $c_1(\theta) > 0$ and that $\{\psi_{\theta_0}(\theta) - \eta_{\theta_0}(\theta)\}c_2(\theta) > 0$. Therefore we obtain

$$(4.11)\qquad c_1(\theta)c_2(\theta)\{\psi_{\theta_0}(\theta) - \eta_{\theta_0}(\theta)\} > 0$$

for any $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0)\backslash\{\theta_0\}$. Suppose that $\tilde{f}(\theta_0) \neq 0$. Then noticing $|\theta_0 - \eta_{\theta_0}(\theta_0)| < \pi/2$, we obtain (4.7). Suppose that $\tilde{f}(\theta_0) = 0$. Then noticing $\eta_{\theta_0}(\theta_0) = \theta_0 + \pi/2$, we see that

$$(4.12)\qquad (\psi_{\theta_0}(\theta) - \eta_{\theta_0}(\theta))\Lambda_{\theta_0}(\theta)\cos(\theta - \eta_{\theta_0}(\theta)) < 0$$

for $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0)\backslash\{\theta_0\}$. By (4.1), we see that $\tilde{g}(\theta)/\lambda^{(1)}_{\theta_0}(\theta) > 0$ for $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0)$. Then from (4.2), we obtain

$$(4.13)\qquad \frac{(\theta-\theta_0)}{\lambda^{(1)}_{\theta_0}(\theta)}\left\langle \mathrm{grad}_f(\theta), \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}\right\rangle > 0$$

for $\theta \in (\theta_0-\varepsilon_0, \theta_0+\varepsilon_0)\backslash\{\theta_0\}$. The following holds:

$$(4.14)\qquad \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = -\sin(\theta-\eta_{\theta_0}(\theta))\boldsymbol{e}^{(1)}_{\theta_0}(\theta) + \cos(\theta-\eta_{\theta_0}(\theta))\boldsymbol{e}^{(2)}_{\theta_0}(\theta).$$

By (4.9) and by (4.14), we see that

$$(4.15)\qquad \frac{1}{\lambda^{(1)}_{\theta_0}(\theta)}\left\langle \mathrm{grad}_f(\theta), \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}\right\rangle$$
$$= \frac{1}{k-1}\cos(\theta-\eta_{\theta_0}(\theta))\sin(\theta-\eta_{\theta_0}(\theta))\{\Lambda_{\theta_0}(\theta)-1\}.$$

Therefore from (4.13) and from (4.15), we obtain

$$(4.16)\qquad (\theta-\theta_0)\{1-\Lambda_{\theta_0}(\theta)\}\cos(\theta-\eta_{\theta_0}(\theta)) > 0$$

for $\theta \in (\theta_0-\varepsilon, \theta_0+\varepsilon)\backslash\{\theta_0\}$. From (4.12) and from (4.16), we obtain (2) of Lemma 4.5. □

Let r be a positive constant such that on $0 < x^2+y^2 \leqq r^2$, there exists no umbilical point.

Lemma 4.6. *Let* θ_0 *be an element of* R^G_f.

(1) *If* $\tilde{f}(\theta_0) \neq 0$, *then for any* $\theta \in (\theta_0-\varepsilon_0, \theta_0+\varepsilon_0)\backslash\{\theta_0\}$, *the following holds*:

$$\{\theta-\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta)\}\{\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta)-\eta_{\theta_0}(\theta)\}\{1-\Lambda_{\theta_0}(\theta)\}\Lambda_{\theta_0}(\theta) < 0;$$

(2) *If* $\tilde{f}(\theta_0) = 0$, *then for any* $\theta \in (\theta_0-\varepsilon_0, \theta_0+\varepsilon_0)\backslash\{\theta_0\}$, *the following holds*:

$$\{\theta-\theta_0\}\{\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta)-\eta_{\theta_0}(\theta)\}\{1-\Lambda_{\theta_0}(\theta)\}\Lambda_{\theta_0}(\theta) > 0.$$

Proof. Noticing (1) of Lemma 2.1, we see that for $\theta \in (\theta_0-\varepsilon_0, \theta_0+\varepsilon_0)\backslash\{\theta_0\}$,

$$(4.17)\qquad \{\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta)-\eta_{\theta_0}(\theta)\}\{\lambda^{(1)}_{\theta_0}(\theta)-\lambda^{(2)}_{\theta_0}(\theta)\}d_f(\theta,\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta)) < 0.$$

Suppose that $\tilde{f}(\theta_0) \neq 0$. Then by (4.9), we obtain

$$\left\langle \mathrm{grad}_f(\theta_0), \begin{pmatrix} \cos\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta_0) \\ \sin\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta_0) \end{pmatrix}\right\rangle = \frac{\cos(\theta_0-\eta_{\theta_0}(\theta_0))\lambda^{(1)}_{\theta_0}(\theta_0)}{k-1}.$$

By (2) of Lemma 2.1, we see that for $\theta \in (\theta_0-\varepsilon_0, \theta_0+\varepsilon_0)\backslash\{\theta_0\}$,

$$(4.18)\qquad \{\theta-\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta)\}\lambda^{(2)}_{\theta_0}(\theta)n_f(\theta,\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta)) < 0.$$

From (4.17) and from (4.18), we obtain (1) of Lemma 4.6.

Suppose that $\tilde{f}(\theta_0) = 0$. Then by (4.2), we see that for $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0) \backslash \{\theta_0\}$,

$$\left\langle \mathrm{grad}_f(\theta), \begin{pmatrix} \cos\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta) \\ \sin\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta) \end{pmatrix} \right\rangle \sin(\theta - \theta_0) \sin(\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta) - \theta_0) \lambda^{(1)}_{\theta_0}(\theta) > 0.$$

Therefore we see from (2) of Lemma 2.1 that for $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0) \backslash \{\theta_0\}$,

$$(\theta - \theta_0)\lambda^{(2)}_{\theta_0}(\theta) n_f(\theta, \phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta)) > 0. \tag{4.19}$$

Therefore by (4.17) and by (4.19), we obtain (2) of Lemma 4.6. □

We shall prove

Proposition 4.7. *Let θ_0 be an element of $R^G_f \backslash R_f$. Then for $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0) \backslash \{\theta_0\}$, the following holds*:

$$\{\theta - \eta_{\theta_0}(\theta)\}\{\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta) - \psi_{\theta_0}(\theta)\}\tilde{K}_f(\theta) < 0.$$

Proof. Since $\theta_0 \neq \eta_{\theta_0}(\theta_0)$, we see that for $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0)$,

$$\{\theta - \eta_{\theta_0}(\theta)\}\{\theta - \phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta)\} > 0.$$

Therefore we see by (1) of Lemma 4.6 that

$$\{\theta - \eta_{\theta_0}(\theta)\}\{\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta) - \eta_{\theta_0}(\theta)\}\tilde{K}_f(\theta) < 0$$

for $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon) \backslash \{\theta_0\}$. Therefore by (4.7), we obtain Proposition 4.7. □

From Proposition 4.7, we obtain Proposition 1.4.

Proposition 4.8. *Let θ_0 be an element of R_f such that $\tilde{f}(\theta_0) = 0$.*

(1) *A root θ_0 is related to the origin and the gradient, and non-related to the curvature*;

(2) *The following holds*:

$$(\mathrm{sign}(\theta_0), \text{g-sign}(\theta_0), \mathrm{sign}[\tilde{K}_f(\theta_0)]) = (+, -, -).$$

Proof. From Lemma 4.2, we see that $\tilde{K}_f(\theta_0) = 0$. Therefore from (1) of Theorem 1.2, we see that θ_0 is related to the origin and satisfies $\mathrm{sign}(\theta_0) = +$. Noticing (2.3) and that f is represented as in (4.1), we see that $\tilde{K}_f(\theta) < 0$ for $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0) \backslash \{\theta_0\}$. Therefore we see that θ_0 is non-related to the curvature and satisfies $\mathrm{sign}[\tilde{K}_f(\theta_0)] = -$. By (2) of Lemma 4.5 and by (2) of Lemma 4.6, we see that for $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon) \backslash \{\theta_0\}$,

$$(\theta - \theta_0)\{\psi_{\theta_0}(\theta) - \eta_{\theta_0}(\theta)\} > 0,$$

$$(\theta - \theta_0)\{\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta) - \eta_{\theta_0}(\theta)\} < 0.$$

Therefore we see that θ_0 is related to the gradient and satisfies $\text{g-sign}(\theta_0) = -$.

Hence we have proved Proposition 4.8. □

We shall prove

THEOREM 4.9. *Let θ_0 be an element of R_f such that $\tilde{f}(\theta_0) \neq 0$. Then for $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0) \backslash \{\theta_0\}$, the following holds*:

$$\{\theta - \phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta)\}\{\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta) - \psi_{\theta_0}(\theta)\}\tilde{K}_f(\theta) < 0. \tag{4.20}$$

PROOF. For a number $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0) \backslash \{\theta_0\}$, one of the following holds:

$$(1)\ \Lambda_{\theta_0}(\theta) > 1, \quad (2)\ \Lambda_{\theta_0}(\theta) \in (0,1), \quad (3)\ \Lambda_{\theta_0}(\theta) < 0.$$

Suppose that $\Lambda_{\theta_0}(\theta) > 1$ for $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0) \backslash \{\theta_0\}$. Then by (1) of Lemma 4.5 and by (1) of Lemma 4.6, we see that

$$\{\theta - \psi_{\theta_0}(\theta)\}\{\psi_{\theta_0}(\theta) - \eta_{\theta_0}(\theta)\} < 0, \tag{4.21}$$

$$\{\theta - \eta_{\theta_0}(\theta)\}\{\psi_{\theta_0}(\theta) - \eta_{\theta_0}(\theta)\} > 0, \tag{4.21}$$

$$\{\theta - \phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta)\}\{\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta) - \eta_{\theta_0}(\theta)\} > 0. \tag{4.23}$$

From (4.23), we see that one of the following holds:

$$(1)\ \eta_{\theta_0}(\theta) < \phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta) < \theta, \quad (2)\ \theta < \phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta) < \eta_{\theta_0}(\theta).$$

Moreover from (4.21), we see that one of the following holds:

(1) $\psi_{\theta_0}(\theta) < \theta < \phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta) < \eta_{\theta_0}(\theta)$,
(2) $\theta < \phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta) < \eta_{\theta_0}(\theta) < \psi_{\theta_0}(\theta)$,
(3) $\psi_{\theta_0}(\theta) < \eta_{\theta_0}(\theta) < \phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta) < \theta$,
(4) $\eta_{\theta_0}(\theta) < \phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta) < \theta < \psi_{\theta_0}(\theta)$.

From (4.22), we see that (1) and (4) may happen, and that (2) and (3) may not happen. If θ satisfies (1) or (4), then we see that (4.20) holds.

Suppose that $\Lambda_{\theta_0}(\theta) \in (0,1)$ for $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0) \backslash \{\theta_0\}$. Then we see that one of the following holds:

(1) $\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta) < \theta < \psi_{\theta_0}(\theta) < \eta_{\theta_0}(\theta)$,
(2) $\theta < \psi_{\theta_0}(\theta) < \eta_{\theta_0}(\theta) < \phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta)$,
(3) $\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta) < \eta_{\theta_0}(\theta) < \psi_{\theta_0}(\theta) < \theta$,
(4) $\eta_{\theta_0}(\theta) < \psi_{\theta_0}(\theta) < \theta < \phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta)$.

If θ satisfies $(1), (2), (3)$ or (4), then we see that (4.20) holds.

Suppose that $\Lambda_{\theta_0}(\theta) < 0$ for $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0) \backslash \{\theta_0\}$. Then we see that one of the following holds:

(1) $\theta < \phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta) < \eta_{\theta_0}(\theta) < \psi_{\theta_0}(\theta)$,
(2) $\psi_{\theta_0}(\theta) < \theta < \phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta) < \eta_{\theta_0}(\theta)$.

If θ satisfies (1) or (2), then we see that (4.20) holds.

Hence we have proved Theorem 4.9. □

From Proposition 4.8 and from Theorem 4.9, we obtain Proposition 1.5. From Theorem 4.9, we see that for $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0) \backslash \{\theta_0\}$,

$$\{[\theta - \phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta)](\theta - \theta_0)\}\{[\phi_{r,\theta_0,\psi_{\theta_0}(\theta_0)}(\theta) - \psi_{\theta_0}(\theta)](\theta - \theta_0)\}\tilde{K}_f(\theta) < 0. \tag{4.24}$$

From Proposition 4.8 and from (4.24), we obtain Theorem 1.6.

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