J. Math. Soc. Japan
Vol. 53, No. 1, 2001

Realization of amenable bicategories and weak amenability of fusion algebras

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(Received Apr. 30, 1999)
(Revised Sept. 16, 1999)

Abstract. In this paper, we consider the realization problem of amenable bicategories by using AFD-bimodules. As a corollary, we can realize an amenable standard lattice as that of an AFD II_1-subfactor. This gives another proof of S. Popa's theorem. Weak amenability of fusion algebras is also discussed.

0. Introduction.

The index theory of subfactors was begun by V. Jones in his celebrated paper [**J**]. One of the most important problems in this theory is the classification of subfactors. In this respect, the standard invariant (or the standard lattice) is useful and important. A. Ocneanu introduced this kind of invariants as paragroups and announced that it is a complete invariant for isomorphism classes of finite depth subfactors. The announcement was shown by S. Popa in [**P1**]. Moreover, in [**P2**] he proved that strongly amenable subfactors are classified by standard invariants, and recently he showed that the invariant remains complete for amenable subfactors (see [**P7**]). In [**P2**] the notion of amenability for subfactors was introduced by S. Popa with some equivalent conditions (also see [**P4**], [**P6**], [**P7**]). In [**HI**] F. Hiai and M. Izumi studied amenability by using fusion algebras.

Another important invariant for subfactors is a tensor category. For a II_1-subfactor $N \subset M$, the bimodule ${}_N L^2(M)_M$ generates a (graded) tensor category and it includes all information of the standard invariant ([**O1**], [**O2**]). In [**Y1**], S. Yamagami proved that tensor categories consisting of bimodules have some rigidity which he called ε-structures. In our previous paper [**HY**], we studied tensor categories by using ε-structures and showed that (non-graded) amenable C^*-tensor categories can be realized by AFD-bimodules, where AFD-bimodules mean bimodules with two-side action of AFD II_1-factors. In this paper, we will show the graded version of this theorem. This type theorem for standard lattices was already established by S. Popa. In [**P5**], he proved that any given standard

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2000 *Mathematics Subject Classification*. 46L37.
Key Words and Phrases. Subfactor, tensor category, fusion algebra.

lattice can be realized by some subfactor which may be non-AFD, and he announced that if standard lattices are amenable, they can be realized by AFD-subfactors. We can give another proof of this theorem as a corollary of the main result in this paper, i.e., we can construct an AFD II_1-subfactor which has a preassigned amenable standard lattice as its higher relative commutants. Here it should be remarked that we cannot realize amenable bicategory only by applying the theorem of [**HY**]. For example, let $N \subset M$ be an extremal non-AFD II_1-subfactor with finite index such that $N \simeq M$. If $N \subset M$ has amenable graphs, its bicategory $\mathscr{C}$ is amenable in our sense. On the other hand, since N is isomorphic to M, we can imbed this bicategory into some non-graded C^*-tensor category $\mathscr{C}'$. Indeed, by using an automorphism $\alpha : N \to M$, N-M bimodules can be regarded as N-N ones. Hence if it is possible to choose α so that $\mathscr{C}'$ becomes amenable in the sense of [**HY**], we can realize it by AFD-bimodules as well as the bicategory $\mathscr{C}$. But it is not known whether such a choice of α is possible or not.

This paper is organized as follows. In Section 1 we review the notion of C^*-bicategories according to [**Y4**]. This is a graded version of C^*-tensor category defined in [**HY**]. A typical example of C^*-bicategories is a bicategory generated by subfactors. In Section 2 we construct commuting squares of AFD II_1-factors from a C^*-bicategory $\mathscr{C} = \bigcup_{i,j \in \{1,2\}} \mathscr{C}_{ij}$ by using an ergodic probability measure on the associated fusion algebra, and in Section 3 we construct bimodules. These constructions are essentially same as those of [**HY**], but some differences occur. For example, thanks to the ergodicity, $A_\infty(X)$ becomes a factor for any $X \in \mathrm{Object}(\mathscr{C}_{11})$. But if X is in $\mathscr{C}_{12}$, $A_\infty(X)$ may not be a factor. (In fact, this can occur.) This corresponds to the fact that the ergodicity of the principal graph of a subfactor does not always imply that of dual one. This kind of examples were found by U. Haagerup ([**Ha**], also see [**HI**, Example 8.11]). S. Popa showed in [**P2**] that the ergodicity of two graphs are mutually equivalent if they are amenable. This fact holds in our situation as well, i.e., if $\mathscr{C}$ is amenable, $A_\infty(X)$ is a factor for any X in $\mathscr{C}_{12}$. In Section 5 this fact is studied in detail and we show that for a probability measure ν on S_{12}, $\nu * \check{\nu}$ is ergodic if and only if $\check{\nu} * \nu$ is ergodic under the assumption of amenability. In Section 4 we investigate the weak amenability of fusion algebras. The notion of weak amenability was introduced by F. Hiai and M. Izumi in [**HI**] as a natural generalization of the amenability for groups. They showed that weak amenability is strictly weaker than amenability and they asked "Find a suitable condition under which weak amenability and amenability are equivalent." We give a partial answer to this problem, i.e., it is shown that for a fusion algebra coming from a C^*-tensor category (in particular, coming from bimodules), if it has "amenable generators", weak amenability and amenability are equivalent.

The author would like to express his hearty gratitude to Professor Shigeru Yamagami and Professor Fumio Hiai for fruitful discussions and comments on this subject. He is also grateful to the referee for careful reading and advice.

1. Preliminaries and notations.

In this section, we define C^*-bicategories and introduce some notations. See [**M**, Chap. XII, Sec. 6] for the definition of abstract bicategories.

1.1. C^*-bicategories.

Definition 1.1.1. A category $\mathscr{C}$ is called a *C^*-bicategory* if it satisfies the following conditions:

(1) The category $\mathscr{C}$ is a disjoint union of categories $\{\mathscr{C}_{ij}\}_{i,j\in L}$ where L is a set.

(2) Hom-sets are considered only for two objects in the same $\mathscr{C}_{ij}$, i.e., $\mathrm{Hom}(X, Y)$ is defined if X and Y are objects in the same $\mathscr{C}_{ij}$.

(3) The category $\mathscr{C}_{ij}$ is a C^*-category. i.e., each hom-set $\mathrm{Hom}(X, Y)$ (X, Y are objects in the same $\mathscr{C}_{ij}$) is a Banach space and there exists a *-operation $* : \mathrm{Hom}(X, Y) \ni f \mapsto f^* \in \mathrm{Hom}(Y, X)$ such that $\|f^* f\| = \|f\|^2$ and $(f^*)^* = f$. In particular, $\mathrm{End}(X) = \mathrm{Hom}(X, X)$ is a C^*-algebra.

(4) For each $i, j, k \in L$, there exist a *-preserving bivariant functor $\otimes : \mathscr{C}_{ij} \times \mathscr{C}_{jk} \to \mathscr{C}_{ik}$ and natural families of unitary isomorphisms $\{a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)\}$ (called *associativity*) which satisfy the following *pentagonal identity*:

$$\begin{tikzcd}
((X \otimes Y) \otimes Z) \otimes W \arrow[r, "a"] \arrow[d, "a\otimes 1"'] & (X \otimes Y) \otimes (Z \otimes W) \arrow[r, equal] & (X \otimes Y) \otimes (Z \otimes W) \arrow[d, "a"] \\
(X \otimes (Y \otimes Z)) \otimes W \arrow[r, "a"'] & X \otimes ((Y \otimes Z) \otimes W) \arrow[r, "1\otimes a"'] & X \otimes (Y \otimes (Z \otimes W)).
\end{tikzcd}$$

(5) For each $i \in L$, there exist an object I_i in $\mathscr{C}_{ii}$ (called *the unit object*) and natural families of unitary isomorphisms $\{l_X : I_i \otimes X \mapsto X$ where X in $\mathscr{C}_{ij}\}$, $\{r_X : X \otimes I_j \mapsto X$ where X in $\mathscr{C}_{ij}\}$ (called *left and right unit constraints* respectively) satisfying the *triangle identity* as follows:

$$\begin{tikzcd}
(X \otimes I_i) \otimes Y \arrow[r, "a"] \arrow[d, "r_X\otimes 1_Y"'] & X \otimes (I_i \otimes Y) \arrow[d, "1_X\otimes l_Y"] \\
X \otimes Y \arrow[r, equal] & X \otimes Y.
\end{tikzcd}$$

In the rest of this paper, we always assume without loss of generality that $L = \{1, 2\}$. In the obvious way, we can extend the notations defined in [**HY**, Section 1] (C^**-tensor functors, conjugations, isomorphisms, strict, simple, semi-simple*, etc.) to C^*-bicategories. We will freely use them.

REMARK. (1) As in [**HY**], for each object X, we denote its conjugation by X^*.

(2) For each intertwiner $f \in \mathrm{Hom}(X, Y)$, we use the notation ${}^t f = \overline{f^*} = (\bar{f})^* \in \mathrm{Hom}(Y^*, X^*)$, where $\bar{f} \in \mathrm{Hom}(X^*, Y^*)$ is the conjugation of f.

(3) We use the same notation "I" for unit objects I_1 and I_2 if no confusions occur.

(4) For X, $Y \in \mathrm{Object}(\mathscr{C}_{ij})$, if there exists $u \in \mathrm{Hom}(X, Y)$ such that $u^* u = 1_X$, we write $X \preceq Y$.

As pointed out in [**HY**, Theorem 1.2], we have the following coherence theorem (see [**Y6**] for details).

THEOREM 1.1.2. *Let $\mathscr{C}$ be a C^*-bicategory with conjugation. Then there exists a C^*-tensor bicategory $\mathscr{C}'$ with conjugation such that $\mathscr{C}'$ is strict and $\mathscr{C} \cong \mathscr{C}'$.*

According to [**Y1**], [**Y4**], we shall work with the following notion as in [**HY**, Definition 1.6].

DEFINITION 1.1.3. Let $\mathscr{C} = \bigcup_{ij} \mathscr{C}_{ij}$ be a strict, semisimple C^*-bicategory with conjugation. A family of self-conjugate morphisms $\{\varepsilon_X = \overline{\varepsilon_X} : X \otimes X^* \to I\}_{X \in \mathrm{Object}}$ is called a *Frobenius duality* (or *ε-structure*) if they satisfy the followings:

(1) (Multiplicativity)

$$\begin{array}{ccc} X \otimes Y \otimes Y^* \otimes X^* & \xrightarrow{\varepsilon_{X \otimes Y}} & I \\ {\scriptstyle 1 \otimes \varepsilon_Y \otimes 1} \downarrow & & \| \\ X \otimes X^* & \xrightarrow[\varepsilon_X]{} & I. \end{array}$$

(2) (Naturality) For a morphism $f : X \to Y$,

$$\begin{array}{ccc} X \otimes Y^* & \xrightarrow{f \otimes 1} & Y \otimes Y^* \\ {\scriptstyle 1 \otimes {}^t f} \downarrow & & \downarrow {\scriptstyle \varepsilon_Y} \\ X \otimes X^* & \xrightarrow[\varepsilon_X]{} & I. \end{array}$$

(3) (Faithfulness) The map

$$\mathrm{Hom}(X, Y) \ni f \mapsto \varepsilon_Y \circ (f \otimes 1) \in \mathrm{Hom}(X \otimes Y^*, I)$$

is injective.

(4) (Minimality) For a morphism $f \in \mathrm{End}(X)$, we have

$$\|\varepsilon_X(f \otimes 1)\varepsilon_X^*\| = \|\varepsilon_{X^*}(1 \otimes f)\varepsilon_{X^*}^*\|.$$

Remark. In [**Y5**] Yamagami proved that a rigid C^*-bicategory has a Frobenius duality uniquely.

Example 1.1.4. Let $N \subset M$ be a II_1-subfactor with finite Jones index. From this inclusion, we can construct a semisimple C^*-bicategory with conjugation as follows: Let ${}_N X_{0M} = {}_N L^2(M)_M$. We consider all unitary equivalence classes of irreducible bimodules which appear as irreducible components of ${}_N(X_0 \otimes_M X_0^*)_N^n$ for some $n \in \boldsymbol{N}$ (where X_0^* is a conjugate bimodule of X_0) and denote it by $S_{N,N}$. Let $\mathscr{C}_{N,N}$ be a category consisting of all finite-type (finite index) N-N bimodules whose irreducible components are contained in $S_{N,N}$. Similarly, by using bimodules ${}_N(X_0 \otimes_M X_0^*)_N^n \otimes_N X_{0M}$, $({}_N(X_0 \otimes_M X_0^*)_N^n \otimes_N X_{0M})^*$ and ${}_M(X_0^* \otimes_N X_0)_M^n$, we get categories $\mathscr{C}_{N,M}$, $\mathscr{C}_{M,N}$ and $\mathscr{C}_{M,M}$. Then it is easy to see that the category $\mathscr{C} = \bigcup_{A,B \in \{N,M\}} \mathscr{C}_{A,B}$ forms a semisimple C^*-bicategory with conjugation. Moreover, if we apply the coherence theorem to $\mathscr{C}$, $\mathscr{C}$ is isomorphic to a strict one $\mathscr{C}'$, which has Frobenius duality (see [**Y1**] for the proof). This bicategory is an invariant for the conjugation classes of $N \subset M$ and is stronger than the standard invariant, i.e., a bicategory includes all the information of standard invariants.

1.2. Minimal traces, quantum dimensions and Frobenius reciprocity.
Let $\mathscr{C}$ be a strict, semisimple C^*-bicategory with conjugation and Frobenius duality.

Definition 1.2.1. For each object X in $\mathscr{C}$, define a linear functional on $\mathrm{End}(X)$ by

$$\mathrm{End}(X) \ni f \mapsto \langle f \rangle_X \in \boldsymbol{C}$$

where $\langle f \rangle_X 1_I = \varepsilon_X(f \otimes I_{X^*})\varepsilon_X^* \in \mathrm{End}(I) = \boldsymbol{C}$.

We call this functional a *minimal trace*.

As in [**HY**, Proposition 1.8], the next proposition holds.

Proposition 1.2.2. (1) *For* $f \in \mathrm{Hom}(X, Y)$ *and* $g \in \mathrm{Hom}(Y, X)$, *we have*

$$\langle fg \rangle_Y = \langle gf \rangle_X.$$

(2) *For* $f \in \mathrm{End}(X)$ *and* $g \in \mathrm{End}(Y)$,

$$\langle f \otimes g \rangle_{X \otimes Y} = \langle f \rangle_X \langle g \rangle_Y.$$

(3) *For* $f \in \mathrm{End}(X)$,

$$\langle {}^t f \rangle_{X^*} = \langle f \rangle_X.$$

Definition 1.2.3. For each object X in $\mathscr{C}$, define

$$d(X) = \langle 1_X \rangle_X$$

$$\text{(or equivalently, } d(X)1_I = \varepsilon_X \varepsilon_X^*).$$

d is called a *quantum* (or *statistical*) *dimension*.

Proposition 1.2.4. (1) $d(X \otimes Y) = d(X)d(Y)$.
(2) $d(X \oplus Z) = d(X) + d(Z)$.

Proof. See [**Y4**]. □

Definition 1.2.5. For $f \in \mathrm{Hom}(X \otimes Y, Z)$, we define *Frobenius transformations* of f by

$$(1_{X^*} \otimes f) \circ (\varepsilon_{X^*}^* \otimes 1_Y) \in \mathrm{Hom}(Y, X^* \otimes Z)$$

$$(f \otimes 1_{Y^*}) \circ (1_X \otimes \varepsilon_Y^*) \in \mathrm{Hom}(X, Z \otimes Y^*).$$

1.3. Fusion algebras and amenability.

Fusion algebras were systematically studied by F. Hiai and M. Izumi in [**HI**]. We recall some notations and properties of fusion algebras.

Let S be a countable set. An algebra $\boldsymbol{C}[S] = \bigoplus_{s \in S} \boldsymbol{C}s$ is called a *fusion algebra* if it satisfies the followings:

(1) The product unit I is contained in the base set S.

(2) There exists a family of non-negative integers $\{N_{s,t}^u\}_{s,t,u \in S}$ such that

$$s \cdot t = \sum_{u \in S} N_{s,t}^u u.$$

(3) There exists a map $S \ni s \mapsto s^* \in S$, called *conjugation*, which is extended to a *-operation in $\boldsymbol{C}[S]$ so that

$$(s \cdot t)^* = t^* \cdot s^*,$$

where $s, t \in S$.

(4) $N_{s,t}^u = N_{s^*,u}^t = N_{u,t^*}^s$.

(5) There exists a map $d : S \to [1, \infty)$, called a *dimension*, such that $d(s) = d(s^*)$ and its linear extension satisfies

$$d(s \cdot t) = d(s)d(t).$$

Definition 1.3.1. Let μ be a probability measure on S. (1) The probability measure μ is said to be *symmetric* if $\mu(s) = \mu(s^*)$ for any $s \in S$.

(2) The probability measure μ is said to be *generating* if its support generates $\boldsymbol{C}[S]$.

(3) For each probability measures μ, ν on S, we define a convolution of them by

$$\mu * \nu(u) = \sum_{s,t \in S} \mu(s)\nu(t)N_{s,t}^{u}\frac{d(u)}{d(s)d(t)}.$$

Here we remark that $\mu * \nu$ is also a probability measure on S.

(4) The probability measure μ is *ergodic* if

$$\lim_{n\to\infty} \|\mu^n * \delta_s - \mu^n\|_1 = 0$$

for any $s \in S$, where μ^n denotes the n-fold convolution $\mu * \cdots * \mu$.

The next lemma is useful (see [**F**] and [**HI**, Lemma 3.2] for the proof).

Lemma 1.3.2 (Foguel). *Let μ be a probability measure on S such that $\mu^k \wedge \mu^{k+1} \neq 0$ for some $k \in \boldsymbol{N}$ (in particular, $I \in \mathrm{support}(\mu)$). Then we have*

$$\lim_{n\to\infty} \|\mu^n - \mu^{n+1}\|_1 = 0.$$

Definition 1.3.3. Let $\boldsymbol{C}[S]$ be a fusion algebra. For each $s \in S$, define a densely defined linear map on the Hilbert space $l^2(S)$ by

$$L_s(\delta_t) = \sum_{u \in S} N_{s,t}^{u}\delta_u.$$

We also define L_x $(x = \sum_{s\in S} x(s)s \in \boldsymbol{C}[S],\ x(s) \in \boldsymbol{C})$ by

$$L_x = \sum_{s \in S} x(s)L_s.$$

The fusion algebra $\boldsymbol{C}[S]$ is said to be *amenable* if for $a = \sum_{s\in S} a(s)s \in \boldsymbol{C}[S]$ with $a(s)$ non-negative integers, we have

$$\|L_a\| = d(a) = \sum_{s \in S} a(s)d(s).$$

Here we remark that the inequality $\|L_a\| \le d(a)$ holds in general.

Let $\mathscr{C}$ be a strict, semisimple C^*-bicategory with conjugation and Frobenius duality. We denote the set of unitary equivalence classes of objects in $\mathscr{C}_{ij}$ by S_{ij} (called the spectrum set) and assume that each S_{ij} is countable or finite. (Throughout this paper, we always assume this.) Then as pointed out in [**HY**, Section 2], $\boldsymbol{C}[S_{ii}]$ has a fusion algebra structure, where $\boldsymbol{C}[S_{ij}]$ is a free vector space over the set S_{ij}. We say that $\mathscr{C}$ is *amenable* if the fusion algebra $\boldsymbol{C}[S_{11}]$ is amenable as in the sense of Definition 1.3.3. Then we have

Proposition 1.3.4. *The fusion algebra $\boldsymbol{C}[S_{11}]$ is amenable if and only if $\boldsymbol{C}[S_{22}]$ is amenable.*

Proof. Assume that $\boldsymbol{C}[S_{11}]$ is amenable. Take an arbitrary $X \in \mathrm{Object}(\mathscr{C}_{12})$. Consider the subfusion algebra of $\boldsymbol{C}[S_{11}]$ generated by XX^* and denote it by $\Xi(XX^*)$. Similarly, define $\Xi(X^*X) \subset \boldsymbol{C}[S_{22}]$. By the amenability of $\boldsymbol{C}[S_{11}]$, we have

$$\begin{aligned} d(X)^2 \geq \|L_{X^*X}\| &= \lim_{n\to\infty} (N^I_{(X^*X)^n})^{1/n} \\ &= \lim_{n\to\infty} (N^I_{X^*(XX^*)^{n-1}X})^{1/n} \\ &= \lim_{n\to\infty} (N^{XX^*}_{(XX^*)^{n-1}})^{1/n} \\ &\geq \lim_{n\to\infty} (N^I_{(XX^*)^{n-1}})^{1/n} = d(X)^2 \end{aligned}$$

where $N^Z_Y = \dim \mathrm{Hom}(Y,Z)$ $(Y,Z \in \mathscr{C}_{ij})$. Then we get $\|L_{X^*X}\| = d(X)^2$ and this implies that $\Xi(X^*X)$ is amenable. For any $Y \in \mathrm{Object}(\mathscr{C}_{22})$, we can take $X \in \mathrm{Object}(\mathscr{C}_{12})$ such that $Y \preceq X^*X$ (for example, take $Z \in \mathrm{Object}(\mathscr{C}_{12})$ and define $X = Z(I+Y)$). Since $Y \in \Xi(X^*X)$ and this fusion algebra is amenable, we have $\|L_Y\| = d(Y)$. This means that $\boldsymbol{C}[S_{22}]$ is amenable. The reverse implication can be shown similarly. $\square$

The disjoint union $\bigcup_{ij} \boldsymbol{C}[S_{ij}]$ forms a graded fusion algebra in the following sense. For each $s \in S_{ij}$ and $t \in S_{jk}$,

(1) $s \cdot t \simeq \bigoplus_{u \in S_{ik}} \dim \mathrm{Hom}(s \cdot t, u)u$ ($s \cdot t$ means $s \otimes t$),

(2) $S_{ij} \ni s \mapsto s^* \in S_{ji}$,

(3) $(s \cdot t)^* \simeq t^* \cdot s^*$, $s \in S_{ij} t \in S_{jk}$,

(4) $\dim \mathrm{Hom}(s \cdot t, u) = \dim \mathrm{Hom}(s^* \cdot u, t) = \dim \mathrm{Hom}(u \cdot t^*, s), s \in S_{ij}, t \in S_{jk}, u \in S_{ik}$,

(5) $d(s)d(t) = \sum_{u \in S_{ik}} \dim \mathrm{Hom}(s \cdot t, u) d(u)$,

(6) $d(s^*) = d(s)$.

For each two probability measures μ on S_{ij} and ν on S_{jk}, we define a probability measure $\mu * \nu$ on S_{ik} by

$$\mu * \nu(u) = \sum_{s \in S_{ij}, t \in S_{jk}} \mu(s)\nu(t) \dim \mathrm{Hom}(s \cdot t, u) \frac{d(u)}{d(s)d(t)}$$

where $u \in S_{ik}$.

1.4. Some notations.

(1) Let X, Y and Z be objects of $\mathscr{C}_{ij}$, $\mathscr{C}_{jk}$ and $\mathscr{C}_{ij}$, respectively. We write

$$XY = X \otimes Y,$$

$$\begin{bmatrix} X \\ Z \end{bmatrix} = \mathrm{Hom}(Z, X),$$

$$N^Z_X = \dim \begin{bmatrix} X \\ Z \end{bmatrix}.$$

(2) For each object X in $\mathscr{C}_{ij}$, we define a probability measure δ_X on the spectrum set S_{ij} by

$$\delta_X = \frac{1}{d(X)} \sum_{s \in S_{ij}} N_X^s d(s) \delta_s.$$

(3) For a subset $\Omega \subset \mathrm{Object}(\mathscr{C}_{11})$, define

$$\Xi_0(\Omega) = \{s \in S_{11} : s \preceq X_1 \cdots X_n \text{ for some } X_1, \dots, X_n \in \Omega \cup \Omega^*\}$$

and let $\Xi(\Omega)$ be the free vector space over $\Xi_0(\Omega)$. Then $\Xi(\Omega)$ is closed under multiplication and conjugation, and it becomes a subfusion algebra of $\boldsymbol{C}[S_{11}]$. We call this the subfusion algebra generated by Ω.

2. Construction of commuting squares.

In this section, we construct commuting squares of AFD II_1-factors from an abstract C^*-bicategory by using a probability measure.

Let $\mathscr{C}$ be a semisimple, strict C^*-bicategory with conjugation and Frobenius duality consisting of $\mathscr{C}_{11}$, $\mathscr{C}_{12}$, $\mathscr{C}_{21}$ and $\mathscr{C}_{22}$. Take a symmetric generating probability measure μ on S_{11} such that $I \in \mathrm{support}(\mu) = S(\mu)$ and fix it throughout this section (μ may not be ergodic). By taking representing elements of S_{ij}, we freely identify elements of S_{ij} with some irreducible objects.

Let R be an AFD II_1-factor and let τ be the unique tracial state on R. Then we can take a mutually orthogonal family of projections $\{e_s\}_{s \in S_{11}}$ of R such that

$$\tau(e_s) = \frac{\mu(s)}{d(s)}$$

for each $s \in S_{11}$ because of the inequality $\sum_{s \in S_{11}} \mu(s)/(d(s)) \leq \sum_{s \in S_{11}} \mu(s) = 1$.

For each integer n and element $x = (x_1, \dots, x_n) \in S_{11}^n = S_{11} \times \cdots \times S_{11}$ (n-times), we define a projection $e_x \in R^{\otimes n}$ by

$$e_x = e_{x_1} \otimes \cdots \otimes e_{x_n}.$$

For each $x, y \in S_{11}^n$, we set

$${}_xR_y = e_x R^{\otimes n} e_y.$$

Definition 2.1. (1) For a positive integer n and an object X in $\mathscr{C}_{1i}$ ($i = 1, 2$), define a von Neumann algebra $A_n(X)$ by

$$A_n(X) = \bigoplus_{x, y \in S_{11}^n} \begin{bmatrix} x_n \cdots x_1 X \\ y_n \cdots y_1 X \end{bmatrix} \otimes {}_xR_y.$$

For $n = 0$, $A_0(X)$ is defined to be $A_0(X) = \begin{bmatrix} X \\ X \end{bmatrix}$. We often use the notation $A_n = A_n(I)$.

(2) For each $s \in S_{1i}$, define

$$A_n^s(X) = \bigoplus_{x, y \in S_{11}^n} \begin{bmatrix} x_n \cdots x_1 X \\ s \end{bmatrix} \otimes \begin{bmatrix} y_n \cdots y_1 X \\ s \end{bmatrix}^* \otimes {}_x R_y.$$

Notice here the following facts:

(1) $A_n(X) \simeq \bigoplus_{s \in S_{1i}} A_n^s(X)$.

(2) For each $s \in S_{1i}$, if $s \preceq x_n \cdots x_1 X$ for some $x \in S(\mu)^n$, then $A_n^s(X) \cong R$. Otherwise, $A_n^s(X) = \{0\}$.

Hence we can define a normal tracial state τ_X^n on $A_n(X)$ by

$$\tau_X^n(I_{A_n^s(X)}) = \mu^n * \delta_X(s).$$

It is easy to check the following.

Lemma 2.2. *Let X be an object in $\mathscr{C}_{1i}$. Then for each $n \in \boldsymbol{N}$, $\rho \in \begin{bmatrix} x_n \cdots x_1 X \\ y_n \cdots y_1 X \end{bmatrix}$ and $a \in {}_x R_y$, we have $\rho \otimes a \in A_n(X)$ and the identity*

$$\tau_X^n(\rho \otimes a) = \frac{1}{d(X)} \delta_{x,y} \langle \rho \rangle_{y_n \cdots y_1 X} \tau(a).$$

Definition 2.3. Let $X \in \mathrm{Object}(\mathscr{C}_{1i})$ and $Y \in \mathrm{Object}(\mathscr{C}_{ij})$. Define an embedding from $A_n(X)$ to $A_{n+1}(X)$ by

$$\begin{bmatrix} x_n \cdots x_1 X \\ y_n \cdots y_1 X \end{bmatrix} \otimes {}_x R_y \ni \rho \otimes a \mapsto \sum_{s \in S} (1_s \otimes \rho) \otimes (e_s \otimes a) \in A_{n+1}(X).$$

We also define an embedding from $A_n(X)$ to $A_n(XY)$ by

$$\begin{bmatrix} x_n \cdots x_1 X \\ y_n \cdots y_1 X \end{bmatrix} \otimes {}_x R_y \ni \rho \otimes a \mapsto (\rho \otimes 1_Y) \otimes a \in A_n(XY).$$

Next three lemmas are easily checked and we omit their proofs (see [**HY**, Section 3]).

Lemma 2.4. *The above two embeddings are mutually compatible and they are also compatible with respect to the tracial states $\{\tau_X^n\}_n$ and $\{\tau_{XY}^n\}_n$.*

By this lemma, $\bigcup_n A_n(X)$ has a tracial state τ_X induced by $\{\tau_X^n\}_n$. We denote by $A_\infty(X)$ the weak closure of the GNS-representation of $\bigcup_n A_n(X)$ with respect to τ_X. Here we remark that the definition of τ_X is slightly different from that of [**HY**] (in [**HY**], τ_X is not a state).

Let X be an object in $\mathscr{C}_{1i}$ and $a \in Z(A_\infty(X))$ (the center of $A_\infty(X)$). Since

$$E_{A_n(X)}(a) \in Z(A_n(X)) = \bigoplus_{s \in S_{1i}} \boldsymbol{C} I_{A_n^s(X)}$$

(where "E" means the trace-preserving conditional expectation), we can write

$$E_{A_n(X)}(a) = \sum_{s \in S_{1i}} f_n(s) \cdot I_{A_n^s(X)}$$

with $f_n(s) \in \boldsymbol{C}$.

Lemma 2.5. *The sequence $\{f_n\}_n$ of functions on S_{1i} satisfies the following properties.*

(1) $$f_n(s) = \sum_{t \in S_{1i}} \mu * \delta_s(t) f_{n+1}(t)$$

for any $s \in S(\mu^n * \delta_X)$.

(2) $\sup_n \|f_n\|_\infty < \infty$.

On the other hand, if $\{f_n\}_n$ $(f_n \in l^\infty(S(\mu^n * \delta_X))$ $n \geq 0)$ satisfies (1) *and* (2), *we can find an element $a \in Z(A_\infty(X))$ such that $\{f_n\}_n$ is associated to a.*

A sequence $\{f_n\}_n$ in the above lemma is called a *harmonic sequence* (see Section 5).

Lemma 2.6. *Let X and Y be objects in $\mathscr{C}_{1i}$ and $\mathscr{C}_{ij}$, respectively. Then the square*

$$\begin{array}{ccc} A_n(X) & \subset & A_{n+1}(X) \\ \cap & & \cap \\ A_n(XY) & \subset & A_{n+1}(XY) \end{array}$$

is commuting with respect to τ_{XY}.

By the above lemma, we obtain an increasing sequence of commuting squares as follows:

$$\begin{array}{ccccccccc} A_0(X) & \subset & A_1(X) & \subset & A_2(X) & \subset & \cdots & \subset & A_\infty(X) \\ \cap & & \cap & & \cap & & & & \cap \\ A_0(XY) & \subset & A_1(XY) & \subset & A_2(XY) & \subset & \cdots & \subset & A_\infty(XY) \end{array}$$

3. Realization of an amenable bicategory.

Let $\mathscr{C}$ be a strict, semisimple, amenable C^*-bicategory with conjugation and Frobenius duality. Then the tensor category $\mathscr{C}_{11}$ is amenable by definition and we can find a symmetric ergodic probability measure μ on S_{11} such that μ does not vanish everywhere (see [**HY**, Theorem 2.5]).

Since $\mathscr{C}_{11}$ is an amenable C^*-tensor category in the sense of [**HY**], the following holds ([**HY**, Section 3 and Section 4]).

Theorem 3.1. *Let X be an object in $\mathscr{C}_{11}$. Then the following holds*:
(1) *For each object X in $\mathscr{C}_{11}$, $A_\infty(X)$ is a factor.*
(2) *We have a standard inclusion of subfactors*

$$A_\infty \subset A_\infty(X) \subset A_\infty(XX^*)$$

with the Jones projection given by

$$\frac{1}{d(X)}\varepsilon_X^*\varepsilon_X \in \begin{bmatrix} XX^* \\ XX^* \end{bmatrix} = A_0(XX^*) \subset A_\infty(XX^*),$$

and the Jones index by $[A_\infty(X) : A_\infty] = d(X)^2$.
(3) $A'_\infty \cap A_\infty(X) = \mathrm{End}(X) = A_0(X)$.

By using this theorem, we can prove the next proposition.

Proposition 3.2. *For each object X in $\mathscr{C}_{12}$, $A_\infty(X)$ is a factor and*

$$A'_\infty \cap A_\infty(X) = A_0(X)\,(= \mathrm{End}(X)).$$

Proof. By Theorem 3.1, we have

$$A'_\infty \cap A_\infty(XX^*) = \mathrm{End}(XX^*)$$

because XX^* is an object in $\mathscr{C}_{11}$. Then, since the squares

$$\begin{array}{ccc} A_0 & \subset & A_\infty \\ \cap & & \cap \\ A_0(X) & \subset & A_\infty(X) \\ \cap & & \cap \\ A_0(XX^*) & \subset & A_\infty(XX^*) \end{array}$$

are commuting, we get

$$\begin{aligned} A'_\infty \cap A_\infty(X) &= E_{A_\infty(X)}(A'_\infty \cap A_\infty(XX^*)) \\ &= E_{A_\infty(X)}(A_0(XX^*)) \\ &= E_{A_0(X)}(A_0(XX^*)) \\ &= \mathrm{End}(X) \end{aligned}$$

where "E" is the trace-preserving conditional expectation. In particular, if X is irreducible, the relative commutant $A'_\infty \cap A_\infty(X)$ is trivial. Then $Z(A_\infty(X)) \subset A'_\infty \cap A_\infty(X)$ implies $Z(A_\infty(X)) = \boldsymbol{C}$. That is, $A_\infty(X)$ is factor if X is irreducible. On the other hand, if $A_\infty(X)$ is a factor for some object in $\mathscr{C}_{12}$, $A_\infty(Y)$ is also a factor for all objects Y in $\mathscr{C}_{12}$ because of Lemma 2.5. Hence the proof is completed. (Any harmonic sequence comes from a harmonic function, see Section 5.) □

Remark. The factoriality of $A_\infty(X)$ (X is an object in $\mathscr{C}_{12}$) is non-trivial without amenability. If we remove the assumption "amenability", we cannot get the factoriality of $A_\infty(X)$ because the ergodicity of μ is only for $\mathscr{C}_{11}$. (In fact, there exists such an example due to U. Haagerup [**Ha**], also see [**HI**, Example 8.11].)

S. Popa proved in [**P2**, Corollary 5.4.5] that if a subfactor is amenable and the canonical probability measure $\delta_{{}_N L^2(M)_N}$ of the fusion algebra generated by ${}_N L^2(M)_N$ is ergodic, then the probability measure $\delta_{{}_M L^2(M) \otimes_N L^2(M)_M}$ is also ergodic in the fusion algebra generated by ${}_M L^2(M) \otimes_N L^2(M)_M$. The above proposition can be considered as an extension of this result (see Section 5 for more details).

We define bimodules as follows.

Definition 3.3. Take an object X_0 in $\mathscr{C}_{12}$ and fix it.

(1) For each object X in $\mathscr{C}_{11}$ and $n \in \boldsymbol{N}$, define

$$X_n = \bigoplus_{x, y \in S_{11}^n} \begin{bmatrix} x_n \cdots x_1 X \\ y_n \cdots y_1 \end{bmatrix} \otimes {}_x R_y.$$

(2) For each object X in $\mathscr{C}_{12}$ and $n \in \boldsymbol{N}$, define

$$X_n = \bigoplus_{x, y \in S_{11}^n} \begin{bmatrix} x_n \cdots x_1 X \\ y_n \cdots y_1 X_0 \end{bmatrix} \otimes {}_x R_y.$$

(3) For each object X in $\mathscr{C}_{21}$ and $n \in \boldsymbol{N}$, define

$$X_n = \bigoplus_{x, y \in S_{11}^n} \begin{bmatrix} x_n \cdots x_1 X_0 X \\ y_n \cdots y_1 \end{bmatrix} \otimes {}_x R_y.$$

(4) For each object X in $\mathscr{C}_{22}$ and $n \in \boldsymbol{N}$, define

$$X_n = \bigoplus_{x, y \in S_{11}^n} \begin{bmatrix} x_n \cdots x_1 X_0 X \\ y_n \cdots y_1 X_0 \end{bmatrix} \otimes {}_x R_y.$$

As observed in [**HY**], for each object X in $\mathscr{C}_{11}$, X_n is an A_n-A_n bimodule where two side actions are defined by

$$(\alpha \otimes b) \cdot (\rho \otimes a) \cdot (\beta \otimes c) = (\alpha \otimes 1_X) \circ \rho \circ \beta \otimes bac$$

$$\alpha \in \begin{bmatrix} x_n \cdots x_1 \\ y_n \cdots y_1 \end{bmatrix}, \quad \rho \in \begin{bmatrix} y_n \cdots y_1 X \\ z_n \cdots z_1 \end{bmatrix}, \quad \beta \in \begin{bmatrix} z_n \cdots z_1 \\ w_n \cdots w_1 \end{bmatrix}, \quad b \in {}_x R_y, \quad a \in {}_y R_z, \quad c \in {}_z R_w.$$

Analogously, for each object Y in $\mathscr{C}_{1i}$, Z in $\mathscr{C}_{i1}$ and W in $\mathscr{C}_{i2}$, we can construct bimodules

$${}_{A_n(Y)}(YZ)_{nA_n(Z^*)}$$

and

$${}_{A_n(Y)}(YW)_{nA_n(X_0 W^*)}.$$

REMARK. (1) For $X \in \mathscr{C}_{12}$, by using the Frobenius reciprocity, we have

$$X_n \simeq (XX_0^*)_n.$$

(2) For $X \in \mathscr{C}_{21}$, since $X_0 X \in \mathrm{Object}(\mathscr{C}_{11})$, we have

$$X_n = (X_0 X)_n.$$

(3) For $X \in \mathscr{C}_{22}$, by using the Frobenius reciprocity, we have

$$X_n \simeq (X_0 X X_0^*)_n.$$

DEFINITION 3.4. For each object X in $\mathscr{C}_{11}$ and $n \in \boldsymbol{N}$, define an inner product on X_n by

$$(\rho \otimes a | \rho' \otimes a') = \delta_{x,x'} \delta_{y,y'} \langle \rho(\rho')^* \rangle_{x_n \cdots x_1 X} \tau(a(a')^*)$$

where $x, x', y, y' \in S_{11}^n$, $\rho \in \begin{bmatrix} x_n \cdots x_1 X \\ y_n \cdots y_1 \end{bmatrix}$, $\rho' \in \begin{bmatrix} x_n' \cdots x_1' X \\ y_n' \cdots y_1' \end{bmatrix}$ and $a \in {}_x R_y$, $a' \in {}_{x'} R_{y'}$. Similarly, by using minimal traces and τ, we define an inner product on X_n for each object X in $\mathscr{C}$.

DEFINITION 3.5. For each object X in $\mathscr{C}_{11}$ and $n \in \boldsymbol{N}$, define an embedding map from X_n to X_{n+1} by

$$\begin{bmatrix} x_n \cdots x_1 X \\ y_n \cdots y_1 \end{bmatrix} \otimes {}_x R_y \ni \rho \otimes a \mapsto \sum_{s \in S_{11}} (1_s \otimes \rho) \otimes (e_s \otimes a) \in X_{n+1}.$$

Similarly, we define embeddings from X_n to X_{n+1} for each object X in $\mathscr{C}$.

For the proofs of the next two lemmas, see [**HY**, Section 3].

LEMMA 3.6. *Embeddings defined in Definition* 3.5 *preserve inner products defined in Definition* 3.4.

LEMMA 3.7. *Above embeddings are compatible with two side actions.*

By completion, for each object X in $\mathscr{C}$, we can define a bimodule X_∞. The next lemma can be shown in the same way as [**HY**, Proposition 3.9].

Lemma 3.8. *For each object Y in $\mathscr{C}_{1i}$, Z in $\mathscr{C}_{i1}$ and W in $\mathscr{C}_{i2}$, we have*

$$\mathrm{End}((YZ)_{\infty A_\infty(Z^*)}) = A_\infty(Y)$$

and

$$\mathrm{End}((YW)_{\infty A_\infty(X_0 W^*)}) = A_\infty(Y).$$

Lemma 3.9. *For each object X in $\mathscr{C}_{12}$ and Y in $\mathscr{C}_{2i}$, the inclusion of factors*

$$A_\infty(X) \subset A_\infty(XY) \subset A_\infty(XYY^*)$$

is standard with the Jones projection

$$1_X \otimes \frac{1}{d(Y)} \varepsilon_Y^* \varepsilon_Y \in \begin{bmatrix} XYY^* \\ XYY^* \end{bmatrix} = A_0(XYY^*) \subset A_\infty(XYY^*),$$

and the Jones index is given by $[A_\infty(XY) : A_\infty(X)] = d(Y)^2$.

Proof. Thanks to Proposition 3.2, $A_\infty(X)$, $A_\infty(XY)$ and $A_\infty(XYY^*)$ are factors. In the canonical way, we can identify

$${}_{A_\infty(X)}L^2(A_\infty(XY))_{A_\infty(X)} \simeq {}_{A_\infty(X)}(XYY^*X^*)_{\infty A_\infty(X)}.$$

(See [**HY**, arguments before Lemma 3.6]).

It is easy to see that under this identification, the Jones projection

$$e_{A_\infty(X)} : L^2(A_\infty(XY)) \to L^2(A_\infty(X))$$

corresponds to $d(Y)^{-1}\varepsilon_Y^*\varepsilon_Y$. Moreover, by Lemma 3.8, we have

$$\mathrm{End}((XYY^*X^*)_{\infty A_\infty(X)}) = A_\infty(XYY^*).$$

Consequently, we get the assertion. □

By the same argument as in [**HY**], it is easy to see that $X \mapsto X_\infty$ is an involutive C^*-tensor functor. To prove that this functor is fully faithful (i.e., this functor is an isomorphism), we need the next proposition.

Proposition 3.10. *For each object X in $\mathscr{C}_{1i}$ and Y in $\mathscr{C}_{ij}$, the equality we have*

$$A_\infty(X)' \cap A_\infty(XY) = 1_X \otimes \mathrm{End}(Y) \subset A_0(XY).$$

For the proof, we use the following simple fact.

Lemma 3.11. *For objects X in $\mathscr{C}_{ij}$, Y in $\mathscr{C}_{jk}$, and Z in $\mathscr{C}_{kl}$, we have*

$$(\mathrm{End}(XY) \otimes 1_Z) \cap (1_X \otimes \mathrm{End}(YZ)) = 1_X \otimes \mathrm{End}(Y) \otimes 1_Z.$$

Proof. For each element $x \in (\mathrm{End}(XY) \otimes 1_Z) \cap (1_X \otimes \mathrm{End}(YZ))$, there exist $f \in \mathrm{End}(XY)$ and $g \in \mathrm{End}(YZ)$ such that

$$x = f \otimes 1_Z = 1_X \otimes g.$$

We compute

$$f = \frac{1}{d(Z)} f \otimes \varepsilon_Z \varepsilon_Z^* = (1_{XY} \otimes \varepsilon_Z) \circ (1_X \otimes g \otimes 1_{Z^*}) \circ (1_{XY} \otimes \varepsilon_Z^*).$$

Let $\tilde{g}$ be $(1_Y \otimes \varepsilon_Z) \circ (g \otimes 1_{Z^*}) \circ (1_Y \otimes \varepsilon_Z^*)$, then $\tilde{g} \in \mathrm{End}(Y)$ and the relation $f = 1/(d(Z))1_X \otimes \tilde{g}$ holds. Hence we get

$$x = \frac{1}{d(Z)} 1_X \otimes \tilde{g} \otimes 1_Z \in 1_X \otimes \mathrm{End}(Y) \otimes 1_Z$$

and

$$(\mathrm{End}(XY) \otimes 1_Z) \cap (1_X \otimes \mathrm{End}(YZ)) \subset 1_X \otimes \mathrm{End}(Y) \otimes 1_Z.$$

The reverse inclusion is obvious. □

Proof of Proposition 3.10. We assume X in $\mathscr{C}_{12}$ and Y in $\mathscr{C}_{22}$. Consider the bimodule ${}_{A_\infty}(XY)_{\infty A_\infty(X_0)}$. Then $\mathrm{End}({}_{A_\infty(X)}(XY)_{\infty A_\infty(X_0)}) = A_\infty(X)' \cap A_\infty(XY)$ holds because $\mathrm{End}((XY)_{\infty A_\infty(X_0)}) = A_\infty(XY)$. Moreover by Proposition 3.2, we have

$$\begin{aligned}\mathrm{End}({}_{A_\infty(X)}(XY)_{\infty A_\infty(X_0)}) &\subset \mathrm{End}({}_{A_\infty}(XY)_{\infty A_\infty(X_0)}) \\ &= A'_\infty \cap A_\infty(XY) = \mathrm{End}(XY)\end{aligned}$$

since XY is an object in $\mathscr{C}_{12}$. On the other hand, via the Frobenius reciprocity, we have

$${}_{A_\infty(X)}(XY)_{\infty A_\infty} \simeq {}_{A_\infty(X)}(XYX_0^*)_{\infty A_\infty}.$$

Thus we get

$$\begin{aligned}A_\infty(X)' \cap A_\infty(XY) &\subset \mathrm{End}({}_{A_\infty(X)}(XY)_{\infty A_\infty}) \\ &\simeq \mathrm{End}({}_{A_\infty(X)}(XYX_0^*)_{\infty A_\infty}).\end{aligned}$$

Then by Lemma 3.8, we have

$$\begin{aligned}\mathrm{End}({}_{A_\infty(X)}(XYX_0^*)_\infty) &\simeq \mathrm{End}((X_0Y^*X^*)_{\infty A_\infty(X)}) \\ &= A_\infty(X_0Y^*)\end{aligned}$$

and

$$\begin{aligned}\mathrm{End}({}_{A_\infty(X)}(XYX_0^*)_{\infty A_\infty}) &\simeq A_\infty' \cap A_\infty(X_0 Y^*) \\ &= \mathrm{End}(X_0 Y^*).\end{aligned}$$

This implies that

$$A_\infty(X)' \cap A_\infty(XY) \subset 1_X \otimes \mathrm{End}(YX_0^*)$$

on $(XYX_0^*)_\infty$. Then we get

$$A_\infty(X)' \cap A_\infty(XY) \subset (\mathrm{End}(XY) \otimes 1_{X_0^*}) \cap (1_X \otimes \mathrm{End}(YX_0^*)).$$

Therefore by using the previous lemma, we get the assertion. In a similar way, we can prove the statement in the case where Y belongs to $\mathscr{C}_{21}$. □

We now get the next theorem.

Theorem 3.12. *A strict, semisimple, amenable C^*-bicategory with conjugation and Frobenius duality can be realized as AFD-bimodules.*

Proof. Thanks to Proposition 3.10, for each object X in $\mathscr{C}_{1i}$, Y in $\mathscr{C}_{ij}$, Z in $\mathscr{C}_{j1}$, we have

$$\begin{aligned}\mathrm{End}({}_{A_\infty(X)}(XYZ)_{\infty A_\infty(Z^*)}) &= A_\infty(X)' \cap A_\infty(XY) \\ &= \mathrm{End}(Y).\end{aligned}$$

This fact and [**HY**, Proposition 1.1] imply that the functor $X \mapsto X_\infty$ is an isomorphism. Here we remark that [**HY**, Proposition 1.1] also holds for C^*-bicategories. □

As an application to subfactor, the following corollary holds.

Corollary 3.13. *Let $N \subset M$ be a II_1-subfactor with finite index and amenable graph. Then there exists an AFD II_1-subfactor $A \subset B$ such that its bicategory is isomorphic to that of $N \subset M$. In particular, their standard invariants* (*paragroups or standard lattices*) *are isomorphic.*

Proof. Let $\mathscr{C} = \mathscr{C}_{N,N} \cup \mathscr{C}_{N,M} \cup \mathscr{C}_{M,N} \cup \mathscr{C}_{M,M}$ be a C^*-bicategory associated with $N \subset M$. Then by the coherence theorem, it is isomorphic to strict one. By applying Theorem 3.12 to this category, we get a C^*-bicategory consisting of AFD-bimodules. Moreover, if we take $X_0 = {}_N L^2(M)_M$, X_0 corresponds to ${}_{A_\infty} L^2(A_\infty(X_0))_{A_\infty(X_0)}$. Thus $A_\infty \subset A_\infty(X_0)$ is a desired subfactor. □

In [**P5**], S. Popa characterized standard invariants of subfactors by some axioms and he called them standard lattices. In [**P5**] he proved that any abstract standard lattice comes from some (not necessarily AFD) subfactor. Furthermore, if the standard lattice is amenable, it gives a complete invariant for amenable AFD II_1-subfactors (see [**P7**]). He also announced that if a given standard lattice is amenable, it can be realized by some AFD-subfactor. We can give an another proof of this fact by using Popa's result in [**P5**] and the above corollary as follows:

Corollary 3.14. *For each amenable standard lattice, there exists an AFD II_1-subfactor which realizes it as higher relative commutants.*

Proof. Let $\{A_{ij}\}_{ij}$ be an amenable standard lattice. By [**P5**], there exists a (non-AFD) II_1-subfactor $N \subset M$ which realizes this lattice. Applying Corollary 3.13 to this inclusion, we get an AFD II_1-subfactor $A \subset B$ whose C^*-bicategory is isomorphic to that of $N \subset M$. If bicategories are isomorphic, standard lattices are also isomorphic. So $A \subset B$ realizes $\{A_{ij}\}_{ij}$. □

4. Some remarks on weak amenability of fusion algebras.

In the paper [**HI**], F. Hiai and M. Izumi introduced the concept of *weak amenability* to fusion algebras. The original definition of weak amenability is defined by the existence of an invariant mean and this is equivalent to the existence of a symmetric ergodic probability measure whose support is the whole set (see [**HI**, Proposition 4.2] and [**HY**, the proof of Theorem 2.5] for the proof). In [**HI**], they proved that weak amenability is strictly weaker than amenability and raised the following question: "Find a suitable condition under which weak amenability and amenability are equivalent." In this section, we will prove that if a fusion algebra which comes from a C^*-tensor category (in particular, if it comes from bimodules, or subfactors) has "amenable generators", weak amenability and amenability are equivalent. This is a partial answer to the above question. Throughout this section, tensor categories are non-graded.

Let $\mathscr{C}$ be a strict semisimple C^*-tensor category with conjugation and Frobenius duality (see [**HY**] for the definition of a C^*-tensor category). We denote the associated fusion algebra of $\mathscr{C}$ by $\boldsymbol{C}[S]$ and assume that $\boldsymbol{C}[S]$ is weakly amenable. Thus there exists a symmetric, ergodic probability measure μ on S such that support$(\mu) = S$. By using μ, we define $A_\infty(X)$, X_∞, etc. as in [**HY**].

For each object X in $\mathscr{C}$, we have a standard inclusion of subfactors

$$A_\infty \subset A_\infty(X) \subset A_\infty(XX^*)$$

with the Jones projection given by

$$\frac{1}{d(X)}\varepsilon_X^*\varepsilon_X \in \begin{bmatrix} XX^* \\ XX^* \end{bmatrix} = A_0(XX^*) \subset A_\infty(XX^*),$$

and the Jones index by $[A_\infty(X) : A_\infty] = d(X)^2$. First we consider a criterion for the extremality of this inclusion. Recall that a II_1-subfactor $N \subset M$ with finite index is called extremal if $E_{M' \cap M_1}(e_N) = [M : N]^{-1} I$ where $N \subset M \subset M_1$ is a basic extension with the Jones projection e_N ([**PP**]).

Lemma 4.1. *The next equality holds.*

$$\frac{1}{d(X)} E_{A_n(X)' \cap A_n(XX^*)}(\varepsilon_X^* \varepsilon_X) = \frac{1}{d(X)^2} \sum_{s,t \in S} \frac{\mu^n(t)}{\mu^n * \delta_X(s)} \frac{d(s)^2}{d(t)^2} I_{A_n^s(X)} \cdot I_{A_n^t(XX^*)}.$$

Proof. By the same argument as in the proof of [**HY**, Lemma 4.6], it is easy to see that $A_n(X)' \cap A_n(XX^*)$ is spanned by the set

$$\left\{ \bigoplus_{x,\rho} (\rho^* \otimes 1_{X^*}) \alpha^* \beta (\rho \otimes 1_{X^*}) \otimes e_x \mid \alpha, \beta : sX^* \to t \;\; s, t \in S \right\}$$

where $x_n, \dots, x_1$, $x = x_n \dots x_1$, and $\{\rho : xX \to s\}$ is a family of coisometries with mutually orthogonal initial spaces. Thus, it suffices to show that

$$\begin{aligned} &\tau_{XX^*}\left(\frac{1}{d(X)} \varepsilon_X^* \varepsilon_X \left\{ \bigoplus_{x,\rho} (\rho^* \otimes 1_{X^*}) \alpha^* \beta (\rho \otimes 1_{X^*}) \otimes e_x \right\} \right) \\ &= \tau_{XX^*}\left(\frac{1}{d(X)^2} \sum_{s,t \in S} \frac{\mu^n(t)}{\mu^n * \delta_X(s)} \frac{d(s)^2}{d(t)^2} I_{A_n^s(X)} \right. \\ &\qquad \left. \cdot I_{A_n^t(XX^*)} \left\{ \bigoplus_{x,\rho} (\rho^* \otimes 1_{X^*}) \alpha^* \beta (\rho \otimes 1_{X^*}) \otimes e_x \right\} \right). \end{aligned}$$

More explicitly, we will show that both two sides are equal to $\langle \alpha^* \beta \rangle_{sX^*} (d(s)^2 \mu^n(t)) / (d(X)^3 d(t)^2)$, where $\alpha, \beta : sX^* \to t$ are coisometries.

The left hand side is computed as follows:

$$\begin{aligned} &\sum_{x,\rho} \langle (1_x \otimes \varepsilon_X)(\rho^* \otimes 1_{X^*}) \alpha^* \beta (\rho \otimes 1_{X^*})(1_x \otimes \varepsilon_X^*) \rangle_x \frac{\mu(x_1) \cdots \mu(x_n)}{d(X)^3 d(x_1) \cdots d(x_n)} \\ &= \sum_{x,u,\sigma,\tau} \langle (1_x \otimes \varepsilon_X)(\sigma^* \otimes 1_{XX^*})(\tau^* \otimes 1_{X^*}) \alpha^* \beta (\tau \otimes 1_{X^*})(\sigma \otimes 1_{XX^*})(1_x \otimes \varepsilon_X^*) \rangle_x \\ &\quad \times \frac{\mu(x_1) \cdots \mu(x_n)}{d(X)^3 d(x_1) \cdots d(x_n)} \end{aligned}$$

(where $u \in S$, $\sigma : x \to u$, $\tau : uX \to s$ are coisometries)

$$= \sum_{x,u,\tau} \langle (1_u \otimes \varepsilon_X)(\tau^* \otimes 1_{X^*})\alpha^*\beta(\tau \otimes 1_{X^*})(1_u \otimes \varepsilon_X^*)\rangle_u N_x^u \frac{\mu(x_1)\cdots\mu(x_n)}{d(X)^3 d(x_1)\cdots d(x_n)}$$

$$= \sum_{u,\tau} \langle (1_u \otimes \varepsilon_X)(\tau^* \otimes 1_{X^*})\alpha^*\beta(\tau \otimes 1_{X^*})(1_u \otimes \varepsilon_X^*)\rangle_u \frac{\mu^n(u)}{d(X)^3 d(u)}$$

$$= \sum_{u,\gamma} \langle (1_u \otimes \varepsilon_X)(\gamma \otimes 1_{XX^*})(1_s \otimes \varepsilon_{X^*}^* \otimes 1_{X^*})$$

$$\times \alpha^*\beta(1_s \otimes \varepsilon_{X^*} \otimes 1_{X^*})(\gamma^* \otimes 1_{XX^*})(1_u \otimes \varepsilon_X^*)\rangle_u$$

$$\times \frac{d(s)\mu^n(u)}{d(X)^3 d(u)^2}$$

(where $\gamma : sX^* \to u$ is a coisometry, then $d(s)^{1/2}/(d(u)^{1/2})(1_s \otimes \varepsilon_{X^*})(\gamma^* \otimes 1_X) : uX \to s$ is also coisometry)

$$= \sum_{u,\gamma} \langle \gamma\alpha^*\beta\gamma^*\rangle_u \frac{d(s)\mu^n(u)}{d(X)^3 d(u)^2}$$

(here we use the hook identity: $(\varepsilon_{X^*} \otimes 1_{X^*})(1_{X^*} \otimes \varepsilon_X^*) = 1_{X^*}$)

$$= \langle \alpha^*\beta\rangle_{sX^*} \frac{d(s)\mu^n(t)}{d(X)^3 d(t)^2}.$$

On the other hand, the right hand side is equal to

$$\sum_{x,\rho} \langle (\rho^* \otimes 1_{X^*})\alpha^*\beta(\rho \otimes 1_{X^*})\rangle_{xXX^*} \frac{\mu^n(t)d(s)^2\mu(x_1)\cdots\mu(x_n)}{d(X)^4\mu^n * \delta_X(s)d(t)^2 d(x_1)\cdots d(x_n)}$$

$$= \sum_x \langle \alpha^*\beta\rangle_{sX^*} N_{xX}^s \frac{\mu^n(t)d(s)^2\mu(x_1)\cdots\mu(x_n)}{d(X)^4\mu^n * \delta_X(s)d(t)^2 d(x_1)\cdots d(x_n)}$$

$$= \langle \alpha^*\beta\rangle_{sX^*} \frac{d(s)\mu^n(t)}{d(X)^3 d(t)^2}. \qquad \square$$

By this lemma, we get the criterion of extremality.

Proposition 4.2. *The subfactor $A_\infty \subset A_\infty(X)$ is extremal if and only if*

$$\lim_{n\to\infty} \sum_{s,t\in S} \left| \frac{d(s)}{d(t)}\mu^n(t) - \frac{d(t)}{d(s)}\mu^n(s) \right| N_{tX}^s = 0.$$

PROOF. By the previous lemma, we get

$$\left\| E_{A_n(X)'\cap A_n(XX^*)}\left(\frac{1}{d(X)}\varepsilon_X^*\varepsilon_X\right) - \frac{1}{d(X)^2}I\right\|_{1,\tau_{XX^*}}$$

$$= \sum_{s,t\in S}\left|\frac{d(s)}{d(t)}\mu^n(t) - \frac{d(t)}{d(s)}\mu^n * \delta_X(s)\right| N_{tX}^s.$$

On the other hand, $A_\infty \subset A_\infty(X)$ is extremal if and only if

$$E_{A_\infty(X)'\cap A_\infty(XX^*)}\left(\frac{1}{d(X)}\varepsilon_X^*\varepsilon_X\right) = \frac{1}{d(X)^2}I.$$

Thus we see that this equality is equivalent to

$$\lim_{n\to\infty}\sum_{s,t\in S}\left|\frac{d(s)}{d(t)}\mu^n(t) - \frac{d(t)}{d(s)}\mu^n * \delta_X(s)\right| N_{tX}^s = 0.$$

Hence the result follows because

$$\lim_{n\to\infty}\sum_{s,t\in S}\frac{d(t)}{d(s)}|\mu^n * \delta_X(s) - \mu^n(s)| N_{tX}^s$$

$$= \lim_{n\to\infty}\sum_{s\in S}\left\{\sum_{t\in S}N_{tX}^s d(t)\right\}\frac{1}{d(s)}|\mu^n * \delta_X(s) - \mu^n(s)|$$

$$= d(X)\lim_{n\to\infty}\sum_{s\in S}|\mu^n * \delta_X(s) - \mu^n(s)| = 0$$

by the ergodicity of μ. □

LEMMA 4.3. *For an object X of $\mathscr{C}$, if the fusion algebra $\Xi(\{XX^*\})$ generated by XX^* is amenable, the subfactor $A_\infty \subset A_\infty(X)$ is extremal.*

PROOF. Since the fusion algebra generated by XX^* is amenable, by the same argument of [**P2**, Theorem 5.3.1, the proof (vi) $\Rightarrow$ (iv)], we get the extremality of $A_\infty \subset A_\infty(XX^*)$. In fact, since

$$\mathrm{End}((XX^*)^n) = A_0((XX^*)^n) \subset A_\infty' \cap A_\infty((XX^*)^n),$$

we get

$$\|L_{XX^*}\| = \lim_{n\to\infty}(N_{(XX^*)^{2n}}^I)^{1/2n}$$

$$= \lim_{n\to\infty}(N_{(XX^*)^n}^{(XX^*)^n})^{1/2n}$$

$$= \lim_{n\to\infty} (\dim A_0((XX^*)^n))^{1/2n}$$

$$\leq \lim_{n\to\infty} (\dim A'_\infty \cap A_\infty((XX^*)^n))^{1/2n}$$

$$= \|\Gamma_{A_\infty, A_\infty(XX^*)}\| \leq d(X)^2.$$

By the amenability of $\Xi(\{XX^*\})$, we have

$$d(X)^2 = \|L_{XX^*}\|.$$

Thus $\|\Gamma_{A_\infty, A_\infty(XX^*)}\|$is equal to the Jones index $d(X)^2$ of $A_\infty \subset A_\infty(XX^*)$. This implies that $A_\infty \subset A_\infty(XX^*)$ is an amenable inclusion, in particular, it is extremal ([**P2**, Corollary 1.3.6 (i)]). Then the extremality of $A_\infty \subset A_\infty(X)$ follows because $A_\infty \subset A_\infty(X)$ is an intermediate subfactor of $A_\infty \subset A_\infty(XX^*)$ ([**P2**, 1.2.5]). $\square$

Theorem 4.4. *Let $\mathscr{C}$ be a strict semisimple C^*-tensor category with conjugation and Frobenius duality. Suppose that there exists a sequence $\{X_k\}_{k=1}^\infty$ ($X_k \in \mathrm{Object}(\mathscr{C})$, X_k may be 0) such that for each k, $\Xi(\{X_k X_k^*\})$ is amenable. Then in the fusion algebra $\Xi(\{X_k\}_{k=1}^\infty)$, weak amenability and amenability are equivalent.*

Proof. We shall prove that in the above setting, weak amenability implies amenability for the fusion algebra $\Xi(\{X_k\}_{k=1}^\infty)$.

Let S be the spectrum set of $\Xi(\{X_k\}_{k=1}^\infty)$. We write

$$\boldsymbol{C}[S] = \Xi(\{X_k\}_{k=1}^\infty).$$

Since $\boldsymbol{C}[S]$ is weakly amenable, there exists a symmetric ergodic probability measure μ on S with the whole support.

Now, we apply the previous considerations to μ and the tensor category generated by $\{X_k\}_{k=1}^\infty$.

By the previous lemma, we see that each subfactor $A_\infty \subset A_\infty(X_i)$ is extremal. Then by using Proposition 4.2, we have for each i,

$$\lim_{n\to\infty} \sum_{s,t\in S} \left| \frac{d(s)}{d(t)}\mu^n(t) - \frac{d(t)}{d(s)}\mu^n(s) \right| N_{tX_i}^s = 0.$$

From this equation, letting $Z_m = I \oplus X_1 \oplus \cdots \oplus X_m$, we get

$$\lim_{n\to\infty} \sum_{s,t\in S} \left| \frac{d(s)}{d(t)}\mu^n(t) - \frac{d(t)}{d(s)}\mu^n(s) \right| N_{tZ_m}^s = 0$$

which in turn implies the extremality of $A_\infty \subset A_\infty(Z_m)$ by Proposition 4.2.

Since

$$A_\infty \subset A_\infty(Z_m) \subset A_\infty(Z_m Z_m^*) \subset A_\infty(Z_m Z_m^* Z_m) \subset \cdots$$

is the Jones tower (see [**HY**]), we see that for any integer k, $A_\infty \subset A_\infty((Z_m Z_m^*)^k)$ is extremal and the equality

$$\lim_{n\to\infty} \sum_{s,t\in S} \left| \frac{d(s)}{d(t)} \mu^n(t) - \frac{d(t)}{d(s)} \mu^n(s) \right| N^s_{t(Z_m Z_m^*)^k} = 0$$

holds by using Proposition 4.2 again.

On the other hand, for any $s_0 \in S$, there is an integer m and k such that $s_0 \preceq (Z_m Z_m^*)^k$. Hence we get

$$\begin{aligned} &\lim_{n\to\infty} \sum_{s,t\in S} \left| \frac{d(s)}{d(t)} \mu^n(t) - \frac{d(t)}{d(s)} \mu^n(s) \right| N^s_{ts_0} \\ &\quad \le \lim_{n\to\infty} \sum_{s,t\in S} \left| \frac{d(s)}{d(t)} \mu^n(t) - \frac{d(t)}{d(s)} \mu^n(s) \right| N^s_{t(Z_m Z_m^*)^k} = 0. \end{aligned}$$

This implies that the condition (D_1) of [**HI**, p. 698] holds for μ. It is easy to see that (D_1) implies the condition (NW_1) [**HI**, p. 688]. Since the implication "$(NW_1) \Rightarrow$ amenability" remains valid in the case of probability measures with infinite supports, $\boldsymbol{C}[S]$ is amenable. ([**HI**, Theorem 4.6, the proof of $(NW_1) \Rightarrow (A)$] works for probability measures with infinite supports.) $\square$

Remark. For an object X, if $\Xi(\{X\})$ is amenable, then by the hereditability of amenability ([**HI**, Proposition 4.8]), $\Xi(\{XX^*\})$ is also amenable since $\Xi(\{XX^*\}) \subset \Xi(\{X\})$. But the reverse implication is not true. For example, let $\boldsymbol{F}_2$ be a free group with 2-generators g and h and let α be an outer action of $\boldsymbol{F}_2$ on an AFD II$_1$-factor R. Define

$${}_R X_R = {}_R \alpha_g L^2(R)_R \oplus {}_R \alpha_h L^2(R)_R.$$

Then $\Xi(\{X\})$ is isomorphic to the group algebra $\boldsymbol{C}[\boldsymbol{F}_2]$ and hence it is non-amenable, whereas

$${}_R XX^*_R = 2I \oplus {}_R \alpha_{gh^{-1}} L^2(R)_R \oplus {}_R \alpha^{-1}_{gh^{-1}} L^2(R)_R$$

shows that $\Xi(\{XX^*\})$ is isomorphic to the group algebra $\boldsymbol{C}[\langle gh^{-1}\rangle]$. Here $\langle gh^{-1}\rangle$ denotes the cyclic group generated by gh^{-1} in $\boldsymbol{F}_2$. Thus $\Xi(\{XX^*\})$ is amenable.

Example 4.6. Let $N \subset P \subset M$ be an inclusion of II$_1$-factors with finite Jones index and we assume both $N \subset P$ and $P \subset M$ are amenable. Then both $\Xi(\{{}_P L^2(P) \otimes_N L^2(P)_P\})$ and $\Xi(\{{}_P L^2(M)_P\})$ are amenable. Thanks to the above theorem, we see that weak amenability and amenability are equivalent in $\Xi(\{{}_P L^2(P) \otimes_N L^2(P)_P, {}_P L^2(M)_P\})$. Therefore $N \subset M$ is amenable if and only if

$\Xi(\{{}_P L^2(P) \otimes_N L^2(P)_P, {}_P L^2(M)_P\})$ is weakly amenable (i.e., has an ergodic probability measure).

If $N \subset P \subset M$ is a free composition subfactor ([**BJ**]), except for $[P : N] = [M : P] = 2$, $N \subset M$ is not amenable ([**HI**, Proposition 8.7]). From this fact and the above argument, we see that if $N \subset P \subset M$ is a free composition subfactor, except for $[P : N] = [M : P] = 2$, $\Xi(\{{}_P L^2(P) \otimes_N L^2(P)_P, {}_P L^2(M)_P\})$ is not weakly amenable.

5. Relations between ergodicity and amenability.

In [**P2**], S. Popa proved that "For a II_1-subfactor $N \subset M$ with amenable graph, the ergodicity of the principal graph is equivalent to that of the dual principal graph." In this section, we generalize this result to the one in C^*-bicategories.

Let $\mathscr{C}$ be a strict, semisimple C^*-bicategory with conjugation and Frobenius duality and $\bigcup_{ij} \boldsymbol{C}[S_{ij}]$ be the associated graded fusion algebra. Let μ be a symmetric generating probability measure on S_{11} such that $I \in \mathrm{support}(\mu)$.

Definition 5.1. For $i \in \{1, 2\}$, (1) Take $X \in \mathrm{Object}(\mathscr{C}_{1i})$ and fix it. For a sequence $\{f_n\}_n$ ($f_n \in l^\infty(\mathrm{support}(\mu^n * \delta_X))$, it is called *$\mu$-harmonic* if it satisfies

$$f_n(s) = \sum_{t \in S_{1i}} \mu * \delta_s(t) f_{n+1}(t)$$

($s \in \mathrm{support}(\mu^n * \delta_X)$, $n \in \boldsymbol{N}$) and $\sup_n \|f_n\|_\infty < \infty$.

(2) For $f \in l^\infty(S_{1i})$, f is called *μ-harmonic* if it satisfies

$$f(s) = \sum_{t \in S_{1i}} \mu * \delta_s(t) f(t)$$

($s \in S_{1i}$).

The next proposition can be shown by using Foguel's lemma.

Proposition 5.2. *For each μ-harmonic sequence $\{f_n\}_n$, there exists a μ-harmonic function f such that $f|_{\mathrm{support}(\mu^n * \delta_X)} = f_n$.*

Proof. Define a linear map $P : l^\infty(S_{1i}) \to l^\infty(S_{1i})$ by

$$(Pg)(s) = \sum_{t \in S_{1i}} \mu * \delta_s(t) g(t)$$

($s \in S_{1i}$, $g \in l^\infty(S_{1i})$). It is easy to see that

$$(P^n g)(s) = \sum_{t \in S_{1i}} \mu^n * \delta_s(t) g(t).$$

Let $\{f_n\}_{n=1}^{\infty}$ $(f_n \in l^{\infty}(\mathrm{support}(\mu^n * \delta_X)) \subset l^{\infty}(S_{1i}))$ be a μ-harmonic sequence. Then we have for any $k, n \in \boldsymbol{N}$,

$$\begin{aligned}
\|P^n f_{n+k+1} - P^{n+1} f_{n+k+1}\|_{\infty} &= \sup_{s \in S_{1i}} |(P^n f_{n+k+1})(s) - (P^{n+1} f_{n+k+1})(s)| \\
&\geq \sup_{s \in \mathrm{support}(\mu^k * \delta_X)} |(P^n f_{n+k+1})(s) - (P^{n+1} f_{n+k+1})(s)| \\
&= \sup_{s \in \mathrm{support}(\mu^k * \delta_X)} |f_{k+1}(s) - f_k(s)| \\
&= \|f_{k+1}|_{\mathrm{support}(\mu^k * \delta_X)} - f_k\|_{\infty}.
\end{aligned}$$

On the other hand,

$$\|P^n f_{n+k+1} - P^{n+1} f_{n+k+1}\|_{\infty} \leq \|f_{n+k+1}\|_{\infty} \|\mu^n - \mu^{n+1}\|_1 \to 0$$

as $n \to \infty$. Hence we get $f_{k+1}|_{\mathrm{support}(\mu^k * \delta_X)} = f_k$ for any $k \in \boldsymbol{N}$. Define f by $f|_{\mathrm{support}(\mu^k * \delta_X)} = f_k$. Then f is a desired μ-harmonic function. □

Proposition 5.3. *Let μ be a symmetric, generating probability measure on S_{11} and $i \in \{1, 2\}$. Then the following three statements are equivalent.*

(1) *Each μ-harmonic sequence consists of constant functions.*

(2) *Each μ-harmonic function is a constant function.*

(3) *For any $s, t \in S_{1i}$,*

$$\lim_{n \to \infty} \|\mu^n * \delta_s - \mu^n * \delta_t\|_1 = 0.$$

Proof. (1) $\Leftrightarrow$ (2) By the previous proposition, this is obvious.

(2) $\Rightarrow$ (3) Take a norm-bounded sequence $\{f_n\}_n$ from $l^{\infty}(S_{1i})$ arbitrarily and fix it. Let ω be a ultrafilter on $\boldsymbol{N}$. Define

$$f(t) = \lim_{n \to \omega} \sum_{s \in S_{1i}} \mu^n * \delta_t(s) f_n(s).$$

Then we have

$$\left| \sum_{s \in S_{1i}} \mu^n * \delta_t(s) f_n(s) - \sum_{s \in S_{1i}} \mu^{n+1} * \delta_t(s) f_n(s) \right| \leq \left(\sup_n \|f_n\|_{\infty} \right) \cdot \|\mu^n - \mu^{n+1}\|_1.$$

This inequality and Foguel's lemma imply

$$\lim_{n \to \omega} \sum_{s \in S_{1i}} \mu^{n+1} * \delta_t(s) f_n(s) = f(t).$$

For $\varepsilon > 0$, there exists a finite set $F \subset S_{1i}$ such that $\sum_{t \in F^c} \mu * \delta_u(t) < \varepsilon$. Then

$$\begin{aligned}
&\left|\sum_{t\in S_{1i}} \mu*\delta_u(t)f(t) - \sum_{t,s\in S_{1i}} \mu*\delta_u(t)\mu^n*\delta_t(s)f_n(s)\right| \\
&\quad\le \sum_{t\in S_{1i}} \mu*\delta_u(t)\left|f(t) - \sum_{s\in S_{1i}} \mu^n*\delta_t(s)f_n(s)\right| \\
&\quad\le 2\sup_n \|f_n\|_\infty \varepsilon + \sum_{t\in F} \mu*\delta_u(t)\left|f(t) - \sum_{s\in S_{1i}} \mu^n*\delta_t(s)f_n(s)\right|.
\end{aligned}$$

Since ε is arbitrary, this implies that

$$\begin{aligned}
\sum_{t\in S_{1i}} \mu*\delta_u(t)f(t) &= \lim_{n\to\omega} \sum_{t,s\in S_{1i}} \mu*\delta_u(t)\mu^n*\delta_t(s)f_n(s) \\
&= \lim_{n\to\omega} \sum_{s\in S_{1i}} \mu^{n+1}*\delta_u(s)f_n(s) \\
&= f(u),
\end{aligned}$$

i.e., f is μ-harmonic. Then, by (2), f is scalar and we have

$$0 = f(s) - f(t) = \lim_{n\to\omega} \sum_{u\in S_{1i}} (\mu^n*\delta_s(u) - \mu^n*\delta_t(u))f_n(u).$$

Take $s,t\in S_{1i}$ and fix them. Since $\{f_n\}_n$ is arbitrary, we can set $f_n(u) = 1$ if $\mu^n*\delta_s(u) \ge \mu^n*\delta_t(u)$, and $f_n(u) = -1$ otherwise. Then we have

$$0 = f(s) - f(t) = \lim_{n\to\omega} \|\mu^n*\delta_s - \mu^n*\delta_t\|_1.$$

Since this equality holds for any ω, we get (3).

(3) $\Rightarrow$ (2) Let $f\in l^\infty(S_{1i})$ be a μ-harmonic function. Then

$$\begin{aligned}
|f(s) - f(t)| &= \left|\sum_{u\in S_{1i}} (\mu^n*\delta_s(u) - \mu^n*\delta_t(u))f(u)\right| \\
&\le \|f\|_\infty \|\mu^n*\delta_s - \mu^n*\delta_t\|_1 \to 0
\end{aligned}$$

(as $n\to\infty$) shows that f is a constant function. □

THEOREM 5.4. *Assume $\mathscr{C}$ is amenable. If μ is ergodic,*

$$\lim_{n\to\infty} \|\mu^n*\delta_s - \mu^n*\delta_t\|_1 = 0$$

holds for any i and $s,t\in S_{1i}$.

Proof. By the ergodicity of μ, $A_\infty(X)$ is a factor for any $X \in \mathrm{Object}(\mathscr{C}_{11})$. Then by the amenability of $\mathscr{C}$, we can apply Proposition 3.2 and see that $A_\infty(Y)$ is also factor for any $Y \in \mathrm{Object}(\mathscr{C}_{12})$. (Here we remark that all results in Section 3 also hold in the case that support$(\mu) \neq S$). Hence by Lemma 2.5, each μ-harmonic sequence is trivial. Then by using Proposition 5.3, we get the assertion. □

Corollary 5.5. *Assume that $\mathscr{C}$ is amenable. Take a generating probability measure ν on S_{12} (where "generating" means that $\nu * \check{\nu}$ is generating in $\boldsymbol{C}[S_{11}]$) and define $\mu = \nu * \check{\nu}$ on S_{11} and $\mu' = \check{\nu} * \nu$ on S_{22} where $\check{\nu}(s) = \nu(s^*)$. Then μ is ergodic if and only if μ' is ergodic.*

Proof. We have only to show that μ' is ergodic if μ is ergodic. Since μ is symmetric, generating, ergodic, $I \in \mathrm{support}(\mu)$ and $\mathscr{C}$ is amenable, we have

$$\lim_{n\to\infty} \|\mu^n * \delta_s - \mu^n * \delta_t\|_1 = 0$$

for each $s, t \in S_{12}$. Then for each $s \in S_{22}$,

$$\begin{aligned}
\|{\mu'}^{n+1} * \delta_s - {\mu'}^{n+1}\|_1 &= \|\check{\nu} * (\mu^n * \nu * \delta_s - \mu^n * \nu)\|_1 \\
&\leq \|\mu^n * \nu * \delta_s - \mu^n * \nu\|_1 \\
&\leq \sum_{t \in S_{12}} \nu(t)\|\mu^n * \delta_t * \delta_s - \mu^n * \delta_t\|_1 \to 0
\end{aligned}$$

as $n \to \infty$. □

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