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A generalization of the Liouville theorem to polyharmonic functions

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Abstract. The aim of this note is to generalize the Liouville theorem to polyharmonic functions u on $\boldsymbol{R}^n$. We give a condition on spherical means to assure that u is a polynomial.

1. Introduction.

Let $\boldsymbol{R}^n$ be the n-dimensional Euclidean space with a point $x = (x_1, x_2, \dots, x_n)$. For a multi-index $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, we set

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n,$$

$$x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$$

and

$$\left(\frac{\partial}{\partial x}\right)^\lambda = \left(\frac{\partial}{\partial x_1}\right)^{\lambda_1} \left(\frac{\partial}{\partial x_2}\right)^{\lambda_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\lambda_n}.$$

We denote by rB^n the open ball centered at the origin with radius $r > 0$, whose boundary is denoted by rS^{n-1}.

A real valued function u is called polyharmonic of order m on $\boldsymbol{R}^n$ if $u \in C^{2m}$ and $\Delta^m u = 0$, where m is a positive integer, Δ denotes the Laplacian and $\Delta^m u = \Delta^{m-1}(\Delta u)$. We denote by $H^m(\boldsymbol{R}^n)$ the space of polyharmonic functions of order m on $\boldsymbol{R}^n$. In particular, u is harmonic on $\boldsymbol{R}^n$ if $u \in H^1(\boldsymbol{R}^n)$. A real valued function u on $\boldsymbol{R}^n$ belongs to $H^m(\boldsymbol{R}^n)$ if and only if there exists a family $\{h_i\}_{i=1}^m \subset H^1(\boldsymbol{R}^n)$ such that

$$u(x) = \sum_{i=1}^m |x|^{2(i-1)} h_i(x) \tag{1}$$

for every $x \in \boldsymbol{R}^n$; this is known as the finite Almansi expansion (cf. [**2**], [**6**]).

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The Liouville theorem for polyharmonic functions is known in several forms (cf. [**1**], [**4**], [**5**]).

THEOREM A. *Let* $u \in H^m(\boldsymbol{R}^n)$ *and* $s > 2(m-1)$. *Then* u *is a polynomial of degree less than* s *if one of the following conditions holds*:

(i) $\lim_{r\to\infty} \frac{1}{r^{s+n-1}} \int_{rS^{n-1}} u^+ \, dS = 0$ (see [**1**]);

(ii) $\lim_{r\to\infty} \frac{1}{r^{s+n}} \int_{rB^n} u^+ \, dx = 0$ (see [**4**]);

(iii) $\limsup_{r\to\infty} \left(\max_{x \in rS^{n-1}} \frac{u(x)}{|x|^s} \right) \le 0$ (see [**5**]).

For harmonic functions, we refer the reader to Brelot [**3**; Appendix].

Now we propose the following theorem.

THEOREM. *Let* $u \in H^m(\boldsymbol{R}^n)$ *and* $s > 2(m-1)$. *Then* u *is a polynomial of degree at most* s *if and only if*

$$\liminf_{r\to\infty} \frac{1}{r^{s+n-1}} \int_{rS^{n-1}} u^+ \, dS < \infty. \tag{2}$$

We here note that each condition of THEOREM A implies (2), so that our theorem gives an improvement of THEOREM A. We also see that if (iii) is replaced by a weaker condition

(iii′) $\liminf_{r\to\infty} \left(\max_{x \in rS^{n-1}} \frac{u(x)}{|x|^s} \right) < \infty,$

then u is a polynomial of degree at most s.

2. The main lemma.

Let us begin by preparing the following lemma, which gives a relation between spherical means and derivatives for harmonic functions.

LEMMA 1. *Suppose* $u \in H^1(\boldsymbol{R}^n)$. *For each multi-index* λ, *there exists a positive constant* $C = C(\lambda)$ *such that*

$$\int_{rS^{n-1}} u x^\lambda \, dS = C r^{2|\lambda|+n-1} \left(\frac{\partial}{\partial x} \right)^\lambda u(0) + P_{2|\lambda|+n-3}(r) \tag{3}$$

for every $r > 0$, *where* $P_k(r)$ *is a polynomial of degree at most* k.

PROOF. We prove this lemma by induction on the length of λ. Assume first that $\lambda_n = 1$ and $\lambda_i = 0$ $(i = 1, \ldots, n-1)$. Using Green's formula and the mean-value property for harmonic functions, we have

$$\begin{aligned}\int_{rS^{n-1}} ux^{\lambda}\,dS &= \int_{rS^{n-1}} ux_n\,dS \\ &= r\int_{rS^{n-1}} u\frac{x_n}{r}\,dS \\ &= r\int_{rB^n} \frac{\partial u}{\partial x_n}\,dx \\ &= \sigma_n r^{n+1}\frac{\partial u}{\partial x_n}(0),\end{aligned}$$

where σ_n is the n-dimensional volume of the unit ball. Hence (3) holds for $|\lambda| = 1$.

Next suppose that (3) holds for $|\lambda| \le k$, where k is a positive integer. Let $\mu = (\mu_1, \dots, \mu_n)$ such that $|\mu| = k+1$. We may assume without loss of generality that $\mu_n \ge 2$, and set $\mu' = (\mu_1, \dots, \mu_{n-1}, \mu_n - 1)$. Then we write

$$\int_{rS^{n-1}} ux^{\mu}\,dS = r\int_{rS^{n-1}} ux^{\mu'}\frac{x_n}{r}\,dS.$$

From Green's formula we obtain

$$\begin{aligned}\int_{rS^{n-1}} ux^{\mu}\,dS &= r\int_{rB^n} \frac{\partial(ux^{\mu'})}{\partial x_n}\,dx \\ &= r\int_{rB^n}\left(x^{\mu'}\frac{\partial u}{\partial x_n} + (\mu_n - 1)ux_1^{\mu_1}\cdots x_n^{\mu_n-2}\right)dx = (*).\end{aligned}$$

Set $\mu'' = (\mu_1, \dots, \mu_{n-1}, \mu_n - 2)$. Since $|\mu'| = k$ and $|\mu''| = k-1$ (if $\mu_n \ge 2$), we find

$$\begin{aligned}(*) &= r\int_0^r\left(\int_{tS^{n-1}}\left(x^{\mu'}\frac{\partial u}{\partial x_n} + (\mu_n - 1)ux^{\mu''}\right)dS\right)dt \\ &= r\int_0^r\left(C(\mu')t^{2|\mu'|+n-1}\left(\frac{\partial}{\partial x}\right)^{\mu'}\left(\frac{\partial u}{\partial x_n}\right)(0) + P_{2|\mu'|+n-3}(t)\right)dt \\ &\quad + r(\mu_n - 1)\int_0^r\left(C(\mu'')t^{2|\mu''|+n-1}\left(\frac{\partial}{\partial x}\right)^{\mu''}u(0) + P_{2|\mu''|+n-3}(t)\right)dt \\ &= C(\mu)r^{2k+n+1}\left(\frac{\partial}{\partial x}\right)^{\mu}u(0) + P_{2k+n-1}(r),\end{aligned}$$

where $C(\mu) = (C(\mu'))/(2k+n) > 0$ and P_ℓ denotes various polynomials of degree at most ℓ which may change from one occurrence to the next; throughout this note, we use this convention. Hence (3) also holds for $|\mu| = k+1$. The induction is completed.

3. Proof of the theorem.

First we show that our theorem is valid under the two sided condition on spherical means for polyharmonic functions.

Lemma 2. *Let $u \in H^m(\mathbf{R}^n)$ and $s > 2(m-1)$. Then u is a polynomial of degree at most s if*

$$\liminf_{r\to\infty} \frac{1}{r^{s+n-1}} \int_{rS^{n-1}} |u|\, dS < \infty. \tag{4}$$

Proof. By (4) we can find a sequence $\{r_j\}_{j=1}^{\infty}$ such that $r_j \to \infty$ and

$$\sup_j \left(r_j^{-s-n+1} \int_{r_jS^{n-1}} |u|\, dS \right) < \infty. \tag{5}$$

Using (1) and Lemma 1, we have

$$\begin{aligned}
\int_{rS^{n-1}} u x^\lambda\, dS &= \int_{rS^{n-1}} \left(\sum_{i=1}^{m} |x|^{2(i-1)} h_i(x) \right) x^\lambda\, dS \\
&= \sum_{i=1}^{m} r^{2(i-1)} \int_{rS^{n-1}} h_i(x) x^\lambda\, dS \\
&= \sum_{i=1}^{m} r^{2(i-1)} \left(C_i r^{2|\lambda|+n-1} \left(\frac{\partial}{\partial x}\right)^\lambda h_i(0) + P_{i,2|\lambda|+n-3}(r) \right),
\end{aligned}$$

where C_i is a positive constant and $P_{i,k}$ denotes various polynomials of degree at most k. Hence it follows that

$$r^{|\lambda|} \int_{rS^{n-1}} |u|\, dS \geq \left| \sum_{i=1}^{m} r^{2(i-1)} \left(C_i r^{2|\lambda|+n-1} \left(\frac{\partial}{\partial x}\right)^\lambda h_i(0) + P_{i,2|\lambda|+n-3}(r) \right) \right|,$$

so that we obtain

$$r_j^{-s-n+1} \int_{r_jS^{n-1}} |u|\, dS \geq r_j^{|\lambda|-s+2(m-1)} \left| C_m \left(\frac{\partial}{\partial x}\right)^\lambda h_m(0) + O(r_j^{-2}) \right|$$

as $r_j \to \infty$. By (5), we find

$$\left(\frac{\partial}{\partial x}\right)^{\lambda} h_m(0) = 0$$

for all $|\lambda| > s - 2(m-1)$. By analyticity of harmonic functions, we see that h_m is a polynomial of degree at most $s - 2(m-1)$. Hence we note that

$$r^{2(m-1)} \int_{rS^{n-1}} h_m(x) x^{\lambda}\, dS = O(r^{s+|\lambda|+n-1}) \quad \text{as } r \to \infty.$$

Consequently,

$$r_j^{-s-n+1} \int_{r_j S^{n-1}} |u|\, dS \ge r_j^{|\lambda|-s+2(m-2)} \left| C_{m-1} \left(\frac{\partial}{\partial x}\right)^{\lambda} h_{m-1}(0) + O(r_j^{-2}) \right| + O(1)$$

as $r_j \to \infty$. This implies that $(\partial/\partial x)^{\lambda} h_{m-1}(0) = 0$ for $|\lambda| > s - 2(m-2)$, so that h_{m-1} is a polynomial of degree at most $s - 2(m-2)$. By repeating this argument, we see that each h_i is a polynomial of degree at most $s - 2(i-1)$ $(i = 1, \ldots, m)$. Thus it follows that u is a polynomial. In view of (1), the degree of u is at most $2(i-1) + s - 2(i-1) = s$.

Proof of the Theorem. If $u \in H^m(\boldsymbol{R}^n)$, then we see from (1) that

$$\frac{1}{\omega_n r^{n-1}} \int_{rS^{n-1}} u\, dS = \sum_{i=1}^{m} r^{2(i-1)} h_i(0),$$

where ω_n denotes the surface measure of S^{n-1}.

Since $|u| = 2u^+ - u$, we have

$$\begin{aligned} &\liminf_{r\to\infty} r^{-s-n+1} \int_{rS^{n-1}} |u|\, dS \\ &\quad = \liminf_{r\to\infty} \left(2r^{-s-n+1} \int_{rS^{n-1}} u^+\, dS - r^{-s-n+1} \int_{rS^{n-1}} u\, dS \right) \\ &\quad = \liminf_{r\to\infty} \left(2r^{-s-n+1} \int_{rS^{n-1}} u^+\, dS - r^{-s} P_{2(m-1)}(r) \right). \end{aligned}$$

Hence (2) implies (4) since $s > 2(m-1)$, so that the present theorem follows from Lemma 2.

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