J. Math. Soc. Japan
Vol. 53, No. 1, 2001

Cardinal invariants associated with predictors II

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(Received Feb. 22, 1999)
(Revised Aug. 9, 1999)

Abstract. We call a function from $\omega^{<\omega}$ to ω a predictor. A function $f \in \omega^\omega$ is said to be constantly predicted by a predictor π, if there is an $n < \omega$ such that $\forall i < \omega \exists j \in [i, i+n)(f(j) = \pi(f \upharpoonright j))$. Let θ_ω denote the smallest size of a set Φ of predictors such that every $f \in \omega^\omega$ can be constantly predicted by some predictor in Φ. In [7], we showed that θ_ω may be greater than $\mathrm{cof}(\mathscr{N})$. In the present paper, we will prove that θ_ω may be smaller than $\mathbf{d}$.

1. Introduction

A. Blass [2] introduced the notion of predictors and several evasion numbers, and studied how large these evasion numbers are compared with cardinals in Cichoń's diagram. After that, J. Brendle [4] extended this notion and studied more closely. Also, he studied the 'dual' cardinals of evasion numbers. Each evasion number can be characterized as the uniformity of a certain subset of $\mathscr{P}(\omega^\omega)$. The 'dual' cardinal of an evasion number means the covering number of the corresponding subset. There are known relations between these cardinals and the cardinals in Cichoń's diagram (for details, see [2], [3], [4], [5], [6]). Concerning this, we [7] introduced a notion of 'constantly predict', and using this notion, defined cardinal invariants θ_K (for $2 \le K \le \omega$), as follows.

Following A. Blass [2], we call a function from $\omega^{<\omega}$ to ω a *predictor*. A function $f \in \omega^\omega$ is said to *be predicted constantly* by a predictor π, if there is an $n < \omega$ such that, for any $i < \omega$, $f(j) = \pi(f \upharpoonright j)$, for some $j \in [i, i+n)$. Let $2 \le K \le \omega$. We denote by θ_K the smallest size of a set of predictors Φ such that every function $f \in K^\omega$ is predicted constantly by some predictor in Φ, and by $\mathrm{Dual}(\theta_K)$ the smallest size of a set of functions $F \subset K^\omega$ such that, for any predictor π, there exists an $f \in F$ which is not predicted constantly by π.

The motivation of θ_K and $\mathrm{Dual}(\theta_K)$ is in some game-theoretical characterizations for cardinals in Cichoń's diagram. F. Galvin gave game-theoretical characterizations for $\mathbf{d}$ and $\mathbf{cov}(\mathscr{M})$, and M. Scheepers for $\mathbf{b}$, $\mathbf{add}(\mathscr{M})$, $\mathbf{non}(\mathscr{M})$

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*The author is partially supported by Grant-in-Aid for Scientific Research (No. 09640288), Ministry of Education, Science, Sports and Culture, Japan

2000 *Mathematics Subject Classification*. Primary 03E35; Secondary 04A20.

Key Words and Phrases. predictor, countable support iteration, rational perfect tree forcing.

and $\mathbf{add}(\mathcal{N})$ (See [**9**], [**10**] for details). After that, M. Kada (in unpublished work) introduced new games in order to characterized other cardinals in Cichoń's diagram. Also, he pointed out the relationship between game-theoretic properties and the notion of predictors. The θ_ω is a translation of the game which corresponds to $\mathrm{cof}(\mathcal{M})$.

It seems to be interesting to decide the size of these θ_K and $\mathrm{Dual}(\theta_K)$ (for $2 \le K \le \omega$) in comparison with the cardinals in Cichoń's diagram and other evasion numbers. Let $\mathcal{M}, \mathcal{N}$ denote the meager ideal and the null ideal on ω^ω, respectively. Concerning these, the followings is a summary of the results of [**7**].

1. If $K \le M \le \omega$ then $\theta_K \le \theta_M$ and $\mathrm{Dual}(\theta_M) \le \mathrm{Dual}(\theta_K)$.
2. $\mathbf{cov}(\mathcal{M}) \le \theta_2$ and $\mathbf{cov}(\mathcal{N}) \le \theta_2$, and $\mathrm{Dual}(\theta_2) \le \mathbf{non}(\mathcal{M})$ and $\mathrm{Dual}(\theta_2) \le \mathbf{non}(\mathcal{N})$.
3. $\mathbf{non}(\mathcal{M}) \le \theta_\omega$ and $\mathrm{Dual}(\theta_\omega) \le \mathbf{cov}(\mathcal{M})$.
4. There is a generic model in which $\mathbf{cof}(\mathcal{N}) = \omega_1$ and $\theta_2 = \omega_2$ hold.
5. There is a generic model in which $\theta_\omega = \omega_2$ and $\theta_K = \omega_1$ hold, for all $K < \omega$.

The purpose of this paper is to give a generic model in which $\theta_\omega = \omega_1$ and $\mathbf{d} = \omega_2$ hold. Before explaining how to get a desired generic model, I mention several questions which I am interested in, but do not know the answers.

Question 0_D. Is it consistent that $\mathbf{b} < \mathrm{Dual}(\theta_\omega)$?

Question 1. For $2 \le K \le \omega$, is it consistent that $\theta_K < \mathbf{non}(\mathcal{N})$?

Question 1_D. For $2 \le K \le \omega$, is it consistent that $\mathbf{cov}(\mathcal{N}) < \mathrm{Dual}(\theta_K)$?

Question 2. Is it consistent that $\theta_2 < \mathbf{non}(\mathcal{M})$?

Question 2_D. Is it consistent that $\mathbf{cov}(\mathcal{M}) < \mathrm{Dual}(\theta_2)$?

Question 3. For $2 \le K < M < \omega$, is it consistent that $\theta_K < \theta_M$?

Question 3_D. For $2 \le K < M \le \omega$, is it consistent that $\mathrm{Dual}(\theta_M) < \mathrm{Dual}(\theta_K)$?

Question 4_D. For $2 \le K \le \omega$, is it consistent that $\mathrm{Dual}(\theta_K) < \mathbf{add}(\mathcal{N})$?

Now, we explain how to get a desired generic model. Let $\boldsymbol{V}$ be a ground model which satisfies CH. In $\boldsymbol{V}$, let $\langle P_\alpha \,|\, \alpha \le \omega_2 \rangle$ be the ω_2-stage countable support iteration of the rational perfect tree forcing. We will show that, in $\boldsymbol{V}^{P_{\omega_2}}$, every $f \in \omega^\omega$ is constantly predicted by some predictor in $\boldsymbol{V}$. Since it is known (see e.g. [**1**]) that $\mathbf{d} = \omega_2$ holds in $\boldsymbol{V}^{P_{\omega_2}}$, the model $\boldsymbol{V}^{P_{\omega_2}}$ is a desired one.

Let $\boldsymbol{PT}$ denote the rational perfect tree forcing. It is not difficult to check that, in $\boldsymbol{V}^{\boldsymbol{PT}}$, every $f \in \omega^\omega$ is constantly predicted by some predictor in $\boldsymbol{V}$. So, if we can show that this property is preserved by countable support iterations, we complete a proof of the result. But I don't know whether the property is preserved by such iterations (even in the case of two step iterations). We consider a somewhat stronger property of a forcing notion P. That is, in $\boldsymbol{V}^P$, for any $f \in \omega^\omega$, there exists a skip branching tree $H \in \boldsymbol{V}$ such that $f \in \mathrm{Lim}(H)$

(for the definition of skip branching trees, see the next section), and we will prove that this property is preserved by the iteration.

In the next section, we give the definitions and notations, and describe the result of this paper (Theorem 2.1). Section 3 is devoted to some technical lemmas. In section 4, we introduce the notion of tentacle trees, and prove some kind of preservation lemmas. Finally, we prove Theorem 2.1 in section 5.

2. Notations and the theorem.

We use the standard set theoretical notions and notations (see [**1**]). For any set A, $[A]^\omega$ denotes the set of countable subsets of A, and $A^{<\omega}$ the set of finite sequences of elements in A. The statement "there exist infinitely many $i \in x$ such that $\cdots x \cdots$" is denoted by $\exists^\infty i \in x(\cdots x \cdots)$. Let P be a forcing notion and $\dot{f}$ a P-name such that $\Vdash_P \dot{f} : \omega \to \boldsymbol{V}$. We say that $g : \omega \to \boldsymbol{V}$ is an interpretation of $\dot{f}$ below $p \in P$, if there exist $p_n \in P$ (for $n < \omega$) such that $p_{n+1} \le p_n \le p$ and $p_n \Vdash \dot{f} \restriction n = g \restriction n$, for all $n < \omega$.

The forcing notion which will be used in this paper is countable support iteration by the rational perfect tree forcing. The rational perfect tree forcing was introduced by A. Miller [**8**]. We start with the definition of the rational perfect tree forcing.

Definition 2.1. *Let $H \subset \omega^{<\omega}$ be a tree. $s \in H$ is a splitting point, if there are distinct $i, j \in \omega$ such that $s^\frown\langle i\rangle, s^\frown\langle j\rangle \in H$. The set of all splitting points of H is denoted by* $\mathrm{split}(H)$. *For any splitting point $s \in H$,* $\mathrm{next}_H(s)$ *denotes the set $\{s^\frown\langle i\rangle \in H \mid i < \omega\}$. H is said to be perfect, if $\forall s \in H\ \exists t \in H$ ($s \subset t$ and t is a splitting point of H). For any perfect tree H,* $\mathrm{stem}(H)$ *denotes the first splitting point of H. H is said to be a rational perfect tree, if it is a perfect tree and* $\mathrm{next}_H(s)$ *is infinite, for all $s \in \mathrm{split}(H)$.*

For any rational perfect tree $H \subset \omega^{<\omega}$, we denote by Γ_H the natural isomorphism from $\omega^{<\omega}$ to $\mathrm{split}(H)$.

Definition 2.2. *The rational perfect tree forcing $\boldsymbol{PT}$ is defined by*

$$\boldsymbol{PT} = \{q \subset \omega^{<\omega} \mid q \text{ is a rational perfect tree}\},$$

and for any $q, q' \in \boldsymbol{PT}$,

$$q \le q' \text{ if and only if } q \subset q'.$$

And, define the orderings $\le_n^*$ on $\boldsymbol{PT}$ (for $n < \omega$) by

$$q \le_n^* q' \text{ if and only if } q \le q' \text{ and } \Gamma_q \restriction \omega^n = \Gamma_{q'} \restriction \omega^n.$$

Note that $\boldsymbol{PT}$ satisfies Baumgartner's Axiom A with these orderings.

For each $q \in \boldsymbol{PT}$ and $s \in \omega^{<\omega}$, $q \restriction s \in \boldsymbol{PT}$ denotes $\{u \in q \mid u \subset \Gamma_q(s) \text{ or } \Gamma_q(s) \subset u\}$.

Definition 2.3. *Let $H \subset \omega^{<\omega}$ be a tree.*

$$\mathrm{Max}(H) \text{ denotes the set } \{s \in H \mid \forall i < \omega(s^\frown\langle i\rangle \notin H)\}.$$

$$B(H) \text{ denotes the set } \{|s| \mid s \in \mathrm{split}(H) \cup \mathrm{Max}(H)\}.$$

$$\mathrm{Lim}(H) \text{ denotes the set } \{f \in \omega^\omega \mid \forall i < \omega(f \restriction i \in H)\}.$$

We say that H is skip branching, *if* $\forall s \in \mathrm{split}(H)$ $(\mathrm{next}_H(s) \cap (\mathrm{split}(H) \cup \mathrm{Max}(H)) = \phi)$.

Now we describe the result of this paper.

Theorem 2.1. *Assume that* CH *holds in V. Let $\langle P_\alpha \mid \alpha \leq \omega_2\rangle$ be the ω_2-stage countable support iteration of the rational perfect tree forcing. Then, in $V^{P_{\omega_2}}$, it holds that, for every $f \in \omega^\omega$, there exists a skip branching tree $H \in V$ such that $f \in \mathrm{Lim}(H)$.*

As a corollary, we have

Corollary 2.2. *Under the assumption of Theorem* 2.1, *in $V^{P_{\omega_2}}$, there exists a set of predictors Φ of size ω_1 such that, for any $f \in \omega^\omega$,*

$$\exists \pi \in \Phi \forall i < \omega(f(i) = \pi(f \restriction i) \quad \text{or} \quad f(i+1) = \pi(f \restriction (i+1))).$$

So, $\theta_\omega = \omega_1$ holds in $V^{P_{\omega_2}}$.

Since $\mathbf{d} = \omega_2$ holds in the generic model, we have that $\theta_\omega < \mathbf{d}$ is consistent.

In order to prove Theorem 2.1, we need several definitions and lemmas.

Definition 2.4. *For any $a \in [\omega]^\omega$, Γ_a denotes the order isomorphism from ω to a.*

*For any $a \in [\omega]^\omega$ and any $g \in \omega^\omega$, we say that a is g-*thin, *if $g(i) < \Gamma_a(i)$, for all $i < \omega$.*

Lemma 2.3. *For any $g \in \omega^\omega$, there exists $\{g_i \mid i < \omega\} \subset \omega^\omega$ such that, for any $\{a_i \mid i < \omega\} \subset [\omega]^\omega$, if $\forall i < \omega$ (a_i is g_i-thin) then $\bigcup_{i<\omega} a_i$ is g-thin.*

Proof. Take a g-thin set a and divide a into countable disjoint infinite sets $\{a_i \mid i < \omega\}$. Then, $g_i = \Gamma_{a_i}$ (for $i < \omega$) are as required. □

Definition 2.5. *A subset I of ω is called* an interval, *if there exist $n, m \in \omega$ such that $n < m$ and $I = [n, m)$.*

A family $\langle I_n \mid n < \omega\rangle$ of intervals is called disjoint intervals, *if it holds that*

$$\forall n < \forall m < \omega(\max(I_n) < \min(I_m)) \quad \text{and} \quad \lim_{n<\omega} |I_n| = \omega.$$

Lemma 2.4. *Let F be an unbounded subset of ω^ω such that $\forall f \in F$ (f is strictly increasing). Then, for any disjoint intervals $\langle I_n \mid n < \omega\rangle$, there exists $f \in F$*

such that

$$\forall a \in [\omega]^{\omega} \ (\text{if } a \text{ is } f\text{-thin, then } \exists^{\infty} n < \omega \ (I_n \cap a = \phi)).$$

Proof. Define $g \in \omega^{\omega}$ by

$$g(j) = \max I_{2j} + 1, \quad \text{for all } j < \omega.$$

Since F is unbounded, take $f \in F$ such that $\exists^{\infty} j < \omega(g(j) < f(j))$. In order to show that f satisfies the requirement of the lemma, let $a \in [\omega]^{\omega}$ be f-thin. We claim that

$$\forall j < \omega \ (\text{if } g(j) < f(j), \text{ then } \exists k \in [j, 2j) \ (I_k \cap a = \phi)).$$

To get a contradiction, assume that

$$g(j) < f(j) \quad \text{and} \quad \forall k \in [j, 2j)(I_k \cap a \neq \phi).$$

Then, since $|a \cap g(j)| \geq 2j - j = j$, it holds that $\Gamma_a(j) < g(j) < f(j)$. This contradicts that a is f-thin. □

Definition 2.6. *For any $\delta, \tau \in \omega^{\omega} \cup \omega^{<\omega}$, $\Delta(\delta, \tau)$ denotes the least $i < \omega$ such that $\delta(i) \neq \tau(i)$, if such i exists, otherwise, $\Delta(\delta, \tau)$ is undefined.*

Definition 2.7. *Let u be a function from ω to $\omega^{<\omega}$. u is called a* <u>type I</u> <u>function</u> *with root $\delta \in \omega^{<\omega}$, if*

$$\forall i < \forall j < \omega(\delta \subset u(i) \text{ and } |\delta| + 2 \leq |u(i)| < |u(j)| \text{ and } u(i)(|\delta|) < u(j)(|\delta|)).$$

u is called a <u>type II</u> function *with limit $h \in \omega^{\omega}$, if*

$$\forall i < \forall j < \omega(u(i) \not\subset h \text{ and } \Delta(u(i), h) + 2 \leq |u(i)| \text{ and } \Delta(u(i), h) + 2 \leq \Delta(u(j), h)).$$

Note that, for any functions $\{f_i \mid i < \omega\} \subset \omega^{\omega}$, if $f_i \neq f_j$ for all $i < j < \omega$, then there exist $a \in [\omega]^{\omega}$ and $k_i < \omega$ (for $i < \omega$) such that $\langle f_{\Gamma_a(i)} \restriction k_i \mid i < \omega \rangle$ is a type I or type II function.

The following lemma will be used in the proof of Theorem 2.1 to handle the successor cases of the induction step. We need the corresponding result which handles the limit cases and will be established the next section.

Lemma 2.5. *Let $h : \omega^{<\omega} \to \omega$, $q \in \boldsymbol{PT}$, and $\dot{f}$ a $\boldsymbol{PT}$-name such that $q \Vdash \dot{f} \in \omega^{\omega} \setminus \boldsymbol{V}$. Then, there exist $q' \leq q$ and $\{\delta_s \mid s \in \omega^{<\omega}\}$ such that, for any $s \in \omega^{<\omega}$,*

(1) $q' \restriction s \Vdash \delta_s \subset \dot{f}$,

(2) $h(s) < |\delta_s| < |\delta_{s^\frown\langle 0 \rangle}|$,

(3) $\langle \delta_{s^\frown\langle i \rangle} \mid i < \omega \rangle$ *is a type I or type II function.*

Proof. It suffices to show that

CLAIM 1. *There exists* $q_n \in \boldsymbol{PT}$ *(for* $n < \omega$*) and* $\delta_s \in \omega^{<\omega}$ *(for* $s \in \omega^{<\omega}$*) which satisfy* (2), (3) *and*

(1)′ $q_n \restriction s \Vdash \delta_s \subset \dot{f}$, *for all* $s \in \omega^n$

(4) $q_{n+1} \leq_n^* q_n \leq q$, *for all* $n < \omega$,

PROOF OF CLAIM 1. By induction on $n < \omega$. We only deal with the cases of that $0 < n$, because the case $n = 0$ can be done by a similar argument. So, assume that $n = m + 1 < \omega$ and q_m and δ_s (for $s \in \omega^m$) have been defined.

Let $s \in \omega^m$. Define $q_n \restriction s$ and $\delta_{s^\frown\langle i\rangle}$, for $i < \omega$, as follows:

Take $f_i \in \omega^\omega$ (for $i < \omega$) such that

f_i is an interpretation of $\dot{f}$ below $q_m \restriction s^\frown\langle i\rangle$ and $\forall i < \forall j < \omega$ $(f_i \neq f_j)$.

This can be taken, since $q_m \restriction s \Vdash \dot{f} \notin \boldsymbol{V}$. Since all f_i's are distinct, there are $a \in [\omega]^\omega$ and $k_i < \omega$ (for $i < \omega$) such that

$\langle f_{\Gamma_a(i)} \restriction k_i \mid i < \omega\rangle$ is a type I or type II function and $h(s^\frown\langle i\rangle) < k_i$, for all $i < \omega$.

For each $i < \omega$, take $r_i \leq q_m \restriction s^\frown\langle \Gamma_a(i)\rangle$ such that $r_i \Vdash f_{\Gamma_a(i)} \restriction k_i \subset \dot{f}$. Set $q_n \restriction s = \bigcup_{i<\omega} r_i$ and $\delta_{s^\frown\langle i\rangle} = f_{\Gamma_a(i)} \restriction k_i$, for $i < \omega$. Note that $q_n \restriction s^\frown\langle i\rangle = r_i$, for all $i < \omega$. So, $\delta_{s^\frown\langle i\rangle}$ (for $s \in \omega^m$ and $i < \omega$) and q_n satisfy the requirements. □

3. Iteration.

In this section, we deal with a countable support(CS, for short) iteration of the rational perfect tree forcing. Throughout this paper, $\langle P_\alpha \mid \alpha \leq \omega_2\rangle$ denotes the ω_2-stage CS iteration of the rational perfect tree forcing. For each $\alpha \leq \omega_2$, the canonical P_α-name of a generic filter is denoted by $\dot{\mathscr{G}}_{P_\alpha}$. For each $p \in P_{\omega_2}$, the support of p is denoted by support(p).

DEFINITION 3.1. *For each* $\xi < \alpha \leq \omega_2$, P_α/P_ξ *denotes the* P_ξ*-name which represents* P_α *in* $\boldsymbol{V}^{P_\xi}$. *That is*

$$\Vdash_\xi P_\alpha/P_\xi = \{p \restriction [\xi, \alpha) \mid p \in P_\alpha\},$$

and for any $r, r' \in P_\alpha/P_\xi$,

$r \leq r'$ *in* P_α/P_ξ *if and only if* $p_0 \cup r \leq p_0 \cup r'$ *in* P_α, *for some* $p_0 \in \dot{\mathscr{G}}_{P_\xi}$.

Let $\xi \leq \beta \leq \omega_2$. It is known that, in $\boldsymbol{V}^{P_\xi}$, it holds that

$\langle P_\alpha/P_\xi \mid \alpha \in [\xi, \beta)\rangle$ is isomorphic to the $(\beta - \xi)$-stage CS iteration of the rational perfect tree forcing.

So, we may identify $\langle P_\alpha/P_\xi \mid \alpha \in [\xi, \beta)\rangle$ with this iteration.

Definition 3.2. *Let $\xi < \alpha \leq \omega_2$, and $p \in P_\alpha$. For each $i < \omega$, define $p[\langle i \rangle_\xi] \in P_\alpha$ by*

$$\text{support}(p[\langle i \rangle_\xi]) = \text{support}(p) \cup \{\xi\},$$

$$p[\langle i \rangle_\xi](\eta) = \begin{cases} p(\eta), & \text{if } \eta \neq \xi, \\ p(\xi) \restriction \langle i \rangle, & \text{if } \eta = \xi \end{cases}$$

Note that $\{p[\langle i \rangle_\xi] \mid i < \omega\}$ is a partition of p.

Definition 3.3. *Let $\xi < \alpha \leq \omega_2$, $p \in P_\alpha$, and $p_i \in P_\alpha$ (for $i < \omega$).*
We say that $\langle p_i \mid i < \omega \rangle$ is a one-point partition of p at ξ, if for all $i < \omega$,

(1) $p_i \restriction \xi = p \restriction \xi$,

(2) $p \restriction \xi \Vdash p_i(\xi) = p(\xi) \restriction \langle i \rangle$,

(3) $p_i \restriction \eta \Vdash p_i(\eta) = p(\eta)$, *for all* $\eta \in [\xi + 1, \alpha)$.

In this case, we say that p is the root of $\langle p_i \mid i < \omega \rangle$.

Note that $\langle p[\langle i \rangle_\xi] \mid i < \omega \rangle$ is a one-point partition of p at ξ.

Lemma 3.1. *Assume that $\langle p_i \mid i < \omega \rangle$ is a one-point partition of p at ξ. Then, it holds that $p_i = p[\langle i \rangle_\xi]$, for all $i < \omega$.*

Proof. Trivial. □

Lemma 3.2. *Let $\alpha \leq \xi < \beta \leq \omega_2$, $p \in P_\beta$, and $p_i \in P_\beta$ (for $i < \omega$). Then, the following* (a) *and* (b) *are equivalent.*

(a) *$\langle p_i \mid i < \omega \rangle$ is a one-point partition of p at ξ.*

(b) *The following* (b.1) *and* (b.2) *hold.*

(b.1) $\forall i < \omega \ (p_i \restriction \alpha = p \restriction \alpha)$.

(b.2) *$p \restriction \alpha \Vdash \langle p_i \restriction [\alpha, \beta) \mid i < \omega \rangle$ is a one-point partition of $p \restriction [\alpha, \beta)$ at ξ in P_β / P_α.*

Proof. Trivial. □

Lemma 3.3. *Let $\xi < \alpha \leq \omega_2$, and $\dot{a}, \dot{q}$ P_ξ-names, and $p_i \in P_\alpha$ (for $i < \omega$). Suppose that*

(1) $\Vdash_\xi \dot{a} \in [\omega]^\omega$ *and* $\dot{q} \in \boldsymbol{PT}$,

(2) $p_i \restriction \xi = p_j \restriction \xi$, *for all* $i, j < \omega$

(3) $p_i \restriction \xi \Vdash_\xi p_i(\xi) \leq \dot{q} \restriction \langle \Gamma_{\dot{a}}(i) \rangle$, *for all* $i < \omega$.

Then, there exists $p \in P_\alpha$ such that $\langle p_i \mid i < \omega \rangle$ is a one-point partition of p at ξ.

PROOF. Note that $\{p_i \upharpoonright \eta \mid i < \omega\}$ is pairwise incompatible, for any $\eta > \xi$. Define a P_ξ-name $\dot{q}_\xi$ by $\Vdash_\xi \dot{q}_\xi = \bigcup_{i<\omega} p_i(\xi)$. By (3), it holds that $p_0 \upharpoonright \xi \Vdash \dot{q}_\xi \in \boldsymbol{PT}$. Let $X = \bigcup_{i<\omega} \operatorname{support}(p_i) \cap [\xi + 1, \alpha)$. For each $\eta \in X$, take a P_η-name $\dot{q}_\eta$ such that

$$p_i \upharpoonright \eta \Vdash \dot{q}_\eta = p_i(\eta), \quad \text{for all } i < \omega.$$

Define $p \in P_\alpha$ by

$$\operatorname{support}(p) = \operatorname{support}(p_0 \upharpoonright \xi) \cup \{\xi\} \cup X,$$

$$p \upharpoonright \xi = p_0 \upharpoonright \xi,$$

$$p(\eta) = \dot{q}_\eta, \quad \text{for } \eta \in \{\xi\} \cup X.$$

Then, p is as required. □

From now on, λ is an arbitrary but fixed and sufficiently large regular cardinal. $H(\lambda)$ denotes the family of sets that are hereditarily of cardinality $< \lambda$. Throughout the rest of this paper, N denotes a countable elementary substructure of $H(\lambda)$, in general. For any forcing notion $P \in N$ and $p \in P$, we say that $p \in P$ is (N, P)-generic, if it holds that

$$D \cap N \text{ is predense below } p, \text{ for any dense subset } D \in N \text{ of } P.$$

LEMMA 3.4. *Let $\xi < \alpha \leq \omega_2$, $p \in P_\alpha$, and ξ, α, $P_\alpha \in N$. Suppose that*

$$p[\langle i \rangle_\xi] \text{ is } (N, P_\alpha)\text{-generic, for all } i < \omega.$$

Then, p is (N, P_α)-generic.

PROOF. Trivial. □

COROLLARY 3.5. *Let $\xi < \alpha \leq \omega_2$, and $\xi, \alpha, P_\alpha \in N$. Suppose that*

$$\bar{p} \in P_\xi \text{ is } (N, P_\xi)\text{-generic and } \bar{p} \Vdash_\xi \dot{q} \in N[\dot{\mathscr{G}}_{P_\xi}] \cap P_\alpha / P_\xi.$$

Then, there exists $p \in P_\alpha$ such that

(1) *p is (N, P_α)-generic and $p \upharpoonright \xi = \bar{p}$,*

(2) *$\bar{p} \Vdash_\xi p \upharpoonright [\xi, \alpha) \leq \dot{q}$ and $p(\xi) \upharpoonright \langle i \rangle \leq \dot{q}(\xi) \upharpoonright \langle i \rangle$, for all $i < \omega$.*

PROOF. We work in $\boldsymbol{V}^{P_\xi}$ below $\bar{p}$. For each $i < \omega$, take $\dot{q}_i \leq \dot{q}[\langle i \rangle_\xi]$ such that

$$\dot{q}_i \text{ is } (N[\dot{\mathscr{G}}_{P_\xi}], P_\alpha / P_\xi)\text{-generic and } \operatorname{support}(\dot{q}_i) \subset N[\dot{\mathscr{G}}_{P_\xi}].$$

Since $\langle \dot{q}_i \mid i < \omega \rangle$ is a one point partition at ξ, let $\dot{r}$ be the root of this. Then, by Lemma 3.4, $\dot{r}$ is $(N[\dot{\mathscr{G}}_{P_\xi}], P_\alpha / P_\xi)$-generic. Return to $\boldsymbol{V}$. Since $\bar{p} \Vdash \operatorname{support}(\dot{r}) \subset$

$N[\dot{\mathscr{G}}_{P_\xi}]$, we can take $p \in P_\alpha$ such that

$$p \restriction \xi = \bar{p} \quad \text{and} \quad \bar{p} \Vdash p \restriction [\xi, \alpha) = \dot{r}.$$

Then, p is as required. $\square$

Definition 3.4. *Let α is a limit ordinal with cofinality ω and $\langle \alpha_n \,|\, n < \omega \rangle$ an increasing sequence of ordinals with limit α. For each $p \in P_\alpha$ and $s \in \omega^{<\omega}$, define $p[[s]] = p[[s]]_{\langle \alpha_n | n < \omega \rangle} \in P_\alpha$ by*

$$\text{support}(p[[s]]) = \text{support}(p) \cup \{\alpha_i | i < |s|\},$$

$$p[[s]](\xi) = p(\xi), \ \textit{if} \ \ \xi \neq \alpha_i \ \textit{for} \ \ i < |s|,$$

$$\Vdash_{\alpha_i} p[[s]](\alpha_i) = p(\alpha_i) \restriction \langle s(i) \rangle, \ \textit{for} \ \ i < |s|.$$

We always omit the subscript $\langle \alpha_n | n < \omega \rangle$ of $[[s]]_{\langle \alpha_n | n < \omega \rangle}$, since there are no confusions throughout this paper.

Note that, for any $s = \langle i_0, \ldots, i_m \rangle \in \omega^{<\omega}$ and any $p \in P_\alpha$, it holds that $p[[s]] = p[\langle i_0 \rangle_{\alpha_0}] \cdots [\langle i_m \rangle_{\alpha_m}]$.

The following hold.

(1) $\{p[[s]] \,|\, s \in \omega^m\}$ is a partition of p, for each $m < \omega$.

(2) If $s, t \in \omega^{<\omega}$ and $s \subset t$, then $p[[t]] \leq p[[s]]$.

(3) If $s, t \in \omega^{<\omega}$ and s, t are incompatible, then $p[[t]]$, $p[[s]]$ are incompatible.

Now, we are ready to establish the result corresponding to the last lemma in the previous section.

Lemma 3.6. *Let α be a limit ordinal with cofinality ω, $\langle \alpha_n \,|\, n < \omega \rangle$ an increasing sequence with limit α, $h : \omega^{<\omega} \to \omega$, $p \in P_\alpha$, and $\dot{f}$ a P_α-name such that $p \Vdash \dot{f} \in \omega^\omega$. Suppose that*

$$p \Vdash_\alpha \dot{f} \notin V^{P_\xi}, \quad \textit{for all} \ \ \xi < \alpha.$$

Then, there exist $p' \leq p$ and $\{\dot{\delta}_s \,|\, s \in \omega^{<\omega}\}$ such that, for all $n < \omega$ and $s \in \omega^n$,

(1) *$\dot{\delta}_{s^\frown\langle i \rangle}$ is a P_{α_n}-name, for all $i < \omega$,*

(2) *$p'[[s]] \Vdash \dot{\delta}_s \subset \dot{f}$,*

(3) *$\Vdash h(s) < |\dot{\delta}_s| < |\dot{\delta}_{s^\frown\langle 0 \rangle}|$,*

(4) *$\Vdash \langle \dot{\delta}_{s^\frown\langle i \rangle} \,|\, i < \omega \rangle$ is a type I or type II function.*

Proof. Take a countable elementary substructure N of $H(\lambda)$ such that h, $\langle \alpha_n | n < \omega \rangle$, P, $\dot{f}$, $p \in N$.

We first show that, by induction on $n < \omega$, we can take $p_n \in P_{\alpha_n}$, a P_{α_n}-name $\dot{q}_n$, and P_{α_n}-names $\dot{\delta}_{s^\frown\langle i \rangle}$ (for $s \in \omega^n$ and $i < \omega$) such that, for all $s \in \omega^n$,

(5) p_n is (N, P_{α_n})-generic and $p_n \leq p \upharpoonright \alpha_n$ and $p_{n+1} \upharpoonright \alpha_n = p_n$,

(6) $p_n \Vdash \dot{q}_n \in N[\dot{\mathscr{G}}_{P_{\alpha_n}}] \cap P_\alpha / P_{\alpha_n}$ and $\dot{q}_n \leq p \upharpoonright [\alpha_n, \alpha)$,

(7) $p_{n+1} \Vdash \dot{q}_{n+1} \leq \dot{q}_n \upharpoonright [\alpha_{n+1}, \alpha)$,

(8) $p_n[[s]] \Vdash \dot{q}_n \Vdash \dot{\delta}_s \subset \dot{f}$,

(9) $p_n[[s]] \Vdash \langle \dot{\delta}_{s\hat{}\langle i\rangle} \,|\, i < \omega \rangle$ is a type I or type II function,

(10) $p_n[[s]] \Vdash h(s\hat{}\langle i\rangle) < |\dot{\delta}_{s\hat{}\langle i\rangle}|$ and $\dot{q}_n[\langle i\rangle_{\alpha_n}] \Vdash \dot{\delta}_{s\hat{}\langle i\rangle} \subset \dot{f}$, for all $i < \omega$.

Case 1. $n = 0$.

In N, take $p' \leq p$ and $\delta_{\langle\,\rangle} \in \omega^{<\omega}$ such that $h(\langle\,\rangle) < |\delta_{\langle\,\rangle}|$ and $p' \Vdash \delta_{\langle\,\rangle} \subset \dot{f}$. Take (N, P_{α_0})-generic condition $p_0 \in P_{\alpha_0}$ such that $p_0 \leq p' \upharpoonright \alpha_0$. Set $r_0 = p' \upharpoonright [\alpha_0, \alpha)$. Then, it holds that

$$p_0 \Vdash r_0 \in N[\dot{\mathscr{G}}_{P_{\alpha_0}}] \cap P_\alpha / P_{\alpha_0}.$$

Work in $N[\dot{\mathscr{G}}_{P_{\alpha_0}}]$. For each $i < \omega$, take an interpretation $\dot{f}_i$ of $\dot{f}$ below $r_0[\langle i\rangle_{\alpha_0}]$ such that whenever $i, j < \omega$ and $i \neq j$, $\dot{f}_i \neq \dot{f}_j$. (This can be done, since $\Vdash \dot{f} \notin V^{P_{\alpha_0}}$.) Take $\dot{a} \in [\omega]^\omega$ and $\dot{k}_i < \omega$ (for $i < \omega$) such that

$$h(\langle i\rangle) < \dot{k}_i \text{ and } \langle \dot{f}_{\Gamma_{\dot{a}}(i)} \upharpoonright \dot{k}_i \,|\, i < \omega \rangle \text{ is a type I or type II function.}$$

Let $\dot{\delta}_{\langle i\rangle} = \dot{f}_{\Gamma_{\dot{a}}(i)} \upharpoonright \dot{k}_i$, for $i < \omega$. For each $i < \omega$, take $\dot{r}_{0,i} \in P_\alpha / P_{\alpha_0}$ such that

$$\dot{r}_{0,i} \leq r_0[\langle \Gamma_{\dot{a}}(i)\rangle_{\alpha_0}] \quad \text{and} \quad \dot{r}_{0,i} \Vdash \dot{\delta}_{\langle i\rangle} \subset \dot{f}.$$

Since $\langle r_0[\langle \Gamma_{\dot{a}}(i)\rangle_{\alpha_0}] \,|\, i < \omega \rangle$ is a one-point partition at α_0, by Lemma 3.3, $\langle \dot{r}_{0,i} \,|\, i < \omega \rangle$ is too. Let $\dot{q}_0 \in P_\alpha / P_{\alpha_0}$ be the root of $\langle \dot{r}_{0,i} \,|\, i < \omega \rangle$.

Note that $p_0 \Vdash \dot{q}_0 \leq r_0$. So, $p_0, \dot{q}_0$, and $\dot{\delta}_{\langle i\rangle}$ (for $i < \omega$) satisfy (5)$\sim$(10).

Case 2. $n = m + 1$.

By induction hypothesis, it holds that $p_m \Vdash \dot{q}_m \in N[\dot{\mathscr{G}}_{P_{\alpha_m}}] \cap P_\alpha / P_{\alpha_m}$. By Corollary 3.5, there exists $p_n \in P_{\alpha_n}$ such that

(11) p_n is (N, P_{α_n})-generic and $p_n \upharpoonright \alpha_m = p_m$,

(12) $p_m \Vdash p_n \upharpoonright [\alpha_m, \alpha_n) \leq \dot{q}_m \upharpoonright [\alpha_m, \alpha_n)$ and $p_n(\alpha_m) \upharpoonright \langle i\rangle \leq \dot{q}_m(\alpha_m) \upharpoonright \langle i\rangle$, for all $i < \omega$.

Note that p_n satisfies (5) and

(8)′ $p_n[[s]] \Vdash \dot{q}_m \upharpoonright [\alpha_m, \alpha) \Vdash \dot{\delta}_s \subset \dot{f}$, for all $s \in \omega^n$.

Let $s \in \omega^n$. Define $\dot{q}_n^s$ and $\langle \dot{\delta}_{s\hat{}\langle i\rangle} \,|\, i < \omega \rangle$ as follows:

Work in $N[\dot{\mathcal{G}}_{P_{\alpha_n}}]$ below $p_n[[s]]$. Similar to the case 1, take $\dot{a} \in [\omega]^{\omega}$ and $\dot{\delta}_{s\hat{}\langle i\rangle}$, $\dot{r}_{n,i}$ (for $i < \omega$) such that

(9)$'$ $\langle \dot{\delta}_{s\hat{}\langle i\rangle} \mid i < \omega\rangle$ is a type I or type II function,

(13) $h(s\hat{}\langle i\rangle) < |\dot{\delta}_{s\hat{}\langle i\rangle}|$ and $\dot{r}_{n,i} \leq \dot{q}_m \restriction [\alpha_n, \alpha)[\langle \Gamma_{\dot{a}}(i)\rangle_{\alpha_n}]$ and $\dot{r}_{n,i} \Vdash \dot{\delta}_{s\hat{}\langle i\rangle} \subset \dot{f}$.

Let $\dot{q}_n^s$ be the root of $\langle \dot{r}_{n,i} \mid i < \omega\rangle$. Then, it holds that

(14) $\dot{q}_n^s \leq \dot{q}_m \restriction [\alpha_n, \alpha)$ and $\dot{q}_n^s[\langle i\rangle_{\alpha_m}] \Vdash \dot{\delta}_{s\hat{}\langle i\rangle} \subset \dot{f}$.

Return to V, since $\langle p_n[[s]] \mid s \in \omega^n\rangle$ is a partition of p_n, we can take a P_{α_n}-name $\dot{q}_n$ which satisfies

$$p_n[[s]] \Vdash \dot{q}_n = \dot{q}_n^s, \quad \text{for all } s \in \omega^n.$$

Then, p_n, $\dot{q}_n$, and $\dot{\delta}_{s\hat{}\langle i\rangle}$ (for $s \in \omega^n$ and $i < \omega$) satisfy (5)$\sim$(10).

Let $p' = \bigcup_{n<\omega} p_n$. It is easy to check that p' and $\langle \dot{\delta}_s \mid s \in \omega^{<\omega}\rangle$ satisfy (1), (2) and

(3)$'$ $p'[[s]] \Vdash h(s\hat{}\langle i\rangle) < |\dot{\delta}_{s\hat{}\langle i\rangle}|$, for all $i < \omega$,

(4)$'$ $p'[[s]] \Vdash \langle \dot{\delta}_{s\hat{}\langle i\rangle} \mid i < \omega\rangle$ is type I or type II function.

Since $\langle p'[[s]] \mid s \in \omega^n\rangle$ is a partition of p', for all $n < \omega$, we can replace $\dot{\delta}_s$'s which satisfy (3) and (4). □

4. Tentacle trees.

In this section, we consider to attach trees $\{H_i \mid i < \omega\}$ on a tree H. In order to define this manipulation, we introduce the notion of tentacle trees. We start with several definitions.

Definition 4.1. *For any $S \subset \omega^{<\omega}$, $\langle S\rangle$ denotes the tree generated by S.*

Definition 4.2. *Let $T \subset \omega^{<\omega}$ be a tree and $\delta \in \omega^{<\omega} \setminus T$.*

$\tilde{\Delta}(T,\delta)$ denotes the maximal element of $T \cap \{\delta \restriction i \mid i < |\delta|\}$ and $\Delta(T,\delta) = |\tilde{\Delta}(T,\delta)|$.

δ can be adjoinable *on T, if it holds that*

(1) $\Delta(T,\delta) + 2 \leq |\delta|$ *and* $|\mathrm{stem}(T)| < \Delta(T,\delta)$

(2) $\tilde{\Delta}(T,\delta) \notin \mathrm{split}(T)$

(3) $\mathrm{next}_T(\tilde{\Delta}(T,\delta)) \cap \mathrm{split}(T) = \phi$

(4) $\delta \restriction (\Delta(T,\delta) - 1) \notin \mathrm{split}(T)$.

T is called a tentacle tree of type I, *if there is a type I function* $u : \omega \to \omega^{<\omega}$ *such that* $T = \langle \mathrm{rang}(u) \rangle$.

T is called a tentacle tree of type II, *if there are a skip branching tree H without a maximal element and a function* $u : \omega \to \omega^{<\omega}$ *such that*

(1) $u(i)$ *is adjoinable on H, for all* $i < \omega$,

(2) $|u(i)| < |u(j)|$ *and* $\Delta(H, u(i)) + 2 \leq \Delta(H, u(j))$, *for all* $i < j < \omega$,

(3) $T = \langle H \cup \mathrm{rang}(u) \rangle$.

In this case, we say that H and u construct *T.*

Note that every tentacle tree is a skip branching tree.

For any tentacle tree T, e_T denotes the enumeration of $\mathrm{Max}(T)$ which is defined by

$$|e_T(i)| < |e_T(j)|, \text{ for all } i < j < \omega.$$

DEFINITION 4.3. $\mathscr{S}$ *denotes the set of all tentacle trees of type I or type II. For each* $g \in \omega^\omega$, $\mathscr{S}(g)$ *denotes the set* $\{H \in \mathscr{S} | B(H) \text{ is } g\text{-thin}\}$.

DEFINITION 4.4. $\mathscr{U}$ *denotes the set*:

$$\{U : \omega \to \omega^\omega | \forall i < \omega(U(i) \text{ is increasing}) \text{ and } \\ \forall i < \forall j < \omega \forall k < \omega(U(i)(k) \leq U(j)(k))\}.$$

DEFINITION 4.5. *For* $K \in \mathscr{S}$ *and* $U \in \mathscr{U}$, $\mathscr{A}(K, U)$ *denotes the set*:

$$\{\varphi | \exists a, b \in [\omega]^\omega(\varphi \in \prod_{i \in a} \mathscr{S}(U(\Gamma_a^{-1}(i))) \text{ and } \forall i < \omega(e_K(\Gamma_b(i)) \subset \mathrm{stem}(\varphi(\Gamma_a(i)))))\}.$$

The next three lemmas can be proved by using easy diagonal arguments. We left proofs to the reader.

LEMMA 4.1. *Let* $g \in \omega^\omega$, $\delta \in \omega^{<\omega}$, *and* u_n *be a type I function with root* δ, *for all* $n < \omega$.

Then, there exists a type I function v with the root δ *such that*

(1) $\{|v(j)| \,|\, j < \omega\}$ *is g-thin,*

(2) $\exists^\infty i < \omega(u_n(i) \in \mathrm{rang}(v))$, *for all* $n < \omega$. □

LEMMA 4.2. *Let* $g \in \omega^\omega$, *and H a skip branching tree without a maximal element, and* $u_n \in (\omega^{<\omega})^\omega$, *for* $n < \omega$. *Assume that*

H and u_n *construct a tentacle tree of type II, for all* $n < \omega$.

Then, there exists a function $v : \omega \to \omega^{<\omega}$ *such that*

(1) *H and v construct a tentacle tree of type II,*

(2) $\exists^\infty i < \omega(u_n(i) \in \mathrm{rang}(v))$, *for all* $n < \omega$,

(3) $\{|v(j)|\,|j < \omega\} \cup \{\Delta(H, v(j))|j < \omega\}$ *is g-thin.* □

Lemma 4.3. *Let $K \in \mathcal{S}$ and $U \in \mathcal{U}$. Then, for any countable subset Ψ of $\mathcal{A}(K, U)$, there exists $\psi \in \mathcal{A}(K, U)$ such that*

$$\forall \varphi \in \Psi \exists^\infty i \in \mathrm{dom}(\varphi) \cap \mathrm{dom}(\psi)(\varphi(i) = \psi(i)). \qquad \square$$

The next lemma is a preservation theorem like those which appeared in [**1**] and is proved almost the same arguments.

Lemma 4.4. *Let $\alpha \le \omega_2$ and $P = P_\alpha \in N$.*

(1) *Let $\delta \in \omega^{<\omega}$. Suppose that v is a type I function with root δ which satisfies $\exists^\infty i < \omega(u(i) \in \mathrm{rang}(v))$, for all type I functions $u \in N$ with root δ. Then, for each $p \in P \cap N$, there exists an (N, P)-generic condition $\tilde{p} \le p$ such that*

$$\tilde{p} \Vdash \exists^\infty i < \omega(u(i) \in \mathrm{rang}(v)), \text{ for all type I functions } u \in N[\dot{\mathcal{G}}_P] \text{ with root } \delta.$$

(2) *Let $H \in N$ be a skip branching tree without a maximal element, and $v : \omega \to \omega^{<\omega}$. Assume that*

$$H \text{ and } v \text{ construct a tentacle tree of type II}$$

and, for any $u \in (\omega^{<\omega})^\omega \cap N$,

$$\text{if } H \text{ and } u \text{ construct a tentacle tree of type } II, \text{ then } \exists^\infty i < \omega(u(i) \in \mathrm{rang}(v)).$$

Then, for each $p \in P \cap N$, there exists an (N, P)-generic condition $\tilde{p} \le p$ such that

$$\tilde{p} \Vdash \text{ for any } u \in (\omega^{<\omega})^\omega \cap N[\dot{\mathcal{G}}_P]$$

$$\begin{pmatrix} \text{if } H \text{ and } u \text{ construct a tentacle tree of type } II, \\ \text{then } \exists^\infty i < \omega(u(i) \in \mathrm{rang}(v)) \end{pmatrix}.$$

(3) *Let $K_n \in \mathcal{S} \cap N$, $U_n \in \mathcal{U} \cap N$, and $\psi_n \in \mathcal{A}(K_n, U_n)$ (for $n < \omega$), and $\eta \le \alpha$, $P^* = P_\alpha / P_\eta$, and $N^* = N[\dot{\mathcal{G}}_{P_\eta}]$. Suppose that, in V^{P_η}, it holds that, for all $n < \omega$,*

$$\forall \varphi \in \mathcal{A}(K_n, U_n) \cap N^* \ (\text{if } \mathrm{rang}(\varphi) \subset N, \text{ then } \exists^\infty i < \omega(\varphi(i) = \psi_n(i))).$$

Then, in V^{P_η}, it holds that, for any $p \in P^* \cap N^*$, there exists a $\tilde{p} \le p$ such that

$$\tilde{p} \text{ is } (N^*, P^*)\text{-generic and } \mathrm{support}(\tilde{p}) \subset N^*, \tag{3.1}$$

and, for any $n < \omega$,

$$\tilde{p} \Vdash \forall \varphi \in \mathcal{A}(K_n, U_n) \cap N^*[\dot{\mathcal{G}}_{P^*}] \ (\text{if } \mathrm{rang}(\varphi) \subset N \text{ then } \exists^\infty i < \omega(\varphi(i) = \psi_n(i))). \tag{3.2}$$

Proof. We only deal with (3), since (1) and (2) can be proved by similar arguments.

Let α' be the unique ordinal such that $\alpha = \eta + \alpha'$. Then, in V^{P_η}, $\langle P_\beta / P_\eta \mid \beta \in [\eta, \alpha) \rangle$ is isomorphic to the α' stage CS iteration of the rational perfect tree forcing. So, we consider that the ground model is V^{P_η} (we will denote this by V in the proof), and the forcing notion is the α'-stage CS iteration. In addition, in the proof, we change notations as follows:

$$N[\dot{\mathscr{G}}_{P_\eta}] \text{ will be denoted by } N,$$

$$\text{the original } N \text{ will be denoted by } \overline{N}.$$

The proof will be done by induction on $\alpha' \le \omega_2$. Let K_n, U_n, ψ_n for $n < \omega$ satisfy the assumption of (3).

Case 1. $\alpha' = \beta + 1$ (cf. the proof of Theorem 7.3.46 (p. 360) in [**1**])

Let $p \in P_{\alpha'} \cap N$. Take an (N, P_β)-generic condition $p' \le p \upharpoonright \beta$ which satisfies the requirements. We work in V^{P_β} below p'. Take an enumeration $\langle \dot{\xi}_i \mid i < \omega \rangle$ of the set $\{\dot{\xi} \in N[\dot{\mathscr{G}}_{P_\beta}] \mid \dot{\xi}$ is a $\boldsymbol{PT}$-name and $\Vdash \dot{\xi} \in \boldsymbol{On}\}$, and, for each $n < \omega$, take an enumeration $\langle \dot{\varphi}_{n,i} \mid i < \omega \rangle$ of the set

$$\{\dot{\varphi} \in N[\dot{\mathscr{G}}_{P_\beta}] \mid \dot{\varphi} \text{ is a } \boldsymbol{PT}\text{-name and } \Vdash_{\boldsymbol{PT}} \dot{\varphi} \in \mathscr{A}(K_n, U_n) \text{ and } \operatorname{rang}(\dot{\varphi}) \subset \overline{N}\}.$$

For each $s \in \omega^{<\omega}$, define the ordering $\le_s$ on $\boldsymbol{PT}$ by

$$q' \le_s q \text{ if and only if } q' \le q \text{ and } \forall t \in \omega^{<\omega}$$

$$(\text{if } s \text{ and } t \text{ is incompatible, then } q' \upharpoonright t = q \upharpoonright t).$$

Take an enumeration $\langle s_j \mid j < \omega \rangle$ of $\omega^{<\omega}$ such that if $s_j \subset s_{j'}$ then $j \le j'$. Note that, for any conditions $q_j \in \boldsymbol{PT}$ (for $j < \omega$),

$$\text{if } q_{j+1} \le_{s_j} q_j, \quad \text{for all } j < \omega \text{ then } \bigcap_{j<\omega} q_j \in \boldsymbol{PT}.$$

Claim 2. *For any $j < \omega$ and any $q \in \boldsymbol{PT} \cap N[\dot{\mathscr{G}}_{P_\beta}]$, there exist $q^+ \le q$ and $k^+ < \omega$ such that $q^+ \in N[\dot{\mathscr{G}}_{P_\beta}]$ and, for each $n, i \le j$,*

(4) *q^+ decides the value of $\dot{\xi}_i$, $\dot{\varphi}_{n,i} \upharpoonright k^+$,*

(5) $\exists m \in [j, k^+) \cap \operatorname{dom}(\psi_n)(q^+ \Vdash m \in \operatorname{dom}(\dot{\varphi}_{n,i})$ *and* $\dot{\varphi}_{n,i}(m) = \psi_n(m))$.

Proof of Claim 2. Work in $N[\dot{\mathscr{G}}_{P_\beta}]$. Take $q' \le q$ such that q' decides the values of $\dot{\xi}_i$, for $i \le j$.

By induction on $k < \omega$, take $l_k < \omega$ and $q_k \in \boldsymbol{PT}$ such that

$$q_{k+1} \le q_k \le q' \quad \text{and} \quad j < l_k < l_{k+1}$$

$$q_k \text{ decides the values of } \dot{\varphi}_{n,i} \upharpoonright l_k, \text{ for } n, i \le j,$$

$$\exists m \in [l_k, l_{k+1})(q_{k+1} \Vdash m \in \operatorname{dom}(\dot{\varphi}_{n,i})), \text{ for each } n, i \le j.$$

For each $n, i \le j$, let $\varphi_{n,i}$ be the function such that

$$q_k \Vdash \dot{\varphi}_{n,i} \restriction l_k = \varphi_{n,i} \restriction l_k, \quad \text{for all } k < \omega.$$

Then it is easy to check that $\varphi_{n,i} \in \mathscr{A}(K_n, U_n)$ and $\mathrm{rang}(\varphi_{n,i}) \subset \overline{N}$, for all $i, n \le j$.

In $\boldsymbol{V}^{P_\beta}$, by induction hypothesis, there exists $\tilde{k} < \omega$ such that

$$\exists m \in [j, l_{\tilde{k}})(m \in \mathrm{dom}(\psi_n) \text{ and } \varphi_{n,i}(m) = \psi_n(m)), \quad \text{for all } n, i \le j.$$

Let $q^+ = q_{\tilde{k}}$ and $k^+ = l_{\tilde{k}}$. Then, q^+ and k^+ are as required. $\square$

Claim 3. *There exist $q_j \in \boldsymbol{PT} \cap N[\dot{\mathscr{G}}_{P_\beta}]$ and intervals I_j (for $j < \omega$) such that*
(6) $q_{j+1} \le_{s_{j+1}} q_j \le p(\beta)$ *and* $\max(I_j) < \min(I_{j+1})$,
(7) $q_j \restriction s_j$ *decides the value of* $\dot{\xi}_i$, *for* $i \le j$,
(8) $\exists m \in I_j \cap \mathrm{dom}(\psi_n)(q_j \restriction s_j \Vdash m \in \mathrm{dom}(\dot{\varphi}_{n,i})$ *and* $\dot{\varphi}_{n,i}(m) = \psi_n(m))$, *for all* $i, n \le j$.

Proof of Claim 3. By induction on $j < \omega$. Assume that $j < \omega$ and $q_{j'}, I_{j'}$ (for $j' < j$) have been defined. Set

$$q = \begin{cases} q_{j-1}, & \text{if } 0 < j, \\ p(\beta), & \text{otherwise,} \end{cases} \quad \text{and} \quad k = \begin{cases} \max(I_{j-1}) + 1, & \text{if } j > 0, \\ 0, & \text{otherwise.} \end{cases}$$

By Claim 2, take an interval $I_j \subset [k, \omega)$ and $q^+ \le q \restriction s_j$ such that

$$q^+ \in N[\dot{\mathscr{G}}_{P_\beta}] \quad \text{and} \quad q^+ \text{ decides the value of } \dot{\xi}_i, \text{ for } i \le j,$$

$$\exists m \in I_j \cap \mathrm{dom}(\psi_n)(q^+ \Vdash m \in \mathrm{dom}(\dot{\varphi}_{n,i}) \text{ and } \dot{\varphi}_{n,i}(m) = \psi_n(m)), \text{ for each } i, n \le j.$$

Define $q_j \le_{s_j} q$ by $q_j \restriction s_j = q^+$. Then, q_j and I_j satisfy (6)$\sim$(8). $\square$

Take q_j and I_j (for $j < \omega$) which satisfy (6)$\sim$(8). Note that it holds that
(7)$'$ $q_j \restriction s_j \Vdash \dot{\xi}_i \in N[\dot{\mathscr{G}}_{P_\beta}]$, for all $i \le j < \omega$.
Let $\tilde{q} = \bigcap_{j<\omega} q_j$.

Claim 4. *For all $i, n < \omega$, it holds that*
(9) $\tilde{q} \Vdash_{\boldsymbol{PT}} \dot{\xi}_i \in N[\dot{\mathscr{G}}_{P_\beta}] \cap \boldsymbol{On}$,
(10) $\tilde{q} \Vdash_{\boldsymbol{PT}} \exists^\infty m \in \mathrm{dom}(\psi_n) \cap \mathrm{dom}(\dot{\varphi}_{n,i})(\psi_n(m) = \dot{\varphi}_{n,i}(m))$.

Proof of Claim 4. Let $q' \le \tilde{q}$ and $i, n < \omega$.

(9) Take $j < \omega$ such that $i \le j$ and $\Gamma_{\tilde{q}}(s_j) \in \mathrm{split}(q')$. Then, q' and $\tilde{q} \restriction s_j$ are compatible. Since $\tilde{q} \restriction s_j \le q_j \restriction s_j$, q' and $q_j \restriction s_j$ are also compatible. By this and by (7)$'$, there exists $q'' \le q'$ such that $q'' \Vdash \dot{\xi}_i \in N[\dot{\mathscr{G}}_{P_\beta}]$.

(10) Similar to the proof of (9). $\square$

Let $\tilde{p} = p'^\frown\langle\tilde{q}\rangle$. By (9) and (10), $\tilde{p}$ is as required.

Case 2. α' is a limit ordinal.

Take an increasing sequence $\langle \alpha_j \mid j < \omega \rangle$ of ordinals such that

$$\alpha_j \in N \cap \alpha', \quad \text{for all } j < \omega \text{ and } \sup_{j<\omega} \alpha_j = \sup(N \cap \alpha').$$

Similar to the case 1, take enumerations $\langle \dot{\xi}_i \mid i < \omega \rangle$ and $\langle \dot{\varphi}_{n,i} \mid i < \omega \rangle$ (for $n < \omega$).

Claim 5. *There exist $p_j \in P_{\alpha_j}$ and P_{α_j}-names $\dot{r}_j$, $\dot{I}_j$ (for $j < \omega$) such that*

(11) *p_j is (N, P_{α_j})-generic and* $\mathrm{support}(p_j) \subset N$ *and* $p_{j+1} \upharpoonright \alpha_j = p_j \leq p \upharpoonright \alpha_j$,

(12) $p_j \Vdash \dot{r}_j \in N[\dot{\mathscr{G}}_{P_{\alpha_j}}] \cap P_{\alpha'}/P_{\alpha_j}$ *and* $\dot{r}_j \leq p \upharpoonright [\alpha_j, \alpha')$,

(13) $p_{j+1} \Vdash \dot{r}_{j+1} \leq \dot{r}_j \upharpoonright [\alpha_{j+1}, \alpha')$ *and* $p_j \Vdash p_{j+1} \upharpoonright [\alpha_j, \alpha_{j+1}) \leq \dot{r}_j \upharpoonright [\alpha_j, \alpha_{j+1})$,

(14) $p_j \Vdash \dot{r}_j$ *decides the value of* $\dot{\xi}_j$,

(15) $p_j \Vdash \dot{I}_j$ *is an interval of* ω *and* $\max(\dot{I}_i) < \min(\dot{I}_j)$, *for all* $i < j$,

(16) $p_j \Vdash \exists m \in \dot{I}_j \cap \mathrm{dom}(\psi_n)(\dot{r}_j \Vdash m \in \mathrm{dom}(\dot{\varphi}_{n,i})$ *and* $\dot{\varphi}_{n,i}(m) = \psi_n(m))$, *for all* $n, i \leq j$.

Proof of Claim 5. Similar to the proof of Claim 3. □

Take p_j, $\dot{r}_j$, $\dot{I}_j$ (for $j < \omega$) which satisfy (11)$\sim$(16) in Claim 5. Let $\tilde{p} = \bigcup_{j<\omega} p_j$. Then, by (11) and (14), $\tilde{p}$ is $(N, P_{\alpha'})$-generic. In order to check that $\tilde{p}$ is as required, let n, i, $m < \omega$ and $p' \leq \tilde{p}$. Take $j < \omega$ such that n, i, $m < j$. Then, since $\tilde{p} \upharpoonright \alpha_j \Vdash \tilde{p} \upharpoonright [\alpha_j, \alpha') \leq \dot{r}_j$, by (16), we have that

$$\exists p'' \leq p'(p'' \Vdash \exists m' \in \mathrm{dom}(\psi_n) \cap \mathrm{dom}(\dot{\varphi}_{n,i}) \setminus m\, (\psi_n(m') = \dot{\varphi}_{n,i}(m'))). \quad \square$$

Corollary 4.5. *Let $\alpha \leq \omega_2$, $P = P_\alpha$, and $g \in \omega^\omega$. Then, the following hold in V^P.*

(1) *Let $\dot{u}$ be a type I function with root $\dot{\delta}$ such that $g(0) < |\dot{\delta}|$. Then, there exists a tentacle tree $T \in V$ of type I such that*

$$\mathrm{stem}(T) = \dot{\delta} \textit{ and } \exists^\infty i < \omega(\dot{u}(i) \in \mathrm{Max}(T)) \textit{ and } B(T) \textit{ is } g\textit{-thin}.$$

(2) *Let $H \in V$ be a skip branching tree without maximal elements, and $\dot{u}$ be a type II function with limit $\dot{h} \in \mathrm{Lim}(H)$. Assume that H and $\dot{u}$ construct a tentacle tree of type II. Then, there exists a tentacle tree $T \in V$ such that*

(2.1) *T is constructed from H and some type II function,*

(2.2) $\{|\delta| \mid \delta \in \mathrm{Max}(T)\} \cup \{\Delta(H, \delta) \mid \delta \in \mathrm{Max}(T)\}$ *is* g*-thin,*

(2.3) $\exists^\infty i < \omega(\dot{u}(i) \in \mathrm{Max}(T))$.

Proof. Let $\alpha \leq \omega_2$, $P = P_\alpha$, and $g \in \omega^\omega$.

(1) Let $p \in P$ and $\dot{u}$, $\dot{\delta}$ be P-names such that

$$p \Vdash \dot{u} \text{ is a type I function with root } \dot{\delta}.$$

Replacing p by a certain stronger condition, if necessary, we may assume that $p \Vdash \dot{\delta} = \delta$, for some δ. Take an elementary substructure N of $H(\lambda)$ such that α, P, g, $\dot{u} \in N$. By Lemma 4.1, there exists a type I function v with root δ such that

(3) $\{|v(j)| \,|\, j < \omega\}$ is g-thin,

(4) $\exists^\infty i < \omega(u(i) \in \operatorname{rang}(v))$, for every type I function $u \in N$ with root δ.

Deleting a certain finite part of v, we may assume that $\{|v(j)| \,|\, j < \omega\} \cup \{|\delta|\}$ is g-thin. By using Lemma 4.4 (1), take $\tilde{p} \le p$ such that $\tilde{p} \Vdash \exists^\infty i < \omega(u(i) \in \operatorname{rang}(v))$, for all type I function $u \in N[\dot{\mathscr{G}}_P]$ with root δ. Especially, it holds that

$$\tilde{p} \Vdash \exists^\infty i < \omega(\dot{u}(i) \in \operatorname{rang}(v)).$$

So, $T = \langle \operatorname{rang}(v) \rangle$ is as required.

(2) Similar to (1) by using Lemma 4.2 and Lemma 4.4 (2). □

5. Proof of the theorem.

Now, we are ready to prove Theorem 2.1. Theorem 2.1 follows from the following lemma.

Lemma 5.1. *Let $\alpha \le \omega_2$, $P = P_\alpha$, $p \in P$, $g \in \omega^\omega$, and $\dot{f}$ be a P-name such that*

$$p \Vdash \dot{f} \in \omega^\omega.$$

Then, there exist $\tilde{p} \le p$ and $H \subset \omega^{<\omega}$ such that

(1) *H is a skip branching tree,*

(2) *$B(H)$ is g-thin,*

(3) $\tilde{p} \Vdash \dot{f} \in \operatorname{Lim}(H)$.

Proof. We prove this lemma by induction on $\alpha \le \omega_2$. So, let $\alpha \le \omega_2$ and assume that the lemma was proved for all $\alpha' < \alpha$. Let $p \in P = P_\alpha$, $g \in \omega^\omega$, and $\dot{f}$ a P-name such that $p \Vdash \dot{f} \in \omega^\omega$.

Claim 6. *Let $\beta < \alpha$ and $g' \in \omega^\omega$. Then, the following holds in V^{P_β}.*

Suppose that $u : \omega \to \omega^{<\omega}$ is a type I function with root δ such that $g'(0) < |\delta|$ or a type II function. Then, there exists a tentacle tree $T \in V$ such that

(4) *$B(T)$ is g'-thin,*

(5) $\exists^\infty i < \omega(u(i) \in \operatorname{Max}(T))$.

Proof of Claim 6. The case that $u : \omega \to \omega^{<\omega}$ is type I was already proved as corollary 4.5 (1). So, it suffices to deal with the case that u is a type II function. We work in V^{P_β}. Let $h \in \omega^\omega$ be the limit of u. Take disjoint intervals $\langle I_n | n < \omega \rangle$ such that

$$\forall n < \omega \exists i < \omega([\varDelta(h, u(i)) - 1, |u(i)| + 2) \subset I_n).$$

By Lemmas 2.3 and 2.4, since $V \cap \omega^\omega$ is unbounded in ω^ω, there is a $g_1 \in \omega^\omega \cap V$

such that

$$\exists^\infty n < \omega (a \cap I_n = \phi) \text{ and } a \cup b \text{ is } g'\text{-thin, for any } g_1\text{-thin sets } a, b \in [\omega]^\omega.$$

By induction hypothesis, take a skip branching tree $H \in V$ such that

$$h \in \mathrm{Lim}(H) \text{ and } B(H) \text{ is } g_1\text{-thin.}$$

Deleting the set of finite maximal branches in H, if necessary, we may assume that H has no maximal elements. Since $B(H)$ is g_1-thin, there exists an $a \in [\omega]^\omega$ such that

$$[\Delta(h, u(i)) - 1, |u(i)| + 2) \cap B(H) = \phi, \quad \text{for all } i \in a.$$

Let $v = u\Gamma_a : \omega \to \omega^{<\omega}$. Then, it holds that

$$H \text{ and } v \text{ construct a tentacle tree of type II.}$$

By Corollary 4.5 (2), there exists a tentacle tree $T \in V$ of type II such that

$$T \text{ is constructed from } H \text{ and some type II function,}$$

$$\{|\delta| \,|\, \delta \in \mathrm{Max}(T)\} \cup \{\Delta(H, \delta) \,|\, \delta \in \mathrm{Max}(T)\} \text{ is } g_1\text{-thin,}$$

$$\exists^\infty i < \omega (v(i) \in \mathrm{Max}(T)).$$

This T is as required. $\square$

Take increasing functions $g_s \in \omega^\omega$ (for $s \in \omega^{<\omega}$) such that, $\bigcup_{s \in \omega^{<\omega}} a_s$ is g-thin, whenever $a_s \in [\omega]^\omega$ is g_s-thin for all $s \in \omega^{<\omega}$, $\forall t \in \omega^{|s|}$(if $\forall i < |s|(t(i) \le s(i))$ then $\forall i < \omega$ $(g_t(i) \le g_s(i))$, for all $s \in \omega^{<\omega}$. For each $s \in \omega^{<\omega}$, set $U_s = \langle g_{s^\frown\langle i \rangle} \,|\, i < \omega \rangle \in \mathscr{U}$.

First, we deal with the case that α is a successor ordinal. So, let $\alpha = \beta + 1$. Without loss of generality, we may assume that $p \Vdash \dot{f} \notin V^{P_\beta}$.

We work in V^{P_β}. By using Lemma 2.5, take $\dot{q} \le p(\beta)$ and $\{\dot{\delta}_s \,|\, s \in \omega^{<\omega}\}$ such that, for all $s \in \omega^{<\omega}$, $g_s(0) < |\dot{\delta}_s|$ and $\langle \dot{\delta}_{s^\frown\langle i \rangle} \,|\, i < \omega \rangle$ is a type I or type II function and $\dot{q} \restriction s \Vdash \dot{\delta}_s \subset \dot{f}$.

Using Claim 6, we can take tentacle trees $\{\dot{T}_s \,|\, s \in \omega^{<\omega}\} \subset V$ such that, for all $s \in \omega^{<\omega}$,

$$B(\dot{T}_s) \text{ is } g_s\text{-thin and } \exists^\infty i < \omega (\dot{\delta}_{s^\frown\langle i \rangle} \in \mathrm{Max}(\dot{T}_s)) \text{ and } \dot{\delta}_s \subset \mathrm{stem}(\dot{T}_s).$$

Set $\dot{\varphi}_s = \langle \dot{T}_{s^\frown\langle i \rangle} \,|\, i < \omega \text{ and } \dot{\delta}_{s^\frown\langle i \rangle} \in \mathrm{Max}(\dot{T}_s) \rangle$, for $s \in \omega^{<\omega}$. Note that it holds that,

$$\dot{\varphi}_s \in \mathscr{A}(\dot{T}_s, U_s) \text{ and } \mathrm{rang}(\dot{\varphi}_s) \subset V, \quad \text{for all } s \in \omega^{<\omega}.$$

Return to V. Take a countable elementary substructure N of $H(\lambda)$ such that the above arguments were done in N. By Lemma 4.3, take $\psi_{K,U} \in \mathscr{A}(K, U)$

(for $K \in \mathcal{S} \cap N$ and $U \in \mathcal{U} \cap N$) such that, for all $K \in \mathcal{S} \cap N$, $U \in \mathcal{U} \cap N$,

$$\forall \varphi \in \mathcal{A}(K, U) \cap N \exists^{\infty} i < \omega(\varphi(i) = \psi_{K,U}(i)).$$

Without loss of generality, we may assume that, for any $K \in \mathcal{S} \cap N$ and any $U \in \mathcal{U} \cap N$, $\mathrm{rang}(\psi_{K,U}) \subset N$. By Lemma 4.4 (3), take $\tilde{p} \leq p \restriction \beta$ such that, for all $K \in \mathcal{S} \cap N$, $U \in \mathcal{U} \cap N$,

$$\tilde{p} \Vdash \forall \varphi \in \mathcal{A}(K, U) \cap N[\dot{\mathcal{G}}_{P_\beta}] \text{ (if } \mathrm{rang}(\varphi) \subset N \text{ then } \exists^{\infty} i(\varphi(i) = \psi_{K,U}(i)).$$

Especially, it holds that

(6) $\tilde{p} \Vdash \exists^{\infty} i(\dot{\varphi}_s(i) = \psi_{\dot{T}_s, U_s}(i))$, for all $s \in \omega^{<\omega}$.

Replacing $\tilde{p}$ by certain stronger condition, if necessary, we may assume that

$$\tilde{p} \Vdash \dot{T}_{\langle\,\rangle} = T, \quad \text{for some } T \in N.$$

By induction on $n < \omega$, define $C_n \subset \omega^n$ and tentacle trees $K_s \in N$ (for $s \in C_n$) by

$$\begin{aligned} C_0 &= \{\langle\,\rangle\}, \\ K_{\langle\,\rangle} &= T \\ C_{n+1} &= \{s^\frown\langle i\rangle \,|\, s \in C_n \text{ and } i \in \mathrm{dom}(\psi_{K_s, U_s})\}, \\ K_{s^\frown\langle i\rangle} &= \psi_{K_s, U_s}(i), \quad \text{for all } s^\frown\langle i\rangle \in C_{n+1}. \end{aligned}$$

Let $C = \bigcup_{n<\omega} C_n$, and $K = \bigcup\{K_s \,|\, s \in C\}$. It is easy to check that K is a skip branching tree. Since it holds that

$$\forall s \in C_n \exists^{\infty} i < \omega(s^\frown\langle i\rangle \in C_{n+1}), \quad \text{for all } n < \omega,$$

C is a perfect rational tree.

Claim 7. *$B(K_{\Gamma_C(s)})$ is g_s-thin, for all $s \in \omega^{<\omega}$.*

Proof of Claim 7. By induction on $|s| < \omega$. The case $s = \langle\,\rangle$ is clear. So, let $s = t^\frown\langle i\rangle$. Set $s' = \Gamma_C(s)$, $t' = \Gamma_C(t)$, $a = \mathrm{dom}(\psi_{K_{t'}, U_{t'}})$, $i' = \Gamma_a(i)$, $u = t'^\frown\langle i\rangle$. Note that $s' = t'^\frown\langle i'\rangle$. So, $K_{\Gamma_C(s)} = K_{s'} = K_{t'^\frown\langle i'\rangle} = \psi_{K_{t'}, U_{t'}}(i')$. Since $\psi_{K_{t'}, U_{t'}}(i') \in \mathcal{S}(U_{t'}(\Gamma_a^{-1}(i'))) = \mathcal{S}(g_u)$, $B(K_{\Gamma_C(s)})$ is g_u-thin. Since $s(j) \leq u(j)$, for all $j < |u|$, $g_s(k) \leq g_u(k)$, for all $k < \omega$. So, $B(K_{\Gamma_C(s)})$ is g_s-thin. □

By Claim 7, since $B(K) \subset \bigcup_{s \in C} B(K_s)$, $B(K)$ is g-thin. Work in V^{P_β} below $\tilde{p}$. By induction on $n < \omega$, define $\dot{D}_n \subset C_n$ by

$$\dot{D}_0 = \{\langle\,\rangle\},$$

$$\dot{D}_{n+1} = \{s^\frown\langle i\rangle \in C_{n+1} \,|\, s \in \dot{D}_n \text{ and } i \in \mathrm{dom}(\dot{\varphi}_s) \text{ and } \dot{\varphi}_s(i) = \psi_{K_s, U_s}(i)\}.$$

Claim 8. *$\tilde{p} \Vdash \forall s \in \dot{D}_n$ $(K_s = \dot{T}_s)$, for all $n < \omega$.*

Proof of Claim 8. Easy. $\square$

By Claim 8 and (6), it holds that
(7) $\tilde{p} \Vdash \forall s \in \dot{D}_n \exists^\infty i < \omega (s^\frown\langle i \rangle \in \dot{D}_{n+1})$, for all $n < \omega$.
Define P_β-name $\dot{r}$ by

$$\Vdash \dot{r} = \bigcap_{n<\omega} \bigcup \{\dot{q} \upharpoonright s \mid s \in \dot{D}_n\}.$$

By (7), it holds that $\tilde{p} \Vdash \dot{r} \in \boldsymbol{PT}$. Note that

$$\tilde{p} \Vdash \dot{r} \leq \dot{q} \text{ and } \{\dot{q} \upharpoonright s \mid s \in \dot{D}_n\} \text{ is predense below } \dot{r}.$$

Since it holds that

$$\tilde{p} \Vdash \{\dot{\delta}_s \mid s \in \bigcup_{n<\omega} \dot{D}_n\} \subset K \text{ and } \dot{q} \upharpoonright s \Vdash \dot{\delta}_s \subset \dot{f}, \quad \text{for all } s \in \omega^{<\omega},$$

we have that $\tilde{p} \Vdash \dot{r} \Vdash \dot{f} \in \mathrm{Lim}(K)$. So, it holds that $\tilde{p}^\frown\langle \dot{r} \rangle \Vdash \dot{f} \in \mathrm{Lim}(K)$. This completes the proof of the case that α is a successor ordinal.

Next, we deal with the case that α is a limit ordinal. Without loss of generality, we may assume that $p \Vdash_\alpha \dot{f} \notin \boldsymbol{V}^{P_\xi}$, for all $\xi < \alpha$ and $\mathrm{cof}(\alpha) = \omega$. Take an increasing sequence $\langle \alpha_n \mid n < \omega \rangle$ of ordinals with the limit α. Replacing p by a certain stronger condition, if necessary, we may assume that there exist $\{\dot{\delta}_s \mid s \in \omega^{<\omega}\}$ which satisfy (1)$\sim$(4) in Lemma 3.6, where $h = \langle g_s(0) \mid s \in \omega^{<\omega} \rangle$. Note that, for any $s, t \in \omega^{<\omega}$, if s and t are incompatible, then $\Vdash \dot{\delta}_s$ and $\dot{\delta}_t$ are incompatible. For each $n < \omega$, by using Claim 6, take P_{α_n}-names $\langle \dot{T}_s \mid s \in \omega^n \rangle$ such that, for each $s \in \omega^n$,
(8) $\Vdash_{\alpha_n} \dot{T}_s \in \boldsymbol{V}$ and $\dot{T}_s$ is a tentacle tree and $B(\dot{T}_s)$ is g_s-thin and $\dot{\delta}_s \subset \mathrm{stem}(\dot{T}_s)$,
(9) $\Vdash_{\alpha_n} \exists^\infty i < \omega (\dot{\delta}_{s^\frown\langle i \rangle} \in \mathrm{Max}(\dot{T}_s))$.
Since $\dot{T}_{\langle\rangle}$ is a P_{α_0}-name, without loss of generality, we may assume that $p \upharpoonright \alpha_0$ decides the value of $\dot{T}_{\langle\rangle}$. Take T such that $p \Vdash \dot{T}_{\langle\rangle} = T$. For notational convenience, we denote T by $\dot{T}^*_{\langle\rangle}$.

Let $n < \omega$ and $s \in \omega^n$. We define P_{α_n}-names $\dot{T}^*_{s^\frown\langle i \rangle}$, $\dot{r}_{s^\frown\langle i \rangle}$ (for $i < \omega$), and $\dot{\varphi}_s$ as follows:

Work in $\boldsymbol{V}^{P_{\alpha_n}}$. Let $i < \omega$. Take $\dot{T}^*_{s^\frown\langle i \rangle} \in \boldsymbol{V}$ and $\dot{r}_{s^\frown\langle i \rangle} \in P_{\alpha_{n+1}}/P_{\alpha_n}$ such that

$$\dot{r}_{s^\frown\langle i \rangle} \leq p \upharpoonright [\alpha_n, \alpha_{n+1})[\langle i \rangle_{\alpha_n}] \text{ and } \dot{r}_{s^\frown\langle i \rangle} \Vdash \dot{T}_{s^\frown\langle i \rangle} = \dot{T}^*_{s^\frown\langle i \rangle}.$$

Note that
(8)$'$ $\dot{T}^*_{s^\frown\langle i \rangle}$ is a tentacle tree and $B(\dot{T}^*_{s^\frown\langle i \rangle})$ is $g_{s^\frown\langle i \rangle}$-thin and $\dot{\delta}_{s^\frown\langle i \rangle} \subset \mathrm{stem}(\dot{T}^*_{s^\frown\langle i \rangle})$,
(9)$'$ $\dot{r}_{s^\frown\langle i \rangle} \Vdash \exists^\infty j < \omega (\dot{\delta}_{s^\frown\langle i,j \rangle} \in \mathrm{Max}(\dot{T}^*_{s^\frown\langle i \rangle}))$.
Let $\dot{\varphi}_s = \langle \dot{T}^*_{s^\frown\langle i \rangle} \mid i < \omega \text{ and } \dot{\delta}_{s^\frown\langle i \rangle} \in \mathrm{Max}(\dot{T}^*_s) \rangle$.

Return to V. Note that, for all $n < \omega$, $s \in \omega^n$,

(10) $\Vdash_{\alpha_n} \mathrm{rang}(\dot{\varphi}_s) \subset V$, and if $\mathrm{dom}(\dot{\varphi}_s) \in [\omega]^\omega$ then $\dot{\varphi}_s \in \mathscr{A}(\dot{T}_s^*, U_s)$.

Take a countable elementary substructure N of $H(\lambda)$ such that the above arguments were done in N. By using the same arguments as in the proof of the successor case, take $\psi_{K,U} \in \mathscr{A}(K, U)$ (for $K \in \mathscr{S} \cap N$ and $U \in \mathscr{U} \cap N$), and define $C_n \subset \omega^n$ (for $n < \omega$), C, and tentacle trees K_s (for $s \in C$) and K. Note that K is a skip branching tree and $B(K)$ is g-thin. We complete the proof by showing that there exists $\tilde{p} \le p$ such that $\tilde{p} \Vdash \dot{f} \in \mathrm{Lim}(K)$. The desired $\tilde{p}$ will be constructed as the union of p_n (for $n < \omega$) in the next claim.

Claim 9. *There exist $p_n \in P_{\alpha_n}$ and $P_{\alpha_{n-1}}$-names $\dot{D}_n$, $\dot{\rho}_n$ (for $n < \omega$) such that, for all $n < \omega$,*

(11) *p_n is (N, P_{α_n})-generic and $p_n \le p \restriction \alpha_n$ and $p_{n+1} \restriction \alpha_n = p_n$,*

(12) *for all $K' \in \mathscr{S} \cap N$ and all $U \in \mathscr{U} \cap N$,*

$p_n \Vdash \forall \varphi \in \mathscr{A}(K', U) \cap N[\dot{\mathscr{G}}_{P_{\alpha_n}}]$ *(if* $\mathrm{rang}(\varphi) \subset N$ *then* $\exists^\infty i(\varphi(i) = \psi_{K,U}(i)))$

(13) *$\Vdash \dot{D}_n \subset C_n$ and $\dot{\rho}_n : \omega^{\le n} \to \bigcup_{k \le n} \dot{D}_k$ is an order isomorphism and $\forall k < n(\dot{\rho}_k \subset \dot{\rho}_n)$,*

(14) *$p_n[[s]] \Vdash \dot{T}^*_{\dot{\rho}_n(s)} = \dot{T}_{\dot{\rho}_n(s)} = K_{\dot{\rho}_n(s)}$, for all $s \in \omega^n$,*

(15) *$\forall p' \le p_n[[s]]$ (if $p' \Vdash \dot{\rho}_n(s) = t$ then $p' \le p \restriction \alpha_n[[t]]$), for all s, $t \in \omega^n$.*

Proof of Claim 9. By induction on $n < \omega$. By using Lemma 4.4 (3), take $p_0 \le p \restriction \alpha_0$ which satisfies (11) and (12). Set $\dot{D}_0 = \{\langle\rangle\}$ and $\dot{\rho}_0$ the unique function from ω^0 to $\dot{D}_0$. Assume that $n = m+1$ and $p_m, \dot{\rho}_m, \dot{D}_m$ have been defined. Let $s \in \omega^m$. Define P_{α_m}-names $\dot{E}_s$, $\dot{\tau}_s$, and $\dot{q}_s$ as follows:

We work in $V^{P_{\alpha_m}}$ below $p_m[[s]]$. Set

$$\dot{t} = \dot{\rho}_m(s) \quad \text{and} \quad \dot{a}_s = \{i < \omega \mid \dot{\varphi}_{\dot{t}}(i) = \psi_{K_{\dot{t}}, U_{\dot{t}}}(i)\}.$$

By induction hypothesis (14), and by (9) and (10), it holds that $\dot{\varphi}_{\dot{t}} \in \mathscr{A}(K_{\dot{t}}, U_{\dot{t}})$. By this and (12), $\dot{a}_s$ is infinite. Set $\dot{E}_s = \{\dot{t}^\frown\langle i\rangle \mid i \in \dot{a}_s\}$ and define $\dot{\tau}_s : \omega \to \dot{E}_s$ by

$$\dot{\tau}_s(i) = \dot{t}^\frown\langle \Gamma_{\dot{a}_s}(i)\rangle, \quad \text{for all } i < \omega.$$

For each $i < \omega$, since $\dot{r}_{\dot{\tau}_s(i)} \in N[\dot{\mathscr{G}}_{P_{\alpha_m}}]$, take $\dot{r}^+_{s^\frown\langle i\rangle} \le \dot{r}_{\dot{\tau}_s(i)}$ such that

$$\dot{r}^+_{s^\frown\langle i\rangle} \text{ is } (N[\dot{\mathscr{G}}_{P_{\alpha_m}}], P_{\alpha_n}/P_{\alpha_m})\text{-generic and } \mathrm{support}(\dot{r}^+_{s^\frown\langle i\rangle}) \subset N[\dot{\mathscr{G}}_{P_{\alpha_m}}],$$

and, for all $K' \in \mathscr{S} \cap N$ and all $U \in \mathscr{U} \cap N$,

$$\dot{r}^+_{s^\frown\langle i\rangle} \Vdash \forall \varphi \in \mathscr{A}(K', U) \cap N[\dot{\mathscr{G}}_{P_{\alpha_n}}] \ (\text{if } \mathrm{rang}(\varphi) \subset N \text{ then } \exists^\infty i(\varphi(i) = \psi_{K,U}(i))).$$

Since $\langle \dot{r}^{+}_{s^\frown\langle i\rangle} \mid i<\omega\rangle$ is a one point partition at α_m, let $\dot{q}_s$ be the root of this. Note that $\mathrm{support}(\dot{q}_s) \subset \{\alpha_m\} \cup \bigcup_{i<\omega} \mathrm{support}(\dot{r}^{+}_{s^\frown\langle i\rangle}) \subset N$ and $\dot{q}_s[\langle i\rangle_{\alpha_m}] = \dot{r}^{+}_{s^\frown\langle i\rangle}$, for all $i<\omega$.

Return to $\boldsymbol{V}$. Take $p_n \in P_{\alpha_n}$ such that

$$p_n \restriction \alpha_m = p_m \text{ and } p_m[[s]] \Vdash p_n \restriction [\alpha_m, \alpha_n) = \dot{q}_s, \quad \text{for all } s \in \omega^m.$$

It is not difficult to check that p_n satisfies (11) and (12). Replacing $\dot{E}_s$, $\dot{\tau}_s$ (for $s \in \omega^m$), if necessary, we may assume that

$$\Vdash \exists^\infty i < \omega(\dot{\rho}_m(s)^\frown\langle i\rangle \in \dot{E}_s) \text{ and } \dot{\tau}_s : \omega \to \dot{E}_s \text{ is a bijection,} \quad \text{for all } s \in \omega^m.$$

Define $\dot{D}_n$ and $\dot{\rho}_n \restriction \omega^n$ by

$$\Vdash \dot{D}_n = \bigcup_{s\in\omega^m} \dot{E}_s \text{ and } \dot{\rho}_n(s^\frown\langle i\rangle) = \dot{\tau}_s(i), \quad \text{for all } s^\frown\langle i\rangle \in \omega^n$$

It is easy to check that $\dot{D}_n$ and $\dot{\rho}_n$ satisfy (13). In order to show (14), let $s = s_0{}^\frown\langle i\rangle \in \omega^n$. Set $\dot{t}_0 = \dot{\rho}_m(s_0)$, $\dot{t} = \dot{\rho}_n(s) = \dot{t}_0{}^\frown\langle \dot{k}\rangle$. Since

$$p_m[[s_0]] \Vdash \dot{r}^{+}_s \le \dot{r}_{\dot{t}} \le p \restriction [\alpha_m, \alpha_n)[\langle \dot{k}\rangle_{\alpha_m}] \quad \text{and} \quad p_n \restriction [\alpha_m, \alpha_n)[\langle i\rangle_{\alpha_m}] = \dot{r}^{+}_{s_0{}^\frown i}$$

and $\dot{r}_{\dot{t}} \Vdash \dot{T}^*_{\dot{t}} = \dot{T}_{\dot{t}}$, it holds that $p_n[[s]] \Vdash \dot{T}^*_{\dot{t}} = \dot{T}_{\dot{t}}$. Since $p_m[[s_0]] \Vdash \dot{k} \in \dot{a}_{s_0}$, we have that $p_m[[s_0]] \Vdash \dot{T}^*_{\dot{t}} = \dot{\varphi}_{\dot{t}_0}(\dot{k}) = \dot{\psi}_{\dot{K}_{\dot{t}_0}, \dot{U}_{\dot{t}_0}}(\dot{k}) = K_{\dot{t}}$. Now, we deal with (15). Suppose that $s, t \in \omega^n$, $p' \le p_n[[s]]$, and $p' \Vdash \dot{\rho}_n(s) = t$. Let $s = s_0{}^\frown\langle i\rangle$ and $t = t_0{}^\frown\langle k\rangle$. By induction hypothesis, it holds that $p' \restriction \alpha_m \le p \restriction \alpha_m[[t_0]]$. Since it hold that

$$p_m[[s_0]] \Vdash p_n \restriction [\alpha_m, \alpha_n)[\langle i\rangle_{\alpha_m}] \le p \restriction [\alpha_m, \alpha_n)[\langle \dot{\rho}_n(s)(m)\rangle_{\alpha_m}] \text{ and}$$

$$p' \restriction \alpha_m \Vdash p' \restriction [\alpha_m, \alpha_n) \le p_n \restriction [\alpha_m, \alpha_n)[\langle i\rangle_{\alpha_m}] \text{ and } \dot{\rho}_n(s)(m) = k,$$

we have that $p' \restriction \alpha_m \Vdash p' \restriction [\alpha_m, \alpha_n) \le p \restriction [\alpha_m, \alpha_n)[\langle k\rangle_{\alpha_m}]$. So, $p' \le p \restriction \alpha_n[[t]]$. □

Let $\tilde{p} = \bigcup_{n<\omega} p_n$. It follws from the next claim that $\tilde{p} \Vdash \dot{f} \in \mathrm{Lim}(K)$.

CLAIM 10. $\tilde{p} \Vdash \forall n < \omega \exists t \in \omega^n$ ($\dot{\delta}_t \in K$ *and* $\dot{\delta}_t \subset \dot{f}$).

PROOF OF CLAIM 10. Let $p' \le \tilde{p}$ and $n < \omega$. Take $s \in \omega^n$ such that p' and $\tilde{p}[[s]]$ are compatible. Without loss of generality, we may assume that

$$p' \le \tilde{p}[[s]] \text{ and } p' \restriction \alpha_n \Vdash \dot{\rho}_n(s) = t, \quad \text{for some } t \in \omega^n.$$

Set $p'' = p' \restriction \alpha_n$. Note that

$$p'' \le \tilde{p} \restriction \alpha_n[[s]] = p_n[[s]].$$

By (14), we have that $p'' \Vdash \dot{\delta}_t \in K_t \subset K$. By (15), $p'' \le p \restriction \alpha_n[[t]]$. So, $p' \le p[[t]]$ and $p' \Vdash \dot{\delta}_t \subset \dot{f}$. □

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