The completions of metric ANR’s and homotopy dense subsets

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Abstract. In this paper, considering the problem when the completion of a metric ANR \( X \) is an ANR and \( X \) is homotopy dense in the completion, we give some sufficient conditions. It is also shown that each uniform ANR is homotopy dense in any metric space containing \( X \) isometrically as a dense subset, and that a metric space \( X \) is a uniform ANR if and only if the metric completion of \( X \) is a uniform ANR with \( X \) a homotopy dense subset. Furthermore, introducing the notions of densely (local) hyper-connectedness and uniformly (local) hyper-connectedness, we characterize of AR’s (ANR’s) and uniform AR’s (uniform ANR’s), respectively.

Introduction.

A subset \( Y \) of a space \( X \) is said to be homotopy dense in \( X \) if there exists a homotopy \( h : X \times [0,1] \to X \) such that \( h_0 = \text{id} \) and \( h_t(Y) \subseteq Y \) for \( t > 0 \). This concept is very important in ANR Theory and Infinite-Dimensional Topology. When \( X \) is an ANR, the concept of the homotopy denseness is dual to the one of local homotopy negligibility introduced by Toruńczyk in [To3]. Actually, \( Y \subseteq X \) is homotopy dense in \( X \) if and only if the complement \( X \setminus Y \) is locally homotopy negligible in \( X \) (cf. [To3, Theorem 2.4]). As well-known, every homotopy dense subset of an ANR is also an ANR and a metrizable space is an ANR if it contains an ANR as a homotopy dense subset. The lack of the homotopy denseness of a metric ANR in its completion often destroys the ANR property of the completion. For instance, the \( \sin 1/x \)-curve in the plane \( \mathbb{R}^2 \) is an ANR but the completion of this curve (= the closure in \( \mathbb{R}^2 \)) is not an ANR. Moreover, even if the completion is an ANR, it is very different from the original ANR. The circle \( S^1 \) is the completion of the space \( S^1 \setminus \{ \text{pt} \} \) and the both spaces are ANR but they are topologically very different from each other. It should be remarked that \( S^1 \setminus \{ \text{pt} \} \) is not homotopy dense in \( S^1 \). It is an interesting problem when a metric ANR is homotopy dense in the metric completion and, in particular, the completion is an ANR.

In [N], Nguyen To Nhu gave a characterization of ANR’s, a variation of which was given in [NS]. In §1 of this paper, we give its alternative proof and apply the technique involved in the proof to find conditions that the completion of a metric space \( X \) is an ANR with \( X \) a homotopy dense subset. In [Mi2], E. Michael introduced

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uniform AR’s and uniform ANR’s, and studied them. The concept of uniform ANR’s is useful since the metric completion of every uniform ANR is also a uniform ANR. In §2, we show that each uniform ANR is homotopy dense in any metric space which contains $X$ isometrically as a dense subset, and that a metric space $X$ is a uniform ANR if and only if the metric completion of $X$ is a uniform ANR with $X$ a homotopy dense subset. By using the notion of (local) hyper-connectedness, C. R. Borges $\text{Bo}$ and R. Cauty $\text{Ca}$ characterized AR’s and ANR’s, respectively. It is shown in §3 that a little weaker notion also characterizes AR’s (or ANR’s). Furthermore, we give a characterization of uniform AR’s (or uniform ANR’s) which is similar to the one of $\text{Bo}$ (or $\text{Ca}$).

The $n$-skeleton of a simplicial complex $K$ is denoted by $K^{(n)}$ and the polyhedron $|K|$ is the space $|K| = \bigcup_{\sigma \in K} \sigma$ endowed with the Whitehead topology. For each simplex $\sigma \in K$, we denote $\sigma^{(n)} = \sigma \cap |K^{(n)}|$, which is the union of all $n$-faces of $\sigma$. The nerve of an open cover $\mathcal{U}$ of a space $X$ is denoted by $N(\mathcal{U})$. Note that $\mathcal{U}$ is the set of vertices of $N(\mathcal{U})$, i.e., $\mathcal{U} = N(\mathcal{U})^{(0)}$. Recall a canonical map $\varphi : X \to |N(\mathcal{U})|$ for $\mathcal{U}$ is a map which sends each $x \in X$ into a simplex $\sigma \in N(\mathcal{U})$, all vertices of which contain $x$. The star of $\mathcal{U}$ is denoted by $\text{st}(\mathcal{U}) = \{\text{st}(U, \mathcal{U}) | U \in \mathcal{U}\}$, where $\text{st}(U, \mathcal{U}) = \bigcup \{V \in \mathcal{U} | U \cap V \neq \emptyset\}$. For a collection $\mathcal{A}$ of subsets of $X$, $\mathcal{A} < \mathcal{U}$ means that each $A \in \mathcal{A}$ is contained in some $U \in \mathcal{U}$. In case $X = (X, d)$ is a metric space, the open ball in $X$ centered at $x \in X$ with radius $r > 0$ is denoted by $B_X(x, r)$ (or $B(x, r)$). For $a \in X$ and $C \subset X$, let dist$(a, C) = \inf \{d(a, x) | x \in C\}$ and diam $C = \sup \{d(x, y) | x, y \in C\}$. For a collection $\mathcal{A}$ of subsets of $X$, let mesh $\mathcal{A} = \sup \{\text{diam} A | A \in \mathcal{A}\}$.

1. A characterization of metric ANR’s.

A sequence $\mathcal{U} = (\mathcal{U}_n)_{n \in \mathbb{N}}$ of open covers of a metric space $X$ is called a zero-sequence if $\lim_{n \to \infty} \text{mesh} \mathcal{U}_n = 0$. For such a sequence, we define the simplicial complex

$$TN(\mathcal{U}) = \bigcup_{n \in \mathbb{N}} N(\mathcal{U}_n \cup \mathcal{U}_{n+1}),$$

where we regard $\mathcal{U}_n \cap \mathcal{U}_m = \emptyset$ ($n \neq m$) as sets of vertices of $TN(\mathcal{U})$ even if $\mathcal{U}_n \cap \mathcal{U}_m \neq \emptyset$ as collections of open sets,\(^1\) whence

$$N(\mathcal{U}_n \cup \mathcal{U}_{n+1}) \cap N(\mathcal{U}_{n+1} \cup \mathcal{U}_{n+2}) = N(\mathcal{U}_{n+1}).$$

For each $\sigma \in TN(\mathcal{U})$, let $n(\sigma) = \max \{n \in \mathbb{N} | \sigma \in N(\mathcal{U}_n \cup \mathcal{U}_{n+1})\}$. Observe that, for a map $f : |TN(\mathcal{U})| \to X$,

$$\lim_{n \to \infty} \text{mesh} \{f(\sigma) | \sigma \in N(\mathcal{U}_n \cup \mathcal{U}_{n+1})\} = 0$$

if and only if $\text{diam} f(\sigma_i) \to 0$ for any sequence $(\sigma_i)_{i \in \mathbb{N}}$ in $TN(\mathcal{U})$ with $n(\sigma_i) \to \infty$. The following is the characterization of ANR’s obtained in [NS, Theorem 1]. Here is given an alternative proof without the assumption that $X$ has no isolated points.

\(^1\)In [NS] we did not regard like this. Considering the set $\bigcup_{n \in \mathbb{N}} \{\text{dist}(U, n) | U \in \mathcal{U}_n\}$ as the set of vertices of $NT(\mathcal{U})$, this is reasonable.
THEOREM 1. A metric space \( X = (X, d) \) is an ANR if and only if \( X \) has a zero-sequence \( \mathcal{U} = (\mathcal{U}_n)_{n \in \mathbb{N}} \) of open covers with a map \( f : |TN(\mathcal{U})| \to X \) satisfying the following conditions:

(i) \( f(U) \in U \) for each \( U \in TN(\mathcal{U})^{(0)} = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n \), and

(ii) \( \lim_{n \to -\infty} \text{mesh}\{f(\sigma) \mid \sigma \in N(\mathcal{U}_n \cup \mathcal{U}_{n+1}) \} = 0. \)

Under the above circumstances, if the image \( f(|TN(\mathcal{U})|) \) is contained in \( Y \subset X \), then \( Y \) is homotopy dense in \( X \).

PROOF. The “only if” part is proved by the same way as [N, Theorem 1-1, (i) \( \Rightarrow \) (ii)], but we give the proof for the reader’s convenience and to make an observation which will be discussed later. Suppose that \( X \) is an ANR. By Arens–Eells’ embedding theorem \([AE]\) (cf. [To]), \( X \) can be isometrically embedded in a normed linear space \( E \) as a closed set. Then, there is a retraction \( r : V \to X \) of an open neighborhood \( V \) of \( X \) in \( E \). For each \( n \in \mathbb{N} \), let \( \mathcal{W}_n \) be a convex open cover of \( V \) such that \( \text{mesh}(\mathcal{W}_n) < 2^{-n} \). We can construct a zero-sequence \( \mathcal{U} = (\mathcal{U}_n)_{n \in \mathbb{N}} \) of open covers of \( X \) so that \( \text{st} \mathcal{U}_n < \mathcal{W}_n \) and \( \mathcal{U}_{n+1} < \mathcal{U}_n \). By choosing a point \( f_0(U) \in U \) for each \( U \in TN(\mathcal{U})^{(0)} = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n \), we define a map \( f_0 : TN(\mathcal{U})^{(0)} \to X \). For each \( \sigma \in TN(\mathcal{U}) \), let \( U_{\sigma} \in \sigma^{(0)} \cap \mathcal{U}_{n(\sigma)} \). Then \( f_0(\sigma^{(0)}) = \text{st}(U_{\sigma}, \mathcal{U}_{n(\sigma)}) \), which is contained in some \( W_{\sigma} \in \mathcal{W}_{n(\sigma)} \). Note that \( W_{\sigma} \) is convex and \( \text{diam}(W_{\sigma}) < 2^{-n(\sigma)} \). By using the linear structure of \( E \), we can extend \( f_0 \) to a map \( f : |TN(\mathcal{U})| \to V \) such that \( f(\sigma) \in W_{\sigma} \) for each \( \sigma \in TN(\mathcal{U}) \), whence \( \text{diam} rf(\sigma) < 2^{-n(\sigma)} \). The map \( rf : |TN(\mathcal{U})| \to X \) clearly satisfies the conditions (i) and (ii).

To prove the “if” part, let \( \mathcal{U} = (\mathcal{U}_n)_{n \in \mathbb{N}} \) be a zero-sequence of open covers of \( X \) with a map \( f : |TN(\mathcal{U})| \to X \) satisfying the conditions (i) and (ii). Then, \( \alpha_n = \text{mesh}\{f(\sigma) \mid \sigma \in N(\mathcal{U}_n \cup \mathcal{U}_{n+1}) \} + \text{mesh} \mathcal{U}_n \to 0 \) as \( n \to \infty \).

For each \( n \in \mathbb{N} \), let \( \varphi_n : X \to |N(\mathcal{U}_n)| \) be a canonical map. Observe that, for each \( x \in X \), we have \( \varphi_n(x) \in N(\mathcal{U}_n \cup \mathcal{U}_{n+1}) \) such that \( \varphi_n(x), \varphi_{n+1}(x) \in \sigma_x \). Then, there is a homotopy \( g^{(n)} : X \times [0, 1] \to |N(\mathcal{U}_n \cup \mathcal{U}_{n+1})| \) such that \( g_0^{(n)} = \varphi_n, g_1^{(n)} = \varphi_{n+1} \) and \( g^{(n)}(\{x\} \times [0, 1]) \subset \sigma_x \) for each \( x \in X \), whence

\[
\text{diam} f g^{(n)}(\{x\} \times [0, 1]) \leq \text{mesh}\{f(\sigma) \mid \sigma \in N(\mathcal{U}_n \cup \mathcal{U}_{n+1}) \} < \alpha_n.
\]

On the other hand, since \( \varphi_n(x) \in \sigma_x \) and \( x, f(U) \in U \) for some \( U \in \sigma_x^{(0)} \), it follows that

\[
d(f(\varphi_n(x), x) \leq d(f(\varphi_n(x), f(U))) + d(f(U), x) \leq \text{diam} f(\sigma_x) + \text{diam} U \leq \alpha_n.
\]

Now, we can define a homotopy \( h : X \times [0, 1] \to X \) as follows:

\[
h(x, t) = \begin{cases} x & \text{if } t = 0; \\
f g^{(n)}(x, 2 - 2^n t) & \text{if } 2^{-n} \leq t \leq 2^{-n+1}.
\end{cases}
\]

The restriction \( h|X \times (0, 1] \) is clearly continuous. For each \( \varepsilon > 0 \), we have \( n \in \mathbb{N} \) such that \( \text{diam} h(\{x\} \times [0, 2^{n+1}]) < \varepsilon \) for every \( x \in X \). In fact, choose \( n \in \mathbb{N} \) so that \( \alpha_m < \varepsilon/2 \) for all \( m \geq n \). For \( 0 < t \leq 2^{-n+1} \), we have \( 2^{-m} < t \leq 2^{-m+1} \) for some \( m \geq n \), whence
\[ d(h(x, t), x) \leq d(f \varphi^{(m)}_n(x, 2 - 2^m t), f \varphi^{(m)}_0(x)) + d(f \varphi_n(x), x) \]
\[ \leq \text{diam } f \varphi^{(m)}(\{x\} \times [0, 1]) + 2\varepsilon < \varepsilon. \]

This implies that \( h \) is continuous at each \((x, 0)\). Moreover, \( f \varphi_n = h_{2^{n+1}} \) is \( \varepsilon \)-homotopic to \( \text{id}_X \), which means that \( X \) is \( \varepsilon \)-homotopy dominated by the simplicial complex \( TN(\mathcal{U}) \). Therefore, \( X \) is an ANR.

In the above argument, if \( f(|TN(\mathcal{U})|) \subseteq Y \) then the homotopy \( h \) constructed above satisfies that \( h(X \times (0, 1]) \subseteq Y \), hence \( Y \) is homotopy dense in \( X \). Thus, we have the additional statement.

\[ \text{Remark. In the above theorem, if } \mathcal{U}_1 = \{ X \} \text{ then } X \text{ is an AR. In fact, } X \text{ is contractible because } f \varphi_1 \text{ is constant.} \]

**Corollary 1.** Let \( X \) be an ANR (resp. AR) contained in a metric space \( M \). Then, there exists a \( G_\delta \)-set \( Z \subseteq M \) such that \( Z \) is an ANR (resp. AR) and \( X \) is homotopy dense in \( Z \).

**Proof.** By [Theorem 1], \( X \) has a zero-sequence \( \mathcal{U} = (\mathcal{U}_n)_{n \in \mathbb{N}} \) of open covers with a map \( f : |TN(\mathcal{U})| \to X \) satisfying the conditions (i) and (ii) of [Theorem 1]. For each open set \( U \) in \( X \), we define
\[ E(U) = \{ x \in M \mid \text{dist}(x, U) < \text{dist}(x, X \setminus U) \}, \]
where \( \text{dist}(x, \emptyset) = \infty \), so \( E(\emptyset) = \emptyset \) and \( E(X) = M \). Then, \( E(U) \) is open in \( M \), \( E(U) \cap X = U \) and \( E(U) \cap E(V) = E(U \cap V) \). The desired \( G_\delta \)-set in \( M \) is defined by
\[ Z = \text{cl } X \cap \bigcap_{n \in \mathbb{N}} \bigcup_{U \in \mathcal{U}_n} E(U). \]
In fact, for each \( n \in \mathbb{N} \), let \( \mathcal{U}_n = \{ Z \cap E(U) \mid U \in \mathcal{U}_n \} \). Since mesh \( \mathcal{U}_n = \text{mesh } \mathcal{U}_n \), \( \mathcal{U}_n = (\bar{U}_n)_{n \in \mathbb{N}} \) is a zero-sequence of open covers of \( Z \). The correspondence \( Z \cap E(U) \mapsto U \) induces the isomorphism from \( TN(\mathcal{U}) \) onto \( TN(\mathcal{U}) \). By the additional statement of [Theorem 1], we have the result.

We can also apply [Theorem 1] to find conditions such that the metric completion of a metric space \( X \) is an ANR with \( X \) a homotopy dense subset. A subset \( D \) of a metric space \( X \) is said to be \( \delta \)-dense in \( X \) if \( \text{dist}(x, D) < \delta \) for every \( x \in X \).

**Corollary 2.** Let \( X \) be a metric space which has a zero-sequence \( \mathcal{U} = (\mathcal{U}_n)_{n \in \mathbb{N}} \) of open covers with a map \( f : |TN(\mathcal{U})| \to X \) satisfying the conditions (i) and (ii) of Theorem 1, where suppose \( \mathcal{U}_n = \{ B_X(x, \delta_n) \mid x \in D_n \} \) for some \( \delta_n \)-dense subset \( D_n \subset X \) and \( 0 < \delta_n < \gamma_n \). Then, any metric space \( Z \) containing \( X \) isometrically as a dense subset is an ANR and \( Z \) is homotopy dense in \( Z \). In particular, the metric completion \( \tilde{X} \) of \( X \) is an ANR and \( X \) is homotopy dense in \( \tilde{X} \).

**Proof.** In this case, each \( \mathcal{U}_n \) extends to the open cover \( \mathcal{U}_n = \{ B_Z(x, \delta_n) \mid x \in D_n \} \) of \( Z \). Thus \( Z \) has a zero-sequence \( \mathcal{U} = (\mathcal{U}_n)_{n \in \mathbb{N}} \). Since \( TN(\mathcal{U}) \) can be identified with \( TN(\mathcal{U}) \), the result follows from the additional statement of [Theorem 1].
In the above, note that the $\gamma_n$-dense subset $D_n$ of $X$ may not be $\delta_n$-dense in $Z$. For example, $D_n = \{i/n | 1 \leq i < n\}$ is $1/n$-dense in $(0,1)$ but it is not $1/n$-dense in $[0,1]$. Now, we consider the following extension property:

$$(e)_k$$ There exist constants $\alpha > 0$ and $\beta > 1$ such that every map $f: |K^{(k)}| \to X$ of the $k$-skeleton of an arbitrary simplicial complex $K$ with $\text{mesh} \{f(\sigma^{(k)}) | \sigma \in K\} < \alpha$ extends to a map $\tilde{f}: |K| \to X$ such that $\text{diam} \tilde{f}(\sigma) \leq \beta \text{diam} f(\sigma^{(k)})$ for each $\sigma \in K$.

The following corollary is motivated by the proof of AR property of hyperspaces (cf. [vM, §5.3]).

**Corollary 3.** Every $LC^{k-1}$ metric space $X$ with the property $(e)_k$ is an ANR.

**Proof.** Without loss of generality, we may assume that $X$ has no isolated points. Since $X$ is $LC^{k-1}$, $X$ has open covers $\mathcal{V}_{(i,n)}$, $0 \leq i \leq k$, $n \in \mathbb{N}$, such that $\text{mesh} \mathcal{V}_{(i,n)} < 2^{-n}\alpha$, $\mathcal{V}_{(i,n+1)} < \mathcal{V}_{(i,n)}$ and each $W \in \text{st} \mathcal{V}_{(i,n)}$ is contained in some $V \in \mathcal{V}_{(i+1,n)}$ such that every map $f: S^i \to W$ extends to a map $\tilde{f}: B^{i+1} \to V$. For each $n \in \mathbb{N}$, let $\mathcal{U}_n = \mathcal{V}_{(0,n)}$. Then, $\mathcal{U} = (\mathcal{U}_n)_{n \in \mathbb{N}}$ is a zero-sequence of open covers of $X$. Let $f_0: TN(\mathcal{U})^{(0)} \to X$ be a map such that $f_0(U) \in U$ for each $U \in TN(\mathcal{U})^{(0)} = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n$. For each $\sigma \in TN(\mathcal{U})$, $f(\sigma^{(0)})$ is contained in some member of $\text{st} \mathcal{U}_{n(\sigma)} = \text{st} \mathcal{V}_{(0,n(\sigma))}$. By the induction, we can extend $f_0$ to a map $f_k: |TN(\mathcal{U})^{(k)}| \to X$ such that $f(\sigma^{(k)})$ is contained in some member of $\text{st} \mathcal{V}_{(k,n(\sigma))}$ for each $\sigma \in TN(\mathcal{U})$, hence

$$\text{mesh} \{f_k(\sigma^{(k)}) | \sigma \in N(\mathcal{U}_n \cup \mathcal{U}_{n+1})\} \leq 2^{-n}\alpha.$$  

By the hypothesis, $f_k$ extends to a map $f: |TN(\mathcal{U})| \to X$ such that

$$\text{mesh} \{f(\sigma) | \sigma \in N(\mathcal{U}_n \cup \mathcal{U}_{n+1})\} \leq 2^{-n}\alpha\beta.$$  

Then, the result follows from Theorem 1. \hfill \Box

**Remark.** The following extension property is stronger than $(e)_k$:  

$$(\hat{e})_k$$ there exists a constant $\beta > 1$ such that every map $f: |K^{(k)}| \to X$ of the $k$-skeleton of an arbitrary simplicial complex $K$ extends to a map $\hat{f}: |K| \to X$ such that $\text{diam} \hat{f}(\sigma) \leq \beta \text{diam} f(\sigma^{(k)})$ for each $\sigma \in K$.

It can be proved that every $C^{k-1}$ and $LC^{k-1}$ metric space $X$ with the property $(\hat{e})_k$ is an AR. Cf. Remark after Theorem 1.

**2. Uniform ANR’s.**

Let $X = (X, d_X)$ and $Y = (Y, d_Y)$ be metric spaces and $A \subset X$. A map $f: X \to Y$ is said to be uniformly continuous at $A$ if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $a \in A$, $x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \varepsilon$. A neighborhood $U$ of $A$ in $X$ is called a uniform neighborhood if $\bigcup_{a \in A} B_X(a, \delta) \subset U$ for some $\delta > 0$.

A uniform ANR is defined in [Mi2] as a metric space $Y$ such that, for an arbitrary metric space $X$ and a closed set $A \subset X$, every uniformly continuous map $f: A \to Y$ extends to a map $\hat{f}: U \to Y$ from some uniform neighborhood $U$ of $A$ in $X$ which is
uniformly continuous at $A$. When $f$ always extends over $X$ (i.e., $U = X$), $Y$ is called a uniform AR. By virtue of [Mi2, Theorem 1.2], a metric space $Y$ is a uniform ANR (resp. a uniform AR) if and only if, for an arbitrary metric space $Z$ which contains $Y$ isometrically as a closed subset, there exists a retraction $r : U \to Y$ for some uniform neighborhood $U$ in $Y$ in $Z$ (resp. $r : Z \to Y$) which is uniformly continuous at $Y$.2

**Lemma 1.** Every uniform ANR $X$ has a zero-sequence $\mathcal{U} = (\mathcal{U}_n)_{n \in \mathbb{N}}$ of open covers with a map $f : |TN(\mathcal{U})| \to X$ such as Corollary 2.

**Proof.** In the proof of the “only if” part of [Theorem 1], since the retraction $r : V \to X$ can be assumed to be a retraction of a uniform open neighborhood of $X$ in $E$ which is uniformly continuous at $X$, we can take as $\mathcal{U}_n$ the open cover $\{B_E(x, r_n) \mid x \in X\}$ for some $r_n > 0$. Let $\delta_n = r_n/3$ and $\gamma_n = r_n/2$. Take a $\delta_n$-dense subset $D_n$ of $X$ and define $\mathcal{U}_n = \{B_X(x, \gamma_n) \mid x \in D_n\}$. By the same argument, we have the result. □

By using this lemma, we can strengthen Proposition 1.4 in [Mi2] as follows:

**Theorem 2.** For an arbitrary metric space $X$, the following conditions are equivalent:

(a) $X$ is a uniform ANR;
(b) Every metric space $Z$ containing $X$ isometrically as a dense subset is a uniform ANR and $X$ is homotopy dense in $Z$;
(c) $X$ is isometrically embedded in some uniform ANR $Z$ as a homotopy dense subset.

**Proof.** The implications (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (a) are obvious.

(a) $\Rightarrow$ (b): By Proposition 1.4 in [Mi2], $Z$ is a uniform ANR. Combining Lemma 1 with Corollary 2, it follows that $X$ is homotopy dense in $Z$.

(c) $\Rightarrow$ (a): By Arens–Eells’ embedding theorem [AE] (cf. [To1]), $Z$ can be isometrically embedded in a normed linear space $E = (E, \|\cdot\|)$ as a closed set which is linearly independent. Let $F$ be the linear subspace of $E$ spanned by $X$. Then $X = Z \cap F$ is closed in $F$. Since $Z$ is a uniform ANR, we have a uniform open neighborhood $U$ of $Z$ in $E$ and a retraction $r : U \to Z$ which is uniformly continuous at $Z$. On the other hand, we have a homotopy $h : Z \times [0, 1] \to Z$ such that $h_0 = \text{id}$ and $h_t(Z) = X$ for all $t > 0$. It is easy to construct maps $\varphi_n : Z \to (0, 1)$, $n \in \mathbb{N}$, such that $\varphi_{n+1}(z) < \varphi_n(z)$ ($\leq 2^{-n}$) and $\text{diam} \{h_t(z) \mid 0 \leq t \leq 1\} < 2^{-n}$. Then we have a homeomorphism $\varphi : Z \times [0, 1] \to Z \times [0, 1]$ such that $\varphi(Z \times [0, 1]) = \{z, \varphi_n(z)\}$. For each $z \in Z$, observe that $d(z, h\varphi(z), t) < 2^{-n}$ if $t < 2^{-n}$. We define a retraction $r' : U \to Z$ by $r'(x) = h\varphi(r(x), \text{dist}(x, Z))$ for each $x \in U$. Note that $r'(U \setminus Z) \subset X$. For each $\varepsilon > 0$, choose $n \in \mathbb{N}$ so that $2^{-n+1} < \varepsilon$. Since $r$ is uniformly continuous at $Z$, there is $\delta > 0$ such that if $x \in U$, $z \in Z$ and $\|x - z\| < \delta$, then $d(r(x), z) < 2^{-n}$. Now, let $x \in U$ and $z \in Z$ with $\|x - z\| < \min\{2^{-n}, \delta\}$. Since $\text{dist}(x, Z) \leq \|x - z\| < 2^{-n}$, it follows that

$$d(r'(x), z) \leq d(h\varphi(r(x), \text{dist}(x, Z)), r(x)) + d(r(x), z) < 2^{-n} + 2^{-n} < \varepsilon.$$ 

Therefore, $r'$ is also uniformly continuous at $Z$. The restriction $r'|U \cap F : U \cap F \to X = Z \cap F$ is a retraction which is uniformly continuous at $X$. By [Mi2, Theorem 1.2], $X$ is a uniform ANR.

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2Such a retraction is called a regular retraction by H. Toruńczyk in [To2].
Theorem 2 above means that a metric space $X$ is a uniform ANR if and only if the metric completion $\tilde{X}$ of $X$ is a uniform ANR and $X$ is homotopy dense in $\tilde{X}$. However, in order that the metric completion of a metric ANR $X$ is an ANR with $X$ a homotopy dense subset, it is not necessary that $X$ is a uniform ANR.

Example. The following subspace $X$ of Euclidean plane $\mathbb{R}^2$ is not a uniform ANR but the metric completion of $X$ is an ANR with $X$ a homotopy dense subset:

$$X = \mathbb{R} \times \{0\} \cup \mathbb{N} \times [0,1) \cup \bigcup_{n \in \mathbb{N}} \{n+2^{-n}\} \times [0,1) \subset \mathbb{R}^2.$$ 

In fact, $X$ is not a uniform neighborhood retract of $\mathbb{R}^2$, but $X$ and the closure of $X$ in $\mathbb{R}^2$ are ANR’s and $X$ is homotopy dense in the closure.

In case $X$ is totally bounded, we have the following:

Proposition 1. A totally bounded metric space $X$ a uniform ANR if and only if the metric completion $\tilde{X}$ of $X$ is an ANR with $X$ a homotopy dense subset.

Proof. It suffices to show the “if” part. Assume that $\tilde{X}$ is an ANR and $X$ is homotopy dense in $\tilde{X}$. Since $\tilde{X}$ is also totally bounded, it is a compact ANR, hence it is a uniform ANR. By Theorem 2 $X$ is also a uniform ANR.

Now, we prove the following theorem:

Theorem 3. Every metric space $Y$ with the property $(e)_0$ is a uniform ANR.

Proof. This can be shown by an alteration of the proof of [Mi$_2$, Theorem 7.1 (c) $\Rightarrow$ (a)] as follows: Let $s_1 > s_2 > \cdots > 0$ be any sequence such that $8s_1 < \alpha$, $\lim_{n \to \infty} s_n = 0$ and $\forall_n \cap \forall_n = \emptyset$ if $m \neq n$, where $\forall_n$ is defined in [Mi$_2$, p. 135]. Then, the map $f$ in the Michael’s proof satisfies the following condition:

$$\text{diam } f(\sigma^{(0)}) < 8s_n \text{ for each } \sigma \in \forall_n.$$ 

Here, instead of extending $f$ step by step, we can apply the property $(e)_0$ to extend $f$ to a map $h : \bigcup_{n \in \mathbb{N}} N(\forall_n) \to Y$ such that $\text{diam } h(\sigma) < 8s_n \beta$ for each $\sigma \in N(\forall_n)$. For each $n \in \mathbb{N}$, let $h_n = h|N(\forall_n)$. By the same definition as in the proof, we can obtain a uniform neighborhood $W$ of $Y$ in $Z$ and a retraction $r : W \to Y$ which is uniformly continuous at $Y$.

By Theorems 2 and 3, we have the following corollary (cf. [SU, Lemma 2]):

Corollary 4. Let $X$ be a metric space and $Y$ a dense subset of $X$. If $Y$ has the property $(e)_0$, then $X$ and $Y$ are ANR’s and $Y$ is homotopy dense in $X$. $\square$

Remark. In Theorem 3 and Corollary 4, if the property $(e)_0$ is replaced by $(\tilde{e})_0$, then “ANR” can be “AR”.

A metric space $Y$ is said to be uniformly $LC^k$ if, for each $\epsilon > 0$, there exists $\delta > 0$ such that any map $f : S^i \to Y$ with $\text{diam } f(S^i) < \delta$ extends to a map $\tilde{f} : B^{i+1} \to Y$ with $\text{diam } f(B^{i+1}) < \epsilon$ for every $i \leq k$. In stead of “uniformly $LC^0$”, we also say “uniformly locally path-connected”. The subspace of $\mathbb{R}^2$ in the example above is not uniformly locally path-connected.
Proposition 2. Every uniformly LC$^{k-1}$ metric space $Y$ with the property $(e)_k$ is a uniform ANR.

Proof. This is also shown by an alteration of the proof of [Mi$_2$, Theorem 7.1 (c) ⇒ (a)]. Here, we can apply the condition (c) of [Mi$_2$, Theorem 7.1] to a simplicial complex $K$ with dim $K \leq k$. In the Michael's proof, replacing 1/n by $\alpha/3n$, the map $f|N(\mathcal{U}_n)^0$ extends to a map $h'_n : |N(\mathcal{U}_n)^{(k)}| \to Y$ such that diam $h'_n(\sigma) < \alpha/3n$ for each $\sigma \in N(\mathcal{U}_n)^{(k)}$. For each $\sigma \in N(\mathcal{U}_n)$, since diam $h'_n(\sigma^{(0)}) < \alpha/3n$, we have diam $h'_n(\sigma^{(k)}) < \alpha/n$. Now, by using the property $(e)_k$, each $h'_n$ can be extended to a map $h_n : |N(\mathcal{U}_n)| \to Y$ such that diam $h_n(\sigma) < \alpha\beta/n$ for each $\sigma \in N(\mathcal{U}_n)$. Then, by the same definition as in the proof, we can obtain a uniform neighborhood $W$ of $Y$ in $Z$ and a retraction $r : W \to Y$ which is uniformly continuous at $Y$.

Combining of Proposition 2 with Theorem 2, we have the following variation of Corollary 3.

Corollary 5. Let $X$ be a metric space and $Y$ a dense subset of $X$. If $Y$ is uniformly LC$^{k-1}$ and has the property $(e)_k$, then $X$ and $Y$ are uniformly ANR's and $Y$ is homotopy dense in $X$.

Remark. In Proposition 2 and Corollary 3, by replacing the property $(e)_k$ with $(e')_k$ and adding the condition that $Y$ is C$^{k-1}$, “uniform ANR” can be “uniform AR”.

3. Dense (or uniform) local hyper-connectedness.

By $A^{n-1}$, we denote the standard $(n - 1)$-simplex in $\mathbb{R}^n$, that is,

$$A^{n-1} = \left\{ (t_1, \ldots, t_n) \in \mathbb{R}^n \mid t_i \geq 0, \sum_{i=1}^{n+1} t_i = 1 \right\}.$$ 

For an open cover $\mathcal{U}$ of a space $X$ and $Y \subset X$, we denote

$$Y^n(\mathcal{U}) = \left\{ (y_1, \ldots, y_n) \in Y^n \mid \exists U \in \mathcal{U} \text{ such that } \{y_1, \ldots, y_n\} \subseteq U \right\}.$$ 

It is said that a space $X$ is densely locally hyper-connected if $X$ has an open cover $\mathcal{V}$, a dense subset $D$ and functions $h_n : D^n(\mathcal{V}) \times A^{n-1} \to X$, $n \in \mathbb{N}$, which satisfy the following conditions:

(i) if $t_i = 0$ then

$$h_n(y_1, \ldots, y_n; t_1, \ldots, t_n) = h_{n-1}(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n; t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n);$$

(ii) $A^{n-1} \ni (t_1, \ldots, t_n) \mapsto h_n(y_1, \ldots, y_n; t_1, \ldots, t_n) \in X$ is continuous for each $(y_1, \ldots, y_n) \in D^n(\mathcal{V})$;

(iii) every open cover $\mathcal{U}$ of $X$ has an open refinement $\mathcal{V}$ such that $\mathcal{V} \prec \mathcal{U}$ (hence $D^n(\mathcal{V}) \subset D^n(\mathcal{U})$) and

$$\{h_n((D \cap V)^n \times A^{n-1}) \mid V \in \mathcal{V} \} \prec \mathcal{U} \text{ for each } n \in \mathbb{N}.$$
It should be noticed that each $h_n$ need not be continuous. If $\mathcal{W}$ can be taken as $\mathcal{W} = \{X\}$ (i.e., $D^n(\mathcal{W}) = D^n$), we say that $X$ is densely hyper-connected. In case $D = X$ (resp. $D = X$ and $\mathcal{W} = \{X\}$), $X$ is locally hyper-connected\(^3\) (resp. hyper-connected). This concept is very similar to Michael's convex structure in [Mi]. In [Bo] and [Ca], AR’s and ANR’s are characterized by the hyper-connectedness and the local hyper-connectedness, respectively. A similar characterization was obtained by Himmelberg [Hi] (cf. Curtis [Cu]). These characterizations can be generalized in terms of the dense hyper-connectedness as follows:

**Theorem 4.** A metrizable space $X$ is an ANR if and only if $X$ is densely locally hyper-connected. Moreover, $X$ is an AR if and only if $X$ is densely hyper-connected.

**Proof.** By the characterization of ANR’s in [Ca] (or AR’s in [Bo]), it suffices to prove the “if” part only. (Or see the proof of Theorem 5 below.)

Assume that $X$ is a densely locally hyper-connected metric space, that is, $X$ has an open cover $\mathcal{W}$, a dense subset $D$ and functions $h_n : D^n(\mathcal{W}) \times A^{n-1} \to X$, $n \in \mathbb{N}$, which satisfy the conditions (i), (ii) and (iii). By the condition (iii), we obtain a sequence $\mathcal{U} = (\mathcal{U}_n)_{n \in \mathbb{N}}$ of open covers of $X$ such that $\mathcal{U}_1 < \mathcal{W}$, $\mathcal{U}_{n+1} < \mathcal{U}_n$, mesh $\mathcal{U}_n < 2^{-n}$ and

$$\text{mesh}\{h_k((D \cap \text{st}(U, \mathcal{U}_n))^k \times A^{k-1}) | k \in \mathbb{N}, U \in \mathcal{U}_n\} < 2^{-n}.$$ 

By choosing a point $f_0(U) \in D \cap U$ for each $U \in TN(\mathcal{U})^{(0)} = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n$, we define a map $f_0 : TN(\mathcal{U})^{(0)} \to D$. For each $\sigma \in TN(\mathcal{U})$, let $\sigma^{(0)} = \{U_1, \ldots, U_k\} \subset \mathcal{U}_n \cup \mathcal{U}_{n+1}$, where we can assume $U_1 \in \mathcal{U}_n$. Then $f_0(\sigma^{(0)}) \subset \text{st}(U_1, \mathcal{U}_n)$ because $\mathcal{U}_{n+1} < \mathcal{U}_n$. By using $h_k$, we can define $f_\sigma : \sigma \to X$ by

$$f_\sigma \left( \sum_{i=1}^k t_i U_i \right) = h_k(f_0(U_1), \ldots, f_0(U_k); t_1, \ldots, t_k).$$

Then $\text{diam} f_\sigma(\sigma) < 2^{-n}$. Observe that $f_\sigma | \sigma \cap \tau = f_\tau | \sigma \cap \tau$ for each $\sigma, \tau \in TN(\mathcal{U})$. Therefore, we can define a map $f : |TN(\mathcal{U})| \to X$ by $f | \sigma = f_\sigma$ for each $\sigma \in TN(\mathcal{U})$. It is easy to verify that $\mathcal{U}$ and $f$ satisfy the conditions (i) and (ii) of Theorem 4, which implies that $X$ is an ANR.

In the above, we may assume that $\text{diam} X < 2^{-1}$. In case $X$ is densely hyper-connected, $\mathcal{W} = \{X\}$, hence we can take $\mathcal{U}_1 = \{X\}$. Then $X$ is an AR by the remark of Theorem 4. \qed

**Remark.** In the definition of densely local hyper-connectedness, if the images of functions $h_n$ are contained in $Y$, then $Y$ is homotopy dense in $X$. In fact, if the images of functions $h_n$ are contained in $Y$, then $f(|TN(\mathcal{U})|) \subset Y$, hence $Y$ is homotopy dense in $X$ by the additional statement of Theorem 4.

For a metric space $X$ and $\eta > 0$, we denote

$$X^n(\eta) = \{(x_1, \ldots, x_n) \in X^n | \text{diam}\{x_1, \ldots, x_n\} < \eta\}.$$ 

A metric space $X$ is said to be **uniformly locally hyper-connected** if there are $\eta > 0$ and

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\(^3\)The local hyper-connectedness is in the sense of [Ca] but not in the sense of [Bo].
functions $h_n : X^n(\eta) \times A^{n-1} \to X$, $n \in \mathbb{N}$, which satisfy the same conditions as (i) and (ii) above, and the following (iii') instead of (iii):

(iii') for each $\varepsilon > 0$, there is $0 < \delta < \varepsilon$ such that

$$\text{diam } h_n(\{x\} \times A^{n-1}) < \varepsilon \text{ for every } n \in \mathbb{N} \text{ and } x \in X^n(\delta).$$

When every $h_n$ is defined on the whole space $X^n \times A^{n-1}$, it is said that $X$ is uniformly hyper-connected.

Now, we give a characterization of uniform ANR’s and uniform AR’s.

**Theorem 5.** A metric space $X = (X, d)$ is a uniform ANR if and only if $X$ is uniformly locally hyper-connected. Moreover, $X$ is a uniform AR if and only if $X$ is uniformly hyper-connected.

**Proof.** First, we see the “only if” part. By Arens–Eells’ embedding theorem \([AE]\) (cf. [To1]), $X$ can be isometrically embedded in a normed linear space $E = (E, \| \cdot \|)$ as a closed set. If $X$ is a uniform ANR, there is a uniform open neighborhood $U$ of $X$ in $E$ with a retraction $r : U \to X$ which is uniformly continuous at $X$. Choose $\eta > 0$ so that $\bigcup_{x \in X} B_E(x, \eta) \subset U$. For each $n \in \mathbb{N}$, we can define a map $h_n : X^n(\eta) \times A^{n-1} \to X$ as follows:

$$h_n(x_1, \ldots, x_n; t_1, \ldots, t_n) = r \left( \sum_{i=1}^{n} t_i x_i \right).$$

It is clear that the maps $h_n$’s satisfy the conditions (i) and (ii). Since the retraction $r$ is uniformly continuous at $X$, for each $\varepsilon > 0$, there is $0 < \delta < \eta$ such that if $x \in X$, $z \in U$ and $\|x - z\| < \delta$ then $d(x, r(z)) < \varepsilon$. For $(x_1, \ldots, x_n) \in X^n(\delta)$ and $(t_1, \ldots, t_n) \in A^{n-1}$, let $z = \sum_{i=1}^{n} t_i x_i \in U$. Since diam$\{x_1, \ldots, x_n\} < \delta$, it follows that $\|x_1 - z\| \leq \sum_{i=1}^{n} t_i \|x_1 - x_i\| < \delta$, which implies that

$$d(x_1, h_n(x_1, \ldots, x_n; t_1, \ldots, t_n)) = d(x_1, r(z)) < \varepsilon.$$

Hence, diam$\{x\} \times A^{n-1} < \varepsilon$ for every $n \in \mathbb{N}$ and $x \in X^n(\delta)$. Thus the condition (iii') is also satisfied. Therefore, $X$ is uniformly locally hyper-connected.

In case $X$ is a uniform AR, since $X^n(\eta)$ can be replaced by $X^n$ in the above, $X$ is uniformly hyper-connected.

Next, to show the “if” part, assume that $X$ is uniformly locally hyper-connected, that is, there are $\eta > 0$ and functions $h_n : X^n(\eta) \times A^{n-1} \to X$, $n \in \mathbb{N}$, which satisfy the conditions (i), (ii) and (iii'). For each $\varepsilon > 0$, we have $\gamma, \delta > 0$ such that diam$\{x\} \times A^{n-1} < \varepsilon/3$ for every $n \in \mathbb{N}$ and $x \in X^n(\gamma)$ and diam$\{x\} \times A^{n-1} < \gamma/2$ for every $n \in \mathbb{N}$ and $x \in X^n(\delta)$. Note that $\delta \leq \gamma/2$ and $\gamma \leq \varepsilon/3$. Let $K$ be a simplicial complex, $L$ a subcomplex of $K$ with $K^{(0)} \subset L$ and $f : |L| \to X$ be a map such that $f(\sigma \cap |L|) < \delta$ for each $\sigma \in K$. Then, by using $h_n$, we can extend $f|K^{(0)}$ to a map $f' : |K| \to X$ such that $f'(\sigma) < \gamma/2$ for each $\sigma \in K$. Each $x \in |L|$ is contained in $\sigma \in L$, whence

$$d(f(x), f'(x)) \leq d(f(x), f(v)) + d(f'(v), f'(x)) < \delta + \gamma/2 < \gamma,$$
where \( v \in \sigma^{(0)} \). By using \( h_1 \), we define a homotopy \( h : [L] \times [0, 1] \to X \) by \( h(x, t) = h_1(f(x), f'(x); t, (1 - t)) \). Then \( h \) is an \( \varepsilon/3 \)-homotopy from \( f \) to \( f' \mid [L] \), that is, \( \text{diam} \; h(\{x\} \times [0, 1]) < \varepsilon/3 \) for each \( x \in [L] \). Since \( X \) is an ANR, we can apply the homotopy extension theorem to extend \( f \) to a map \( \tilde{f} : [K] \to X \) which is \( \varepsilon/3 \)-homotopic to \( f' \). Then \( \text{diam} \; \tilde{f}(\sigma) < \varepsilon \) for each \( \sigma \in K \). In fact, for each \( x, x' \in \sigma \),

\[
d(\tilde{f}(x), \tilde{f}(x')) \leq d(\tilde{f}(x), f'(x)) + d(f'(x), f'(x')) + d(f'(x'), \tilde{f}(x')) < \varepsilon/3 + \gamma/2 + \varepsilon/3 < \varepsilon/2 + \varepsilon/6 < \varepsilon.
\]

By [Mi₂, Theorem 7.1], this means that \( X \) is a uniform ANR.

In case \( X \) is uniformly hyper-connected, since it is an AR and a uniform ANR, \( X \) is a uniform AR by [Mi₂, Proposition 1.3].

The following is a combination of Theorems 2 and 5:

**Corollary 6.** Let \( X \) be a uniformly (locally) hyper-connected metric space and \( Z \) a metric space which contains \( X \) isometrically as a dense subset. Then, \( X \) and \( Z \) are uniform AR's (uniform ANR's) and \( X \) is homotopy dense in \( Z \). In particular, the metric completion \( \hat{X} \) of \( X \) is a uniform AR (uniform ANR) and \( X \) is homotopy dense in \( \hat{X} \).

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