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A semilinear elliptic equation in a thin network-shaped domain

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Abstract. We consider a semilinear elliptic equation in a varying thin domain of $\boldsymbol{R}^n$. This thin domain degenerates into a geometric graph when a certain parameter tends to zero. We determine a limit equation on the graph and we prove that a solution of the PDE converges to a solution of the limit equation. Conversely, when a solution of the limit equation is given, we construct a solution of the PDE approaching a solution of the limit equation.

§1. Introduction.

We consider a situation that a domain $\Omega(\zeta)$ of $\boldsymbol{R}^n$ $(n \geqq 2)$ is a varying thin domain whose size in some directions vanishes when ζ tends to zero. We assume the boundary $\partial\Omega(\zeta)$ is decomposed into two portions $\Sigma(\zeta)$ and $\Gamma(\zeta)$ and the size of $\Omega(\zeta)$ in the normal direction on $\Sigma(\zeta)$ vanishes as $\zeta \to 0$. In this situation, we consider a boundary value problem

$$(1.1) \qquad \begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega(\zeta), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma(\zeta), \\ u = a_\zeta & \text{on } \Gamma(\zeta) \end{cases}$$

where ν denotes the unit outward normal vector on $\partial\Omega(\zeta)$, f is a function on $\boldsymbol{R}$ and a_ζ is a function on $\Gamma(\zeta)$. For some domains, we can determine a limit problem of (1.1) on a low dimensional domain.

Many researchers have studied PDEs on thin domains and associated low dimensional equations. Among them, Yanagida [**8**] has studied the existence of a stable stationary solution of reaction-diffusion equations when an associated one-dimensional equation has a stable stationary solution. Hale and Raugel [**3**] have studied the upper semi-continuity at $\zeta = 0$ of the attractors of reaction-diffusion equations on a thin L-shaped domain of $\boldsymbol{R}^2$. Yanagida [**9**] classified graphs according to stability on non-constant steady states of a reaction-diffusion equation.

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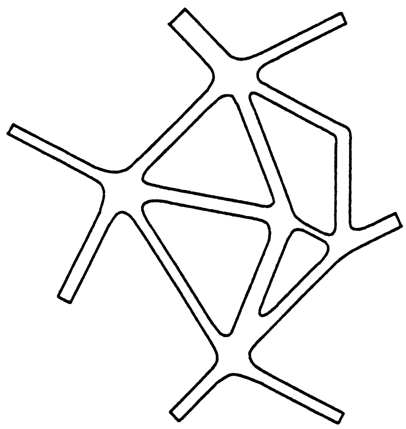

Figure 1

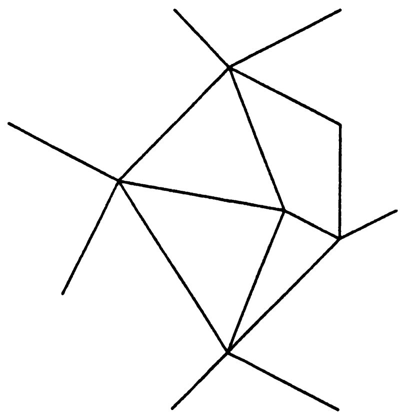

Figure 2

In this paper, we specify some varying thin network-shaped domains (see Figure 1) which approach some geometric graphs (see Figure 2) and we consider (1.1) on such a domain $\Omega(\zeta)$ and an associated equation on a graph $\mathscr{G}$. The first purpose is to prove a solution of (1.1) on $\Omega(\zeta)$ converges uniformly to a solution of the limit equation on $\mathscr{G}$ as ζ tends to zero. The second purpose is, when a solution of the limit equation on $\mathscr{G}$ exists, to prove the existence of a solution of (1.1) on $\Omega(\zeta)$ which approaches it as $\zeta \to 0$.

An outline of this paper is as follows: In §2, we deal with a simple graph $\mathscr{G}$ and a varying thin domain $\Omega(\zeta)$ which degenerates into $\mathscr{G}$ and we describe a result in this special case and prove it. In §3, we deal with a more general graph $\mathscr{G}$ and a network-shaped domain $\Omega(\zeta)$ and describe a similar result to §2 (cf. Theorem 2). In §4, we consider a certain inverse problem of Theorem 2. We prove that if the linearized equation around a solution of the limit equation has no zero eigenvalue, then the PDE has a solution which approaches it.

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§2. A simple case.

We consider a simple graph $\mathscr{G}$ such that several line segments meet one point, that is, $\mathscr{G}$ is a set which consists of a point O and line segments $E_j = \overline{OV_j}$ $(j = 1, \dots, N, N \geqq 2)$ (see Figure 3). To simplify an argument, O is the origin and $l_j > 0$ denotes the length of E_j. Let $x = (x_1, \dots, x_n) = (x_1, x') \in \boldsymbol{R}^n$. We define thin cylinder regions $D_j(\zeta) \subset \boldsymbol{R}^n$ $(j = 1, \dots, N)$ as

$$D_j(\zeta) = \{R_j x : \zeta l \leqq x_1 < l_j, \ |x'| < \zeta d_j\} \quad \text{for } \zeta \in (0, \zeta_*]$$

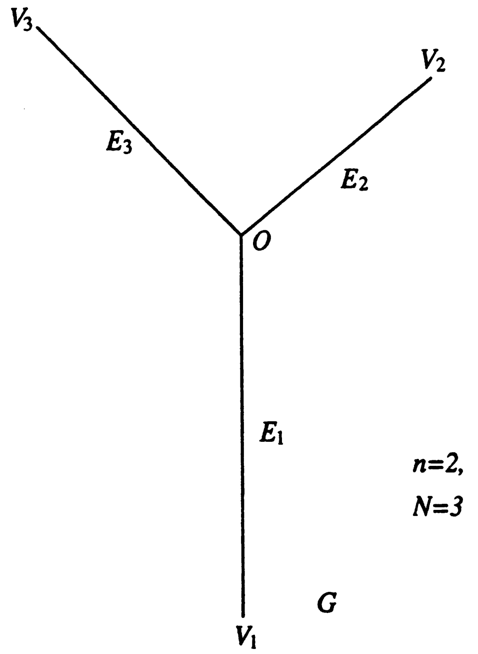

Figure 3

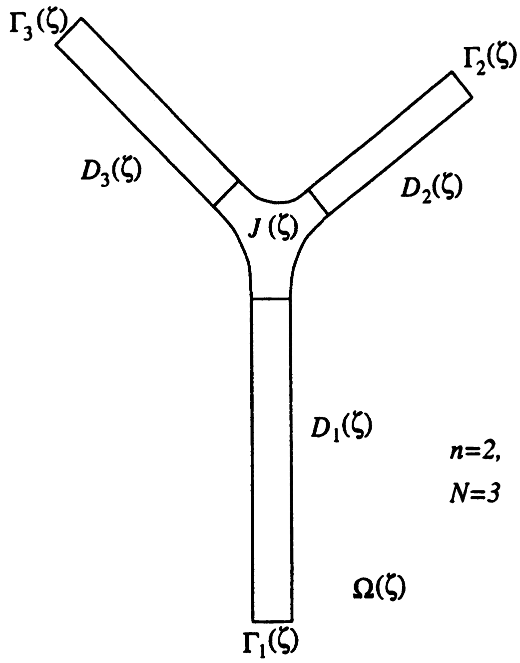

Figure 4

where each d_j is a positive constant and R_j is an orthogonal transformation satisfying $\det R_j = 1$ and $R_j e_1 = l_j^{-1} V_j$ for $e_1 = (1, 0, \dots, 0)$. We take constants $l > 0$ and $\zeta_* > 0$ such that $D_j(\zeta) \neq \emptyset$ and $D_j(\zeta) \cap D_{j'}(\zeta) = \emptyset$ for $j \neq j'$ and $\zeta \in (0, \zeta_*]$. We denote by $\Gamma_j(\zeta)$ a portion of the boundary $\partial D_j(\zeta)$ which approaches V_j and by $\tilde{\Gamma}_j(\zeta)$ a portion which approaches O. Namely,

$$\Gamma_j(\zeta) = \{R_j x : x_1 = l_j, |x'| \leqq \zeta d_j\},$$

$$\tilde{\Gamma}_j(\zeta) = \{R_j x : x_1 = \zeta l, |x'| \leqq \zeta d_j\}.$$

Let J be an open set of $\boldsymbol{R}^n$ which contains O and satisfies $J \cap D_j(\zeta_*) = \emptyset$ and $\partial J \cap \partial D_j(\zeta_*) = \tilde{\Gamma}_j(\zeta_*)$ for $1 \leqq j \leqq N$ and $\partial((\bigcup_{j=1}^N D_j(\zeta_*)) \cup J) \backslash (\bigcup_{j=1}^N \Gamma_j(\zeta_*))$ is C^3 (if $n = 2$, each connected component is C^3). We define a varying region $J(\zeta) \subset \boldsymbol{R}^n$ as

$$J(\zeta) = \{(\zeta/\zeta_*)x : x \in J\} \quad \text{for } \zeta \in (0, \zeta_*].$$

Now, we define a varying domain $\Omega(\zeta) \subset \boldsymbol{R}^n$ $(0 < \zeta \leqq \zeta_*)$ as

$$\Omega(\zeta) = \left(\bigcup_{j=1}^N D_j(\zeta) \right) \cup J(\zeta) \quad \text{for } \zeta \in (0, \zeta_*]$$

(see Figure 4). We remark that $\partial\Omega(\zeta) \backslash (\bigcup_{j=1}^N \Gamma_j(\zeta))$ is C^3 and $\bigcap_{\zeta > 0} \Omega(\zeta) = \mathscr{G}$. We will call such domains as simple network-shaped domains in this paper.

In this situation, we study the convergence of a solution of a boundary value problem

$$(2.1)\qquad \begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega(\zeta), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma(\zeta), \\ u = a_{j,\zeta} & \text{on } \Gamma_j(\zeta) \text{ for } j = 1, \dots, N \end{cases}$$

where $\Sigma(\zeta)$ is a set $\Sigma(\zeta) = \partial\Omega(\zeta) \backslash (\bigcup_{j=1}^{N} \Gamma_j(\zeta))$, f is a real valued function on $\boldsymbol{R}$ such that

$$(2.2)\qquad f \in C^2(\boldsymbol{R}), \quad \limsup_{\xi \to \infty} f(\xi) < 0, \quad \liminf_{\xi \to -\infty} f(\xi) > 0,$$

and each $a_{j,\zeta}$ is a real valued continuous function on $\Gamma_j(\zeta)$ which approaches a certain constant a_j, that is,

$$(2.3)\qquad \lim_{\zeta \to 0} \sup_{\Gamma_j(\zeta)} |a_{j,\zeta}(x) - a_j| = 0 \quad \text{for } j = 1, \dots, N.$$

By the assumption (2.2) and (2.3), we can easily to show a-priori bound of solutions of (2.1) because of Hopf's maximum principle (see Protter and Weinberger [**7**]). By an argument similar to the monotone method (see Sattinger [**10**]), we obtain a solution of (2.1).

Now, we prepare a certain system of ordinary differential equations used in main results. The system of ODEs is

$$(2.4)\qquad \begin{cases} \psi_j''(s) + f(\psi_j(s)) = 0 \quad \text{on } 0 < s < l_j & \text{for } j = 1, \dots, N, \\ \psi_1(0) = \cdots = \psi_N(0), & \\ \displaystyle\sum_{j=1}^{N} d_j^{n-1} \psi_j'(0) = 0, & \\ \psi_j(l_j) = a_j & \text{for } j = 1, \dots, N, \end{cases}$$

where each ψ_j is a function on an interval $[0, l_j]$, the second condition of (2.4) implies that the solution is continuous at O and the third condition implies that the sum of flux vanishes at O (see Yanagida [**9**]).

The equation (2.4) is not a usual 2-points boundary value problem. However, we can prove the existence of solutions by using the Green function. Indeed, applying the maximum principle with the assumption (2.2) and second and third condition of (2.4), we have a-priori bound of solutions of (2.4). From easy calculation, any solution of (2.4) is a fixed point of a map $\mathscr{F} : (\psi_1, \dots, \psi_N) \to (\phi_1, \dots, \phi_N)$ on $C([0, l_1]) \times \cdots \times C([0, l_N])$

$$\phi_j(s) = \sum_{k=1}^{N} \int_0^{l_k} G_{j,k}(s,t) f^*(\psi_k(t))\, dt + w_j(s) \quad (1 \leqq j \leqq N)$$

where $G_{j,k}$ is the Green function

$$G_{j,j}(s,t) = \begin{cases} \dfrac{l_j - s}{l_j a}\left(d_j^{n-1} + \left(a - \dfrac{d_j^{n-1}}{l_j}\right)t\right) & 0 \leqq t \leqq s \leqq l_j, \\ \dfrac{l_j - t}{l_j a}\left(d_j^{n-1} + \left(a - \dfrac{d_j^{n-1}}{l_j}\right)s\right) & 0 \leqq s \leqq t \leqq l_j, \end{cases}$$

$$G_{j,k}(s,t) = \frac{l_j - s}{l_j a} \frac{d_k^{n-1}(l_k - t)}{l_k} \quad 0 \leqq s \leqq l_j, \quad 0 \leqq t \leqq l_k, \quad j \neq k,$$

$$a = \sum_{k=1}^{N} \frac{d_j^{n-1}}{l_j}$$

and w_j is a harmonic function on $[0, l_j]$

$$w_j(s) = \left(a_j - \frac{1}{a}\left(\sum_{k=1}^{N} \frac{d_k^{n-1} a_k}{l_k}\right)\right)\frac{s}{l_j} + \frac{1}{a}\left(\sum_{k=1}^{N} \frac{d_k^{n-1} a_k}{l_k}\right)$$

and f^* is a continuous function

$$f^*(\xi) = \begin{cases} f(\tilde{\xi}) & \xi \geqq \tilde{\xi}, \\ f(\xi) & -\tilde{\xi} \leqq \xi \leqq \tilde{\xi}, \\ f(-\tilde{\xi}) & \xi \leqq -\tilde{\xi}, \end{cases}$$

$$\tilde{\xi} = \max\{|a_j|, |\xi| : f(\xi) = 0\}.$$

It is easy to show that the map $\mathscr{F}$ is a compact map on a certain bounded ball. Therefore, we obtain a solution of (2.4).

Now, we present one of the main results as follows:

Theorem 1. *Suppose that a sequence* $\{\zeta_m\}_{m=1}^{\infty} \subset (0, \zeta_*]$ *satisfies* $\lim_{m\to\infty} \zeta_m = 0$ *and that* u_m *is any solution of* (2.1) *at* $\zeta = \zeta_m$. *Then, there exist a subsequence* $\{\zeta_{m(k)}\}_{k=1}^{\infty} \subset \{\zeta_m\}_{m=1}^{\infty}$ *and a solution* $\psi = (\psi_1, \ldots, \psi_N)$ *of* (2.4) *such that*

$$\begin{cases} \lim\limits_{k\to\infty} \sup\limits_{x \in J(\zeta_{m(k)})} |u_{m(k)}(x) - b(\psi)| = 0, \\ \lim\limits_{k\to\infty} \sup\limits_{x \in D_j(\zeta_{m(k)})} |u_{m(k)}(x) - \psi_j(\pi_1 \circ R_j^{-1} x)| = 0 \quad \textit{for } 1 \leqq j \leqq N \end{cases} \tag{2.5}$$

where π_1 *is the orthogonal projection to the first coordinate* $\pi_1 x = x_1$ *and* $b(\psi)$ *is the value of* ψ *at* O, *that is,* $b(\psi) = \psi_1(0) = \cdots = \psi_N(0)$.

We first prove a proposition which is necessary in the proof. The following proposition is proved by the maximum principle.

Proposition 2.1. *Let* $\Omega \subset \boldsymbol{R}^n$ *be a bounded domain with a piecewise* C^3 *boundary* $\partial\Omega$ *which is decomposed into the sets* Σ *and* Γ, *that is,* $\partial\Omega = \Sigma \cup \Gamma$ *and* $\Sigma \cap \Gamma = \emptyset$. *Let* $\lambda_1 = \lambda_1(\Omega)$ *be the first eigenvalue of the eigenvalue problem*

$$\begin{cases} \Delta\phi + \lambda\phi = 0 & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial\nu} = 0 & \text{on } \Sigma, \\ \phi = 0 & \text{on } \Gamma. \end{cases}$$

Assume $h(x) < \lambda_1$ *in* $\bar{\Omega}$ *and* $u \in C^2(\Omega) \cap C^1(\Omega \cup \Sigma) \cap C^0(\Omega \cup \Gamma)$ *satisfies*

$$\begin{cases} \Delta u + h(x)u \geqq 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial\nu} = 0 & \text{on } \Sigma, \\ u \leqq 0 & \text{on } \Gamma. \end{cases}$$

Then, $u \leqq 0$ *in* $\bar{\Omega}$.

Proof of Proposition 2.1. We take the first eigenfunction $\phi_1 > 0$ in Ω. We define $\delta' \geqq 0$ as

$$\delta' = \inf\{\delta > 0 : u(x) - \delta\phi_1(x) \leqq 0 \text{ in } \Omega \cup \Sigma\}.$$

Suppose u attains its positive maximum $u(x') > 0$ at some points $x' \in \Omega \cup \Sigma$. Then, $u - \delta'\phi_1$ attains its maximum 0 at $x' \in \Omega \cup \Sigma$ and we obtain $\delta' > 0$ and

$$\begin{aligned} &\Delta(u - \delta'\phi_1) + (h(x) - \lambda_1)(u - \delta'\phi_1) \\ &\quad = -\lambda_1(u - \delta'\phi_1) + (\lambda_1 - h(x))\delta'\phi_1 \\ &\quad \geqq 0 \quad \text{in } \Omega. \end{aligned}$$

Applying the maximum principle and E. Hopf's lemma (see Gilbarg and Trudinger [**1**]), we obtain $u(x) - \delta'\phi_1(x) \equiv 0$ in Ω. Therefore u is also the first eigenfunction. This is contrary to the assumption $h(x) < \lambda_1$. We complete the proof of Proposition 2.1. □

Proof of Theorem 1. We take a positive constant

$$c_1 = \max\{\max\{|\xi| : f(\xi) = 0, \xi \in \boldsymbol{R}\}, \max\{|a_j| + 1 : 1 \leqq j \leqq N\}\}.$$

Applying the maximum principle to (2.1) with (2.2) and (2.3), any solution u_m of (2.1) at $\zeta = \zeta_m$ satisfies

$$\sup_{x \in \Omega(\zeta_m)} |u_m(x)| \leqq c_1 \quad \text{for } m \geqq 1.$$

Let $\varepsilon_0 > 0$ be small so that

$$\pi^2 \varepsilon_0^{-2} > \sup_{|\xi| < 2c_1 + 1} |f'(\xi)| \tag{2.6}$$

and let $0 < \delta_1 < \delta_2 < \varepsilon_0$. Without loss of generality, we may take $\zeta_* > 0$ small so that $\zeta_* l < \delta_1$ and $\zeta_* < \delta_2 - \delta_1$.

To see the behavior of u_m on a thin portion $D_j(\zeta)$, we define a cylinder domain $Q(\alpha, \beta, \gamma) \subset \boldsymbol{R}^n$ $(\alpha < \beta,\ \gamma > 0)$ as

$$Q(\alpha, \beta, \gamma) = \{y = (y_1, y') \in \boldsymbol{R}^n : \alpha < y_1 < \beta, |y'| < \gamma\}$$

and we define functions $w_{j,m}$ on $Q(\zeta l, l_j, d_j)$ as

$$w_{j,m}(y) = u_m(R_j(y_1, \zeta y')) \quad (y \in Q(\zeta l, l_j, d_j),\ m \geqq 1, 1 \leqq j \leqq N).$$

To see the behavior of u_m on $J(\zeta)$, we define a portion $J_{\varepsilon_0}(\zeta) \subset \Omega(\zeta)$ which contains $J(\zeta)$ as

$$J_{\varepsilon_0}(\zeta) = \left(\bigcup_{j=1}^{N} \{R_j x : \zeta l \leqq x_1 < \varepsilon_0, |x'| < \zeta d_j\}\right) \cup J(\zeta)$$

(see Figure 5) and we define functions v_m on a fixed domain $J_{\varepsilon_0}(\zeta_*)$ as

$$v_m(y) = u_m((\zeta_m/\zeta_*)y) \quad (y \in J_{\varepsilon_0}(\zeta_*)).$$

We define functions $\psi_{j,m}(s)$ on $[\delta_1, l_j - \delta_1]$ as

$$\psi_{j,m}(s) = \frac{1}{|B_{d_j}^{n-1}|} \int_{|y'| < d_j} w_{j,m}(s, y')\, dy' \quad (\delta_1 \leqq s \leqq l_j - \delta_1)$$

for $m \geqq 1$ and $1 \leqq j \leqq N$ where $B_{d_j}^{n-1}$ is an $n-1$ dimensional ball of a radius d_j and $|B_{d_j}^{n-1}|$ is its $n-1$ dimensional measure.

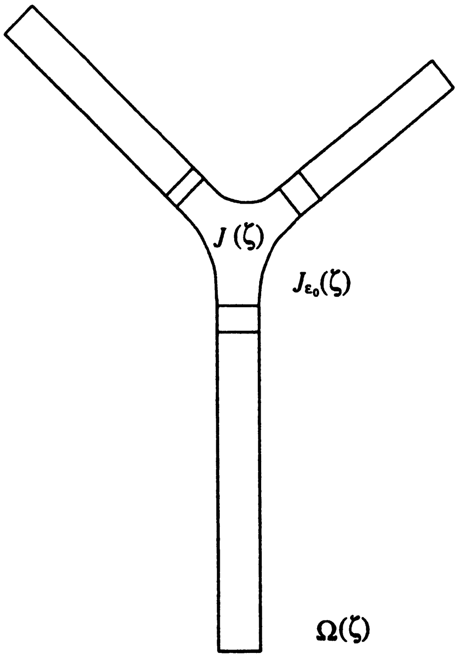

Figure 5

It is easy to see that the function v_m satisfies

$$\text{(2.7)} \qquad \begin{cases} \Delta_y v_m + (\zeta_m/\zeta^*)^2 f(v_m) = 0 & \text{in } J_{\varepsilon_0}(\zeta_*), \\ \dfrac{\partial v_m}{\partial \nu} = 0 & \text{on } \partial\Omega(\zeta_*) \cap \overline{J_{\varepsilon_0}(\zeta_*)} \end{cases}$$

for $m \geqq 1$ and the function $w_{j,m}$ satisfies

$$\text{(2.8)} \qquad \begin{cases} P_{\zeta_m} w_{j,m} + f(w_{j,m}) = 0 & \text{in } Q(\zeta l, l_j, d_j), \\ \dfrac{\partial w_{j,m}}{\partial \nu} = 0 & \text{on } \partial Q(\zeta l, l_j, d_j) \cap \partial Q(-\infty, \infty, d_j) \end{cases}$$

for $m \geqq 1$ and $1 \leqq j \leqq N$ where Δ_y is the Laplacian

$$\Delta_y = \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2}$$

and P_ζ denotes the differential operator

$$P_\zeta = \frac{\partial^2}{\partial y_1^2} + \frac{1}{\zeta^2} \sum_{i=2}^{n} \frac{\partial^2}{\partial y_i^2}.$$

In this situation, Jimbo [**4**] has proved that the partial derivative $\partial w_{j,\zeta}/\partial y_1$ of the solution of (2.8) is bounded in such a cylinder domain and that the restriction of $\partial w_{j,\zeta}/\partial y_i$ to a certain portion of the boundary is bounded.

Lemma 2.2. There exists a constant $c_2 = c_2(c_1, \delta_1) > 0$ such that

$$\left|\frac{\partial w_{j,m}}{\partial y_1}(y)\right| \leqq c_2 \quad \text{for } y \in \overline{Q(\delta_1, l_j - \delta_1, d_j)},$$

$$\sum_{i=2}^{n}\left|\frac{\partial w_{j,m}}{\partial y_i}(y)\right|^2 \leqq c_2\zeta_m^4 \quad \text{for } y \in \partial Q(\delta_1, l_j - \delta_1, d_j) \cap \partial Q(-\infty, \infty, d_j)$$

for $m \geqq 1$ and $1 \leqq j \leqq N$.

We omit the proof (see Jimbo [**4**; Lemma 3.7, 3.8]).

Lemma 2.3. *There exists a constant* $c_3 > 0$ *such that*

$$\int_{Q(\delta_1, l_j - \delta_1, d_j)} |w_{j,m}(y) - \psi_{j,m}(y_1)|^2\, dy \leqq c_3\zeta_m^2 \quad (m \geqq 1, 1 \leqq j \leqq N).$$

Proof of Lemma 2.3. From the Poincaré inequality, there exists a constant $c_4 > 0$ such that

$$\int_{|y'|<d_j} |w_{j,m}(y_1, y') - \psi_{j,m}(y_1)|^2\, dy'$$

$$= \int_{|y'|<d_j} \left| w_{j,m}(y_1, y') - \frac{1}{|B_{d_j}^{n-1}|}\int_{|x'|<d_j} w_{j,m}(y_1, x')\, dx'\right|^2 dy'$$

$$\leqq c_4 \int_{|y'|<d_j} |\nabla_{y'} w_{j,m}(y_1, y')|^2\, dy' \quad (\delta_1 \leqq y_1 \leqq l_j - \delta_1).$$

From (2.8) and Lemma 2.2, we have

$$\int_{Q(\delta_1, l_j - \delta_1, d_1)} |\nabla_{y'} w_{j,m}(y_1, y')|^2\, dy = -\int_{Q(\delta_1, l_j - \delta_1, d_1)} w_{j,m}(y) \Delta_{y'} w_{j,m}(y)\, dy$$

$$= \zeta_m^2 \int_{Q(\delta_1, l_j - \delta_1, d_1)} w_{j,m}(y) \frac{\partial^2 w_{j,m}}{\partial y_1^2}(y)\, dy$$

$$+ \zeta_m^2 \int_{Q(\delta_1, l_j - \delta_1, d_1)} w_{j,m}(y) f(w_{j,m}(y))\, dy$$

$$= \zeta_m^2 \int_{|y'|<d_j} w_{j,m}(l_j - \delta_1, y') \frac{\partial w_{j,m}}{\partial y_1}(l_j - \delta_1, y')\, dy'$$

$$- \zeta_m^2 \int_{|y'|<d_j} w_{j,m}(\delta_1, y') \frac{\partial w_{j,m}}{\partial y_1}(\delta_1, y')\, dy'$$

$$- \zeta_m^2 \int_{Q(\delta_1, l_j - \delta_1, d_1)} \left(\frac{\partial w_{j,m}}{\partial y_1}(y)\right)^2 dy$$

$$+ \zeta_m^2 \int_{Q(\delta_1, l_j - \delta_1, d_1)} w_{j,m}(y) f(w_{j,m}(y))\, dy$$

$$\leqq \zeta_m^2 |B_{d_j}^{n-1}| \left\{ 2c_1c_2 + l_j c_2^2 + l_j c_1 \sup_{|\xi|<c_1} |f(\xi)| \right\}.$$

Thus, we complete the proof of Lemma 2.3. □

LEMMA 2.4. *There exist a subsequence* $\{m(k)\}_{k=1}^{\infty}$, $\psi_{j,\infty} \in C^0([\delta_1, l_j - \delta_1])$ $(1 \leqq j \leqq N)$ *and a constant* b *such that*

$$\lim_{k\to\infty} \sup_{y \in J(\zeta_*)} |v_{m(k)}(y) - b| = 0, \tag{2.9}$$

$$\lim_{k\to\infty} \sup_{\delta_1 \leqq s \leqq l_j - \delta_1} |\psi_{j,m(k)}(s) - \psi_{j,\infty}(s)| = 0, \tag{2.10}$$

$$\lim_{k\to\infty} \sup_{y \in \partial Q(\delta_2, l_j - \delta_2, d_j)} |w_{j,m(k)}(y) - \psi_{j,\infty}(y_1)| = 0 \tag{2.11}$$

for $1 \leqq j \leqq N$.

PROOF OF LEMMA 2.4. For $j = 1, \ldots, N$, we define a pair of functions $w_{j,m}^{\mathrm{s}}$ and $w_{j,m}^{\mathrm{e}}$ as

$$w_{j,m}^{\mathrm{s}}(y) = w_{j,m}(\zeta_m y_1 + \delta_2, y') \quad (y \in Q(-1, 1, d_j)),$$

$$w_{j,m}^{\mathrm{e}}(y) = w_{j,m}(\zeta_m y_1 + l_j - \delta_2, y') \quad (y \in Q(-1, 1, d_j)).$$

It is easy to see that $w_{j,m}^{\mathrm{s}}$ and $w_{j,m}^{\mathrm{e}}$ satisfy an equation

$$\begin{cases} \Delta_y w + \zeta_m^2 f(w) = 0 & \text{in } Q(-1, 1, d_j), \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on } \partial Q(-1, 1, d_j) \cap \partial Q(-\infty, \infty, d_j) \end{cases} \tag{2.12}$$

for $m \geqq 1$ and $1 \leqq j \leqq N$.

Applying the Schauder interior estimates and boundary estimates (see Gilbarg and Trudinger [**1**]) to (2.7) and (2.12), there exists a constant $c_5 > 0$ such that $\|v_m\|_{C^2(\overline{J(\zeta_*)})} \leqq c_5$, $\|w^{\mathrm{s}}_{j,m}\|_{C^2(\overline{Q(-1/2,1/2,d_j)})} \leqq c_5$, $\|w^{\mathrm{e}}_{j,m}\|_{C^2(\overline{Q(-1/2,1/2,d_j)})} \leqq c_5$ for $m \geqq 1$ and $1 \leqq j \leqq N$. We have also $\|\psi_{j,m}\|_{C^1([\delta_1, l_j - \delta_1])} \leqq c_1 + c_2$ and $\|w_{j,m}\|_{C^1(\partial Q(\delta_2, l_j - \delta_2, d_j) \cap \partial Q(-\infty,\infty,d_j))} \leqq c_1 + c_2 + ((n-1)c_2)^{1/2}\zeta_*$ for $1 \leqq j \leqq N$ by Lemma 2.2. From the Ascoli-Arzelà theorem, there exist a subsequence $\{m(k)\}_{k=1}^{\infty}$ and functions

$$v_\infty \in C^1(\overline{J(\zeta_*)}), \quad w^{\mathrm{s}}_{j,\infty}, w^{\mathrm{e}}_{j,\infty} \in C^1(\overline{Q(-1/2,1/2,d_j)}),$$

$$w_{j,\infty} \in C^0(\partial Q(\delta_2, l_j - \delta_2, d_j) \cap \partial Q(-\infty,\infty,d_j)),$$

$$\psi_{j,\infty} \in C^0([\delta_1, l_j - \delta_1]),$$

such that

$$\lim_{k\to\infty} \|v_{m(k)} - v_\infty\|_{C^1(\overline{J(\zeta_*)})} = 0,$$

$$\lim_{k\to\infty} \|w^{\mathrm{s}}_{j,m(k)} - w^{\mathrm{s}}_{j,\infty}\|_{C^1(\overline{Q(-1/2,1/2,d_j)})} = 0,$$

$$\lim_{k\to\infty} \|w^{\mathrm{e}}_{j,m(k)} - w^{\mathrm{e}}_{j,\infty}\|_{C^1(\overline{Q(-1/2,1/2,d_j)})} = 0,$$

$$\lim_{k\to\infty} \|w_{j,m(k)} - w_{j,\infty}\|_{C^0(\partial Q(\delta_2,l_j-\delta_2,d_j)\cap\partial Q(-\infty,\infty,d_j))} = 0,$$

$$\lim_{k\to\infty} \|\psi_{j,m(k)} - \psi_{j,\infty}\|_{C^0([\delta_1,l_j-\delta_1])} = 0,$$

for $1 \leqq j \leqq N$. Thus, we obtain (2.10).

By the definition of v_m and $w_{j,m}$ and Lemma 2.2, we have

$$\begin{aligned}
\int_{J(\zeta_*)} |\nabla_y v_m(y)|^2\, dy &= \frac{\zeta_*^{n-2}}{\zeta_m^{n-2}} \int_{J(\zeta_m)} |\nabla_x u_m(x)|^2\, dx \\
&\leqq \frac{\zeta_*^{n-2}}{\zeta_m^{n-2}} \int_{J_{\varepsilon_0}(\zeta_m)} |\nabla_x u_m(x)|^2\, dx \\
&= \frac{\zeta_*^{n-2}}{\zeta_m^{n-2}} \left\{ \int_{\partial J_{\varepsilon_0}(\zeta_m)} u_m(x) \frac{\partial u_m}{\partial \nu}(x)\, ds_x + \int_{J_{\varepsilon_0}(\zeta_m)} u_m(x) f(u_m(x))\, dx \right\} \\
&\leqq \zeta_m \zeta_*^{n-2} c_1 c_2 \sum_{j=1}^{N} |B_{d_j}^{n-1}| + \frac{|J_{\varepsilon_0}(\zeta_m)|}{\zeta_m^{n-2}} \zeta_*^{n-2} c_1 \sup_{|\xi|<c_1} |f(\xi)| \\
&\to 0 \quad (m \to \infty).
\end{aligned}$$

Thus, $|\nabla_y v_\infty| = 0$ in $\overline{J(\zeta_*)}$ and we obtain (2.9).

To prove (2.11), we show $w^{\mathrm{s}}_{j,\infty} \equiv \psi_{j,\infty}(\delta_2)$, $w^{\mathrm{e}}_{j,\infty} \equiv \psi_{j,\infty}(l_j - \delta_2)$ and $w_{j,\infty}(y) = \psi_{j,\infty}(y_1)$ on $y = (y_1, y') \in \partial Q(\delta_1, l_j - \delta_1, d_j) \cap \partial Q(-\infty, \infty, d_j)$ for $1 \leqq j \leqq N$. By a similar argument to the proof of that v_∞ is a constant function, $w^{\mathrm{s}}_{j,\infty}$ and $w^{\mathrm{e}}_{j,\infty}$ are constant functions. Thus, we have

$$\begin{aligned} &|\psi_{j,\infty}(\delta_2) - w^{\mathrm{s}}_{j,\infty}| \\ &\leqq |\psi_{j,\infty}(\delta_2) - \psi_{j,m(k)}(\delta_2)| + \frac{1}{|B^{n-1}_{d_j}|}\int_{|y'|<d_j} |w^{\mathrm{s}}_{j,m(k)}(0, y') - w^{\mathrm{s}}_{j,\infty}|\, dy' \\ &\to 0 \quad (k \to \infty) \end{aligned}$$

by the definition of $w^{\mathrm{s}}_{j,m}$ and (2.10) and we obtain $w^{\mathrm{s}}_{j,\infty} \equiv \psi_{j,\infty}(\delta_2)$. In a similar way, we obtain $w^{\mathrm{e}}_{j,\infty} \equiv \psi_{j,\infty}(l_j - \delta_2)$.

From Lemma 2.2 and (2.8), we have

$$\begin{aligned} &\int_{Q(\delta_1, l_j-\delta_1, d_j)} |\nabla_y(w_{j,m}(y) - \psi_{j,m}(y_1))|^2\, dy \\ &= \int_{Q(\delta_1, l_j-\delta_1, d_j)} \left(\frac{\partial w_{j,m}}{\partial y_1}(y) - \psi'_{j,m}(y_1)\right)^2 dy \\ &\quad + \int_{Q(\delta_1, l_j-\delta_1, d_j)} |\nabla_{y'} w_{j,m}(y_1, y')|^2\, dy \\ &\leqq |B^{n-1}_{d_j}| \left\{ 2c_2^2 l_j + \zeta_m^2 (2c_1c_2 + l_j c_2^2 + l_j c_1 \sup_{|\xi|<c_1} |f(\xi)|) \right\}. \end{aligned}$$

Applying the trace theorem with Lemma 2.3 and (2.10), we obtain

$$\int_{\partial Q(\delta_1, l_j-\delta_1, d_j)\cap \partial Q(-\infty, \infty, d_j)} |w_{j,\infty}(y) - \psi_{j,\infty}(y_1)|^2\, ds_y = 0 \quad (1 \leqq j \leqq N).$$

Thus, we obtain (2.11). $\square$

Lemma 2.5. *Functions $\psi_{j,m}$ and $\psi_{j,\infty}$ satisfy*

$$\psi''_{j,m}(s) + \frac{1}{|B^{n-1}_{d_j}|}\int_{|y'|<d_j} f(w_{j,m}(s, y'))\, dy' = 0 \tag{2.13}$$

$$(\delta_1 < s < l_j - \delta_1,\ \ m \geqq 1),$$

$$\psi''_{j,\infty}(s) + f(\psi_{j,\infty}(s)) = 0 \quad (\delta_1 < s < l_j - \delta_1), \tag{2.14}$$

$$\lim_{k\to\infty} \sup_{\delta_1 \leqq s \leqq l_j - \delta_1} |\psi'_{j,m(k)}(s) - \psi'_{j,\infty}(s)| = 0 \tag{2.15}$$

for $1 \leqq j \leqq N$.

Proof of Lemma 2.5. We take an arbitrary $\phi(s) \in C_0^\infty((\delta_1, l_j - \delta_1))$. Then, we have

$$\begin{aligned} 0 &= \int_{Q(\delta_1, l_j - \delta_1, d_j)} \{P_\zeta w_{j,m}(y) + f(w_{j,m}(y))\}\phi(y_1)\,dy \\ &= \int_{\delta_1}^{l_j-\delta_1} \left\{ |B_{d_j}^{n-1}|\psi_{j,m}(y_1)\phi''(y_1) + \int_{|y'|<d_j} f(w_{j,m}(y_1, y'))\,dy'\phi(y_1) \right\} dy_1 \end{aligned}$$

by the equation (2.8). Thus, we obtain (2.13).

By the above equation, Lemma 2.3 and (2.10), applying Schwarz's inequality, we have

$$\begin{aligned} &\left| |B_{d_j}^{n-1}| \int_{\delta_1}^{l_j-\delta_1} \{\psi_{j,\infty}(y_1)\phi''(y_1) + f(\psi_{j,\infty}(y_1))\phi(y_1)\}\,dy_1 \right| \\ &= \left| \int_{Q(\delta_1, l_j-\delta_1, d_j)} \{\psi_{j,\infty}(y_1)\phi''(y_1) + f(\psi_{j,\infty}(y_1))\phi(y_1)\}\,dy \right. \\ &\qquad \left. - \int_{Q(\delta_1, l_j-\delta_1, d_j)} (P_\zeta w_{j,m}(y) + f(w_{j,m}(y)))\phi(y_1)\,dy \right| \\ &= \left| \int_{Q(\delta_1, l_j-\delta_1, d_j)} \{\psi_{j,\infty}(y_1) - w_{j,m(k)}(y)\}\phi''(y_1)\,dy \right. \\ &\qquad \left. + \int_{Q(\delta_1, l_j-\delta_1, d_j)} \{f(\psi_{j,\infty}(y_1)) - f(w_{j,m(k)}(y))\}\phi(y_1)\,dy \right| \\ &\leqq \left(\|\phi''\|_{L^\infty} + \|\phi\|_{L^\infty} \sup_{|\xi| \leqq 2c_1} |f'(\xi)| \right) \cdot \\ &\qquad \{l_j|B_{d_j}^{n-1}|\, \|\psi_{j,\infty} - \psi_{j,m(k)}(y_1)\|_{C([\delta_1, l_j-\delta_1])} \\ &\qquad + (l_j|B_{d_j}^{n-1}|)^{1/2} \|\psi_{j,m(k)} \circ \pi_1 - w_{j,m(k)}\|_{L^2(Q(\delta_1, l_j-\delta_1, d_j))}\} \\ &\to 0 \quad (k \to \infty) \end{aligned}$$

where $\psi_{j,m(k)} \circ \pi_1$ denotes a composite function $\psi_{j,m(k)} \circ \pi_1(y) = \psi_{j,m(k)}(y_1)$. Thus, we obtain (2.14).

We have $\psi_{j,m(k)}(\delta_1) \to \psi_{j,\infty}(\delta_1)$ and $\psi_{j,m(k)}(l_j - \delta_1) \to \psi_{j,\infty}(l_j - \delta_1)$ as $k \to \infty$ for $1 \leqq j \leqq N$ by (2.10). Thus, from (2.13) and (2.14), we obtain (2.15). □

Lemma 2.6.

$$\lim_{k\to\infty} \sup_{y \in Q(\delta_2, l_j - \delta_2, d_j)} |w_{j,m(k)}(y) - \psi_{j,\infty}(y_1)| = 0 \quad \text{for } 1 \leqq j \leqq N.$$

Proof of Lemma 2.6. For $j = 1, \dots, N$, we define a pair of comparison functions $\Theta_{j,m}^{+}$ and $\Theta_{j,m}^{-}$ as

$$\Theta_{j,m}^{\pm}(y) = \psi_{j,\infty}(y_1) \pm \frac{1}{n-1} \sup_{|\xi| \leqq c_1} |f(\xi)|(d_j^2 - |y'|^2)\zeta_m^2$$
$$\pm \sup_{y \in \partial Q(\delta_2, l_j - \delta_2, d_j)} |w_{j,m}(y) - \psi_{j,\infty}(y_1)| \quad (y \in Q(\delta_2, l_j - \delta_2, d_j)).$$

Then, we have

$$\begin{cases} P_{\zeta_{m(k)}}(w_{j,m(k)} - \Theta_{j,m(k)}^{+})(y) \\ \qquad = -f(w_{j,m(k)}(y)) + f(\psi_{j,\infty}(y_1)) + 2 \sup\limits_{|\xi| \leqq c_1} |f(\xi)| \\ \qquad \geqq 0 \quad \text{in } Q(\delta_2, l_j - \delta_2, d_j), \\ w_{j,m(k)} - \Theta_{j,m(k)}^{+} \leqq 0 \quad \text{on } \partial Q(\delta_2, l_j - \delta_2, d_j). \end{cases}$$

Applying the maximum principle, we have $w_{j,m(k)} \leqq \Theta_{j,m(k)}^{+}$ in $Q(\delta_2, l_j - \delta_2, d_j)$. In a similar way, we have $\Theta_{j,m(k)}^{-} \leqq w_{j,m(k)}$ in $Q(\delta_2, l_j - \delta_2, d_j)$. Thus, we obtain Lemma 2.6. □

In order to see the asymptotic behavior of $w_{j,m(k)}$ in $Q(\zeta_{m(k)} l, \delta_2, d_j) \cup Q(l_j - \delta_2, l_j, d_j)$, we define functions $\psi_{j,m(k)}^{s,+}$ and $\psi_{j,m(k)}^{s,-}$ on an interval $[\zeta_{m(k)} l, \varepsilon_0]$ and $\psi_{j,m(k)}^{e,+}$ and $\psi_{j,m(k)}^{e,-}$ on an interval $[l_j - \varepsilon_0, l_j]$ $(1 \leqq j \leqq N)$ as follows:

Each of $\psi_{j,m(k)}^{s,+}$ and $\psi_{j,m(k)}^{s,-}$ is a unique solution of

$$\begin{cases} (\psi_{j,m(k)}^{s,\pm})''(s) + f(\psi_{j,m(k)}^{s,\pm}(s)) = 0 \quad (\zeta_{m(k)} l < s < \varepsilon_0), \\ \psi_{j,m(k)}^{s,\pm}(\zeta_{m(k)} l) = b \pm \sup\limits_{x \in J(\zeta_{m(k)})} |u_{m(k)}(x) - b|, \\ \psi_{j,m(k)}^{s,\pm}(\varepsilon_0) = \psi_{j,\infty}(\varepsilon_0) \pm \sup\limits_{y \in Q(\delta_2, l_j - \delta_2, d_j)} |w_{j,m(k)}(y) - \psi_{j,\infty}(y_1)|, \end{cases}$$

respectively, and each of $\psi_{j,m(k)}^{e,+}$ and $\psi_{j,m(k)}^{e,-}$ is also a unique solution of

$$\begin{cases} (\psi^{\mathrm{e},\pm}_{j,m(k)})''(s) + f(\psi^{\mathrm{e},\pm}_{j,m(k)}(s)) = 0 \quad (l_j - \varepsilon_0 < s < l_j), \\ \psi^{\mathrm{e},\pm}_{j,m(k)}(l_j - \varepsilon_0) = \psi_{j,\infty}(l_j - \varepsilon_0) \pm \sup_{y \in Q(\delta_2, l_j - \delta_2, d_j)} |w_{j,m(k)}(y) - \psi_{j,\infty}(y_1)|, \\ \psi^{\mathrm{e},\pm}_{j,m(k)}(l_j) = a_j \pm \sup_{x \in \Gamma_j(\zeta_{m(k)})} |a_{j,\zeta_{m(k)}}(x) - a_j|, \end{cases}$$

respectively. It comes from (2.2) and (2.6) that each equation has a unique solution. Then, we obtain

$$\begin{cases} \psi^{\mathrm{s},-}_{j,m(k)}(y_1) \leqq w_{j,m(k)}(y) \leqq \psi^{\mathrm{s},+}_{j,m(k)}(y_1) \quad (y \in Q(\zeta_{m(k)} l, \varepsilon_0, d_j)), \\ \psi^{\mathrm{e},-}_{j,m(k)}(y_1) \leqq w_{j,m(k)}(y) \leqq \psi^{\mathrm{e},+}_{j,m(k)}(y_1) \quad (y \in Q(l_j - \varepsilon_0, l_j, d_j)). \end{cases} \tag{2.16}$$

Indeed, we can see the function $w(x) = w_{j,m(k)}(x_1, {\zeta_{m(k)}}^{-1} x') - \psi^{\mathrm{s},+}_{j,m(k)}(x_1)$ satisfies

$$\begin{cases} \Delta w(x) + h(x) w(x) = 0 \quad \text{in } Q(\zeta_{m(k)} l, \varepsilon_0, \zeta_{m(k)} d_j), \\ \dfrac{\partial w}{\partial \nu}(x) = 0 \qquad \text{on } \partial Q(\zeta_{m(k)} l, \varepsilon_0, \zeta_{m(k)} d_j) \backslash (\{x_1 = \zeta_{m(k)} l\} \cup \{x_1 = \varepsilon_0\}), \\ w(x) \leqq 0 \qquad \text{on } \partial Q(\zeta_{m(k)} l, \varepsilon_0, \zeta_{m(k)} d_j) \cap (\{x_1 = \zeta_{m(k)} l\} \cup \{x_1 = \varepsilon_0\}) \end{cases}$$

where h is a function $h(x) = \int_0^1 f'(t w_{j,m(k)}(x_1, \zeta_{m(k)}^{-1} x') + (1-t) \psi^{\mathrm{s},+}_{j,m(k)}(x_1))\, dt$. Let λ_1 be the first eigenvalue of the following eigenvalue problem

$$\begin{cases} \Delta \phi + \lambda \phi = 0 \quad \text{in } Q(\zeta_{m(k)} l, \varepsilon_0, \zeta_{m(k)} d_j), \\ \dfrac{\partial \phi}{\partial \nu} = 0 \qquad \text{on } \partial Q(\zeta_{m(k)} l, \varepsilon_0, \zeta_{m(k)} d_j) \backslash (\{x_1 = \zeta_{m(k)} l\} \cup \{x_1 = \varepsilon_0\}), \\ \phi = 0 \qquad \text{on } \partial Q(\zeta_{m(k)} l, \varepsilon_0, \zeta_{m(k)} d_j) \cap (\{x_1 = \zeta_{m(k)} l\} \cup \{x_1 = \varepsilon_0\}). \end{cases}$$

We have $\lambda_1 = \pi^2 (\varepsilon_0 - \zeta_{m(k)} l)^{-2} > h(x)$ by (2.6). Applying Proposition 2.1, we obtain $w(x) \leqq 0$ in $Q(\zeta_{m(k)} l, \varepsilon_0, \zeta_{m(k)} d_j)$. In a similar way, we obtain (2.16).

Let $\psi^{\mathrm{s}}_{j,\infty} = \lim_{k \to \infty} \psi^{\mathrm{s},\pm}_{j,m(k)}$ and $\psi^{\mathrm{e}}_{j,\infty} = \lim_{k \to \infty} \psi^{\mathrm{e},\pm}_{j,m(k)}$ for $1 \leqq j \leqq N$ and we define functions $\psi_j(s)$ on $[0, l_j]$ $(j = 1, \ldots, N)$ as

$$\psi_j(s) = \begin{cases} \psi^{\mathrm{s}}_{j,\infty}(s) \quad (0 \leqq s \leqq \varepsilon_0), \\ \psi_{j,\infty}(s) \quad (\varepsilon_0 < s < l_j - \varepsilon_0), \\ \psi^{\mathrm{e}}_{j,\infty}(s) \quad (l_j - \varepsilon_0 \leqq s \leqq l_j). \end{cases}$$

Because of $\psi^{\mathrm{s}}_{j,\infty}(s) = \psi_{j,\infty}(s)$ on $\delta_2 < s < \varepsilon_0$ and $\psi_{j,\infty}(s) = \psi^{\mathrm{e}}_{j,\infty}(s)$ on $l_j - \varepsilon_0 < s < l_j - \delta_2$, $(\psi_1, \ldots, \psi_n)$ satisfies (2.5) and (2.4) except the compatibility condition

$$\sum_{j=1}^{N} d_j^{n-1}\psi_j'(0) = 0. \tag{2.17}$$

Therefore, we check the above condition.

By the definition of $w_{j,m}$ and (2.15), we get

$$\lim_{k\to\infty} \zeta_{m(k)}^{1-n} \int_{J_{\varepsilon_0}(\zeta_{m(k)})} \Delta_x u_{m(k)}(x)\,dx = \lim_{k\to\infty} \sum_{j=1}^{N} \int_{|y'|<d_j} \frac{\partial w_{j,m(k)}}{\partial y_1}(\varepsilon_0, y')\,dy'$$
$$= \sum_{j=1}^{N} |B_{d_j}^{n-1}|\psi_j'(\varepsilon_0).$$

On the other hand, by Lemma 2.6 and (2.16), we have

$$\lim_{k\to\infty} \zeta_{m(k)}^{1-n} \int_{J_{\varepsilon_0}(\zeta_{m(k)})} f(u_{m(k)}(x))\,dx = \sum_{j=1}^{N} |B_{d_j}^{n-1}| \int_0^{\varepsilon_0} f(\psi_j(s))\,ds.$$

Since (2.1) and each ψ_j satisfies $\psi_j''(s) + f(\psi_j(s)) = 0$ on $0 < s < l_j$, we obtain

$$\sum_{j=1}^{N} |B_{d_j}^{n-1}|\psi_j'(\varepsilon_0) = -\sum_{j=1}^{N} |B_{d_j}^{n-1}| \int_0^{\varepsilon_0} f(\psi_j(s))\,ds$$
$$= \sum_{j=1}^{N} |B_{d_j}^{n-1}| \int_0^{\varepsilon_0} \psi_j''(s)\,ds$$
$$= \sum_{j=1}^{N} |B_{d_j}^{n-1}| \{\psi_j'(\varepsilon_0) - \psi_j'(0)\}.$$

Thus, we obtain (2.17) and we complete the proof of Theorem 1. □

§3. Network-shaped domains.

In this section, we consider a more general network-shaped domain $\Omega(\zeta)$ for $\zeta \in (0, \zeta_*]$. We assume $\Omega(\zeta)$ is a union of simple network-shaped domains $\Omega_i(\zeta)$ $(i = 1, \dots, N')$ defined in §2 (see Figure 6). Namely, we assume

$$\Omega(\zeta) = \bigcup_{i=1}^{N'} \Omega_i(\zeta)$$

where each $\Omega_i(\zeta)$ satisfies the following:

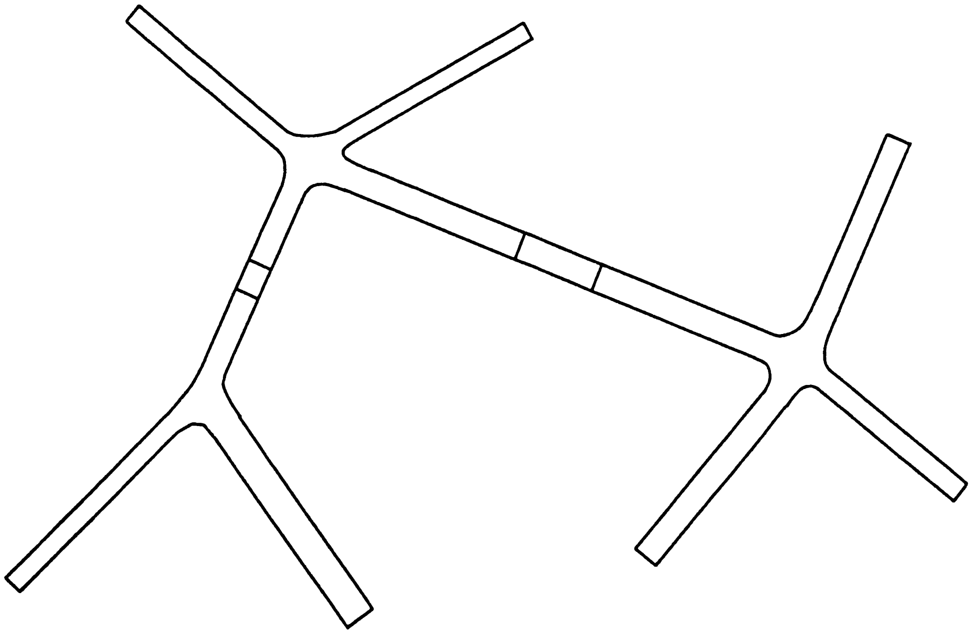

Figure 6

$\Omega_i(\zeta)$ is a union of a junction region $J_i(\zeta)$ and thin cylinder regions $D_{i,q}(\zeta)$ $(q = 1, \dots, N_i)$, that is,

$$\Omega_i(\zeta) = \left(\bigcup_{q=1}^{N_i} D_{i,q}(\zeta) \right) \cup J_i(\zeta)$$

and each $\bigcap_{\zeta>0} \Omega_i(\zeta)$ is a union of straight line segments which meet one point. If the intersection of $\Omega_i(\zeta)$ and $\Omega_{i'}(\zeta)$ $(i \neq i')$ is not empty, then there is a pair of thin cylinder regions $D_{i,q}(\zeta)$ and $D_{i',q'}(\zeta)$ such that $\Omega_i(\zeta) \cap \Omega_{i'}(\zeta) = D_{i,q}(\zeta) \cap D_{i',q'}(\zeta)$ and that $D_{i,q}(\zeta) \cup D_{i',q'}(\zeta)$ is a cylinder region for any $0 < \zeta < \zeta_*$.

Let $N > 0$ be the number of connected components of $\bigcup_{i,p} D_{i,p}(\zeta)$. We denote by $D_j(\zeta)$ one of the connected components of $\bigcup_{i,p} D_{i,p}(\zeta)$ for $1 \leqq j \leqq N$ and we denote by ζd_j the radius of the circular cross section of each cylinder region $D_j(\zeta)$. We remark $\Omega(\zeta)$ is represented by

$$\Omega(\zeta) = \left(\bigcup_{i=1}^{N'} J_i(\zeta) \right) \cup \left(\bigcup_{j=1}^{N} D_j(\zeta) \right).$$

We denote by $\mathscr{G}$ the geometric graph $\bigcap_{\zeta>0} \Omega(\zeta)$. Let V_i be a point $\bigcap_{\zeta>0} J_i(\zeta)$ $(i = 1, \dots, N')$ or a extreme point of $\mathscr{G}$ $(i = N' + 1, \dots, N'')$. Let E_j be a line segment $\bigcap_{\zeta>0} D_j(\zeta)$ $(j = 1, \dots, N)$. We remark $\mathscr{G}$ is a union of V_i $(i = 1, \dots, N'')$ and E_j $(j = 1, \dots, N)$. We assume each E_j has its direction and we denote by l_j its length. We denote by $\imath(j)$ and $\kappa(j)$ numbers of the startpoint and the endpoint of E_j

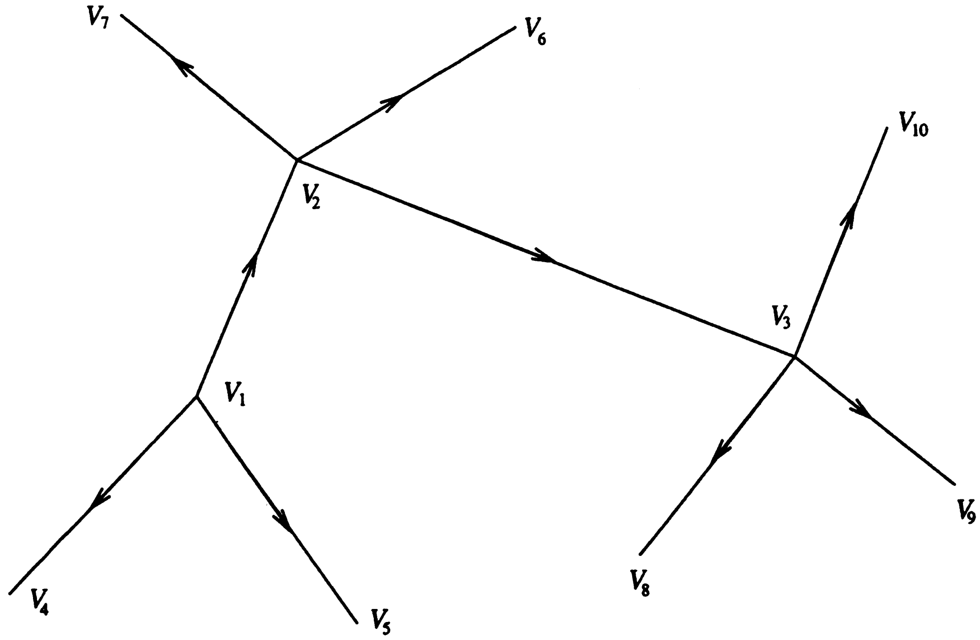

Figure 7

respectively, that is, $V_{\iota(j)}$ denotes the startpoint of E_j and $V_{\kappa(j)}$ denotes the endpoint of it. Without loss of generality, we may assume $\iota(j) < \kappa(j)$ for $j = 1, \dots, N$ (see Figure 7).

We define $C^0(\mathscr{G})$ as a set of continuous functions on $\mathscr{G}$, that is,

$$\begin{aligned} C^0(\mathscr{G}) = \{\phi = (\phi_1, \dots, \phi_N) : \phi_j \in C^0([0, l_j]) \quad (1 \leqq j \leqq N), \\ \text{any } \phi_j(l_j) \text{ and } \phi_{j'}(0) \text{ with } \kappa(j) = \iota(j') = i \text{ have} \\ \text{an equal value for each } i = 1, \dots, N'\}. \end{aligned}$$

We denote by $b_i(\phi)$ the value of $\phi \in C^0(\mathscr{G})$ at V_i, that is,

$$b_i(\phi) = \begin{cases} \phi_j(0) & \text{if } \iota(j) = i, \\ \phi_j(l_j) & \text{if } \kappa(j) = i. \end{cases}$$

We define mappings T_j on $\boldsymbol{R}^n$ $(j = 1, \dots, N)$ as

$$T_j x = R_j x + V_{\iota(j)} \quad x \in \boldsymbol{R}^n$$

where each R_j is an orthogonal transformation satisfying $\det R_j = 1$ and $R_j e_1 = l_j^{-1}(V_{\kappa(j)} - V_{\iota(j)})$ for $e_1 = (1, 0, \dots, 0)$. By using T_j, we have $T_j^{-1} D_j(\zeta) \subset Q(0, l_j, \zeta d_j)$. For $\Phi \in C^0(\Omega(\zeta))$ and $\phi \in C^0(\mathscr{G})$, we define $d(\Omega(\zeta); \Phi, \phi)$ as

$$d(\Omega(\zeta); \Phi, \phi) = \sum_{j=1}^{N} \sup_{x \in D_j(\zeta)} |\Phi(x) - \phi_j(\pi_1 \circ T_j^{-1} x)| + \sum_{i=1}^{N'} \sup_{x \in J_i(\zeta)} |\Phi(x) - b_i(\phi)|.$$

Now, we consider a boundary value problem,

$$(3.1)\qquad \begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega(\zeta), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma(\zeta), \\ u = a_{i,\zeta} & \text{on } \Gamma_i(\zeta) \ \text{ for } \ N'+1 \leqq i \leqq N'' \end{cases}$$

where f satisfies (2.2), each $\Gamma_i(\zeta)$ is a portion of $\partial\Omega(\zeta)$ which degenerates into V_i, that is,

$$\Gamma_i(\zeta) = \{T_j x : \kappa(j) = i, x_1 = l_j, \ |x'| \leqq \zeta d_j\} \quad (N'+1 \leqq i \leqq N''),$$

each $a_{i,\zeta}$ is a continuous function on $\Gamma_i(\zeta)$ which converges to a constant a_i, that is,

$$(3.2)\qquad \lim_{\zeta\to 0} \sup_{\Gamma_i(\zeta)} |a_{i,\zeta}(x) - a_i| = 0 \quad (N'+1 \leqq i \leqq N''),$$

and $\Sigma(\zeta) = \partial\Omega(\zeta) \backslash (\bigcup \Gamma_i(\zeta))$.

In this situation, we consider that the limit problem associated with (3.1) is

$$(3.3)\qquad \begin{cases} \psi_j'' + f(\psi_j) = 0 \quad \text{on } \ 0 < s < l_j & \text{for } 1 \leqq j \leqq N, \\ \psi = (\psi_1, \dots, \psi_N) \in C^0(\mathscr{G}), & \\ \displaystyle\sum_{\kappa(j)=i} d_j^{n-1} \psi_j'(l_j) = \sum_{\iota(j)=i} d_j^{n-1} \psi_j'(0) & \text{for } 1 \leqq i \leqq N', \\ b_i(\psi) = a_i & \text{for } N'+1 \leqq i \leqq N''. \end{cases}$$

By a similar argument to the proof of Theorem 1 in §2, we obtain that a solution u_m of (3.1) at $\zeta = \zeta_m$ approaches a solution $\psi \in C^0(\mathscr{G})$ of (3.3) as $m \to \infty$ in the following sense:

Theorem 2. *Suppose that a sequence* $\{\zeta_m\}_{m=1}^\infty \subset (0, \zeta_*]$ *satisfies* $\lim_{m\to\infty} \zeta_m = 0$ *and that* u_m *is any solution of* (3.1) *at* $\zeta = \zeta_m$. *Then, there exist a subsequence* $\{\zeta_{m(k)}\}_{k=1}^\infty \subset \{\zeta_m\}_{m=1}^\infty$ *and a solution* ψ *of* (3.3) *such that*

$$\lim_{k\to\infty} d(\Omega(\zeta_{m(k)}); u_{m(k)}, \psi) = 0.$$

Similarly, we have the following corollary:

Corollary 3.1. *Let* $\{\zeta_m\}_{m=1}^\infty$ *be a sequence with* $\lim_{m\to\infty} \zeta_m = 0$. *Suppose that* $a_{i,\zeta}$ *satisfies* (3.2) *and sequences of functions* $\{H_m\}_{m=1}^\infty, \{\tilde{H}_m\}_{m=1}^\infty \subset C^0(\overline{\Omega(\zeta_m)})$ *approach functions* $h, \tilde{h} \in C^0(\mathscr{G})$ *as* $m \to \infty$ *respectively, that is,*

$$\lim_{m\to\infty} d(\Omega(\zeta_m); H_m, h) = 0 \quad \text{and} \quad \lim_{m\to\infty} d(\Omega(\zeta_m); \tilde{H}_m, \tilde{h}) = 0.$$

If functions u_m $(m \geqq 1)$ *satisfy*

$$\begin{cases} \Delta u_m + H_m(x) u_m = \tilde{H}_m(x) & \text{in } \Omega(\zeta_m), \\ \dfrac{\partial u_m}{\partial \nu} = 0 & \text{on } \Sigma(\zeta_m), \\ u_m = a_{i,\zeta_m} & \text{on } \Gamma_i(\zeta_m) \text{ for } N'+1 \leqq i \leqq N'', \end{cases}$$

and $\sup_{x \in \Omega(\zeta_m)} |u_m(x)| \leqq M$ *where the positive constant* M *is independent of* ζ_m. *Then there exist a subsequence* $\{\zeta_{m(k)}\}_{k=1}^{\infty} \subset \{\zeta_m\}_{m=1}^{\infty}$ *and* $\psi = (\psi_1, \dots, \psi_n) \in C^0(\mathscr{G})$ *such that*

$$\begin{cases} \psi_j'' + h_j(s)\psi_j = \tilde{h}_j(s) \quad 0 < s < l_j & \text{for } 1 \leqq j \leqq N, \\ \sum_{\iota(j)=i} d_j^{n-1} \psi_j'(0) = \sum_{\kappa(j)=i} d_j^{n-1} \psi_j'(l_j) & \text{for } 1 \leqq i \leqq N', \\ b_i(\psi) = a_i & \text{for } N'+1 \leqq i \leqq N'' \end{cases}$$

and that $\lim_{k \to \infty} d(\Omega(\zeta_{m(k)}); u_{m(k)}, \psi) = 0$.

Remark. In the preceding theorem, when we replace the boundary condition on $\Gamma_i(\zeta)$ of (3.1) by the Neumann boundary condition and we replace $b_i(\phi) = a_i$ of the system (3.3) by $\phi_j'(l_j) = 0 \quad (\kappa(j) = i)$, similar results hold by an argument similar to the proof of Theorem 1.

We may naturally consider the case that the thin domain converges to a smooth curve instead of a straight line. In that generalized case, we can expect similar mathematical phenomena, while several technical difficulties arise.

§4. Inverse problem.

In this section, we consider a certain inverse problem. We have stated a solution of PDE (3.1) approaches to a solution of an associated limit equation (3.3) as ζ tends to zero. In that situation, conversely, the following problem occurs naturally:

When a solution of (3.3) is given, can we prove the existence of a solution of (3.1) which approaches it?

We have a positive answer. Namely, we can prove that (3.1) has a solution which approaches a solution of (3.3) when the solution of (3.3) satisfies a certain condition. Using the same notation as §3, we present a main result in this section.

Theorem 3. *Suppose that there exists a solution* $\psi = (\psi_1, \dots, \psi_n)$ *of* (3.3) *such that the linearized equation*

$$
(4.1)\qquad \begin{cases} \phi_j'' + f'(\psi_j)\phi_j = 0 \quad on\ 0 < s < l_j \quad for\ 1 \leqq j \leqq N, \\ \phi = (\phi_1, \dots, \phi_n) \in C^0(\mathscr{G}), \\ \sum_{\iota(j)=i} d_j^{n-1}\phi_j'(0) = \sum_{\kappa(j)=i} d_j^{n-1}\phi_j'(l_j) \quad for\ 1 \leqq i \leqq N', \\ b_i(\phi) = 0 \qquad for\ N'+1 \leqq i \leqq N'', \end{cases}
$$

has no solution except the trivial solution $(\phi_1, \dots, \phi_n) = (0, \dots, 0)$. *Namely, we suppose the eigenvalue problem of the linearized equation around the solution* ψ *has no zero eigenvalue. Then, there exists a constant* $\zeta_* > 0$ *such that the equation* (3.1) *has a solution* Ψ_ζ *for any* $\zeta \in (0, \zeta_*]$ *and that* $\{\Psi_\zeta : 0 < \zeta < \zeta_*\}$ *satisfies*

$$
(4.2)\qquad \lim_{\zeta \to 0} d(\Omega(\zeta); \Psi_\zeta, \psi) = 0.
$$

Proof of Theorem 3. We construct an approximate solution of (3.1). Let a solution $\psi = (\psi_1, \dots, \psi_n)$ of (3.3) satisfy the assumption of Theorem 3. We define a Lipschitz continuous function $\Psi_\zeta^{(0)}$ as

$$
\Psi_\zeta^{(0)}(x) = \begin{cases} b_i(\psi) \quad x \in \overline{J_i(\zeta)} \qquad \text{for } 1 \leqq i \leqq N', \\ \psi_j((l_j - \zeta l)^{-1} l_j(\pi_1 \circ T_j^{-1}x - \zeta l)) \quad x \in \overline{D_j(\zeta)} \\ \qquad \text{for } \iota(j) \leqq N' \text{ and } \kappa(j) \geqq N'+1, \\ \psi_j((l_j - 2\zeta l)^{-1} l_j(\pi_1 \circ T_j^{-1}x - \zeta l)) \quad x \in \overline{D_j(\zeta)} \\ \qquad \text{for } \iota(j) \leqq N' \text{ and } \kappa(j) \leqq N'. \end{cases}
$$

We define a function $\Psi_\zeta^{(1)}$ as the unique solution of

$$
\begin{cases} \Delta \Psi_\zeta^{(1)} = -f(\Psi_\zeta^{(0)}(x)) \quad \text{in } \Omega(\zeta), \\ \dfrac{\partial \Psi_\zeta^{(1)}}{\partial \nu} = 0 \qquad \text{on } \Sigma(\zeta), \\ \Psi_\zeta^{(1)} = a_{i,\zeta} \qquad \text{on } \Gamma_i(\zeta) \text{ for } N'+1 \leqq i \leqq N''. \end{cases}
$$

Applying Corollary 3.1, we obtain

$$
(4.3)\qquad \lim_{\zeta \to 0} d(\Omega(\zeta); \Psi_\zeta^{(0)}, \psi) = 0,
$$

$$
(4.4)\qquad \lim_{\zeta \to 0} d(\Omega(\zeta); \Psi_\zeta^{(1)}, \psi) = 0.
$$

Let c_1 be an upper bound of $\Psi_\zeta^{(1)}$, that is,

$$\sup_{x \in \Omega(\zeta)} |\Psi_\zeta^{(1)}(x)| \leqq c_1 \quad \text{for any } \zeta > 0.$$

After this, let $\|\cdot\|_\zeta$ denote a norm $\|g\|_\zeta = \sup_{x \in \Omega(\zeta)} |g(x)|$ of $C^0(\overline{\Omega(\zeta)})$.

Lemma 4.2. *There exists a constant $\zeta' > 0$ such that if Φ satisfies*

$$\begin{cases} \Delta\Phi + f'(\Psi_\zeta^{(1)}(x))\Phi = 0 & \text{in } \Omega(\zeta), \\ \dfrac{\partial \Phi}{\partial \nu} = 0 & \text{on } \Sigma(\zeta), \\ \Phi = 0 & \text{on } \Gamma_i(\zeta) \text{ for } N'+1 \leqq i \leqq N'' \end{cases} \tag{4.5}$$

for any $\zeta \in (0, \zeta']$, then $\Phi \equiv 0$ in $\Omega(\zeta)$.

Proof of Lemma 4.2. Suppose there exists a sequence $\{\zeta_m\}_{m=1}^\infty$ with $\lim_{m\to\infty} \zeta_m = 0$ such that the equation (4.5) at $\zeta = \zeta_m$ has a nontrivial solution $W_m \not\equiv 0$ in $\Omega(\zeta_m)$. Let $\widetilde{W}_m(x) = W_m(x)/\|W_m\|_{\zeta_m}$. Then, we obtain $\widetilde{W}_m$ satisfies (4.5) and $\|\widetilde{W}_m\|_{\zeta_m} = 1$ for any $m \geqq 1$. By (4.4) and applying Corollary 3.1, we obtain a nontrivial solution of (4.1). This contradicts the assumption of Theorem 3. Thus we complete the proof of Lemma 4.2. □

We consider the equation

$$\begin{cases} \Delta u + f'(\Psi_\zeta^{(1)})u = \Phi & \text{in } \Omega(\zeta), \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Sigma(\zeta), \\ u = 0 & \text{on } \Gamma(\zeta), \end{cases} \tag{4.6}$$

where $\Phi \in L^2(\Omega(\zeta))$ and $\Gamma(\zeta) = \bigcup \Gamma_i(\zeta)$. Because of Lemma 4.2, the equation (4.6) has a unique solution for each Φ. We denote by $A_\zeta\Phi$ the solution of (4.6) for Φ.

Lemma 4.3. *There exist constants $c_2 > 0$ and $\zeta'' > 0$ such that*

$$\|A_\zeta \Phi\|_\zeta \leqq c_2 \|\Phi\|_\zeta$$

for any $\zeta \in (0, \zeta'']$ and $\Phi \in C^0(\overline{\Omega(\zeta)})$ satisfying $A_\zeta\Phi \in C^2(\Omega(\zeta))$.

Proof of Lemma 4.3. We assume the contrary. Namely, we assume there exist a sequence $\{\zeta_m\}_{m=1}^\infty$ with $\lim_{m\to\infty} \zeta_m = 0$ and C^0 functions Θ_m such that $\|\Theta_m\|_{\zeta_m} = 1$ and $\|A_{\zeta_m}\Theta_m\|_{\zeta_m} \geqq m$ for $m \geqq 1$. Let

$$\Phi_m(x) = \frac{A_{\zeta_m}\Theta_m(x)}{\|A_{\zeta_m}\Theta_m\|_{\zeta_m}},$$

$$\tilde{\Theta}_m(x) = \frac{\Theta_m(x)}{\|A_{\zeta_m}\Theta_m\|_{\zeta_m}}.$$

Then, Φ_m and $\tilde{\Theta}_m$ satisfy

$$\begin{cases} \Delta\Phi_m + f'(\Psi^{(1)}_{\zeta_m})\Phi_m = \tilde{\Theta}_m & \text{in } \Omega(\zeta_m), \\ \dfrac{\partial\Phi_m}{\partial\nu} = 0 & \text{on } \Sigma(\zeta_m), \\ \Phi_m = 0 & \text{on } \Gamma(\zeta_m), \end{cases}$$

$$\|\Phi_m\|_{\zeta_m} = 1,$$

$$\|\tilde{\Theta}_m\|_{\zeta_m} \leqq \frac{1}{m}.$$

Applying Corollary 3.1, we obtain a nontrivial solution of (4.1). This contradicts the assumption of Theorem 3. Thus we complete the proof of Lemma 4.3. $\square$

Let W_ζ be a harmonic function

$$\begin{cases} \Delta W_\zeta = 0 & \text{in } \Omega(\zeta), \\ \dfrac{\partial W_\zeta}{\partial\nu} = 0 & \text{on } \Sigma(\zeta), \\ W_\zeta = a_{i,\zeta} & \text{on } \Gamma_i(\zeta) \text{ for } N'+1 \leqq i \leqq N'', \end{cases}$$

and let $U^{(1)}_\zeta = \Psi^{(1)}_\zeta - W_\zeta$. We define a sequence $\{U^{(p)}_\zeta\}_{p=1}^\infty \subset C^0(\overline{\Omega(\zeta)})$ as

$$\begin{aligned} U^{(p+1)}_\zeta &= F_\zeta(U^{(p)}_\zeta) \\ &= A_\zeta(f'(\Psi^{(1)}_\zeta)U^{(p)}_\zeta - f(U^{(p)}_\zeta + W_\zeta)) \quad \text{for } p \geqq 1. \end{aligned}$$

By the definition of A_ζ, each $U^{(p)}_\zeta$ is a C^2 function.

We take a constant $\delta > 0$ such that

$$\delta < \min\left\{1/2, \left(2c_2 \sup_{|\xi|<c_1+1} |f''(\xi)|\right)^{-1}\right\}. \tag{4.7}$$

For this δ, we can take a constant $\zeta_* > 0$ small so that

$$\|f(\Psi_\zeta^{(0)}) - f(\Psi_\zeta^{(1)})\|_\zeta \leqq \frac{\delta}{4c_2} \quad \text{for } \zeta \in (0, \zeta_*] \tag{4.8}$$

by (4.3) and (4.4). Then, we have the following:

Lemma 4.4.

$$\|U_\zeta^{(p)} - U_\zeta^{(1)}\|_\zeta \leqq \delta \tag{4.9}$$

for any $p \geqq 1$ *and* $\zeta \in (0, \zeta_*]$.

Proof of Lemma 4.4. We prove Lemma 4.4 for each ζ by the induction. It is trivial that (4.9) is satisfied at $p = 1$. We assume (4.9) is satisfied at $p = p'$. We have

$$\|U_\zeta^{(p'+1)} - U_\zeta^{(1)}\|_\zeta \leqq \|F_\zeta(U_\zeta^{(p')}) - F_\zeta(U_\zeta^{(1)})\|_\zeta + \|F_\zeta(U_\zeta^{(1)}) - U_\zeta^{(1)}\|_\zeta.$$

The estimation of the first term of the right-hand side is

$$\begin{aligned}
&\|F_\zeta(U_\zeta^{(p')}) - F_\zeta(U_\zeta^{(1)})\|_\zeta \\
&\quad \leqq c_2 \left\| \int_0^1 (f'(\Psi_\zeta^{(1)}) - f'(tU_\zeta^{(p')} + (1-t)U_\zeta^{(1)} + W_\zeta))\, dt (U_\zeta^{(p')} - U_\zeta^{(1)}) \right\|_\zeta \\
&\quad \leqq c_2 \delta \sup_{|\xi| \leqq c_1 + 2\delta} |f''(\xi)| \, \|U_\zeta^{(p')} - U_\zeta^{(1)}\|_\zeta \\
&\quad \leqq \frac{1}{2}\delta
\end{aligned}$$

by (4.7). The estimation of the last term is

$$\begin{aligned}
&\|F_\zeta(U_\zeta^{(1)}) - U_\zeta^{(1)}\|_\zeta \\
&\qquad = \|A_\zeta(f'(\Psi_\zeta^{(1)})U_\zeta^{(1)} - f(\Psi_\zeta^{(1)})) - A_\zeta(f'(\Psi_\zeta^{(1)})U_\zeta^{(1)} - f(\Psi_\zeta^{(0)}))\|_\zeta \\
&\qquad \leqq c_2 \|f(\Psi_\zeta^{(0)}) - f(\Psi_\zeta^{(1)})\|_\zeta \\
&\qquad \leqq \frac{\delta}{4}
\end{aligned}$$

by (4.8). Therefore $\|U_\zeta^{(p'+1)} - U_\zeta^{(1)}\| \leqq \delta$. We complete the proof of Lemma 4.4. □

From Lemma 4.4, we have $\|U_\zeta^{(p+1)} - U_\zeta^{(p)}\|_\zeta \leqq 2^{-1}\|U_\zeta^{(p)} - U_\zeta^{(p-1)}\|_\zeta$ for any $p \geqq 1$. We have immediately that the sequence $\{U_\zeta^{(p)}\}_{p=1}^\infty$ is a Cauchy sequence in $C^0(\overline{\Omega(\zeta)})$. We denote by $U_\zeta^{(\infty)}$ the limit of $U_\zeta^{(p)}$ as $p \to \infty$. We obtain $U_\zeta^{(\infty)} = F_\zeta(U_\zeta^{(\infty)}) \in C^2(\Omega(\zeta))$ by the definition of F_ζ. Let $\Psi_\zeta = U_\zeta^{(\infty)} + W_\zeta$. Then, Ψ_ζ satisfies (3.1) and

$$\begin{aligned}\|\Psi_\zeta - \Psi_\zeta^{(1)}\|_\zeta &= \|U_\zeta^{(\infty)} - U_\zeta^{(1)}\|_\zeta \\ &= \|F_\zeta(U_\zeta^{(\infty)}) - F_\zeta(U_\zeta^{(1)}) + F_\zeta(U_\zeta^{(1)}) - U_\zeta^{(1)}\|_\zeta \\ &\leqq \frac{1}{2}\|\Psi_\zeta - \Psi_\zeta^{(1)}\|_\zeta + c_2\|f(\Psi_\zeta^{(0)}) - f(\Psi_\zeta^{(1)})\|_\zeta.\end{aligned}$$

Thus, we obtain

$$\|\Psi_\zeta - \Psi_\zeta^{(1)}\|_\zeta \leqq 2c_2\|f(\Psi_\zeta^{(1)}) - f(\Psi_\zeta^{(0)})\|_\zeta.$$

By (4.3) and (4.4), we obtain (4.2). We complete the proof of Theorem 3. □

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