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Dimension theory of group C^*-algebras of connected Lie groups of type I

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Abstract. In this paper we determine isomorphism classes of connected solvable Lie groups with some conditions such that their group C^*-algebras have stable rank one, and give its applications. Also, we show that stable rank of group C^*-algebras of connected Lie groups of type I is estimated in terms of their closed normal subgroups and quotient groups.

§1. Introduction.

Stable rank of C^*-algebras was introduced by M. A. Rieffel [**Rf**], who raised the interesting problem of describing stable rank of group C^*-algebras of Lie groups in terms of the structure of groups. In this direction, H. Takai and the author [**ST1**], [**ST2**] estimated stable rank of the C^*-algebras of solvable Lie groups of type I by the complex dimension of the spaces of all 1-dimensional representations of groups. Moreover, the author [**Sd1**], [**Sd2**] considered both amenable and nonamenable cases for connected Lie groups of type I.

In this paper, first of all, we give a lemma which states isomorphism classes of connected, solvable Lie groups with the centers of their universal covering groups connected such that their group C^*-algebras have stable rank one. To show the lemma, we use the technical lemma in [**ST2**] which states isomorphism classes of simply connected, solvable Lie groups such that their group C^*-algebras have stable rank one. Also, we show a similar result in the case of connected nilpotent Lie groups. Applying these results, we give some generalizations of the results in [**ST1**], [**ST2**].

Secondly, combining some main results obtained in [**Sd1**], [**Sd2**], we estimate stable rank of the reduced C^*-algebras of connected Lie groups of type I by the complex dimension of the spaces of all 1-dimensional representations in the reduced duals of these groups. Moreover, we estimate stable rank of the reduced C^*-algebras of these groups in terms of their closed normal subgroups and quotient groups.

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§2. Stable rank of group C^*-algebras of connected solvable Lie groups of type I.

We first review some notations used in this paper. Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{A}^n$ its n-direct sum. For a unital C^*-algebra $\mathfrak{A}$, we denote by $L_n(\mathfrak{A})$ the set of all elements $(a_i)_{i=1}^n$ of $\mathfrak{A}^n$ such that $\sum_{i=1}^n a_i^* a_i$ is invertible in $\mathfrak{A}$. Then the stable rank of $\mathfrak{A}$, denoted by $\mathrm{sr}(\mathfrak{A})$, is defined by

$$\{\infty\} \wedge \inf\{n \in \boldsymbol{N} \mid L_n(\mathfrak{A}) \text{ is dense in } \mathfrak{A}^n\}$$

where $\wedge$ means minimum. For a nonunital C^*-algebra, its stable rank is defined by that of its unitization.

Let G be a Lie group. We denote by $\hat{G}$ the space of all equivalent classes of irreducible unitary representations of G equipped with the hull-kernel topology and by $\hat{G}_1$ the space of all 1-dimensional representations of G. We use the facts that $\hat{G}_1$ is closed in $\hat{G}$ (cf. [**ST2**; Lemma 2.6]), and $\hat{G}_1$ is isomorphic to $(G/[G,G])^\wedge$ as a topological group, where $[G,G]$ is the commutator subgroup of G (cf. [**ST2**; Lemma 2.3]). Let $\hat{G}_r$ be the reduced dual of G. Put $\hat{G}_{r,1} = \hat{G}_r \cap \hat{G}_1$. By definition, $\hat{G}_{r,1} = \hat{G}_1$ if G is amenable, i.e. $\hat{G} = \hat{G}_r$. And $\hat{G}_{r,1} = \varnothing$ if G is nonamenable. Let $C^*(G)$, $C_r^*(G)$ be the full, reduced group C^*-algebra of G respectively.

Let $\boldsymbol{R}^n$, $\boldsymbol{Z}^n$ be the n-direct product of the group of real numbers, integers respectively, and $\boldsymbol{T}^s$ the s-torus.

First of all, we give the following lemma:

Lemma 2.1. *Let G be a connected solvable Lie group, and $\tilde{G}$ its universal covering group. If the center Z of $\tilde{G}$ is connected, then* $\mathrm{sr}(C^*(G)) = 1$ *if and only if G is isomorphic to either $\boldsymbol{R}$ or $\boldsymbol{T}^s$ or the direct product $\boldsymbol{T}^s \times \boldsymbol{R}$.*

Proof. Let Γ be the central discrete subgroup of $\tilde{G}$ such that $\tilde{G}/\Gamma \cong G$. Since $\Gamma \subset Z$, then by the homomorphism theorem of groups,

$$\tilde{G}/Z \cong (\tilde{G}/\Gamma)/(Z/\Gamma) \cong G/(Z/\Gamma).$$

Note that $C^*(G) = C_r^*(G)$ since G is solvable, and Z/Γ is amenable. Thus $C^*(\tilde{G}/Z)$ is considered as a quotient C^*-algebra of $C^*(G)$ (cf. [**Kn**; p. 1349]). On the other hand, we have the following exact sequence of abelian groups (cf. [**OV**; p. 47]):

$$\pi_1(Z) \to \pi_1(\tilde{G}) \to \pi_1(\tilde{G}/Z) \to Z/Z_0 \to 0$$

where $\pi_1(\cdot)$ and Z_0 respectively mean the fundamental group and the connected component of the identity of Z. Since Z is connected and $\tilde{G}$ is simply connected, i.e, $\pi_1(\tilde{G}) = 0$, we obtain that $\tilde{G}/Z$ is simply connected.

If $\tilde{G} = Z$, then G is commutative. Hence $\mathrm{sr}(C^*(G)) = 1$ if and only if G is isomorphic to either $\boldsymbol{R}$ or $\boldsymbol{T}^s$ or $\boldsymbol{T}^s \times \boldsymbol{R}$.

If $\tilde{G}/Z$ is not isomorphic to $\boldsymbol{R}$, then by [**ST2**; Lemma 3.7] we see $\mathrm{sr}(C^*(\tilde{G}/Z)) \geq 2$. Hence $\mathrm{sr}(C^*(G)) \geq 2$.

If $\tilde{G}/Z \cong \boldsymbol{R}$, then $\tilde{G}$ is isomorphic to the semi-direct product $\boldsymbol{R}^s \rtimes_\alpha \boldsymbol{R}$ with $s = \dim Z$ by simply connectedness of $\tilde{G}$ (cf. [**OV**; p. 57 Exercises 15]). Then one can check from calculation of product that $\tilde{G}$ is commutative. So is G. □

Remark. If $G = \boldsymbol{R}^2 \rtimes_\alpha \boldsymbol{R}$ with α the rotation action of $\boldsymbol{R}$ on $\boldsymbol{R}^2$, then its center is isomorphic to $\boldsymbol{Z}$. It is known that G is the nonexponential, simply connected, solvable Lie group unique up to isomorphisms with dimension ≤ 3 ([**LL**]).

It is also known that connected is the center of any connected, nilpotent Lie group (cf. [**Hc**; XVI. Theorem 1.1]).

For a topological space X, we denote by $\dim X$ its covering dimension. We let $\dim_{\boldsymbol{C}} X = [\dim X/2] + 1$ with $[\cdot]$ the Gauss symbol. Then we have the following:

Proposition 2.2. *Let G be a connected nilpotent Lie group. Then the following are equivalent*:

(1) $\mathrm{sr}(C^*(G)) = 1$.

(2) *G is isomorphic to either $\boldsymbol{R} \times \boldsymbol{T}^k$ or $\boldsymbol{R}$ or $\boldsymbol{T}^k$.*

(3) $\dim_{\boldsymbol{C}} \hat{G}_1 = 1$.

Proof. (1) $\Leftrightarrow$ (2): This follows from Lemma 2.1 and the fact [**Hc**; XVI. Theorem 1.1] that connected is the center of any connected nilpotent Lie group.

(2) $\Rightarrow$ (3): It is well known that if $G = \boldsymbol{R} \times \boldsymbol{T}^k$, then $\hat{G}$ is isomorphic to $\boldsymbol{R} \times \boldsymbol{Z}^k$.

(3) $\Rightarrow$ (2): If G is commutative, it is isomorphic to $\boldsymbol{T}^k \times \boldsymbol{R}^t$ for some $k, t \geq 0$. So the implication is clear.

Suppose that G is noncommutative. We take a maximal compact commutative subgroup K of G such that G is homeomorphic to the product space $K \times \boldsymbol{R}^t$ for some $t \geq 0$ (cf. [**Cv**; Theorem 9]). Moreover, since G is nilpotent, K is contained in the center of G (cf. [**GOV**; Theorem 1.6]). Thus K is a normal subgroup of G. Then G/K is a simply connected nilpotent Lie group. Put $H = G/K$.

If $H \cong \boldsymbol{R}$, then $G \cong K \rtimes_\alpha \boldsymbol{R}$. Note that $K \cong \boldsymbol{T}^k$ for some $k \geq 0$. Since K is contained in the center of G, we have $G \cong \boldsymbol{T}^k \times \boldsymbol{R}$.

Otherwise, we have that $H/[H,H] \cong \boldsymbol{R}^n$ for some $n \geq 2$ [**ST1**; Lemma 3.5] (cf. [**ST2**; Lemma 2.1]). Hence we obtain that

$$\dim_{\boldsymbol{C}} \hat{G}_1 \geq \dim_{\boldsymbol{C}} \hat{H}_1 \geq 2.$$

□

REMARK. It is known that for a simply connected, nilpotent Lie group G, the following are equivalent (cf. [**ST1**; Lemma 3.5]):

(1) G is isomorphic to $\boldsymbol{R}$.

(2) $\dim_{\boldsymbol{C}} \hat{G}_1 = 1$.

(3) $\mathrm{sr}(C^*(G)) = 1$.

Using Proposition 2.2, we have the following:

THEOREM 2.3. *Let G be a connected nilpotent Lie group. Then*

$$\mathrm{sr}(C^*(G)) = \dim_{\boldsymbol{C}} \hat{G}_1.$$

PROOF. It is known ([**Sd2**], cf. [**ST2**; Proposition 3.3]) that for any connected amenable Lie group G of type I, one has that

$$\dim_{\boldsymbol{C}} \hat{G}_1 \leq \mathrm{sr}(C^*(G)) \leq 2 \vee \dim_{\boldsymbol{C}} \hat{G}_1$$

where $\vee$ means maximum. By Proposition 2.2, the proof is complete. □

REMARK. This result generalizes the main theorem in [**ST1**] which states that the above equality holds for simply connected, nilpotent Lie groups.

By Lemma 2.1 and the same reason in Proposition 2.2, we have the following:

THEOREM 2.4. *Let G be a connected, solvable Lie group of type I. If the center of $\tilde{G}$ is connected, then*

$$\mathrm{sr}(C^*(G)) = \begin{cases} 1 & \text{if } G \cong \boldsymbol{R} \text{ or } \boldsymbol{T}^s \text{ or } \boldsymbol{R} \times \boldsymbol{T}^s \\ 2 \vee \dim_{\boldsymbol{C}} \hat{G}_1 & \text{otherwise} \end{cases}$$

where $\vee$ means maximum.

REMARK. H. Takai and the author [**ST2**] obtained the following formula:

$$\mathrm{sr}(C^*(G)) = (2 \vee \dim_{\boldsymbol{C}} \hat{G}_1) \wedge \dim G$$

for any simply connected, solvable Lie group G of type I.

We next give some examples as follows:

EXAMPLE 2.5. Let $\tilde{G} = \boldsymbol{R}^n$ and $G = \boldsymbol{T}^n \cong \boldsymbol{R}^n/\boldsymbol{Z}^n$. Then by Fourier transform, $C^*(G) \cong C_0(\boldsymbol{Z}^n)$. Moreover, we obtain that

$$\mathrm{sr}(C^*(G)) = 1 = \dim_{\boldsymbol{C}} \hat{G}_1 \leq \dim_{\boldsymbol{C}}(\tilde{G})^\wedge_1 = [n/2] + 1 = \mathrm{sr}(C^*(\tilde{G})).$$

Example 2.6. Let $\tilde{G}$ be the 3-dimensional real Heisenberg group of all matrices g

$$g = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad a, b, c \in \boldsymbol{R}.$$

We let $g = (c, b, a)$. Then the center of $\tilde{G}$ consists of all elements of the form $(c, 0, 0)$. Then $\Gamma = \{(t, 0, 0) | t \in \boldsymbol{Z}\}$ is a central discrete subgroup of G. Put $G = \tilde{G}/\Gamma \cong (\boldsymbol{T} \times \boldsymbol{R}) \rtimes_\alpha \boldsymbol{R}$ whose action α is defined by $\alpha_a(e^{it}, b) = (e^{i(t+ab)}, b)$ for $(e^{it}, b) \in \boldsymbol{T} \times \boldsymbol{R}$. Then $[G, G] \cong \boldsymbol{T}$ so that $G/[G, G] \cong \boldsymbol{R}^2$. Hence $\hat{G}_1 \cong \boldsymbol{R}^2$. Since $(\tilde{G})^\wedge_1 \cong \tilde{G}/[\tilde{G}, \tilde{G}] \cong \boldsymbol{R}^2$, and by Theorem 2.3, we get that

$$\mathrm{sr}(C^*(G)) = 2 = \mathrm{sr}(C^*(\tilde{G})).$$

§3. Dimension theory of group C^*-algebras of connected Lie groups of type I.

In this section, first of all, we estimate stable rank of the reduced C^*-algebras of connected Lie groups of type I by combining some results obtained in [**Sd1**] and [**Sd2**]. Next, we estimate stable rank of the reduced C^*-algebras of these groups in terms of their closed normal subgroups and quotient groups.

We first show the following estimation of stable rank of the reduced C^*-algebras of connected Lie groups of type I:

Theorem 3.1. *Let G be a connected Lie group of type I. Then*

$$\dim_{\boldsymbol{C}} \hat{G}_{r,1} \leq \mathrm{sr}(C_r^*(G)) \leq 2 \vee \dim_{\boldsymbol{C}} \hat{G}_{r,1}$$

where $\hat{G}_{r,1}$ is the space of all 1-dimensional representations in the reduced dual $\hat{G}_r$.

Proof. Note that if G is amenable, we have $\hat{G}_{r,1} = \hat{G}_1$. On the other hand, if G is nonamenable, then $\hat{G}_{r,1} = \varnothing$ so that $\dim_{\boldsymbol{C}} \hat{G}_{r,1} = 0$ since by definition $\dim \varnothing = -1$. Thus, by [**Sd2**; Proposition 3.5] and [**Sd1**; Proposition 2.3], we have the conclusion. □

Remark. In the case that $G = \boldsymbol{R}$, the above formula gives $1 = \mathrm{sr}(C^*(G)) < 2$. If G is the real $ax + b$ group, then we have $1 < \mathrm{sr}(C^*(G)) = 2$. Consequently, the above estimation is optimal.

Next we show the product formula of stable rank in the case of the reduced C^*-algebras of connected Lie groups of type I as follows:

Theorem 3.2. *If G, H are two connected Lie groups of type I, then*

$$\mathrm{sr}(C_r^*(G) \otimes C_r^*(H)) \le \mathrm{sr}(C_r^*(G)) + \mathrm{sr}(C_r^*(H)).$$

Proof. Note that $G \times H$ is amenable if and only if so are both G and H. This case was considered in [**Sd2**; Corollary 3.6]. If $G \times H$ is nonamenable, then by [**Sd1**; Proposition 2.3], $\mathrm{sr}(C_r^*(G \times H)) \le 2$. The same methods in [**Sd1**; Corollary 2.4] implies the conclusion. □

Remark. If $G = \boldsymbol{R}$ and $H = \boldsymbol{R}$, then

$$\mathrm{sr}(C_r^*(G) \otimes C_r^*(H)) = 2 = \mathrm{sr}(C_r^*(G)) + \mathrm{sr}(C_r^*(H)).$$

Hence, the above inequality is optimal.

Finally, we estimate stable rank of the reduced C^*-algebras of connected Lie groups of type I in terms of their closed normal subgroups and quotient groups as follows:

Theorem 3.3. *Let G be a connected Lie group of type I and H any closed normal subgroup. Then*

$$\mathrm{sr}(C_r^*(G)) \le \mathrm{sr}(C_r^*(H)) + \mathrm{sr}(C_r^*(G/H)).$$

Proof. If G is nonamenable, then by [**Sd1**; Proposition 2.3], $\mathrm{sr}(C_r^*(G)) \le 2$. Thus the claim of theorem is established.

Next suppose that G is amenable. Then so are H, G/H. Since $[H, H]$, $[G/H, G/H]$ are also amenable, we have that $C_r^*(H/[H, H])$, $C_r^*((G/H)/[G/H, G/H])$ are quotient C^*-algebras of $C_r^*(H)$, $C_r^*(G/H)$ respectively (cf. [**Kn**; p. 1349]). Then we choose a closed normal subgroup K of G such that the next sequence is exact:

$$1 \to K/[G, G] \to G/[G, G] \to (G/H)/[G/H, G/H] \to 1.$$

By Pontryagin's corresponding theorem, one has the following exact sequence:

$$1 \leftarrow (K/[G, G])^\wedge \leftarrow (G/[G, G])^\wedge \leftarrow (G/K)^\wedge \leftarrow 1$$

as commutative Lie groups. Note that $K/[G, G]$ is a quotient group of $H/[H, H]$ via the map $H/[H, H] \ni h[H, H] \mapsto h[G, G] \in K/[G, G]$. In fact, if $k[G, G] \in K/[G, G]$, then $kH \in [G/H, G/H]$. So $kH = gH$ for some $g \in [G, G]$. Thus, $g^{-1}k \in H$ so that $g^{-1}k[G, G] = g^{-1}[G, G]k[G, G] = k[G, G]$. Hence we have that

$$\begin{aligned}\dim(G/[G, G])^\wedge &= \dim(K/[G, G])^\wedge + \dim(G/K)^\wedge \\ &\le \dim(H/[H, H])^\wedge + \dim((G/H)/[G/H, G/H])^\wedge.\end{aligned}$$

Therefore, using [**Sd2**; Proposition 3.5] and [**ST2**; Lemma 3.2], we obtain that

$$\begin{aligned}\mathrm{sr}(C^*(G)) &\le 2 \vee \dim_{\boldsymbol{C}} \hat{G}_1 = 2 \vee \dim_{\boldsymbol{C}}(G/[G,G])^\wedge \\ &\le \dim_{\boldsymbol{C}}(H/[H,H])^\wedge + \dim_{\boldsymbol{C}}((G/H)/[G/H,G/H])^\wedge \\ &= \mathrm{sr}(C_r^*(H/[H,H])) + \mathrm{sr}(C_r^*((G/H)/[G/H,G/H])) \\ &\le \mathrm{sr}(C^*(H)) + \mathrm{sr}(C^*(G/H)). \qquad \square\end{aligned}$$

Remark. Note that the above inequality is optimal. In fact, we let $G = \boldsymbol{R}^2$ and take its closed normal subgroup H isomorphic to $\boldsymbol{R}$. Then we have that

$$\mathrm{sr}(C^*(G)) = 2 = \mathrm{sr}(C^*(H)) + \mathrm{sr}(C^*(G/H)).$$

Example 3.4. We denote by G_{n+2} the simply connected, nilpotent Lie group of all upper triangular $(n+2) \times (n+2)$ matrices over real numbers with one on the diagonal. Let H_{2n+1} be the $(2n+1)$-dimensional generalized Heisenberg group consisting of all matrices

$$\begin{pmatrix} 1 & a_1 & \cdots & a_n & c \\ & \ddots & & 0 & b_1 \\ & & \ddots & & \vdots \\ & & & \ddots & b_n \\ 0 & & & & 1 \end{pmatrix} \quad a_i, b_i, c \in \boldsymbol{R} \ \ (1 \le i \le n).$$

By direct computation, one can see that H_{2n+1} is a closed normal subgroup of G_{n+2}. Moreover, we have that $\dim_{\boldsymbol{C}} \hat{G}_{n+2} = [(n+1)/2] + 1$ and $\dim_{\boldsymbol{C}} \hat{H}_{2n+1} = n+1$. Using Theorem 2.3, we obtain that

$$\begin{cases} \mathrm{sr}(C^*(G_3)) = \mathrm{sr}(C^*(H_3)) & \text{with } G_3 = H_3, \\ \mathrm{sr}(C^*(G_{n+2})) < \mathrm{sr}(C^*(H_{2n+1})) & \text{if } n \ge 2. \end{cases}$$

Example 3.5. In Theorem 3.3, we note that H is of non type I in general. For example, let $M = \boldsymbol{C}^2 \rtimes_\alpha \boldsymbol{R}$ be the Mautner group where $\alpha_t(z,w) = (e^{it}z, e^{i\theta t}w)$ for $t \in \boldsymbol{R}$, $(z,w) \in \boldsymbol{C}^2$, $\theta \in \boldsymbol{R} \backslash \boldsymbol{Q}$, and let $G = M \rtimes_{\hat{\alpha}} \boldsymbol{R}$ where $\hat{\alpha}_s(z,w,t) = (z, e^{2\pi i s}w, t)$ for $s \in \boldsymbol{R}$. Then G is of type I, but M is of non type I (cf. [**Tk**; 9. Appendix]).

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