

On the characteristics for convolution equations in tube domains

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Abstract. We study holomorphic solutions for convolution equations in tube domains. Let \mathcal{O}^τ be the sheaf of holomorphic functions in tube domains on the purely imaginary space $\sqrt{-1}\mathbf{R}^n$ and \mathcal{S} the complex $0 \rightarrow \mathcal{O}^\tau \xrightarrow{\mu^*} \mathcal{O}^\tau \rightarrow 0$ generated by the convolution operator μ^* with hyperfunction kernel μ . In this paper, we give a new definition of “the characteristic set” $\text{Char}(\mu^*)$ using terms of zeros of the total symbol of μ^* , and show, under the abstract condition (S), the equivalence between two notions of characteristics outside of the zero section $T_{\sqrt{-1}\mathbf{R}^n}^*(\sqrt{-1}\mathbf{R}^n)$. Moreover we conclude that the micro-support $\text{SS}(\mathcal{S})$ of \mathcal{S} coincides with the characteristics $\text{Char}(\mu^*)$.

0. Introduction.

Convolution equations are a natural extension of linear partial differential equations with constant coefficients, and have been studied in various situations. The solvability and the continuation of holomorphic solutions have attracted many researchers.

In the case where the kernels are analytic functionals, the existence of holomorphic solutions in the complex domain was considered by Malgrange [11]. In particular, Korobeĭnik [8] and Epifanov [2] gave a complete answer to the problem of the existence in one-dimensional case.

For the equations with analytic functional kernels supported by the origin, the results of Kawai [10] are outstanding. Such equations become differential equations of finite or infinite order with constant coefficients. In the case of finite-order differential equations, Kiselman [6] considered the continuation of holomorphic solutions. Refer also to Sebbar [13] for the case of infinite-order operators with constant coefficients, and to Aoki [1] for the case of infinite-order operators with variable coefficients.

In the case where the kernels are supported by the real axis, that is, where the kernels are hyperfunctions, the convolution operators act on the space of hyperfunctions or of Fourier hyperfunctions. Kawai [10] proved surjectivity of such operators under some natural condition (called Condition (S) in [10]). He also constructed parametrix

and gave estimates of singularities. In this direction, we also recall Kaneko [7] for a structure theorem of hyperfunctions, the author [12] for the surjectivity of convolution operators, and Ishimura [4] for the estimate of characteristics.

Ishimura and the author in [5] studied the existence and the continuation of holomorphic solutions in tube domains for convolution equations with hyperfunction kernels. These two problems can be formulated in a unified way by means of the complex \mathcal{S} on the purely imaginary space $\sqrt{-1}\mathbf{R}^n$ generated by the convolution operator μ^* which operates on the spaces of holomorphic functions in tube domains. Then the characteristics are defined in terms of the exponential behavior of the total symbol of μ^* . They showed, under condition (S) due to Kawai [10], that the micro-support of the complex is included in the characteristics.

In the present paper, we will give another definition of characteristics in terms of the zeros of the total symbol (§2). The new definition given in this paper is essentially equivalent to our previous one (Corollary 2.6), but the new one is much simpler and convenient for checking examples. We will also show that the characteristic set of μ^* coincides with the micro-support of the complex \mathcal{S} (§3).

1. Preliminaries.

Let \mathcal{O} be the sheaf of germs of holomorphic functions on \mathbf{C}^n . We make the identification $\mathbf{C}^n \cong \mathbf{R}^n \times \sqrt{-1}\mathbf{R}^n$. Then we denote by τ the natural projection \mathbf{C}^n to the purely imaginary space $\sqrt{-1}\mathbf{R}^n$. We introduce the sheaf \mathcal{O}^τ on $\sqrt{-1}\mathbf{R}^n$ by

$$\mathcal{O}^\tau := \tau_*\mathcal{O}.$$

For a hyperfunction $\mu \in \mathcal{B}_{\mathbf{R}^n}$ with compact support, the convolution operator $P := \mu^*$ becomes a sheaf morphism of \mathcal{O}^τ and induces the complex

$$\mathcal{S} : 0 \rightarrow \mathcal{O}^\tau \xrightarrow{P} \mathcal{O}^\tau \rightarrow 0,$$

on $\sqrt{-1}\mathbf{R}^n$. The aim of the present note is to clarify how the micro-support $\text{SS}(\mathcal{S})$ of \mathcal{S} is estimated by the kernel μ . We refer to Kashiwara-Schapira [9] for notions such as sheaves, complexes, derived functors, and micro-supports.

For a holomorphic function f defined on an open convex subset U in \mathbf{C}^n and a compact subset K in U , we define the semi-norm $\|f\|_K$ by

$$\|f\|_K := \sup_{z \in K} |f(z)|.$$

Note that the system of semi-norms $\{\|\cdot\|_K\}_{K \subset\subset U}$ defines (FS) topology on the space $\mathcal{O}(U)$.

For an analytic functional $T \in \mathcal{O}(\mathbf{C}^n)'$, we denote by $\hat{T}(\zeta)$ its Fourier-Borel transform

$$\hat{T}(\zeta) = \langle T, e^{z \cdot \zeta} \rangle_z.$$

According to the theorem of Polyà-Ehrenpreis-Martineau, $\hat{T}(\zeta)$ is an entire function of exponential type satisfying the following estimate. If T is supported by a compact set K in \mathbf{C}^n , then for every $\varepsilon > 0$, we can take a constant $C_\varepsilon > 0$ satisfying

$$|\hat{T}(\zeta)| \leq C_\varepsilon \exp(H_K(\zeta) + \varepsilon|\zeta|).$$

Here $H_K(\zeta) := \sup_{z \in K} \operatorname{Re}\langle z, \zeta \rangle$ is the supporting function of K . In particular, if μ is a hyperfunction with compact support, its Fourier-Borel transform $\hat{\mu}(\zeta)$ has an infra-exponential growth on $\sqrt{-1}\mathbf{R}^n$: for any $\varepsilon > 0$, we have

$$\sup_{\sqrt{-1}\eta \in \sqrt{-1}\mathbf{R}^n} |\hat{\mu}(\sqrt{-1}\eta)e^{-\varepsilon|\eta|}| < \infty.$$

The system of coordinates of the dual complex space \mathbf{C}^n is denoted by $\zeta = \xi + \sqrt{-1}\eta$ with $\xi, \eta \in \mathbf{R}^n$. For $R > 0$ and $\zeta_0 \in \mathbf{C}^n$, we denote by $B(\zeta_0; R)$ the open ball centered at ζ_0 with radius R in \mathbf{C}^n .

In this paper, we suppose the following condition (S) due to T. Kawai [10] for the entire function $f(\zeta) = \hat{\mu}(\zeta)$.

$$(S) \quad \begin{cases} \text{For every } \varepsilon > 0, \text{ there exists } N > 0 \text{ such that} \\ \text{for any } \sqrt{-1}\eta \in \sqrt{-1}\mathbf{R}^n \text{ with } |\eta| > N, \\ \text{we can find } \zeta \in B(\sqrt{-1}\eta; \varepsilon|\eta|) \text{ satisfying } |f(\zeta)| \geq e^{-\varepsilon|\eta|}. \end{cases}$$

We recall the definition of the characteristic set $\operatorname{Char}_\infty(P)$ at infinity. We define the sphere at infinity S_∞^{2n-1} by $(\mathbf{C}^n \setminus \{0\})/\mathbf{R}_+$. Then ζ_∞ denotes the equivalent class of $\zeta \in \mathbf{C}^n \setminus \{0\}$. We define a natural compactification $\mathbf{D}^{2n} = \mathbf{C}^n \sqcup S_\infty^{2n-1}$ of \mathbf{C}^n . We denote by $\sqrt{-1}S_\infty^{n-1}$ the pure imaginary sphere at infinity $\{(\xi + \sqrt{-1}\eta)_\infty \in S_\infty^{2n-1}; \xi = 0\}$, which is a closed subset in S_∞^{2n-1} . For an entire function f and $\varepsilon > 0$, we set

$$V_f(\varepsilon) := \{\zeta \in \mathbf{C}^n; e^{\varepsilon|\zeta|}|f(\zeta)| < 1\},$$

$$W_f(\varepsilon) := \sqrt{-1}S_\infty^{n-1} \cap (\text{the closure of } V_f(\varepsilon) \text{ in } \mathbf{D}^{2n}).$$

DEFINITION 1.1. ([5, Definition 4.3]). Under the above notation, we define the *characteristics* of μ^* (at infinity)

$$\operatorname{Char}_\infty(\mu^*) := \text{the closure of } \bigcup_{\varepsilon > 0} W_\mu(\varepsilon),$$

which is a closed set in $\sqrt{-1}S_\infty^{n-1}$.

2. A new definition of characteristics.

In this section, we will give a new definition of characteristics in terms of zeros of the total symbol $\hat{\mu}$, and give an estimate of the modulus of $\hat{\mu}$ from below outside characteristic directions, which shows the equivalence between two definitions of characteristics. To give the definition, we utilize the topological spaces $X := \mathbf{R} \times \mathbf{C}^n$ and $X_+ := \{(t, \zeta) \in X; t > 0\}$, and also utilize the diagram:

$$\mathbf{C}^n \xhookrightarrow{\iota} X \xhookrightarrow{j} X_+ \xrightarrow{\varpi} \mathbf{C}^n.$$

Here ι is the closed embedding $\iota(\zeta) = (0, \zeta)$, j the natural inclusion, and ϖ the map defined by $\varpi(t, \zeta) = t^{-1}\zeta$.

DEFINITION 2.1. For an entire function $f \in \mathcal{O}(\mathbf{C}^n)$, we define the set $Z_\infty(f) \subset \mathbf{C}^n$ by

$$Z_\infty(f) = \iota^{-1}(\text{the closure of } j(\varpi^{-1}(\{f = 0\})) \text{ in } X).$$

Let μ be a hyperfunction with compact support and $P = \mu*$ the convolution operator with kernel μ . We define the characteristics $\text{Char}(P) \subset T^*(\sqrt{-1}\mathbf{R}^n)$ of P by

$$(2.1) \quad \text{Char}(P) = \{(\sqrt{-1}y, \sqrt{-1}\eta); \sqrt{-1}\eta \in Z_\infty(\hat{\mu}) \cap \sqrt{-1}\mathbf{R}^n\}.$$

REMARK 2.2. Note that $Z_\infty(f)$ is a closed and that $\text{Char}(P)$ is a closed subset of $T^*(\sqrt{-1}\mathbf{R}^n)$. For a non-zero vector $\dot{\zeta} \in \sqrt{-1}\mathbf{R}^n$, $\dot{\zeta}$ does not belong to $Z_\infty(f)$ if and only if there exist an open cone $\gamma \subset \mathbf{C}^n$ containing $\dot{\zeta}$ and a constant $N > 0$ satisfying $\{|\zeta| > N\} \cap \gamma \cap \{f = 0\} = \emptyset$. Conversely, $\dot{\zeta}$ belongs to $Z_\infty(f)$ if and only if there exists a sequence $\{\zeta_k\}_k \subset \{f = 0\}$ with $|\zeta_k| \rightarrow \infty$ and $\zeta_k/|\zeta_k| \rightarrow \dot{\zeta}/|\dot{\zeta}|$ as $k \rightarrow \infty$. The condition $0 \in Z_\infty(f)$ is equivalent to $\{f = 0\} \neq \emptyset$.

In order to show the equivalence between the two definitions of characteristics, we need a minimum modulus estimate for holomorphic functions of infra-exponential growth on a direction.

LEMMA 2.3. *Let f be a holomorphic function defined in an open cone $\gamma \subset \mathbf{C}^n$. We assume that $f \neq 0$ on γ . Let $\rho \in \gamma$ be a vector with $|\rho| = 1$. We assume that f satisfies the estimate of infra-exponential type with respect to the direction ρ :*

$$(2.2) \quad \begin{cases} \text{for any } \varepsilon > 0, \text{ there exists an open cone } \gamma' \subset \gamma \\ \text{containing } \rho \text{ with } \sup_{\zeta \in \gamma'} |f(\zeta)| \exp(-\varepsilon|\zeta|) < \infty. \end{cases}$$

We also assume the following localized condition of (S) with respect to the direction ρ :

$$(S_\rho) \quad \begin{cases} \text{for every } \varepsilon > 0, \text{ there exists } N > 0 \text{ such that for any } r > N \\ \text{we can find } \zeta \in \gamma \cap B(rp; \varepsilon r) \text{ satisfying } |f(\zeta)| \geq e^{-\varepsilon r}. \end{cases}$$

Then for any $\varepsilon > 0$, we can find an open cone $\Gamma_\varepsilon \subset \gamma$ containing ρ and constants N'_ε and C'_ε with the estimate

$$(2.3) \quad |f(\zeta)| \geq C'_\varepsilon \exp(-\varepsilon|\zeta|) \quad \text{if } \zeta \in \Gamma_\varepsilon, \quad |\zeta| > N'_\varepsilon.$$

We remark that the condition (2.2) is always satisfied for the Fourier-Borel transform $\hat{\mu}$ of a hyperfunction μ with compact support and a purely imaginary vector $\rho \in \sqrt{-1}\mathbf{R}^n$ with $|\rho| = 1$. Moreover for an entire function f , the condition (S) implies the condition (S_ρ) for any $\rho \in \sqrt{-1}\mathbf{R}^n$.

PROOF OF LEMMA 2.3. For a given constant $\varepsilon > 0$, take $\varepsilon' > 0$ with the properties

$$\frac{9\varepsilon'}{1 - \varepsilon'} < \varepsilon,$$

and

$$B(\rho; 4\varepsilon') \subset \gamma.$$

On account of (2.2), we can take constants δ with $0 < \delta < \varepsilon'$ and $M_{\varepsilon'}$ so that for any $r > 0$ f satisfies

$$(2.4) \quad |f(\zeta)| < M_{\varepsilon'} \exp(\varepsilon' r)$$

on the ball $B(rp; 4\delta r)$. The condition (S_ρ) let us choose such a constant $N_\delta > 0$ that for any $r > N_\delta$, there exist $\zeta' \in B(rp; \delta r)$ satisfying

$$|f(\zeta')| > \exp(-\delta r).$$

Here we recall a lemma of Harnack-Malgrange-Hörmander ([3, Lemma 3.1]).

LEMMA 2.4. Let $F(\zeta)$, $H(\zeta)$ and $G(\zeta) = H(\zeta)/F(\zeta)$ be three holomorphic functions in the open ball $B(0; R)$. If $|F(\zeta)| < A$ and $|H(\zeta)| < B$ hold on $B(0; R)$, then the estimate

$$|G(\zeta)| \leq BA^{2|\zeta|/(R-|\zeta|)} |F(0)|^{-(R+|\zeta|)/(R-|\zeta|)}$$

holds for all $\zeta \in B(0; R)$.

We apply this lemma to $F = f$, $G = 1/f$, and $H = 1$ on the ball $B(\zeta'; 3\delta r)$. By the inclusions $B(rp; \delta r) \subset B(\zeta'; 2\delta r) \subset B(\zeta'; 3\delta r) \subset B(rp; 4\delta r)$, we have, by (2.4),

$$\sup_{\zeta \in B(\zeta'; 3\delta r)} |f(\zeta)| \leq M_{\varepsilon'} \exp(\varepsilon' r).$$

Hence we get the estimate for $r > N_\delta$

$$\begin{aligned}
& \sup_{\zeta \in B(rp; \delta r)} |1/f(\zeta)| \\
& \leq \sup_{\zeta \in B(\zeta'; 2\delta r)} |1/f(\zeta)| \\
& \leq (M_{\varepsilon'} \exp(\varepsilon' r))^{(2 \cdot 2\delta r)/(3\delta r - 2\delta r)} (\exp(-\delta r))^{-(3\delta r + 2\delta r)/(3\delta r - 2\delta r)} \\
& = M_{\varepsilon'}^4 \exp(4\varepsilon' r + 5\delta r) \\
& \leq M_{\varepsilon'}^4 \exp(9\varepsilon' r).
\end{aligned}$$

Since $\zeta \in B(rp; \delta r)$ satisfies the estimate $(1 - \delta)r < |\zeta|$, we get

$$\begin{aligned}
|f(\zeta)| & \geq M_{\varepsilon'}^{-4} \exp\left(-\frac{9\varepsilon'}{1 - \delta} |\zeta|\right) \\
& \geq M_{\varepsilon'}^{-4} \exp\left(-\frac{9\varepsilon'}{1 - \varepsilon'} |\zeta|\right)
\end{aligned}$$

where $\zeta \in B(rp; \delta r)$.

We set

$$C'_\varepsilon = M_{\varepsilon'}^{-4},$$

$$N'_\varepsilon = N_\delta,$$

$$\Gamma_\varepsilon = \bigcup_{r>0} B(rp; \delta r).$$

Then we have the desired result (2.3). (q.e.d. for Lemma 2.3). □

Now we give

THEOREM 2.5. *Let μ be a hyperfunction with compact support, and $\sqrt{-1}\eta$ a vector in $\sqrt{-1}\mathbf{R}^n$ with $\eta \neq 0$. Assume that $\hat{\mu}$ satisfies the condition (S). Then $\sqrt{-1}\eta$ belongs to $Z_\infty(\hat{\mu})$ if and only if $\sqrt{-1}\eta_\infty$ belongs to $\text{Char}_\infty(\mu^*)$.*

COROLLARY 2.6. *Under the same hypothesis, we have*

$$\text{Char}(\mu^*) \setminus T_{\sqrt{-1}\mathbf{R}^n}^*(\sqrt{-1}\mathbf{R}^n) = \sqrt{-1}\mathbf{R}^n \times p^{-1}(\text{Char}_\infty(\mu^*)).$$

Here p is the projection $\sqrt{-1}\mathbf{R}^n \setminus \{0\} \rightarrow \sqrt{-1}S_\infty^{n-1}$ defined by $p(\sqrt{-1}\eta) = \sqrt{-1}\eta_\infty$.

PROOF OF THEOREM 2.5. We set $f = \hat{\mu}$.

First we show that if $\sqrt{-1}\dot{\eta} \in Z_\infty(f)$ then $\sqrt{-1}\dot{\eta}_\infty \in \text{Char}_\infty(\mu^*)$. There exists a sequence $\{\zeta_k\}_k$ in $\mathbf{C}^n \setminus \{0\}$ satisfying

$$(2.5) \quad f(\zeta_k) = 0 \quad \text{for any } k,$$

$$(2.6) \quad \lim_{k \rightarrow \infty} |\zeta_k| = \infty,$$

$$(2.7) \quad \lim_{k \rightarrow \infty} \zeta_k / |\zeta_k| = \sqrt{-1}\dot{\eta} / |\dot{\eta}|.$$

From (2.5) we have $\{\zeta_k\}_k \subset V_f(\varepsilon)$ for any $\varepsilon > 0$, and from (2.6) and (2.7) it follows that $\{\zeta_k\}_k$ converges to $\sqrt{-1}\dot{\eta}_\infty$ in \mathbf{D}^{2n} . Thus $\sqrt{-1}\dot{\eta}_\infty$ belongs to $\text{Char}_\infty(f)$.

Next we show that if $\sqrt{-1}\dot{\eta} \notin Z_\infty(f)$ then $\sqrt{-1}\dot{\eta}_\infty \notin \text{Char}_\infty(\mu^*)$. Since $Z_\infty(f)$ is closed, we can take an open neighborhood $U \subset \sqrt{-1}\mathbf{R}^n \setminus \{0\}$ of $\sqrt{-1}\dot{\eta}$ which does not meet $Z_\infty(f)$. For any vector $\sqrt{-1}\eta \in U$, we can take an open cone $\gamma \subset \mathbf{C}^n$ containing $\sqrt{-1}\eta$ and a constant $r > 0$ with $(\sqrt{-1}r\eta + \gamma) \cap \{f = 0\} = \emptyset$. By applying Lemma 2.3 to the function $g(\zeta) := f(\zeta + \sqrt{-1}r\eta)$, a vector $\rho = \sqrt{-1}\eta / |\eta|$, and the cone γ , we have

$$(2.8) \quad \sqrt{-1}\eta_\infty \notin W_f(\varepsilon) \quad \text{for any } \varepsilon > 0.$$

From (2.8), we can deduce

$$\{\sqrt{-1}\eta_\infty; \sqrt{-1}\eta \in U\} \cap W_f(\varepsilon) = \emptyset \quad \text{for any } \varepsilon > 0,$$

which shows $\sqrt{-1}\dot{\eta} \notin \text{Char}_\infty(f)$. □

3. The inverse inclusion between the micro-support and the characteristics.

In this section, we denote by \mathcal{N} , the sheaf of \mathcal{O}^τ -solutions of the homogeneous equation $\mu * g = 0$. Namely, for any open set $\sqrt{-1}\omega \subset \sqrt{-1}\mathbf{R}^n$, we set

$$\mathcal{N}(\sqrt{-1}\omega) := \{g \in \mathcal{O}^\tau(\sqrt{-1}\omega); \mu * g = 0\}.$$

For a convex subset $\sqrt{-1}\omega \subset \sqrt{-1}\mathbf{R}^n$, the space $\mathcal{N}(\sqrt{-1}\omega)$ is a closed subspace of $\mathcal{O}^\tau(\sqrt{-1}\omega) = \mathcal{O}(\mathbf{R}^n \times \sqrt{-1}\omega)$. Thus $\mathcal{N}(\sqrt{-1}\omega)$ is an (FS) space by the induced topology.

PROPOSITION 3.1. *Let μ be a hyperfunction with compact support and $\sqrt{-1}\dot{\eta}$ a vector in $Z_\infty(\hat{\mu})$ with $|\dot{\eta}| = 1$. For any open convex subset $\sqrt{-1}\Omega \subset \sqrt{-1}\mathbf{R}^n$ and any point $\sqrt{-1}\dot{y} \in \sqrt{-1}\Omega$, we set*

$$\sqrt{-1}\Omega' := \{\sqrt{-1}y \in \sqrt{-1}\Omega; (y - \dot{y}) \cdot \dot{\eta} < 0\}.$$

Then the restriction map

$$r : \mathcal{N}(\sqrt{-1}\Omega) \rightarrow \mathcal{N}(\sqrt{-1}\Omega')$$

is not surjective.

Note that this proposition holds without assuming the condition (S).

PROOF. We will prove this proposition by contradiction. Assume that the restriction map r is surjective. Since r is injective and continuous, we can deduce, from the open mapping theorem, that r must be a topological isomorphism.

Take a sequence $\{\zeta_k\}_k$ in $\mathbf{C}^n \setminus \{0\}$ satisfying

$$\hat{\mu}(\zeta_k) = 0 \quad \text{for any } k,$$

$$\lim_{k \rightarrow \infty} |\zeta_k| = \infty,$$

$$\lim_{k \rightarrow \infty} \zeta_k / |\zeta_k| = \sqrt{-1}\dot{\eta} / |\dot{\eta}|,$$

and set

$$\varphi_k(z) := |\zeta_k| \exp(-\zeta_k \cdot (z - \sqrt{-1}\dot{y})).$$

Then we can easily show that each φ_k satisfies $\mu * \varphi_k = 0$.

CLAIM 3.2. *The sequence $\{\varphi_k\}_k$ converges to 0 in $\mathcal{N}(\sqrt{-1}\Omega')$.*

PROOF. For any compact subset K in $\mathbf{R}^n \times \sqrt{-1}\Omega'$, we can take a constant $\varepsilon > 0$ and an open cone $\gamma \subset \mathbf{C}^n$ containing $\sqrt{-1}\dot{\eta}$ which enjoy the estimate

$$\operatorname{Re} -\zeta \cdot (z - \sqrt{-1}\dot{y}) < -\varepsilon|\zeta|$$

for any $\zeta \in \gamma$ and any $z \in K$. If k is sufficiently large, then ζ_k belongs to γ and we get the estimate

$$\|\varphi_k\|_K < |\zeta_k| \exp(-\varepsilon|\zeta_k|).$$

Since the right hand side converges to 0, we deduce

$$\lim_{k \rightarrow \infty} \|\varphi_k\|_K = 0.$$

(q.e.d. for Claim). □

CLAIM 3.3. *The sequence $\{\varphi_k\}_k$ is not a convergent series in $\mathcal{N}(\sqrt{-1}\Omega)$.*

PROOF. Take $K := \{\sqrt{-1}j\} \subset \Omega$. Then we get

$$\|\varphi_k\|_K = |\varphi_k(\sqrt{-1}j)| = |\zeta_k|$$

for any k . Since the right hand side diverges, $\{\varphi_k\}_k$ can not be a convergent series in $\mathcal{N}(\sqrt{-1}\Omega)$. (q.e.d. for Claim). \square

From the above two Claims, it follows that the sequence $\{\varphi_k\}_k$ is a convergent series in $\mathcal{N}(\sqrt{-1}\Omega')$ but not a convergent series in $\mathcal{N}(\sqrt{-1}\Omega)$, which is a contradiction. (q.e.d. for Proposition 3.1). \square

Now we state our main theorem.

THEOREM 3.4. *Let $P = \mu*$ be a convolution operator with kernel μ and \mathcal{S} the complex $0 \rightarrow \mathcal{O}^\tau \xrightarrow{P} \mathcal{O}^\tau \rightarrow 0$. Assume that $\hat{\mu}$ satisfies the condition (S). Then we have*

$$\text{SS}(\mathcal{S}) = \text{Char}(P).$$

PROOF. Outside the zero section $T_{\sqrt{-1}\mathbf{R}^n}^*(\sqrt{-1}\mathbf{R}^n) = \sqrt{-1}\mathbf{R}^n$, the inclusion $\text{SS}(\mathcal{S}) \setminus \sqrt{-1}\mathbf{R}^n \subset \text{Char}(P) \setminus \sqrt{-1}\mathbf{R}^n$ is proved in [5, Theorem 5.2], and the inverse inclusion can be deduced from Proposition 3.1.

The equality $\text{SS}(\mathcal{S}) \cap \sqrt{-1}\mathbf{R}^n = \text{Char}(P) \cap \sqrt{-1}\mathbf{R}^n$ is an easy corollary of the following lemma. \square

LEMMA 3.5. *$\mathcal{N} = 0$ if and only if $\hat{\mu}$ has no zeros.*

PROOF. If there exists $\check{\zeta} \in \mathbf{C}^n$ with $\hat{\mu}(\check{\zeta}) = 0$, the entire function $\exp(-\check{\zeta} \cdot z)$ is a solution of $\mu * g = 0$.

Conversely, assume that $\hat{\mu}$ has no zeros. Then there exist a constant $C \neq 0$ and a real vector $\dot{x} \in \mathbf{R}^n$ with $\hat{\mu}(\zeta) = C \exp(\dot{x} \cdot \zeta)$, which implies $\mu = C\delta(x - \dot{x})$. Here δ denotes Dirac's delta function. Thus the operator $\mu*$ is the composition of the multiplication by the constant C and the translation $g(z) \mapsto g(z - \dot{x})$, which shows $\mathcal{N} = 0$. \square

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