

Quantum double construction for subfactors arising from periodic commuting squares

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Abstract. By generalizing Erlijman's method, we construct a subfactor from a fusion rule algebra with quantum $6j$ -symbols which produce periodic commuting squares. This construction produces the same subfactor as Ocneanu's asymptotic inclusion for the subfactor which is generated by the original periodic commuting square. This result can be applied to the quantum $SU(n)_k$ subfactors which is the same as Hecke algebra subfactors of type A of Wenzl for example, which shows that Erlijman's construction gives the same subfactor as the asymptotic inclusion.

1. Introduction.

V. F. R. Jones introduced the index for subfactors in [10] and he found his celebrated polynomial invariant for knots by using the subfactor theory in [11]. This work has revealed an unexpected relation between the subfactor theory and the 3-dimensional topology. A. Ocneanu's paragroup theory (see [17], [18], [19], [14], [30] for the definition of *paragroup* for example) also has revealed the deep relation between the subfactor theory and the quantum group theory, 3-dimensional topology and rational conformal field theory etc. (See [1], [8], [9], [19], [20], [22], [24], [29] etc. for these topics.)

A. Ocneanu ([17], [18]) introduced the *asymptotic inclusion* $M \vee (M' \cap M_\infty) \subset M_\infty$ for a factor inclusion $N \subset M$, where $N \subset M = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_\infty = \bigvee_{n=0}^\infty M_n$ is the Jones tower. In the following we will also use the notation $M \supset N = M_{-1} \supset M_{-2} \supset \cdots$ for the downward basic construction. The notion "asymptotic inclusion" is regarded as the right analogue of the quantum double construction of Drinfel'd [3] (see [23] and [9]). In [21] he claimed that the combinatorial data satisfying Moore–Seiberg axiom ([16]) in the rational conformal field theory can be constructed after passing to the asymptotic inclusion from a given paragroup (see [9, Section 5]). He also says in [24] that if the fusion graph of the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$ is connected then the system of $M_\infty - M_\infty$ bimodules are braided and non-degenerate. And if we have a braided and non-degenerate system of bimodules then we get Reshetikhin–Turaev type invariant based on surgery ([26]). So the asymptotic inclusions are important for these reasons and others.

Most fundamental examples of the asymptotic inclusions are subfactors generated by commuting squares of two-sided sequence of Jones projections;

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$$\begin{array}{ccc}
\langle e_{-n}, \dots, e_{-1}, e_1, \dots, e_n \rangle & \subset & \langle e_{-n-1}, \dots, e_{-1}, e_1, \dots, e_{n+1} \rangle \\
\cap & & \cap \\
\langle e_{-n}, \dots, e_{-1}, e_0, e_1, \dots, e_n \rangle & \subset & \langle e_{-n-1}, \dots, e_{-1}, e_0, e_1, \dots, e_{n+1} \rangle.
\end{array}$$

Here Jones projections $\{e_i\}_{i \in \mathbb{Z}}$ satisfy the following relations;

$$\begin{aligned}
e_i e_{i \pm 1} e_i &= \beta^{-2} e_i, \quad \text{for } i \in \mathbb{Z}, \\
e_i e_j &= e_j e_i, \quad \text{whenever } |i - j| \neq 2,
\end{aligned}$$

where $\beta = 2 \cos(\pi/N)$. We remark that the above commuting squares have periodicity 2 in the sense of Wenzl (see [27, page 357] for the definition of periodicity of commuting squares). The indices of the subfactors were first computed by M. Choda in [2]. The above subfactor is easily shown to be isomorphic to the asymptotic inclusion of Jones' subfactor with the principal graph A_n . A proof of this fact follows from a general fact that the commuting square

$$\begin{array}{ccc}
M'_{-k} \cap M_0 \vee M'_0 \cap M_k & \subset & M'_{-k-1} \cap M_0 \vee M'_0 \cap M_{k+1} \\
\cap & & \cap \\
M'_{-k} \cap M_k & \subset & M'_{-k-1} \cap M_{k+1}
\end{array}$$

generates the asymptotic inclusion for a large enough k .

Another fundamental example is group case, which was first claimed in [18, III.3] (see [15, Appendix] for a complete proof). That is, if we start with a finite group G and consider the subfactor $R^G \subset R$, where R is the AFD II_1 factor, then the asymptotic inclusion is of the form $R^{G \times G} \subset R^G$, where $G \times G$ acts freely on R with G embedded into $G \times G$ with a map $g \mapsto (g, g)$. This example gives one of the reason why the asymptotic inclusion is analogous to the quantum double construction.

Recently J. Erlijman showed in [4] that the following commuting squares of two-sided sequence of generators $\{g_i\}_{i \in \mathbb{Z}}$ of Hecke algebra of type B, C, D produce the asymptotic inclusion for the Hecke algebra subfactors of type B, C, D of Wenzl [28]:

$$\begin{array}{ccc}
\langle g_{-n}, \dots, g_{-1}, g_1, \dots, g_n \rangle & \subset & \langle g_{-n-1}, \dots, g_{-1}, g_1, \dots, g_{n+1} \rangle \\
\cap & & \cap \\
\langle g_{-n}, \dots, g_{-1}, g_0, g_1, \dots, g_n \rangle & \subset & \langle g_{-n-1}, \dots, g_{-1}, g_0, g_1, \dots, g_{n+1} \rangle.
\end{array}$$

In these cases the commuting squares have the period 2. But in the Hecke algebra of type A case, they have the period n ($n \geq 2$) in general. And the period 2 case is nothing but the above examples of two-sided sequence of Jones projections. J. Erlijman also showed in [5] that even in the Hecke algebra of type A case the subfactor generated by the above commuting square is isomorphic to the asymptotic inclusion for the Hecke algebra subfactors of type A of Wenzl.

In this paper we generalize her construction of subfactors to a fusion rule algebra with quantum $6j$ -symbols which produce periodic commuting squares. We prove that this construction produces the same subfactor as the asymptotic inclusion for the subfactor generated by the original periodic commuting square. We also give some

examples and if we apply this result in the case of fusion rule algebras of $SU(n)_k$ WZW models, which is the same as Hecke algebra subfactors of type A of Wenzl [27], we get the same result as in [5]. That is, the above two-sided sequence of Hecke algebra of type A generators produces the asymptotic inclusion of the Hecke algebra subfactor of type A .

According to the A. Ocneanu's theory as in [24] and [9], we can get a lot of combinatorial data of RCFT and Reshetikhin–Turaev type topological invariants by our method.

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2. Quantum double construction for subfactors arising from periodic commuting squares.

We start with a finite fusion rule algebra \mathcal{A} with quantum $6j$ -symbols satisfying unitarity, tetrahedral symmetry, and the pentagon equation (see [19] [7] for the definitions of these). We denote the standard basis of \mathcal{A} by $\{a_j\}_{j \in J}$. Fix an element $h = a_j$ for some $j \in J$ and take finite powers h^k of h and decompose them into sums of basis in \mathcal{A} . Thus we get a fusion rule subalgebra \mathcal{B} of \mathcal{A} generated by h and quantum $6j$ -symbols restricted to the subalgebra \mathcal{B} .

We assume that $h^n \succ * = id$, that is, h^n contains an element $* = id$ in the expression of h^n as a sum of elements in the basis of \mathcal{B} . Here id represents the identity element of the fusion rule algebra \mathcal{A} . We take the smallest n satisfying the condition $h^n \succ * = id$ and denote it by the same n . Then a natural finite grading arises in the elements of the basis of \mathcal{B} as follows. We set Ω_k for all $k \pmod{n}$ a subset of \mathcal{B} consist of elements of the basis of \mathcal{B} which appear in the expression of h^k as a sum of the elements in the basis of \mathcal{B} . Then the basis of \mathcal{B} is decomposed into a union of finite subsets $\Omega_0, \Omega_1, \dots, \Omega_{n-1}$ so that they satisfy the following.

1. $* = id \in \Omega_0$, $h \in \Omega_1$,
2. if x is in Ω_k , then $x \cdot h$ can be decomposed into a sum of elements in Ω_{k+1} for $k, k+1 \in \mathbb{N} \pmod{n}$.

DEFINITION 2.1. We call a fusion rule subalgebra \mathcal{B} generated by an element $h \in \mathcal{A}$ of a fusion rule algebra \mathcal{A} *periodic* when the generator h of \mathcal{B} satisfies $h^n \succ id$. The smallest such n is called the *period* of the subalgebra \mathcal{B} .

We remark that if the system \mathcal{B} is periodic with the period n , then it produces a periodic commuting square with the same period n in the following way. Hence we get a subfactor with the finite index. (See [27] Lemma 1.4.) We make a double sequence of string algebras $\{A_{k,l}\}_{k,l}$ which is a modified version of the original string algebra construction from a paragroup as follows. First we put $*$ in the two upper left corners. Then we pass to the right by multiplying the generator h from the right and pass to the downward direction by multiplying h from the left. In this way we get periodic commuting squares of string algebras with the period n both in the horizontal and vertical directions.

$$\begin{array}{ccccccc}
& & A_{-1,1} & \subset & A_{-1,2} & \subset & A_{-1,3} & \cdots & \rightarrow & P \\
& & \cap & & \cap & & \cap & & & \cap \\
A_{0,0} & \subset & A_{0,1} & \subset & A_{0,2} & \subset & A_{0,3} & \cdots & \rightarrow & Q \\
& \cap & \cap & & \cap & & \cap & & & \cap \\
A_{1,0} & \subset & A_{1,1} & \subset & A_{1,2} & \subset & A_{1,3} & \cdots & \rightarrow & R_1 \\
& \cap & \cap & & \cap & & \cap & & & \cap \\
\vdots & & \vdots & & \vdots & & \vdots & & & \vdots
\end{array}$$

Here we used the notations $P \subset Q \subset R_1 \subset R_2 \cdots$ because the tower is *not* obtained by the iteration of the Jones' basic constructions. Note that the grading of the vertices of these commuting squares are illustrated as follows:

$$\begin{array}{ccccccc}
& & 0 & 1 & \cdots & n-2 & n-1 & 0 & \cdots \\
0 & 1 & 2 & \cdots & n-1 & 0 & 1 & 2 & \cdots \\
1 & 2 & 3 & \cdots & 0 & 1 & 2 & 3 & \cdots \\
2 & 3 & 4 & \cdots & 1 & 2 & 3 & 4 & \cdots \\
3 & 4 & 5 & \cdots & 2 & 3 & 4 & 5 & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
n-1 & 0 & 1 & \cdots & n-2 & n-1 & 0 & 1 & \cdots \\
0 & 1 & 2 & \cdots & n-1 & 0 & 1 & 2 & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{array}$$

We can show that the connection on the above commuting squares is *flat* in the following sense, that is, the horizontal string algebra $A_{0,k}$ (resp. $A_{-1,k}$) commutes with the vertical string algebra $A_{l,0}$ (resp. $A_{l,1}$). (See [17], [18], [12] or [13] for the definition of the flatness in the usual period 2 case.) To prove it, we first remark that the above definition of the flatness in the case of double sequence of periodic commuting squares is equivalent to the next condition, that is, the following identity holds for any choice of bimodules x_1, \dots, x_{2n-1} , y_1, \dots, y_{2m-1} and intertwiners ξ_1, \dots, ξ_{2n} , η_1, \dots, η_{2m} (cf. [13, Theorem 2.1]).

$$\begin{array}{ccccccc}
& & \xrightarrow{\cdot \otimes h} & \xrightarrow{\cdot \otimes h} & & \xrightarrow{\cdot \otimes \bar{h}} & \\
& & \xi_1 & \xi_2 & \cdots & \xi_{2n} & \\
& * & \xrightarrow{\quad} x_1 & \xrightarrow{\quad} x_2 & \cdots & x_{2n-1} & \xrightarrow{\quad} * \\
h \otimes \cdot \downarrow & \eta_1 \downarrow & & & & & \eta_1 \downarrow \\
& y_1 & & & & & y_1 \\
h \otimes \cdot \downarrow & \eta_2 \downarrow & & & & & \eta_2 \downarrow \\
& y_2 & & & & & y_2 \\
& \vdots & & & & & \vdots \\
& y_{2m-1} & & & & & y_{2m-1} \\
\bar{h} \otimes \cdot \downarrow & \eta_{2m} \downarrow & & & & & \eta_{2m} \downarrow \\
& * & \xrightarrow{\quad} x_1 & \xrightarrow{\quad} x_2 & \cdots & x_{2n-1} & \xrightarrow{\quad} * \\
& \xi_1 & \xi_2 & & \xi_{2n} & &
\end{array} = 1.$$

In the above diagram we pass to the right by tensoring h (resp. \bar{h}) from the right in the left (resp. right) half of the horizontal paths and pass to the downward by tensoring h (resp. \bar{h}) from the left in the upper (resp. lower) half of the vertical paths. Then this identity is shown by a slight modification of the original proof in the canonical period two case ([7, Section 4]) if we have the pentagon relation which is one of the axioms for the quantum $6j$ -symbols. This flatness condition is used later to define a *surface bimodule* which has the generator h as the labels of the edges on the boundary.

Because the vertical string algebra has the period n , the vertical graphs are not the (dual) principal graph of the subfactor $P \subset Q$ in spite of the flatness when the period n is greater than 2. But we can get the principal graph of the subfactor by using the following “orientation reversing” method of [6, page 459] (cf. [6, Corollary 3.4, Corollary 3.6]). We change the construction in the vertical direction from multiplying only h to multiplying h and \bar{h} alternately. Note that the system \mathcal{B} contains \bar{h} by the definition of periodicity of \mathcal{B} and the Frobenius reciprocity, i.e., we have $h^{n-1} \succ \bar{h}$. So we obtain a subsystem of \mathcal{B} consisting of elements of the basis appearing in the finite alternating products of h and \bar{h} . We denote this subsystem by \mathcal{C} . In this way we get the grading of the commuting squares changed as follows:

$$\begin{array}{cccccccc}
 & 0 & 1 & 2 & \cdots & n-1 & 0 & 1 & \cdots \\
 0 & 1 & 2 & 3 & \cdots & 0 & 1 & 2 & \cdots \\
 n-1 & 0 & 1 & 2 & \cdots & n-1 & 0 & 1 & \cdots \\
 0 & 1 & 2 & 3 & \cdots & 0 & 1 & 2 & \cdots \\
 n-1 & 0 & 1 & 2 & \cdots & n-1 & 0 & 1 & \cdots \\
 \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots
 \end{array}$$

We denote these modified commuting squares as follows:

$$\begin{array}{ccccccc}
 B_{-1,1} & \subset & B_{-1,2} & \subset & B_{-1,3} & \cdots & \rightarrow N \\
 \cap & & \cap & & \cap & & \cap \\
 B_{0,0} & \subset & B_{0,1} & \subset & B_{0,2} & \subset & B_{0,3} \cdots \rightarrow M \\
 \cap & & \cap & & \cap & & \cap \\
 B_{1,0} & \subset & B_{1,1} & \subset & B_{1,2} & \subset & B_{1,3} \cdots \rightarrow M_1 \\
 \cap & & \cap & & \cap & & \cap \\
 \vdots & & \vdots & & \vdots & & \vdots
 \end{array}$$

Here we remark that the subfactor $N \subset M$ is identical to $P \subset Q$.

Because of the flatness of the connection and Wenzl’s dimension estimate [27, Theorem 1.6], we conclude that the vertical graphs are the (dual) principal graph of this subfactor. (See [6, Corollary 3.4, Corollary 3.6].) So we obtain the canonical double sequence of higher relative commutants by applying the “orientation reversing” method to both horizontal and vertical directions. (See [6, Theorem 3.5].) The grading of the vertices again changes as follows:

$$\begin{array}{cccccccc}
& 0 & n-1 & 0 & \cdots & n-1 & 0 & n-1 & \cdots \\
0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & \cdots \\
n-1 & 0 & n-1 & 0 & \cdots & n-1 & 0 & n-1 & \cdots \\
0 & 1 & 0 & 1 & \cdots & 0 & 1 & 0 & \cdots \\
n-1 & 0 & n-1 & 0 & \cdots & n-1 & 0 & n-1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots
\end{array}$$

We denote these canonical commuting squares as follows:

$$\begin{array}{ccccccc}
C_{-1,1} & \subset & C_{-1,2} & \subset & C_{-1,3} & \cdots & \rightarrow N \\
\cap & & \cap & & \cap & & \cap \\
C_{0,0} & \subset & C_{0,1} & \subset & C_{0,2} & \subset & C_{0,3} \cdots \rightarrow M \\
\cap & & \cap & & \cap & & \cap \\
C_{1,0} & \subset & C_{1,1} & \subset & C_{1,2} & \subset & C_{1,3} \cdots \rightarrow M_1 \\
\cap & & \cap & & \cap & & \cap \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

By using this canonical Jones tower $N \subset M \subset M_1 \subset M_2 \cdots \subset M_\infty$, we can construct the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$. We also get another subfactor $Q \vee (Q' \cap Q_\infty) \subset Q_\infty$ from the previous tower $P \subset Q \subset R_1 \subset R_2 \cdots \subset Q_\infty$. Again we remark that this tower is different from the Jones tower for the inclusion $P \subset Q$.

We remark that the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$ is generated by the following commuting squares:

$$\begin{array}{ccc}
C_{n,0} \vee C_{0,n} & \subset & C_{n+1,0} \vee C_{0,n+1} \\
\cap & & \cap \\
C_{n,n} & \subset & C_{n+1,n+1},
\end{array}$$

and the subfactor $Q \vee (Q' \cap Q_\infty) \subset Q_\infty$ is also generated by the commuting squares:

$$\begin{array}{ccc}
A_{n,0} \vee A_{0,n} & \subset & A_{n+1,0} \vee A_{0,n+1} \\
\cap & & \cap \\
A_{n,n} & \subset & A_{n+1,n+1}.
\end{array}$$

Now we give a graphical expression of bimodules arising from these subfactors $M \vee (M' \cap M_\infty) \subset M_\infty$ and $Q \vee (Q' \cap Q_\infty) \subset Q_\infty$. First remark that the generator h can be identified with M as an M - N bimodule by using canonical commuting squares as above and the graphical expression as in Figure 2.1 (see [23] and [9] for a graphical expression of bimodules).

So the graphical expressions of the bimodules arising from the subfactor $M \vee (M' \cap M_\infty) \subset M_\infty$ are exactly the same as the original ones in [23], [9] by this identification of $h = {}_M M_N$ and $\bar{h} = {}_N M_M$.

Next we give graphical expressions of the bimodules arising from the subfactor $Q \vee (Q' \cap Q_\infty) \subset Q_\infty$. The algebras $Q = M$, $Q \vee (Q' \cap Q_\infty)$ and Q_∞ are expressed as in Figure 2.2, 2.3 and 2.4 respectively. Here the point is that we change all the labels of edges on the boundary from ${}_M M_N$ and ${}_N M_M$ to the generator h . In particular Figure 2.2 is exactly the 2-dimensional expression of the string algebras with period n .

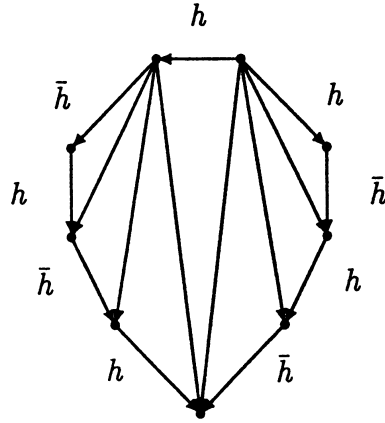


Figure 2.1

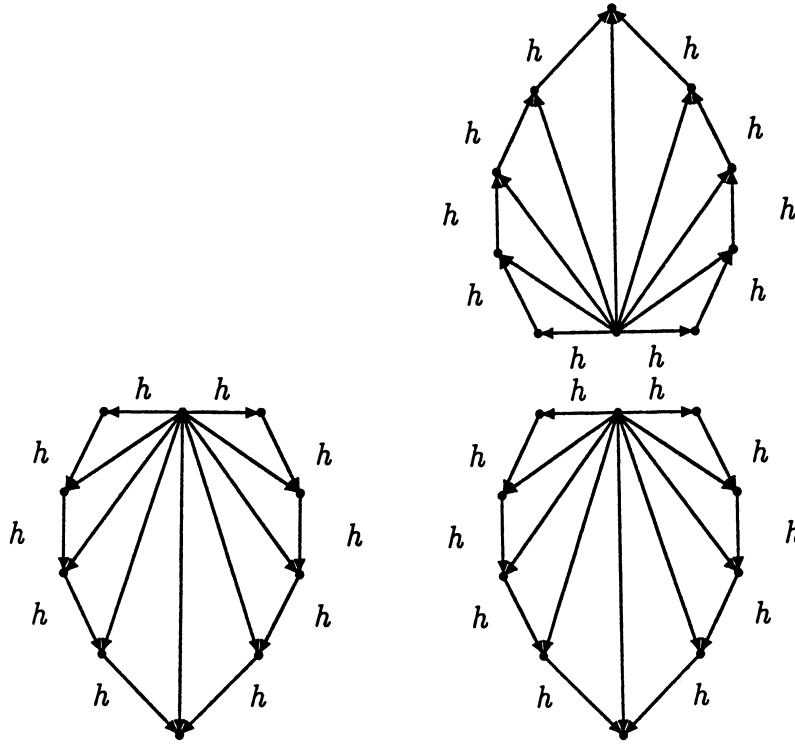


Figure 2.2

Figure 2.3

Here Figure 2.2 actually represents $Q^{\text{op}} = M^{\text{op}}$ but we use this expression instead of the upside-down picture for simplicity.

Similarly if we change the labels of the edges on the boundary from ${}_M M_N$ and ${}_N M_M$ to h , we get a graphical expression of Q - Q bimodule as in Figure 2.5.

Here x is one of the elements in the basis of \mathcal{B} . We denote this Q - Q bimodule by K_x and call it a *surface bimodule*. In the following we use the notation $\mathcal{I} \equiv \Omega_0 \cap \mathcal{C}$ which is a subset of the basis in the system \mathcal{C} .

THEOREM 2.2. *The bimodule K_x above is irreducible. And the set of Q - Q bimodules $\{K_x\}_{x \in \mathcal{I}}$ makes an isomorphic system of Q - Q bimodules arising from the subfactor $(N \subset M) \cong (P \subset Q)$, i.e., the system $\{A_x\}_{x \in \mathcal{I}}$ of M - M bimodules as above.*

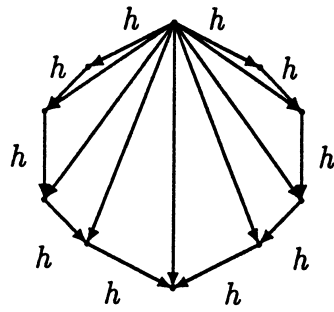


Figure 2.4

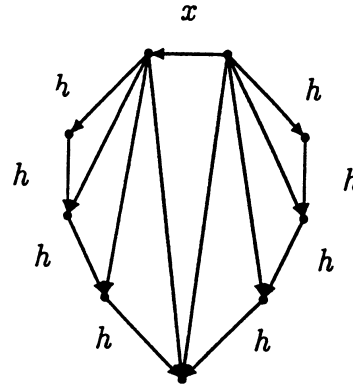


Figure 2.5

PROOF. From the commuting squares with the following gradings

$$\begin{array}{ccccccccc}
 0 & 1 & 2 & 3 & \cdots & 0 & 1 & 2 & \cdots & \rightarrow & M = Q \\
 n-1 & 0 & 1 & 2 & \cdots & n-1 & 0 & 1 & \cdots & \rightarrow & M_1 \\
 0 & 1 & 2 & 3 & \cdots & 0 & 1 & 2 & \cdots & \rightarrow & M_2 \\
 n-1 & 0 & 1 & 2 & \cdots & n-1 & 0 & 1 & \cdots & \rightarrow & M_3 \\
 \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots
 \end{array}$$

and the graphical expression of $Q = M$ and Q - Q bimodule, the set of bimodules $\{K_x\}_{x \in \mathcal{J}}$ really makes a system of Q - Q bimodules. The same method as in the proof of Theorem 2.1 in [9] also works by using Wenzl's dimension estimate [27, Theorem 1.6] in order to show the irreducibility of the inclusion $Q \subset R_x$. Here R_x denotes a von Neumann algebra corresponding to Figure 2.6.

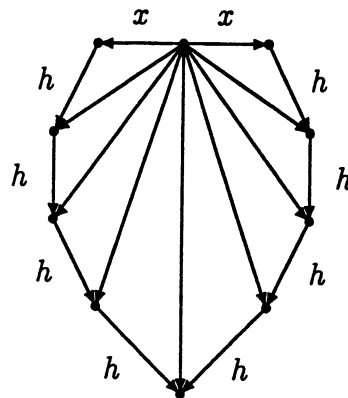


Figure 2.6

Here the edges of the boundaries are labelled by h except for the top two x 's.

And a graphical inspection as in the proof of Theorem 2.1 in [9] shows that the fusion rule and quantum $6j$ -symbols of the system of surface bimodules depend only on the labels on the top edges $x \in \mathcal{J}$ of the surface bimodules and do not depend on the labels of edges on the boundaries. So the above system of Q - Q bimodules $\{K_x\}_{x \in \mathcal{J}}$ have the same fusion rule and quantum $6j$ -symbols as the system of Q - Q bimodule arising

from the subfactor $P \subset Q$. □

From the above theorem we may and do use the notation \mathcal{Q} for the two isomorphic system of Q - Q bimodules as in the theorem.

Similarly we get an irreducible $Q \vee (Q' \cap Q_\infty)$ - $Q \vee (Q' \cap Q_\infty)$ bimodules expressed as in Figure 2.7.

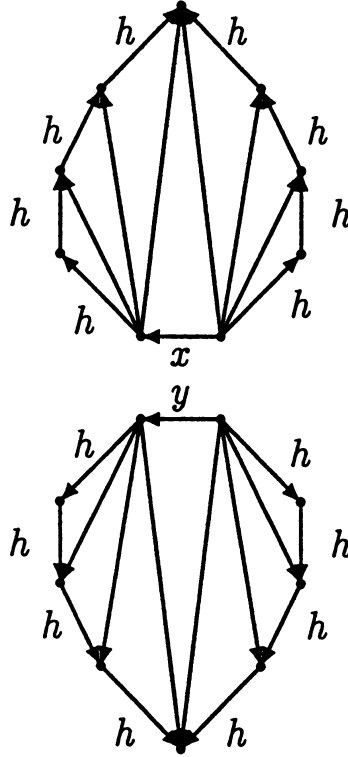


Figure 2.7

Here x and y are any pair of basis in \mathcal{J} and the edges on the boundaries are labelled by h . We denote this bimodule by $L_{x,y}$. The above theorem shows that this system $\{L_{x,y}\}$ has the same fusion rule and quantum $6j$ -symbols as the system $\{B_{x,y}\}$ of $M \vee (M' \cap M_\infty)$ - $M \vee (M' \cap M_\infty)$ bimodules.

We can also get the sets of *irreducible* $Q \vee (Q' \cap Q_\infty)$ - Q_∞ bimodules $\{L_z\}$ and *irreducible* Q_∞ - Q_∞ bimodules $\{L_{\pi_i}\}$ as in Figure 2.8 and Figure 2.9 respectively by changing the labels of the edges on the boundaries from ${}_M M_N$ and ${}_N M_M$ to h . The irreducibility of these $Q \vee (Q' \cap Q_\infty)$ - Q_∞ and Q_∞ - Q_∞ bimodules are shown by the same way as in [9, Theorem 4.1, Theorem 4.2].

Here the labellings π_i are given by the elements of the subset of the minimal central projections of Tube \mathcal{Q} , the *tube algebra* of the system \mathcal{Q} of Q - Q bimodules (see [24], [9] for the definition of the tube algebra) which are reachable from $*$ by the fusion graph of the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$ (see [24]).

Now we can get the following theorem similarly.

THEOREM 2.3. *All the systems $\{L_{x,y}\}$, $\{L_z\}$, $\{\bar{L}_z\}$ and $\{L_{\pi_i}\}$ consisting of four kinds of bimodules arising from the subfactor $Q \vee (Q' \cap Q_\infty) \subset Q_\infty$ have the same (\mathbb{Z}_2 graded) fusion rules and quantum $6j$ -symbols as those of four kinds of bimodules arising from the subfactor $M \vee (M' \cap M_\infty) \subset M_\infty$. Hence the two systems make the same paragroups.*

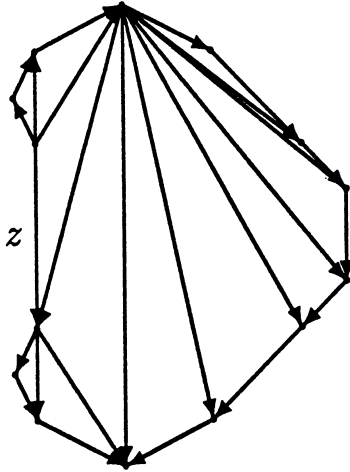


Figure 2.8

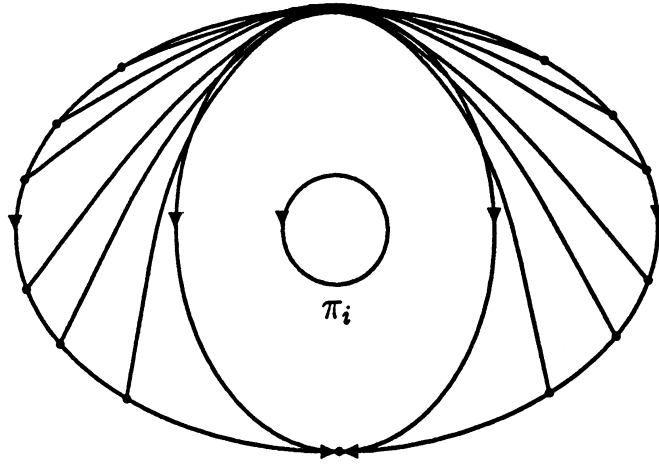


Figure 2.9

PROOF. By the graphical inspection as in the proof of Theorem 2.1 in [9] we can easily see that the fusion rules and quantum $6j$ -symbols of the system of these four kinds of bimodules depend only on the labelling of x, y, z and π_i . So we get the result.

COROLLARY 2.4. *Two subfactors $Q \vee (Q' \cap Q_\infty) \subset Q_\infty$ and $M \vee (M' \cap M_\infty) \subset M_\infty$ are isomorphic.*

PROOF. Because the two subfactors have finite index and finite depth, and have the same paragroups, these are isomorphic by Popa's generating theorem for strongly amenable subfactors [25].

In the following we give some applications of this result.

EXAMPLE 2.5. We start with $SU(n)$ WZW model with level k . We can get a commutative fusion rule algebra with quantum $6j$ -symbols. If we take the fundamental generator h , then the resulting subfactor $(P \subset Q) \cong (N \subset M)$ is the same as Hecke algebra subfactor of type A of Wenzl. (See [6].) The above corollary shows that the following commuting squares

$$\begin{array}{ccc} \langle g_{-n}, \dots, g_{-1}, g_1, \dots, g_n \rangle & \subset & \langle g_{-n-1}, \dots, g_{-1}, g_1, \dots, g_{n+1} \rangle \\ \cap & & \cap \\ \langle g_{-n}, \dots, g_{-1}, g_0, g_1, \dots, g_n \rangle & \subset & \langle g_{-n-1}, \dots, g_{-1}, g_0, g_1, \dots, g_{n+1} \rangle \end{array}$$

generate the asymptotic inclusion $M \vee (M' \cap M_\infty) \subset M_\infty$. Here g_i 's are the standard generators of Hecke algebras satisfying the following relations:

$$\begin{array}{ll} g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} & \text{for } i \in \mathbb{Z}, \\ g_i g_j = g_j g_i & \text{whenever } |i - j| \geq 2, \\ g_i^2 = (q - 1)g_i + q & \text{for } i \in \mathbb{Z}, \end{array}$$

where $q = e^{\pm i\pi/n}$.

The isomorphism for this example has been obtained by J. Erlijman [5] independently by a different method.

EXAMPLE 2.6. Again we start with $SU(n)$ WZW model with level k and take another generator h different from the fundamental one, then the resulting subfactor is

isomorphic to $pPp \subset pQ_kp$, where $P \subset Q$ denotes Hecke algebra subfactor of type A of Wenzl and p is a projection in $P' \cap Q_k$ for some k . Here Q_k is one of the von Neumann algebras in the tower $P \subset Q \subset Q_1 \subset \cdots \subset Q_k \subset \cdots$ which are generated by the double sequences of the period n commuting squares as in [6] (see also [27]). In this case we do not have natural generators for the commuting squares as in the previous example. But our method also works in such cases and we can construct the asymptotic inclusions for many such subfactors. This is an advantage of our method.

EXAMPLE 2.7. We start with a fusion rule algebra which consists of N - N bimodules of a subfactor $N \subset M$ with principal graph D_{2m} for $m \geq 2$. If we take a bimodule corresponding to one of the two tails of the principal graph D_{2m} as a generator, then the subsystem $\mathcal{B} = \langle h \rangle$ has period 2 if m is even and period greater than 3 if m is odd. This is because the contragredient map for D_{2m} changes by mod 4, i.e., if m is even, we have $\bar{h} = h$ and if m is odd, we have $\bar{h} \neq h$.

REMARK 2.8. We can easily modify this method to the case when the subsystem \mathcal{B} has more than two generators. For example if we have m generators h_1, h_2, \dots, h_m which are elements in the basis of \mathcal{A} , then we take $h = h_1 + h_2 + \cdots + h_m$ as a new generator. And in this case we have to change the definition (Definition 2.1) of the periodicity. We say that the fusion rule subalgebra \mathcal{B} generated by the above h is *periodic* with the *period* n if the Bratteli diagram for $\{h^k\}_{k=0,1,2,\dots}$ has the period n .

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