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Standard weights on algebras of unbounded operators

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Abstract. The purpose of this paper is to define and study an important class in weights on O^*-algebras which is possible to develop the Tomita-Takesaki theory in O^*-algebras. The Connes cocycle theorem for weights on von Neumann algebras is generalized to the case of O^*-algebras.

1. Introduction.

Weights on O^*-algebras (that is, linear functionals that take positive, but not necessarily finite values) appear naturally in the studies of the unbounded Tomita-Takesaki theory and the quantum physics. Thus the investigations of weights on O^*-algebras are important both for study of structure of the O^*-algebras and their physical applications. Further, the weights on O^*-algebras exhibit some pathological phenomena which don't occur for weights on C^*- and W^*-algebras. From this viewpoint we defined and studied systematically weights and quasi-weights on O^*-algebras in the previous paper [**12**]. In particular, we have investigated the regularity of (quasi-) weights; that is the question when a (quasi-)weight can be represented as $\sup_\alpha f_\alpha$ where $\{f_\alpha\}$ is a net of positive linear functionals. We also defined and studied an important class of regular (quasi-)weights suitable for developing the Tomita-Takesaki theory for O^*-algebras.

In this paper we shall continue the study of standard (quasi-)weights. Let $\mathscr{M}$ be a closed O^*-algebra on a dense subspace $\mathscr{D}$ in a Hilbert space $\mathscr{H}$. After defining the algebraic positive cone $\mathscr{P}(\mathscr{M})$, the operational positive cone $\mathscr{M}_+$ and the corresponding (quasi-)weights we come to the problem of the GNS-construction. Then we face the following problem: If φ is a (quasi-)weight then in the bounded case the set $\mathfrak{N}_\varphi^\circ \equiv \{X \in \mathscr{M}; \varphi(X^\dagger X) < \infty\}$ is a left ideal of $\mathscr{M}$. This is not so in general. To circumvent this difficulty we introduce the set $\mathfrak{N}_\varphi \equiv \{X \in \mathscr{M}; \varphi((AX)^\dagger(AX)) < \infty$ for all $A \in \mathscr{M}\}$ which is always a left ideal of $\mathscr{M}$. Then we construct the GNS-representation π_φ and the vector representation λ_φ by the method similar to that used for positive linear functionals. That is, π_φ is a $*$-homomorphism of $\mathscr{M}$ onto the O^*-algebra $\pi_\varphi(\mathscr{M})$ on the dense subspace $\mathscr{D}(\pi_\varphi)$ in the Hilbert space $\mathscr{H}_\varphi$, and λ_φ is a linear map of $\mathfrak{N}_\varphi$ into $\mathscr{D}(\pi_\varphi)$ satisfying $\lambda_\varphi(AX) = \pi_\varphi(A)\lambda_\varphi(X)$ for all $A \in \mathscr{M}$ and $X \in \mathfrak{N}_\varphi$. In order that π_φ carries enough structure of $\mathscr{M}$ the left ideal $\mathfrak{N}_\varphi$ must be sufficiently rich. This is not at all

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guaranteed. In the extreme there are non-zero (quasi-)weights φ such that $\mathfrak{N}_\varphi^\circ$ has many elements but $\mathfrak{N}_\varphi = \{0\}$. To avoid situations leading to noninteresting representations we define notions of faithfulness, semifiniteness and σ-weak continuity of (quasi-) weights. If φ is a faithful semifinite (quasi-)weight on $\mathscr{P}(\mathscr{M})$ such that $\pi_\varphi(\mathscr{M})'_{\mathrm{w}}\mathscr{D}(\pi_\varphi) \subset \mathscr{D}(\pi_\varphi)$, then the map Λ_φ defined by $\Lambda_\varphi(\pi_\varphi(X)) = \lambda_\varphi(X), X \in \mathfrak{N}_\varphi$ is a generalized vector for the O*-algebra $\pi_\varphi(\mathscr{M})$ i.e. Λ_φ is a linear map of the left ideal $\mathscr{D}(\Lambda_\varphi) \equiv \pi_\varphi(\mathfrak{N}_\varphi)$ into $\mathscr{D}(\pi_\varphi)$ satisfying $\Lambda_\varphi(\pi_\varphi(A)\pi_\varphi(X)) = \pi_\varphi(A)\Lambda_\varphi(\pi_\varphi(X))$ for all $A \in \mathscr{M}$ and $X \in \mathfrak{N}_\varphi$. Using (quasi-)standard generalized vectors defined and studied in [**10**], we define the notion of (quasi-)standardness of φ as follows: φ is said to be *standard* (resp. *quasi-standard*) if the generalized vector Λ_φ is standard (resp. quasi-standard). We demonstrate that if φ is standard, then the modular automorphism group $\{\sigma_t^\varphi\}_{t\in\boldsymbol{R}}$ of $\mathfrak{N}_\varphi^\dagger \cap \mathfrak{N}_\varphi$ is defined and φ is a $\{\sigma_t^\varphi\}$-KMS (quasi-)weight. If φ is quasi-standard, then it can be uniquely extended to a standard quasi-weight $\bar{\varphi}$ on the positive cone $\mathscr{P}(\pi_\varphi(\mathscr{M})''_{\mathrm{wc}})$ of the generalized von Neumann algebra $\pi_\varphi(\mathscr{M})''_{\mathrm{wc}}$. We shall generalize the Connes cocycle theorem for (quasi-)-weights on von Neumann algebras to O*-algebras. In [**10**] we have generalized the Connes cocycle theorem for standard generalized vectors. As the notion of generalized vectors is spatial, such a generalization is possible to a certain extent, but the notion of (quasi-)weights is purely algebraic and the algebraic properties don't reflect the topological properties in general, and so such a generalization for weights have some difficult problems. Let φ and ψ be faithful, σ-weakly continuous and semifinite (quasi-) weights on $\mathscr{P}(\mathscr{M})$ such that π_φ and π_ψ are self-adjoint. We consider the matrix algebra $\mathscr{M} \otimes M_2(\boldsymbol{C})$ on $\mathscr{D} \otimes \mathscr{D}$:

$$\left\{X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix};\quad X_{ij} \in \mathscr{M}\right\}$$

and a faithful, σ-weakly continuous semifinite (quasi-)weight θ on $\mathscr{P}(\mathscr{M} \otimes M_2(\boldsymbol{C}))$ by

$$\theta(X^\dagger X) = \varphi(X_{11}^\dagger X_{11} + X_{21}^\dagger X_{21}) + \psi(X_{12}^\dagger X_{12} + X_{22}^\dagger X_{22}),\quad X = (X_{ij}) \in \mathscr{M} \otimes M_2(\boldsymbol{C}).$$

In case of von Neumann algebras, $\Lambda_\theta^c((\mathscr{D}(\Lambda_\theta^c)^* \cap \mathscr{D}(\Lambda_\theta^c))^2)$ is total in $\mathscr{H}_\theta$, and π_φ and π_ψ are unitarily equivalent, and further $\pi_\varphi(\mathscr{M})$ and $\pi_\psi(\mathscr{M})$ are von Neumann algebras, which imply the Connes cocycle theorem [**24**]. In case of O*-algebras these properties do not hold automatically. Hence we should consider the following questions:

A. When is $\Lambda_\theta^c((\mathscr{D}(\Lambda_\theta^c)^* \cap \mathscr{D}(\Lambda_\theta^c))^2)$ total in $\mathscr{H}_\theta$? When are π_φ and π_ψ unitarily equivalent?

B. Let $\mathscr{M}$ be a generalized von Neumann algebra. When is $\pi_\varphi(\mathscr{M})$ a generalized von Neumann algebra?

For Question A we have the result that if $\Lambda_\varphi^c((\mathscr{D}(\Lambda_\varphi^c)^* \cap \mathscr{D}(\Lambda_\varphi^c))^2)$ is total in $\mathscr{H}_\varphi$ and $\Lambda_\psi^c((\mathscr{D}(\Lambda_\psi^c)^* \cap \mathscr{D}(\Lambda_\psi^c))^2)$ is total in $\mathscr{H}_\psi$, then the following statements are equivalent:

(i) π_φ and π_ψ are unitarily equivalent.

(ii) $\mathrm{II}(\pi_\varphi, \pi_\psi)^*\mathrm{II}(\pi_\varphi, \pi_\psi)$ and $\mathrm{II}(\pi_\psi, \pi_\varphi)^*\mathrm{II}(\pi_\psi, \pi_\varphi)$ are nondegenerate $*$-subalgebras of the von Neumann algebras $\pi_\varphi(\mathscr{M})'_{\mathrm{w}}$ and $\pi_\psi(\mathscr{M})'_{\mathrm{w}}$, respectively, where $\mathrm{II}(\pi_1, \pi_2)$ is the intertwining space for $*$-representations π_1 and π_2.

(iii) $\Lambda_{\psi,\varphi}^c(\mathscr{D}(\Lambda_{\psi,\varphi}^c))$ is dense in $\mathscr{H}_\varphi$ and $\Lambda_{\varphi,\psi}^c(\mathscr{D}(\Lambda_{\varphi,\psi}^c))$ is dense in $\mathscr{H}_\psi$, where $\Lambda_{\psi,\varphi}^c$ and $\Lambda_{\varphi,\psi}^c$ are generalized vectors for $\mathrm{II}(\pi_\psi, \pi_\varphi)$ and $\mathrm{II}(\pi_\varphi, \pi_\psi)$, respectively.

(iv) $\Lambda_\theta^c((\mathscr{D}(\Lambda_\theta^c)^* \cap \mathscr{D}(\Lambda_\theta^c))^2)$ is total in $\mathscr{H}_\theta$.

In this case, we obtain that φ and ψ are quasi-standard if and only if θ is quasi-standard, and then the cocycle $[D\bar{\psi} : D\bar{\varphi}]$ associated with the (quasi-)weight $\bar{\psi}$ on $\mathscr{P}(\pi_\psi(\mathscr{M})''_{\mathrm{wc}})$ with respect to the (quasi-)weight $\bar{\varphi}$ on $\mathscr{P}(\pi_\varphi(\mathscr{M})''_{\mathrm{wc}})$ is defined, but $\pi_\varphi(\mathscr{M})$ is not a generalized von Neumann algebra in general even if $\mathscr{M}$ is a generalized von Neumann algebra, and so the cocycle $[D\bar{\psi} : D\bar{\varphi}]$ for the generalized von Neumann algebra $\pi_\varphi(\mathscr{M})''_{\mathrm{wc}}$ does not necessarily induce the cocycle $[D\psi : D\varphi]$ associated with the (quasi-)weight ψ on $\mathscr{P}(\mathscr{M})$ with respect to the (quasi-)weight φ on $\mathscr{P}(\mathscr{M})$.

We also consider Question B and show that if $\mathscr{M}$ is a generalized von Neumann algebra with strongly dense bounded part and φ is strongly faithful, then $\pi_\varphi(\mathscr{M})$ is spatially isomorphic to $\mathscr{M}$, and so it is a generalized von Neumann algebra and the cocycle $[D\psi : D\varphi]$ for the generalized von Neumann algebra $\mathscr{M}$ is well-defined.

2. Preliminaries.

Here we state some definitions and the basic properties concerning O*-algebras [**4**, **14**, **17**, **19**, **21**] and generalized vectors for O*-algebras [**1**, **9**, **10**, **11**].

Let $\mathscr{D}$ be a dense subspace in a Hilbert space $\mathscr{H}$. We denote by $\mathscr{L}^\dagger(\mathscr{D})$ the set of all linear operators X from $\mathscr{D}$ into $\mathscr{D}$ such that $\mathscr{D}(X^*) \supset \mathscr{D}$ and $X^*\mathscr{D} \subset \mathscr{D}$. Then $\mathscr{L}^\dagger(\mathscr{D})$ is a $*$-algebra with the usual operations and the involution $X \to X^\dagger \equiv X^* \lceil_{\mathscr{D}}$. A $*$-subalgebra of $\mathscr{L}^\dagger(\mathscr{D})$ is called an *O*-algebra* on $\mathscr{D}$ in $\mathscr{H}$ according to the Schmüdgen book [**21**] though it is also called by an Op*-algebra in many papers. Throughout this paper we assume that an O*-algebra has always an identity operator. Let $\mathscr{M}$ be an O*-algebra on $\mathscr{D}$. The locally convex topology on $\mathscr{D}$ defined by the family $\{\| \ \|_X; X \in \mathscr{M}\}$ of seminorms: $\|\xi\|_X = \|X\xi\|$ $(\xi \in \mathscr{D})$ is called the *graph topology* on $\mathscr{D}$, which is denoted by $t_{\mathscr{M}}$. If the locally convex space $\mathscr{D}[t_{\mathscr{M}}]$ is complete, then $\mathscr{M}$ is said to be *closed*. We put

$$\tilde{\mathscr{D}}(\mathscr{M}) = \bigcap_{X \in \mathscr{M}} \mathscr{D}(\bar{X}) \quad \text{and} \quad \tilde{X} = \bar{X} \lceil_{\tilde{\mathscr{D}}(\mathscr{M})} (X \in \mathscr{M}).$$

Then $\tilde{\mathscr{D}}(\mathscr{M})$ equals the completion of $\mathscr{D}[t_{\mathscr{M}}]$ and $\tilde{\mathscr{M}} \equiv \{\tilde{X}; X \in \mathscr{M}\}$ is a closed O*-algebra on $\tilde{\mathscr{D}}(\mathscr{M})$ which is the smallest closed extension of $\mathscr{M}$ and it is called the *closure* of $\mathscr{M}$. Hence $\mathscr{M}$ is closed if and only if $\mathscr{D} = \tilde{\mathscr{D}}(\mathscr{M})$. If $\mathscr{D}^*(\mathscr{M}) \equiv \bigcap_{X \in \mathscr{M}} \mathscr{D}(X^*) = \tilde{\mathscr{D}}(\mathscr{M})$, then $\mathscr{M}$ is said to be *essentially self-adjoint*, and if $\mathscr{D}^*(\mathscr{M}) = \mathscr{D}$, then $\mathscr{M}$ is said to be *self-adjoint*. We define the *weak commutant* $\mathscr{M}'_{\mathrm{w}}$ of $\dagger$-invariant subset $\mathscr{M}$ of $\mathscr{L}^\dagger(\mathscr{D})$ as follows:

$$\mathscr{M}'_{\mathrm{w}} = \{C \in \mathscr{B}(\mathscr{H}); (CX\xi|\eta) = (C\xi|X^\dagger\eta) \quad \text{for each } \xi, \eta \in \mathscr{D} \text{ and } X \in \mathscr{M}\},$$

where $\mathscr{B}(\mathscr{H})$ is the set of all bounded linear operators on $\mathscr{H}$. Then $\mathscr{M}'_{\mathrm{w}}$ is a $*$-invariant weakly closed subspace of $\mathscr{B}(\mathscr{H})$, but it is not necessarily an algebra. Further, if $\mathscr{M}$ is self-adjoint, then $\mathscr{M}'_{\mathrm{w}}\mathscr{D} \subset \mathscr{D}$, and $\mathscr{M}'_{\mathrm{w}}\mathscr{D} \subset \mathscr{D}$ if and only if $\mathscr{M}'_{\mathrm{w}}$ is a von Neumann algebra and $\bar{X}$ is affiliated with $(\mathscr{M}'_{\mathrm{w}})'$ for each $X \in \mathscr{M}$. Let $\mathscr{M}$ be an O*-algebra on $\mathscr{D}$ in $\mathscr{H}$. We call the locally convex topology defined by the family $\{P_{\xi,\eta}; \xi, \eta \in \mathscr{D}\}$ (resp.

$\{P_\xi; \xi \in \mathscr{D}\}, \{P^*_\xi; \xi \in \mathscr{D}\})$ of seminorms: $P_{\xi,\eta}(X) = |(X\xi|\eta)|$ (resp. $P_\xi(X) = \|X\xi\|$, $P^*_\xi(X) = \|X\xi\| + \|X^\dagger\xi\|$), $X \in \mathscr{M}$ the *weak topology* (resp. *strong topology, strong* topology*) on $\mathscr{M}$ and denote it by τ_w (resp. τ_s, τ_s^*). We put

$$\mathscr{D}^\infty(\mathscr{M}) = \left\{ \{\xi_n\} \subset \mathscr{D}; \sum_{n=1}^\infty \|X\xi_n\|^2 < \infty \text{ for each } X \in \mathscr{M} \right\}$$

and call the locally convex topology defined by the family $\{P_{\{\xi_n\},\{\eta_n\}}; \{\xi_n\}, \{\eta_n\} \in \mathscr{D}^\infty(\mathscr{M})\}$ (resp. $\{P_{\{\xi_n\}}; \{\xi_n\} \in \mathscr{D}^\infty(\mathscr{M})\}, \{P^*_{\{\xi_n\}}; \{\xi_n\} \in \mathscr{D}^\infty(\mathscr{M})\}$) of seminorms:

$$P_{\{\xi_n\},\{\eta_n\}}(X) = \left| \sum_{n=1}^\infty (X\xi_n|\eta_n) \right|$$

$$\left(\text{resp. } P_{\{\xi_n\}}(X) = \left(\sum_{n=1}^\infty \|X\xi_n\|^2 \right)^{1/2}, \quad P^*_{\{\xi_n\}}(X) = P_{\{\xi_n\}}(X) + P_{\{\xi_n\}}(X^\dagger) \right)$$

the *σ-weak topology* (resp. *σ-strong topology*, *σ-strong* topology*) on $\mathscr{M}$ and denote it by $\tau_{\sigma w}$ (resp. $\tau_{\sigma s}, \tau^*_{\sigma s}$). A closed O*-algebra $\mathscr{M}$ on $\mathscr{D}$ in $\mathscr{H}$ is said to be a *generalized von Neumann algebra* on $\mathscr{D}$ if $\mathscr{M}'_w\mathscr{D} \subset \mathscr{D}$ and $\mathscr{M} = \mathscr{M}''_{wc} \equiv \{X \in \mathscr{L}^\dagger(\mathscr{D}); CX \subset XC\ C \in \mathscr{M}'_w\}$. It is known that $\mathscr{M}$ is a generalized von Neumann algebra on $\mathscr{D}$ if and only if $\mathscr{M}$ equals the strong*-closure of the O*-algebra $(\mathscr{M}'_w)'\lceil_\mathscr{D}$ on $\mathscr{D}$ in $\mathscr{L}^\dagger(\mathscr{D})$ [**7**].

A $(*$-)homomorphism π of a $*$-algebra $\mathscr{A}$ onto an O*-algebra on $\mathscr{D}$ in $\mathscr{H}$ is said to be a (*-)representation of $\mathscr{A}$. We here denote $\mathscr{D}$ and $\mathscr{H}$ by $\mathscr{D}(\pi)$ and $\mathscr{H}_\pi$, respectively. A $*$-representation π of $\mathscr{A}$ is said to be *closed* (resp. *self-adjoint*) if the O*-algebra $\pi(\mathscr{A})$ is closed (resp. self-adjoint). Let π be a $*$-representation of $\mathscr{A}$. We put

$$\mathscr{D}(\tilde{\pi}) = \bigcap_{x \in \mathscr{A}} \mathscr{D}(\overline{\pi(x)}), \quad \tilde{\pi}(x) = \overline{\pi(x)}\lceil_{\mathscr{D}(\tilde{\pi})},$$

$$\mathscr{D}(\pi^*) = \bigcap_{x \in \mathscr{A}} \mathscr{D}(\pi(x)^*), \quad \pi^*(x) = \pi(x^*)^*\lceil_{\mathscr{D}(\pi^*)}, \ x \in \mathscr{A}.$$

Then $\tilde{\pi}$ is a closed $*$-representation of $\mathscr{A}$ such that $\tilde{\pi}(\mathscr{A}) = \widetilde{\pi(\mathscr{A})}$ and it is called the *closure* of π, and π^* is a closed representation of $\mathscr{A}$ and it is called the *adjoint* of π. Let π_1 and π_2 be $*$-representations of $\mathscr{A}$. We define the *intertwining space* $\mathrm{II}(\pi_1, \pi_2)$ for π_1 and π_2 as follows:

$$\mathrm{II}(\pi_1, \pi_2) = \{C \in \mathscr{B}(\mathscr{H}_{\pi_1}, \mathscr{H}_{\pi_2}); \quad C\mathscr{D}(\pi_1) \subset \mathscr{D}(\pi_2) \quad \text{and} \quad C\pi_1(x)\xi = \pi_2(x)C\xi \quad \text{for each } x \in \mathscr{A} \quad \text{and} \quad \xi \in \mathscr{D}(\pi_1)\},$$

and this is an important tool in representation theory [**21**].

We next introduce the notion of generalized vectors which is a generalization of cyclic vectors for O*-algebras [**11**]. Let $\mathscr{M}$ be an O*-algebra on $\mathscr{D}$ such that $\mathscr{M}'_w\mathscr{D} \subset \mathscr{D}$. A map λ of $\mathscr{M}$ into $\mathscr{D}$ is said to be a *generalized vector* for $\mathscr{M}$ if the domain $\mathscr{D}(\lambda)$ of λ is a left ideal of $\mathscr{M}, \lambda$ is a linear map of $\mathscr{D}(\lambda)$ into $\mathscr{D}$ and $\lambda(XA) = X\lambda(A)$ for all $X \in \mathscr{M}$ and $A \in \mathscr{D}(\lambda)$. Suppose that a generalized vector λ for $\mathscr{M}$ satisfies the condition:

(i) $\lambda((\mathscr{D}(\lambda)^{\dagger} \cap \mathscr{D}(\lambda))^2)$ is total in $\mathscr{H}$.

Then we define the *commutant* λ^c of λ which is a generalized vector for the von Neumann algebra $\mathscr{M}_{\mathrm{w}}'$ as follows:

$$\begin{cases} \mathscr{D}(\lambda^c) = \{K \in \mathscr{M}_{\mathrm{w}}'; {}^{\exists}\xi_K \in \mathscr{D} \text{ s.t. } K\lambda(X) = X\xi_K \text{ for all } X \in \mathscr{D}(\lambda)\}, \\ \lambda^c(K) = \xi_K, \quad K \in \mathscr{D}(\lambda^c). \end{cases}$$

A generalized vector λ for $\mathscr{M}$ is said to be *cyclic and separating* if the above condition (i) and the following condition (ii) hold:

(ii) $\lambda^c((\mathscr{D}(\lambda^c)^* \cap \mathscr{D}(\lambda^c))^2)$ is total in $\mathscr{H}$.

Suppose λ is a cyclic and separating generalized vector for $\mathscr{M}$ and put

$$\begin{cases} \mathscr{D}(\lambda^{cc}) = \{A \in (\mathscr{M}_{\mathrm{w}}')'; {}^{\exists}\xi_A \in \mathscr{H} \text{ s.t. } A\lambda^c(K) = K\xi_A \text{ for all } K \in \mathscr{D}(\lambda^c)\}, \\ \lambda^{cc}(A) = \xi_A, \quad A \in \mathscr{D}(\lambda^{cc}). \end{cases}$$

Then λ^{cc} is a cyclic and separating generalized vector for the von Neumann algebra $(\mathscr{M}_{\mathrm{w}}')'$. So the maps $\lambda(X) \mapsto \lambda(X^{\dagger}), X \in \mathscr{D}(\lambda)^{\dagger} \cap \mathscr{D}(\lambda)$ and $\lambda^{cc}(A) \mapsto \lambda^{cc}(A^*), A \in \mathscr{D}(\lambda^{cc})^* \cap \mathscr{D}(\lambda^{cc})$ are closable in $\mathscr{H}$ and their closures are denoted by S_λ and $S_{\lambda^{cc}}$, respectively. Let $S_\lambda = J_\lambda \Delta_\lambda^{1/2}$ and $S_{\lambda^{cc}} = J_{\lambda^{cc}} \Delta_{\lambda^{cc}}^{1/2}$ be the polar decompositions of S_λ and $S_{\lambda^{cc}}$, respectively. Then we see that $S_\lambda \subset S_{\lambda^{cc}}$, and $J_{\lambda^{cc}}(\mathscr{M}_{\mathrm{w}}')'J_{\lambda^{cc}} = \mathscr{M}_{\mathrm{w}}'$ and $\Delta_{\lambda^{cc}}^{it}(\mathscr{M}_{\mathrm{w}}')'\Delta_{\lambda^{cc}}^{-it} = (\mathscr{M}_{\mathrm{w}}')'$ for all $t \in \boldsymbol{R}$ by the Tomita fundamental theorem [**25**]. But, we don't know how the unitary group $\{\Delta_{\lambda^{cc}}^{it}\}_{t \in \boldsymbol{R}}$ acts on the O*-algebra $\mathscr{M}$, and so we define a system which has the best properties:

A generalized vector λ for $\mathscr{M}$ is said to be *standard* if the following conditions hold:

$(S)_1$ λ is cyclic and separating.

$(S)_2$ $\Delta_{\lambda^{cc}}^{it}\mathscr{D} \subset \mathscr{D}$ and $\Delta_{\lambda^{cc}}^{it}\mathscr{M}\Delta_{\lambda^{cc}}^{-it} = \mathscr{M}$ for each $t \in \boldsymbol{R}$.

$(S)_3$ $\Delta_{\lambda^{cc}}^{it}(\mathscr{D}(\lambda)^{\dagger} \cap \mathscr{D}(\lambda))\Delta_{\lambda^{cc}}^{-it} = \mathscr{D}(\lambda)^{\dagger} \cap \mathscr{D}(\lambda)$ for each $t \in \boldsymbol{R}$.

For standard generalized vectors we have obtained the following

Theorem 2.1 ([**11**] Theorem 5.5). *Suppose λ is a standard generalized vector for $\mathscr{M}$. Then the following statements hold*:

(1) *$S_\lambda = S_{\lambda^{cc}}$, and so $J_\lambda = J_{\lambda^{cc}}$ and $\Delta_\lambda = \Delta_{\lambda^{cc}}$.*

(2) *$\{\sigma_t^\lambda\}_{t \in \boldsymbol{R}}$ is a one-parameter group of $*$-automorphisms of $\mathscr{M}$, where $\sigma_t^\lambda(X) = \Delta_\lambda^{it}X\Delta_\lambda^{-it}$ for $X \in \mathscr{M}$ and $t \in \boldsymbol{R}$.*

(3) *λ satisfies the* KMS-*condition with respect to $\{\sigma_t^\lambda\}$, that is, for each $X, Y \in \mathscr{D}(\lambda)^{\dagger} \cap \mathscr{D}(\lambda)$ there exists an element $f_{X,Y}$ of $A(0,1)$ such that*

$$f_{X,Y}(t) = (\lambda(\sigma_t^\lambda(X))|\lambda(Y)) \quad \textit{and} \quad f_{X,Y}(t+i) = (\lambda(Y^{\dagger})|\lambda(\sigma_t^\lambda(X^{\dagger}))$$

for all $t \in \boldsymbol{R}$, where $A(0,1)$ is the set of all complex-valued functions, bounded and continuous on $0 \leq \operatorname{Im} z \leq 1$ and analytic in the interior.

Weakening the above conditions $(S)_2$ and $(S)_3$, we define and study the notion of quasi-standard generalized vectors which enable us to extend the Tomita-Takesaki theory. A generalized vector λ for $\mathscr{M}$ is said to be *quasi-standard* if the above condition $(S)_1$ and the following condition hold:

(QS) $\Delta_{\lambda^{cc}}^{it}\mathscr{D} \subset \mathscr{D}$ for each $t \in \boldsymbol{R}$.

For quasi-standard generalized vectors we have the following

Theorem 2.2. *Suppose λ is a quasi-standard generalized vector for $\mathcal{M}$ and then put*

$$\begin{cases} \mathcal{D}(\bar{\lambda}) = \{X \in \mathcal{M}''_{\mathrm{wc}}; {}^{\exists}\xi_X \in \mathcal{D} \text{ s.t. } X\lambda^c(K) = K\xi_X, \ {}^{\forall}K \in \mathcal{D}(\lambda^c)\} \\ \bar{\lambda}(X) = \xi_X, \quad X \in \mathcal{D}(\bar{\lambda}). \end{cases}$$

Then $\bar{\lambda}$ is a standard generalized vector for the generalizd von Neumann algebra $\mathcal{M}''_{\mathrm{wc}}$ such that $\lambda \subset \bar{\lambda}, \lambda^c = \bar{\lambda}^c$ and

$$\begin{cases} \mathcal{D}(\bar{\lambda}) = \{X \in \mathcal{M}''_{\mathrm{wc}}; {}^{\exists}\{A_\alpha\} \subset \mathcal{D}(\lambda^{cc}) \textit{ and } {}^{\exists}\xi_X \in \mathcal{D} \text{ s.t. } A_\alpha\xi \to X\xi, \ {}^{\forall}\xi \in \mathcal{D} \\ \qquad \textit{and } \lambda^{cc}(A_\alpha) \to \xi_X\} \\ \bar{\lambda}(X) = \xi_X, \quad X \in \mathcal{D}(\bar{\lambda}). \end{cases}$$

Proof. It is shown similarly to the proof of ([**11**] Theorem 5.11) that $\bar{\lambda}$ is a generalized vector for $\mathcal{M}''_{\mathrm{wc}}$ such that $\lambda \subset \bar{\lambda}, \lambda^c = \bar{\lambda}^c$ and

$$\begin{cases} \mathcal{D}(\bar{\lambda}) = \{X \in \mathcal{M}''_{\mathrm{wc}}; {}^{\exists}\{A_\alpha\} \subset \mathcal{D}(\lambda^{cc}) \text{ and } {}^{\exists}\xi_X \in \mathcal{D} \text{ s.t. } A_\alpha\xi \to X\xi, \ {}^{\forall}\xi \in \mathcal{D} \\ \qquad \text{and } \lambda^{cc}(A_\alpha) \to \xi_X\} \\ \bar{\lambda}(X) = \xi_X, \quad X \in \mathcal{D}(\bar{\lambda}). \end{cases}$$

Hence $\bar{\lambda}$ is a cyclic and separating generalized vector for $\mathcal{M}''_{\mathrm{wc}}$. Further, since $\Delta^{it}_{\lambda^{cc}}\mathcal{D} \subset \mathcal{D}$ and $\sigma^{\lambda^{cc}}_t(\mathcal{M}'_{\mathrm{w}}) \subset \mathcal{M}'_{\mathrm{w}}$ for each $t \in \boldsymbol{R}$ where $\sigma^{\lambda^{cc}}_t(A) = \Delta^{it}_{\lambda^{cc}} A \Delta^{-it}_{\lambda^{cc}}$, it follows that

$$\Delta^{it}_{\lambda^{cc}} X \Delta^{-it}_{\lambda^{cc}} C\xi = \Delta^{it}_{\lambda^{cc}} X \sigma^{\lambda^{cc}}_{-t}(C) \Delta^{-it}_{\lambda^{cc}}\xi = C\Delta^{it}_{\lambda^{cc}} X \Delta^{-it}_{\lambda^{cc}}\xi$$

for each $X \in \mathcal{M}''_{\mathrm{wc}}, C \in \mathcal{M}'_{\mathrm{w}}, \xi \in \mathcal{D}$ and $t \in \boldsymbol{R}$. This implies $\Delta^{it}_{\lambda^{cc}} X \Delta^{-it}_{\lambda^{cc}} \in \mathcal{M}''_{\mathrm{wc}}$ for each $X \in \mathcal{M}''_{\mathrm{wc}}$ and $t \in \boldsymbol{R}$. Hence we have $\sigma^{\lambda^{cc}}_t(\mathcal{M}''_{\mathrm{wc}}) = \mathcal{M}''_{\mathrm{wc}}$ for each $t \in \boldsymbol{R}$. It follows from the definition of $\bar{\lambda}$ that $\sigma^{\lambda^{cc}}_t(\mathcal{D}(\bar{\lambda})^{\dagger} \cap \mathcal{D}(\bar{\lambda})) = \mathcal{D}(\bar{\lambda})^{\dagger} \cap \mathcal{D}(\bar{\lambda})$ for all $t \in \boldsymbol{R}$. Thus $\bar{\lambda}$ is a standard generalized vector for M''_{wc}. This completes the proof.

3. Standard weights.

In this section we define and study the notions of standard (quasi-)weights and quasi-standard (quasi-)weights on O*-algebras. Let $\mathcal{M}$ be a closed O^*-algebra on $\mathcal{D}$ in $\mathcal{H}$. For a subspace $\mathcal{N}$ of $\mathcal{M}$ we put

$$\mathcal{P}(\mathcal{N}) = \left\{ \sum_{k=1}^{n} X_k^{\dagger} X_k; \ X_k \in \mathcal{N} \ (k = 1, 2, \ldots, n), \ n \in \boldsymbol{N} \right\}$$

and call it the *positive cone generated by* $\mathcal{N}$. A map φ of $\mathcal{P}(\mathcal{M})$ into $\boldsymbol{R}_+ \cup \{+\infty\}$ is said to be a *weight on* $\mathcal{P}(\mathcal{M})$ if

$(\mathrm{W})_1$ $\varphi(A+B) = \varphi(A) + \varphi(B), \ A, B \in \mathcal{P}(\mathcal{M})$;

$(\mathrm{W})_2$ $\varphi(\alpha A) = \alpha\varphi(A), \ A \in \mathcal{P}(\mathcal{M}), \ \alpha \geq 0$,

where $0 \cdot (+\infty) = 0$. A map φ of the positive cone $\mathcal{P}(\mathfrak{N})$ generated by a left ideal $\mathfrak{N}$ of $\mathcal{M}$ into $\boldsymbol{R}_+$ is said to be a *quasi-weight* on $\mathcal{P}(\mathcal{M})$ if it satisfies the above conditions $(\mathrm{W})_1$ and $(\mathrm{W})_2$ for $\mathcal{P}(\mathfrak{N})$, and then $\mathfrak{N}$ is denoted by $\mathfrak{N}_\varphi$. Let φ be a quasi-weight on

$\mathscr{P}(\mathscr{M})$. We denote by $\mathscr{D}(\varphi)$ the subspace of $\mathscr{M}$ generated by $\{X^\dagger X; X \in \mathfrak{N}_\varphi\}$. Since $\mathfrak{N}_\varphi$ is a left ideal of $\mathscr{M}$, we have

$$\mathscr{D}(\varphi) = \text{the linear span of } \{Y^\dagger X;\ X, Y \in \mathfrak{N}_\varphi\},$$

and so each $\sum_k \alpha_k Y_k^\dagger X_k$ $(\alpha_k \in \boldsymbol{C}, X_k, Y_k \in \mathfrak{N}_\varphi)$ is represented as $\sum_j \beta_j Z_j^\dagger Z_j$ for some $\beta_j \in \boldsymbol{C}$ and $Z_j \in \mathfrak{N}_\varphi$. Then we can define a linear functional $\dot{\varphi}$ on $\mathscr{D}(\varphi)$ by

$$\dot{\varphi}\left(\sum_k \alpha_k Y_k^\dagger X_k\right) = \sum_j \beta_j \varphi(Z_j^\dagger Z_j).$$

It is easily shown that

$$|\dot{\varphi}(Y^\dagger X)|^2 \le \varphi(Y^\dagger Y)\varphi(X^\dagger X), \quad X, Y \in \mathfrak{N}_\varphi. \tag{3.1}$$

We put

$$\mathscr{N}_\varphi = \{X \in \mathfrak{N}_\varphi; \varphi(X^\dagger X) = 0\}, \quad \lambda_\varphi(X) = X + \mathscr{N}_\varphi \in \mathfrak{N}_\varphi/\mathscr{N}_\varphi, \quad X \in \mathfrak{N}_\varphi.$$

Then it follows from (3.1) that $\mathscr{N}_\varphi$ is a left ideal of $\mathfrak{N}_\varphi$ and $\lambda_\varphi(\mathfrak{N}_\varphi) \equiv \mathfrak{N}_\varphi/\mathscr{N}_\varphi$ is a pre-Hilbert space with the inner product

$$(\lambda_\varphi(X)|\lambda_\varphi(Y)) = \dot{\varphi}(Y^\dagger X), \quad X, Y \in \mathfrak{N}_\varphi.$$

We denote by $\mathscr{H}_\varphi$ the Hilbert space obtained by the completion of the pre-Hilbert space $\lambda_\varphi(\mathfrak{N}_\varphi)$. We define a $*$-representation π_φ^0 of $\mathscr{M}$ by

$$\pi_\varphi^0(A)\lambda_\varphi(X) = \lambda_\varphi(AX), \quad A \in \mathscr{M}, \quad X \in \mathfrak{N}_\varphi,$$

and denote by π_φ the closure of π_φ^0. We call the triple $(\pi_\varphi, \lambda_\varphi, \mathscr{H}_\varphi)$ the GNS-*construction* for φ. Let φ be a weight on $\mathscr{P}(\mathscr{M})$ and put

$$\mathfrak{N}_\varphi = \{X \in \mathscr{M}; \varphi((AX)^\dagger(AX)) < \infty \text{ for all } A \in \mathscr{M}\}.$$

Then $\mathfrak{N}_\varphi$ is a left ideal of $\mathscr{M}$ and the restriction $\varphi\lceil\mathscr{P}(\mathfrak{N}_\varphi)$ of φ to the positive cone $\mathscr{P}(\mathfrak{N}_\varphi)$ is a quasi-weight on $\mathscr{P}(\mathscr{M})$ and it is called the *quasi-weight on* $\mathscr{P}(\mathscr{M})$ *generated by* φ and is denoted by φ_q. We denote by $(\pi_\varphi, \lambda_\varphi, \mathscr{H}_\varphi)$ the GNS-construction for the quasi-weight φ_q generated by φ. We need the notions of faithfulness and semifiniteness of (quasi-)weights:

Definition 3.1. Let φ be a (quasi-)weight on $\mathscr{P}(\mathscr{M})$. If $\varphi(A^\dagger A) = 0, A \in \mathscr{M}$ implies $A = 0$, then φ is said to be *faithful*. If there exists a net $\{U_\alpha\}$ in $\mathfrak{N}_\varphi^\dagger \cap \mathfrak{N}_\varphi$ such that $\|\overline{U_\alpha}\| \le 1$ for each α and $\{U_\alpha\}$ converges strongly to I, then φ is said to be *semifinite*.

We have defined in [**12**] the notion of semifiniteness of (quasi-)weights which is stronger than that of semifiniteness defined above. Let φ be a faithful semifinite (quasi-) weight on $\mathscr{P}(\mathscr{M})$. Then it is easily shown that π_φ is a $*$-isomorphism and the generalized vector Λ_φ for the O*-algebra $\pi_\varphi(\mathscr{M})$ is defined by

$$\Lambda_\varphi(\pi_\varphi(X)) = \lambda_\varphi(X), \quad X \in \mathfrak{N}_\varphi.$$

Suppose

(S)$_1$ $\pi_\varphi(\mathscr{M})'_{\rm w}\mathscr{D}(\pi_\varphi) \subset \mathscr{D}(\pi_\varphi)$,

(S)$_2$ $\lambda_\varphi((\mathfrak{N}_\varphi^\dagger \cap \mathfrak{N}_\varphi)^2)$ is total in $\mathscr{H}_\varphi$.

Then we can define a generalized vector Λ_φ^c for the von Neumann algebra $\pi_\varphi(\mathscr{M})'_{\rm w}$ by

$$\begin{cases} \mathscr{D}(\Lambda_\varphi^c) = \{K \in \pi_\varphi(\mathscr{M})'_{\rm w};\ ^\exists \xi_K \in \mathscr{D}(\pi_\varphi) \\ \qquad\qquad \text{s.t. } K\Lambda_\varphi(\pi_\varphi(X)) = \pi_\varphi(X)\xi_K,\ ^\forall X \in \mathfrak{N}_\varphi\} \\ \Lambda_\varphi^c(K) = \xi_K, \quad K \in \mathscr{D}(\Lambda_\varphi^c). \end{cases}$$

Further, suppose

(S)$_3$ $\Lambda_\varphi^c((\mathscr{D}(\Lambda_\varphi^c)^* \cap \mathscr{D}(\Lambda_\varphi^c))^2)$ is total in $\mathscr{H}_\varphi$.

Then, the generalized vector Λ_φ^{cc} for the von Neumann algebra $(\pi_\varphi(\mathscr{M})'_{\rm w})'$ is defined by

$$\begin{cases} \mathscr{D}(\Lambda_\varphi^{cc}) = \{A \in (\pi_\varphi(\mathscr{M})'_{\rm w})';\ ^\exists \xi_A \in \mathscr{H}_\varphi \\ \qquad\qquad \text{s.t. } A\Lambda_\varphi^c(K) = K\xi_A,\ ^\forall K \in \mathscr{D}(\Lambda_\varphi^c)\} \\ \Lambda_\varphi^{cc}(A) = \xi_A, \quad A \in \mathscr{D}(\Lambda_\varphi^{cc}) \end{cases}$$

and $\Lambda_\varphi^{cc}((\mathscr{D}(\Lambda_\varphi^{cc})^* \cap \mathscr{D}(\Lambda_\varphi^{cc}))^2)$ is total in $\mathscr{H}_\varphi$. Hence, the maps $\lambda_\varphi(X) \mapsto \lambda_\varphi(X^\dagger)$, $X \in \mathfrak{N}_\varphi^\dagger \cap \mathfrak{N}_\varphi$ and $\Lambda_\varphi^{cc}(A) \mapsto \Lambda_\varphi^{cc}(A^*)$, $A \in \mathscr{D}(\Lambda_\varphi^{cc})^* \cap \mathscr{D}(\Lambda_\varphi^{cc})$ are closable in $\mathscr{H}_\varphi$ and their closures are denoted by S_φ and $S_{\Lambda_\varphi^{cc}}$, respectively. Let $S_\varphi = J_\varphi \Delta_\varphi^{1/2}$ and $S_{\Lambda_\varphi^{cc}} = J_{\Lambda_\varphi^{cc}} \Delta_{\Lambda_\varphi^{cc}}^{1/2}$ be the polar decompositions of S_φ and $S_{\Lambda_\varphi^{cc}}$, respectively. Then we see that $S_\varphi \subset S_{\Lambda_\varphi^{cc}}$, and by the Tomita fundamental theorem $J_{\Lambda_\varphi^{cc}}(\pi_\varphi(\mathscr{M})'_{\rm w})' J_{\Lambda_\varphi^{cc}} = \pi_\varphi(\mathscr{M})'_{\rm w}$ and $\Delta_{\Lambda_\varphi^{cc}}^{it}(\pi_\varphi(\mathscr{M})'_{\rm w})'\Delta_{\Lambda_\varphi^{cc}}^{-it} = (\pi_\varphi(\mathscr{M})'_{\rm w})'$ for all $t \in \boldsymbol{R}$. But, we don't know how the unitary group $\{\Delta_{\Lambda_\varphi^{cc}}^{it}\}_{t \in \boldsymbol{R}}$ acts on the O*-algebra $\pi_\varphi(\mathscr{M})$, and so we define a system which has the best properties:

Definition 3.2. A faithful semifinite (quasi-)weight φ on $\mathscr{P}(\mathscr{M})$ is said to be *quasi-standard* if the above conditions (S)$_1$, (S)$_2$, (S)$_3$ and the following condition (S)$_4$ hold:

(S)$_4$ $\Delta_{\Lambda_\varphi^{cc}}^{it}\mathscr{D}(\pi_\varphi) \subset \mathscr{D}(\pi_\varphi)$ for all $t \in \boldsymbol{R}$.

Further, if

(S)$_5$ $\Delta_{\Lambda_\varphi^{cc}}^{it}\pi_\varphi(\mathscr{M})\Delta_{\Lambda_\varphi^{cc}}^{-it} = \pi_\varphi(\mathscr{M})$ for all $t \in \boldsymbol{R}$,

then φ is said to be *essentially standard*, and in addition if

(S)$_6$ $\Delta_{\Lambda_\varphi^{cc}}^{it}\pi_\varphi(\mathfrak{N}_\varphi^\dagger \cap \mathfrak{N}_\varphi)\Delta_{\Lambda_\varphi^{cc}}^{-it} = \pi_\varphi(\mathfrak{N}_\varphi^\dagger \cap \mathfrak{N}_\varphi)$ for all $t \in \boldsymbol{R}$,

then φ is said to be *standard*.

We remark that a faithful semifinite (quasi-)weight φ is standard (resp. essentially standard, quasi-standard) if and only if the generalized vector Λ_φ for $\pi_\varphi(\mathscr{M})$ induced by φ is standard (resp. essentially standard, quasi-standard). Hence by Theorem 2.1 we have the following results for standard (quasi-)weights:

THEOREM 3.3. *Suppose φ is a faithful semifinite standard (quasi-)weight on $\mathscr{P}(\mathscr{M})$. Then the following statements hold*:

(1) $S_\varphi = S_{\Lambda_\varphi^{cc}}$, *and so* $J_\varphi = J_{\Lambda_\varphi^{cc}}$ *and* $\Delta_\varphi = \Delta_{\Lambda_\varphi^{cc}}$.

(2) *There exists a one-parameter group $\{\sigma_t^\varphi\}_{t\in\boldsymbol{R}}$ of $*$-automorphisms of $\mathscr{M}$ such that $\pi_\varphi(\sigma_t^\varphi(X)) = \Delta_\varphi^{it}\pi_\varphi(X)\Delta_\varphi^{-it}$ for all $X \in \mathscr{M}$ and $t \in \boldsymbol{R}$.*

(3) *φ is a $\{\sigma_t^\varphi\}$-KMS (quasi-)weight, that is, for any $X, Y \in \mathfrak{N}_\varphi^\dagger \cap \mathfrak{N}_\varphi$ there exists an element $f_{X,Y}$ of $A(0,1)$ such that $f_{X,Y}(t) = \dot{\varphi}(Y\sigma_t^\varphi(X))$ and $f_{X,Y}(t+i) = \dot{\varphi}(\sigma_t^\varphi(X)Y)$ for all $t \in \boldsymbol{R}$, where $A(0,1)$ is the set of all complex-valued functions, bounded and continuous on $0 \le \operatorname{Im} z \le 1$ and analytic in the interior.*

We next consider quasi-standard (quasi-)weights. Let φ be a faithful semifinite quasi-standard (quasi-)weight on $\mathscr{P}(\mathscr{M})$. We put

$$\begin{cases} \mathscr{D}(\overline{\Lambda_\varphi}) = \{A \in \pi_\varphi(\mathscr{M})''_{\mathrm{wc}};\ {}^\exists\xi_A \in \mathscr{D}(\pi_\varphi) \\ \qquad \text{s.t. } A\Lambda_\varphi^c(K) = K\xi_A,\ {}^\forall K \in \mathscr{D}(\Lambda_\varphi^c)\}, \\ \overline{\Lambda_\varphi}(A) = \xi_A, \quad A \in \mathscr{D}(\overline{\Lambda_\varphi}). \end{cases}$$

Then it is easily shown that $\overline{\Lambda_\varphi}$ is a generalized vector for the generalized von Neumann algebra $\pi_\varphi(\mathscr{M})''_{\mathrm{wc}}$ such that

$$\Lambda_\varphi \subset \overline{\Lambda_\varphi} \quad \text{and} \quad \Lambda_\varphi^c = \overline{\Lambda_\varphi}^c. \tag{3.2}$$

We now put

$$\bar{\varphi}\left(\sum_k A_k^\dagger A_k\right) = \sum_k \|\overline{\Lambda_\varphi}(A_k)\|^2, \quad \{A_k\} \subset \mathscr{D}(\overline{\Lambda_\varphi}).$$

Then $\bar{\varphi}$ is a faithful semifinite quasi-weight on $\mathscr{P}(\pi_\varphi(\mathscr{M})''_{\mathrm{wc}})$ such that

$$(\pi_{\bar{\varphi}}(\pi_\varphi(\mathscr{M})''_{\mathrm{wc}}), \lambda_{\bar{\varphi}}) \text{ is unitarily equivalent to } (\pi_\varphi(\mathscr{M})''_{\mathrm{wc}}, \overline{\Lambda_\varphi}), \tag{3.3}$$

that is, there exists a unitary operator U of $\mathscr{H}_\varphi$ onto $\mathscr{H}_{\bar{\varphi}}$ such that $U\overline{\Lambda_\varphi}(A) = \lambda_{\bar{\varphi}}(A)$ for each $A \in \mathscr{D}(\overline{\Lambda_\varphi})$ and $\pi_{\bar{\varphi}}(B) = UBU^*$ for each $B \in \pi_\varphi(\mathscr{M})''_{\mathrm{wc}}$. The above $\bar{\varphi}$ is said to be the *quasi-weight on $\mathscr{P}(\pi_\varphi(\mathscr{M})''_{\mathrm{wc}})$ induced by φ*. By (3.2), (3.3) and Theorem 2.2 we have the following

THEOREM 3.4. *Suppose φ is a faithful semifinite quasi-standard (quasi-)weight on $\mathscr{P}(\mathscr{M})$. Then the quasi-weight $\bar{\varphi}$ on $\mathscr{P}(\pi_\varphi(\mathscr{M})''_{\mathrm{wc}})$ induced by φ is standard, and so it is a $\{\sigma_t^{\bar{\varphi}}\}_{t\in\boldsymbol{R}}$-KMS quasi-weight, where $\sigma_t^{\bar{\varphi}}(A) = \Delta_{\Lambda_\varphi^{cc}}^{it} A \Delta_{\Lambda_\varphi^{cc}}^{-it}$, $A \in \pi_\varphi(\mathscr{M})''_{\mathrm{wc}}, t \in \boldsymbol{R}$.*

Conversely we consider when a KMS (quasi-)weight is standard.

THEOREM 3.5. *Let $\{\alpha_t\}_{t\in\boldsymbol{R}}$ be a one-parameter group of $*$-automorphisms of $\mathscr{M}$. Suppose φ is a $\{\alpha_t\}$-KMS (quasi-)weight on $\mathscr{P}(\mathscr{M})$ such that $\lambda_\varphi((\mathfrak{N}_\varphi^\dagger \cap \mathfrak{N}_\varphi)^2)$ is total in $\mathscr{H}_\varphi$. Then the following statements hold*:

(1) *The map $\lambda_\varphi(X) \mapsto \lambda_\varphi(X^\dagger), X \in \mathfrak{N}_\varphi^\dagger \cap \mathfrak{N}_\varphi$ is a closable conjugate-linear operator in $\mathscr{H}_\varphi$. Let S_φ be the closure of the above operator $\lambda_\varphi(X) \mapsto \lambda_\varphi(X^\dagger)$ and $S_\varphi = J_\varphi\Delta_\varphi^{1/2}$ the polar decomposition of S_φ.*

(2) $\Delta_\varphi^{it}\lambda_\varphi(X) = \lambda_\varphi(\alpha_t(X))$, $^\forall X \in \mathfrak{N}_\varphi$, $^\forall t \in \boldsymbol{R}$.
(3) *φ is standard if and only if the following statements hold*:
 (i) *Λ_φ is well-defined.*
 (ii) *$\Lambda_\varphi^c((\mathscr{D}(\Lambda_\varphi^c)^* \cap \mathscr{D}(\Lambda_\varphi^c))^2)$ is total in $\mathscr{H}_\varphi$.*
 (iii) *$J_\varphi\Lambda_\varphi^c(\mathscr{D}(\Lambda_\varphi^c)^* \cap \mathscr{D}(\Lambda_\varphi^c)) \subset \Lambda_\varphi^{cc}(\mathscr{D}(\Lambda_\varphi^{cc})^* \cap \mathscr{D}(\Lambda_\varphi^{cc}))$.*
 (iv) *$(J_\varphi\Lambda_\varphi^{cc}(A)|\Lambda_\varphi^{cc}(A^*)) \geq 0$, $^\forall A \in \mathscr{D}(\Lambda_\varphi^{cc})^* \cap \mathscr{D}(\Lambda_\varphi^{cc})$.*

Proof. We put

$$U_t\lambda_\varphi(X) = \lambda_\varphi(\alpha_t(X)), \quad X \in \mathfrak{N}_\varphi.$$

Since φ is $\{\alpha_t\}$-KMS (quasi-)weight on $\mathscr{P}(\mathscr{M})$, for any $X, Y \in \mathfrak{N}_\varphi^\dagger \cap \mathfrak{N}_\varphi$ there exists an element $f_{X,Y}$ of $A(0,1)$ such that

$$f_{X,Y}(t) = \dot{\varphi}(\alpha_t(X)Y) \quad \text{and} \quad f_{X,Y}(t+i) = \dot{\varphi}(Y\alpha_t(X)), \quad ^\forall t \in \boldsymbol{R}.$$

We now have

$$\begin{aligned}\lim_{t\to 0}\|U_t\lambda_\varphi(X) - \lambda_\varphi(X)\|^2 &= \lim_{t\to 0}\{\varphi(\alpha_t(X)^\dagger\alpha_t(X)) - \dot{\varphi}(\alpha_t(X)^\dagger X)\\ &\qquad - \dot{\varphi}(X^\dagger\alpha_t(X)) + \varphi(X^\dagger X)\}\\ &= \lim_{t\to 0}\{2\varphi(X^\dagger X) - f_{X^\dagger,X}(t) - f_{X,X^\dagger}(t+i)\}\\ &= 0\end{aligned}$$

for each $X \in \mathfrak{N}_\varphi^\dagger \cap \mathfrak{N}_\varphi$, which implies that $\{\overline{U_t}\}_{t\in\boldsymbol{R}}$ is a strongly continuous one-parameter group of unitary operators on $\mathscr{H}_\varphi$. Let $\{X_n\}$ be any sequence in $\mathfrak{N}_\varphi^\dagger \cap \mathfrak{N}_\varphi$ such that $\lim_{n\to\infty}\lambda_\varphi(X_n) = 0$ and $\lim_{n\to\infty}\lambda_\varphi(X_n^\dagger) = \xi$. For any $Y \in \mathfrak{N}_\varphi^\dagger \cap \mathfrak{N}_\varphi$ we have

$$\begin{aligned}\lim_{n\to\infty}\sup_{t\in\boldsymbol{R}}|f_{X_n,Y}(t) - (\lambda_\varphi(Y)|\overline{U_t}\xi)| &= \lim_{n\to\infty}\sup_{t\in\boldsymbol{R}}|(\lambda_\varphi(Y)|\overline{U_t}(\lambda_\varphi(X_n^\dagger) - \xi))|\\ &\leq \lim_{n\to\infty}\|\lambda_\varphi(Y)\|\,\|\lambda_\varphi(X_n^\dagger) - \xi\|\\ &= 0,\end{aligned}$$

$$\lim_{n\to\infty}\sup_{t\in\boldsymbol{R}}|f_{X_n,Y}(t+i)| = 0,$$

and hence there exists an element f of $A(0,1)$ such that $f(t) = (\lambda_\varphi(Y)|\overline{U_t}\xi)$ and $f(t+i) = 0$ for all $t \in \boldsymbol{R}$. Hence we have $f = 0$, and so $\xi = 0$. Thus the statement (1) holds. The statement (2) is shown similarly to the proof of ([7] Lemma 3.8). We show the statement (3). It is clear that if φ is standard, then the statements (i) $\sim$ (iv) hold. Conversely suppose the statements (i) $\sim$ (iv) hold. We put

$$T\Lambda_\varphi^{cc}(A) = J_\varphi\Lambda_\varphi^{cc}(A^*), \quad A \in \mathscr{D}(\Lambda_\varphi^{cc})^* \cap \mathscr{D}(\Lambda_\varphi^{cc}).$$

Then T is a well-defined from (iii) that T is positive and $\overline{T} = J_\varphi S_{\Lambda_\varphi^{cc}} = J_\varphi J_{\Lambda_\varphi^{cc}}\Delta_{\Lambda_\varphi^{cc}}^{1/2}$. We put $U = J_\varphi J_{\Lambda_\varphi^{cc}}$. Then U is a unitary operator on $\mathscr{H}_\varphi$. Since $T^* = S_{\Lambda_\varphi^{cc}}^* J_\varphi$ and $J_\varphi\Lambda_\varphi^c(\mathscr{D}(\Lambda_\varphi^c)^* \cap \mathscr{D}(\Lambda_\varphi^c))$ is a core for $S_{\Lambda_\varphi^{cc}}^*$, it follows from (iv) that $\overline{T}$ is a positive self-adjoint operator in $\mathscr{H}_\varphi$, and so $U = I$ and $J_\varphi = J_{\Lambda_\varphi^{cc}}$. Hence we have $\Delta_\varphi = \Delta_{\Lambda_\varphi^{cc}}$, which implies by (2) that φ is standard. This completes the proof.

4. Generalized Connes cocycle theorem for weights.

In this section we generalize the Connes cocycle theorem for weights on O*-algebras. In [**10**] we studied to generalize the Connes cocycle theorem and the Pedersen-Takesaki Radon-Nikodym theorem to generalized von Neumann algebras in case of standard generalized vectors. As the notion of generalized vectors is spatial, such a generalization is possible to a certain extent, but the notion of (quasi-)weights is purely algebraic and not spatial and the algebraic properties don't reflect to the topological properties in general (for example, $\pi_\varphi(\mathscr{M})$ is not necessarily a generalized von Neumann algebra when $\mathscr{M}$ is a generalized von Neumann algebra), and so such generalizations for (quasi-)weights have some difficult problems. We first need the notion of σ-weak continuity of (quasi-)weights. Let $\mathscr{M}$ be a closed O*-algebra on $\mathscr{D}$ in $\mathscr{H}$.

Definition 4.1. For any $X \in \mathfrak{N}_\varphi$ we put

$$\varphi_X(A) = \dot{\varphi}(X^\dagger A X), \quad A \in \mathscr{M}.$$

Then φ_X is a positive linear functional on $\mathscr{M}$. If φ_X is σ-weakly continuous for each $X \in \mathfrak{N}_\varphi$, then φ is said to be *σ-weakly continuous*.

Lemma 4.2. *Let φ be a (quasi-)weight on $\mathscr{P}(\mathscr{M})$. Then the following statements hold*:

(1) *φ is σ-weakly continuous if and only if $\varphi_{X,Y}$ is a σ-weakly continuous linear functional on $\mathscr{M}$ for each $X, Y \in \mathfrak{N}_\varphi$, where*

$$\varphi_{X,Y}(A) = \dot{\varphi}(Y^\dagger A X), \quad A \in \mathscr{M}.$$

(2) *Suppose $\dot{\varphi}$ is σ-weakly continuous on $\mathscr{D}(\varphi)$, then φ is σ-weakly continuous.*

(3) *Suppose φ is faithful, σ-weakly continuous and semifinite. Then Λ_φ is a semifinite generalized vector for $\pi_\varphi(\mathscr{M})$ such that $\Lambda_\varphi((\mathscr{D}(\Lambda_\varphi)^\dagger \cap \mathscr{D}(\Lambda_\varphi))^2)$ is total in $\mathscr{H}_\varphi$.*

Proof. (1) This follows since any $\varphi_{X,Y}$ is a linear combination of $\{\varphi_{X_k}; X_k \in \mathfrak{N}_\varphi\}$.

(2) This is almost trivial.

(3) Since φ is semifinite, there exists a net $\{U_\alpha\}$ in $\mathfrak{N}_\varphi^\dagger \cap \mathfrak{N}_\varphi$ such that $\|\overline{U_\alpha}\| \le 1$ for all α and $\{U_\alpha\}$ converges strongly to I. Take an arbitrary $X \in \mathfrak{N}_\varphi$. Since φ_X is a σ-weakly continuous positive linear functional on the bounded part $\mathscr{M}_b$ of $\mathscr{M}$, it follows that φ_X can be extended to a σ-weakly continuous positive linear functional φ_X'' on the von Neumann algebra $\overline{\mathscr{M}_b}''$. Hence we have

$$\begin{aligned}
&\pi_\varphi(U_\alpha) \in \mathscr{D}(\Lambda_\varphi)^\dagger \cap \mathscr{D}(\Lambda_\varphi), \quad {}^\forall\alpha, \\
&\|\pi_\varphi(U_\alpha)\lambda_\varphi(X)\|^2 = \varphi_X(U_\alpha^\dagger U_\alpha) \\
&\qquad = \varphi_X''(\overline{U_\alpha}^*\overline{U_\alpha}) \\
&\qquad \le \|\overline{U_\varphi}\|^2 \varphi_X''(I) \\
&\qquad \le \|\lambda_\varphi(X)\|^2
\end{aligned}$$

for all α and

$$\|\pi_\varphi(U_\alpha)\lambda_\varphi(X) - \lambda_\varphi(X)\|^2 = \varphi_X((U_\alpha - I)^\dagger(U_\alpha - I)) \xrightarrow{\alpha} 0,$$

which implies that Λ_φ is semifinite. Further, it follows that $\pi_\varphi(U_\beta U_\alpha X) \in (\mathscr{D}(\Lambda_\varphi)^\dagger \cap \mathscr{D}(\Lambda_\varphi))^2$ and

$$\begin{aligned}\lim_{\alpha,\beta} \Lambda_\varphi(\pi_\varphi(U_\beta U_\alpha X)) &= \lim_{\alpha,\beta} \pi_\varphi(U_\beta)\pi_\varphi(U_\alpha)\lambda_\varphi(X)\\ &= \lambda_\varphi(X),\\ &= \Lambda_\varphi(\pi_\varphi(X)),\end{aligned}$$

which implies that $\Lambda_\varphi((\mathscr{D}(\Lambda_\varphi)^\dagger \cap \mathscr{D}(\Lambda_\varphi))^2)$ is total in $\mathscr{H}_\varphi$.

Let φ and ψ be faithful, σ-weakly continuous semifinite (quasi-)weights on $\mathscr{P}(\mathscr{M})$ such that π_φ and π_ψ are self-adjoint. Let $M_2(\boldsymbol{C})$ be the 2×2-matrix algebra on $\boldsymbol{C}$ and put

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Every element X of $\mathscr{M} \otimes M_2(\boldsymbol{C})$ is represented as

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = X_{11} \otimes E_{11} + X_{12} \otimes E_{12} + X_{21} \otimes E_{21} + X_{22} \otimes E_{22}.$$

We put

$$\begin{aligned}\theta(X^\dagger X) &= \varphi(X_{11}^\dagger X_{11} + X_{21}^\dagger X_{21}) + \psi(X_{12}^\dagger X_{12} + X_{22}^\dagger X_{22}),\\ X &= (X_{ij}) \in \mathscr{M} \otimes M_2(\boldsymbol{C}).\end{aligned}$$

Then we have the following

Lemma 4.3. (1) *θ is a faithful, σ-weakly continuous, semifinite (quasi-)weight on $\mathscr{P}(\mathscr{M} \otimes M_2(\boldsymbol{C}))$ such that π_θ is self-adjoint and*

$$\mathfrak{N}_\theta = \{X = (X)_{ij} \in \mathscr{M} \otimes M_2(\boldsymbol{C});\ X_{11}, X_{21} \in \mathfrak{N}_\varphi \text{ and } X_{12}, X_{22} \in \mathfrak{N}_\psi\}.$$

(2) *$\lambda_\varphi(\mathfrak{N}_\varphi \cap \mathfrak{N}_\psi^\dagger)$ is dense in $\mathscr{H}_\varphi$ and $\lambda_\psi(\mathfrak{N}_\psi \cap \mathfrak{N}_\varphi^\dagger)$ is dense in $\mathscr{H}_\psi$.*

Proof. (1) It is easily shown that θ is a faithful, σ-weakly continuous (quasi-) weight on $\mathscr{P}(\mathscr{M} \otimes M_2(\boldsymbol{C}))$ such that π_θ is self-adjoint and

$$\mathfrak{N}_\theta = \{X = (X)_{ij} \in \mathscr{M} \otimes M_2(\boldsymbol{C});\ X_{11}, X_{21} \in \mathfrak{N}_\varphi \text{ and } X_{12}, X_{22} \in \mathfrak{N}_\psi\}.$$

Let $\{U_\alpha\}$ and $\{V_\beta\}$ be nets in $\mathfrak{N}_\varphi^\dagger \cap \mathfrak{N}_\varphi$ and $\mathfrak{N}_\psi^\dagger \cap \mathfrak{N}_\psi$, respectively such that $\|\overline{U_\alpha}\| \le 1$ for all α and $\|\overline{V_\beta}\| \le 1$ for all β and $\{U_\alpha\}$ and $\{V_\beta\}$ converge strongly to I. Considering

$$\begin{pmatrix} U_\alpha & 0 \\ 0 & V_\beta \end{pmatrix} \in \mathfrak{N}_\theta^\dagger \cap \mathfrak{N}_\theta, \quad {}^\forall\alpha, \beta,$$

we can show that θ is semifinite.

(2) Take an arbitrary $X \in \mathfrak{N}_\varphi$. We have $V_\beta X \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\psi^\dagger$,

$$\|\lambda_\varphi(V_\beta X) - \lambda_\varphi(X)\|^2 = \varphi_X((V_\beta - I)^\dagger(V_\beta - I))$$

for each β, and hence it follows from the σ-weak continuity of φ_X that $\lambda_\varphi(\mathfrak{N}_\varphi \cap \mathfrak{N}_\psi^\dagger)$ is dense in $\mathscr{H}_\varphi$. Similarly, $\lambda_\psi(\mathfrak{N}_\psi \cap \mathfrak{N}_\varphi^\dagger)$ is dense in $\mathscr{H}_\psi$.

By Lemma 4.2, (4) and Lemma 4.3, (1) we have

$$\lambda_\theta((\mathfrak{N}_\theta^\dagger \cap \mathfrak{N}_\theta)^2) \text{ is total in } \mathscr{H}_\theta. \tag{4.1}$$

Hence we can define the generalized vector Λ_θ^c for the von Neumann algebra $\pi_\theta(\mathscr{M} \otimes M_2(\boldsymbol{C}))_{\mathrm{w}}'$, and so to decide it we first define the following map $\Lambda_{\varphi,\psi}^c$:

$$\begin{cases} \mathscr{D}(\Lambda_{\varphi,\psi}^c) = \{K \in \mathrm{II}(\pi_\varphi, \pi_\psi);\ {}^\exists\eta \in \mathscr{D}(\pi_\psi) \\ \qquad\qquad \text{s.t. } K\lambda_\varphi(X) = \pi_\psi(X)\eta,\ {}^\forall X \in \mathfrak{N}_\varphi\}, \\ \Lambda_{\varphi,\psi}^c(K) = \eta, \quad K \in \mathscr{D}(\Lambda_{\varphi,\psi}^c). \end{cases}$$

Then we have the following

Lemma 4.4. *$\Lambda_{\varphi,\psi}^c$ is a linear map of $\mathscr{D}(\Lambda_{\varphi,\psi}^c)$ into $\mathscr{D}(\pi_\psi)$ satisfying*

(i) *$CK \in \mathscr{D}(\Lambda_{\varphi,\psi}^c)$ and $\Lambda_{\varphi,\psi}^c(CK) = C\Lambda_{\varphi,\psi}^c(K)$ for each $C \in \pi_\psi(\mathscr{M})_{\mathrm{w}}'$ and $K \in \mathscr{D}(\Lambda_{\varphi,\psi}^c)$;*

(ii) *$CK \in \mathscr{D}(\Lambda_\varphi^c)$ and $\Lambda_\varphi^c(CK) = C\Lambda_{\varphi,\psi}^c(K)$ for each $C \in \mathrm{II}(\pi_\psi, \pi_\varphi)$ and $K \in \mathscr{D}(\Lambda_{\varphi,\psi}^c)$.*

We here put

$$\mathscr{H}_1 = \overline{\lambda_\theta(\mathfrak{N}_\varphi \otimes E_{11})}, \quad \mathscr{H}_2 = \overline{\lambda_\theta(\mathfrak{N}_\varphi \otimes E_{21})},$$

$$\mathscr{H}_3 = \overline{\lambda_\theta(\mathfrak{N}_\psi \otimes E_{12})}, \quad \mathscr{H}_4 = \overline{\lambda_\theta(\mathfrak{N}_\psi \otimes E_{22})},$$

and

$$U_1\lambda_\varphi(X) = \lambda_\theta(X \otimes E_{11}), \quad X \in \mathfrak{N}_\varphi,$$

$$U_2\lambda_\varphi(X) = \lambda_\theta(X \otimes E_{21}), \quad X \in \mathfrak{N}_\varphi,$$

$$U_3\lambda_\psi(X) = \lambda_\theta(X \otimes E_{12}), \quad X \in \mathfrak{N}_\psi,$$

$$U_4\lambda_\psi(X) = \lambda_\theta(X \otimes E_{22}), \quad X \in \mathfrak{N}_\psi.$$

Then $\{\mathscr{H}_i\}_{i=1,\dots,4}$ is a set of mutually orthogonal closed subspaces of $\mathscr{H}_\theta$ such that $\mathscr{H}_\theta = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \mathscr{H}_3 \oplus \mathscr{H}_4$, and U_1 and U_2 (resp. U_3 and U_4) can be extended to the isometries from $\mathscr{H}_\varphi$ (resp. $\mathscr{H}_\psi$) to $\mathscr{H}_1$ and $\mathscr{H}_2$ (resp. $\mathscr{H}_3$ and $\mathscr{H}_4$), and they are also denoted by U_1 and U_2 (resp. U_3 and U_4). For $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \in \mathscr{M} \otimes M_2(\boldsymbol{C})$, $\pi_\theta(X)$ is given by the matrix:

$$\begin{pmatrix} U_1\pi_\varphi(X_{11})U_1^* & U_1\pi_\varphi(X_{12})U_2^* & 0 & 0 \\ U_2\pi_\varphi(X_{21})U_1^* & U_2\pi_\varphi(X_{22})U_2^* & 0 & 0 \\ 0 & 0 & U_3\pi_\psi(X_{11})U_3^* & U_3\pi_\psi(X_{12})U_4^* \\ 0 & 0 & U_4\pi_\psi(X_{21})U_3^* & U_4\pi_\psi(X_{22})U_4^* \end{pmatrix}.$$

We now have the following results for the von Neumann algebras $\pi_\theta(\mathscr{M} \otimes M_2(\boldsymbol{C}))'_{\mathrm{w}}$ and $(\pi_\theta(\mathscr{M} \otimes M_2(\boldsymbol{C}))'_{\mathrm{w}})'$ and the generalized vector Λ^c_θ:

Lemma 4.5.

$$\pi_\theta(\mathscr{M} \otimes M_2(\boldsymbol{C}))'_{\mathrm{w}}$$

$$= \left\{ \begin{pmatrix} U_1 C_1 U_1^* & 0 & U_1 C_2 U_3^* & 0 \\ 0 & U_2 C_1 U_2^* & 0 & U_2 C_2 U_4^* \\ U_3 C_3 U_1^* & 0 & U_3 C_4 U_3^* & 0 \\ 0 & U_4 C_3 U_2^* & 0 & U_4 C_4 U_4^* \end{pmatrix} ; \begin{array}{l} C_1 \in \pi_\varphi(\mathscr{M})'_{\mathrm{w}}, \\ C_2 \in \mathrm{II}(\pi_\psi, \pi_\varphi), \\ C_3 \in \mathrm{II}(\pi_\varphi, \pi_\psi), \\ C_4 \in \pi_\psi(\mathscr{M})'_{\mathrm{w}} \end{array} \right\},$$

$$(\pi_\theta(\mathscr{M} \otimes M_2(\boldsymbol{C}))'_{\mathrm{w}})'$$

$$= \left\{ \begin{pmatrix} U_1 A_{11} U_1^* & U_1 A_{12} U_2^* & 0 & 0 \\ U_2 A_{21} U_1^* & U_2 A_{22} U_2^* & 0 & 0 \\ 0 & 0 & U_3 B_{11} U_3^* & U_3 B_{12} U_4^* \\ 0 & 0 & U_4 B_{21} U_3^* & U_4 B_{22} U_4^* \end{pmatrix} ; \ (A_{ij}, B_{ij}) \in \mathfrak{A}_{\varphi,\psi}, \ \ (i,j = 1,2) \right\},$$

where

$$\mathfrak{A}_{\varphi,\psi} = \left\{ (A,B); \ \begin{array}{c} A \in (\pi_\varphi(\mathscr{M})'_{\mathrm{w}})', \ \ B \in (\pi_\psi(\mathscr{M})'_{\mathrm{w}})', \\ AC = CB \ \ \text{and} \ \ A^*C = CB^* \\ \text{for all} \ \ C \in \mathrm{II}(\pi_\psi, \pi_\varphi) \end{array} \right\}$$

and

$$\mathscr{D}(\Lambda^c_\theta) = \left\{ C = \begin{pmatrix} U_1 C_1 U_1^* & 0 & U_1 C_2 U_3^* & 0 \\ 0 & U_2 C_1 U_2^* & 0 & U_2 C_2 U_4^* \\ U_3 C_3 U_1^* & 0 & U_3 C_4 U_3^* & 0 \\ 0 & U_4 C_3 U_2^* & 0 & U_4 C_4 U_4^* \end{pmatrix} ; \begin{array}{l} C_1 \in \mathscr{D}(\Lambda^c_\varphi), \\ C_2 \in \mathscr{D}(\Lambda^c_{\psi,\varphi}), \\ C_3 \in \mathscr{D}(\Lambda^c_{\varphi,\psi}), \\ C_4 \in \mathscr{D}(\Lambda^c_\psi) \end{array} \right\},$$

$$\Lambda^c_\theta(C) = \begin{pmatrix} U_1 \lambda^c_\varphi(C_1) \\ U_2 \lambda^c_{\psi,\varphi}(C_2) \\ U_3 \lambda^c_{\varphi,\psi}(C_3) \\ U_4 \lambda^c_\psi(C_4) \end{pmatrix}, \quad C \in \mathscr{D}(\Lambda^c_\theta).$$

In bounded case $\Lambda^c_\theta((\mathscr{D}(\Lambda^c_\theta)^* \cap \mathscr{D}(\Lambda^c_\theta))^2)$ is total in $\mathscr{H}_\theta$, but in unbounded case this fact doesn't necessarily hold even if φ and ψ are standard. We have the following result for this problem:

Theorem 4.6. *Let φ and ψ be faithful, σ-weakly continuous, semifinite (quasi-) weights on $\mathscr{P}(\mathscr{M})$ such that π_φ and π_ψ are self-adjoint. Suppose $\Lambda^c_\varphi((\mathscr{D}(\Lambda^c_\varphi)^* \cap \mathscr{D}(\Lambda^c_\varphi))^2)$ is total in $\mathscr{H}_\varphi$ and $\Lambda^c_\psi((\mathscr{D}(\Lambda^c_\psi)^* \cap \mathscr{D}(\Lambda^c_\psi))^2)$ is total in $\mathscr{H}_\psi$. The following statements are equivalent*:

(i) *π_φ and π_ψ are unitarily equivalent.*

(ii) $\mathrm{II}(\pi_\varphi, \pi_\psi)^*\mathrm{II}(\pi_\varphi, \pi_\psi)$ and $\mathrm{II}(\pi_\psi, \pi_\varphi)^*\mathrm{II}(\pi_\psi, \pi_\varphi)$ *are nondegenerate $*$-subalgebras of the von Neumann algebra $\pi_\varphi(\mathscr{M})'_{\mathrm{w}}$ and $\pi_\psi(\mathscr{M})'_{\mathrm{w}}$, respectively.*

(iii) *$\Lambda^c_{\psi,\varphi}(\mathscr{D}(\Lambda^c_{\psi,\varphi}))$ is dense in $\mathscr{H}_\varphi$ and $\Lambda^c_{\varphi,\psi}(\mathscr{D}(\Lambda^c_{\varphi,\psi}))$ is dense in $\mathscr{H}_\psi$.*

(iv) *$\Lambda^c_\theta((\mathscr{D}(\Lambda^c_\theta)^* \cap \mathscr{D}(\Lambda^c_\theta))^2)$ is total in $\mathscr{H}_\theta$.*

Proof. (i) $\Leftrightarrow$ (ii) This follows from ([**6**] Theorem 3.2).

(i) $\Rightarrow$ (iii) There exists a unitary transform W of $\mathscr{H}_\varphi$ onto $\mathscr{H}_\psi$ such that $W\mathscr{D}(\pi_\varphi) = \mathscr{D}(\pi_\psi)$ and $\pi_\varphi(X) = W^*\pi_\psi(X)W$ for all $X \in \mathscr{M}$. Then we have

$$\begin{cases} \mathscr{D}(\Lambda^c_{\varphi,\psi}) = \{WC;\ C \in \mathscr{D}(\Lambda^c_\varphi)\}, \\ \Lambda^c_{\varphi,\psi}(WC) = W\Lambda^c_\varphi(C), \quad C \in \mathscr{D}(\Lambda^c_\varphi). \end{cases}$$

Hence, $\Lambda^c_{\varphi,\psi}(\mathscr{D}(\Lambda^c_{\varphi,\psi}))$ is dense in $\mathscr{H}_\psi$. Similarly, $\Lambda^c_{\psi,\varphi}(\mathscr{D}(\Lambda^c_{\psi,\varphi}))$ is dense in $\mathscr{H}_\varphi$.

(iii) $\Rightarrow$ (iv) By Lemma 4.5 we have

$$\Lambda^c_\theta((\mathscr{D}(\Lambda^c_\theta)^* \cap \mathscr{D}(\Lambda^c_\theta))^2) = \left\{ \begin{pmatrix} U_1C_1\Lambda^c_\varphi(D_1) + U_1C_2\Lambda^c_{\varphi,\psi}(D_3) \\ U_2C_1\Lambda^c_{\psi,\varphi}(D_2) + U_2C_2\Lambda^c_\psi(D_4) \\ U_3C_3\Lambda^c_\varphi(D_1) + U_3C_4\Lambda^c_{\varphi,\psi}(D_3) \\ U_4C_3\Lambda^c_{\psi,\varphi}(D_2) + U_4C_4\Lambda^c_\psi(D_4) \end{pmatrix};\ \begin{array}{l} C_1, D_1 \in \mathscr{D}(\Lambda^c_\varphi)^* \cap \mathscr{D}(\Lambda^c_\varphi), \\ C_2, D_2, C_3^*, D_3^* \in \mathscr{D}(\Lambda^c_{\psi,\varphi}), \\ C_3, D_3, C_2^*, D_2^* \in \mathscr{D}(\Lambda^c_{\varphi,\psi}), \\ C_4, D_4 \in \mathscr{D}(\Lambda^c_\psi)^* \cap \mathscr{D}(\Lambda^c_\psi) \end{array} \right\}, \tag{4.2}$$

which implies since $\mathscr{D}(\Lambda^c_\varphi)^* \cap \mathscr{D}(\Lambda^c_\varphi)$ and $\mathscr{D}(\Lambda^c_\psi)^* \cap \mathscr{D}(\Lambda^c_\psi)$ are nondegenerate that $\lambda^c_\theta((\mathscr{D}(\Lambda^c_\theta)^* \cap \mathscr{D}(\Lambda^c_\theta))^2)$ is total in $\mathscr{H}_\theta$.

(iv) $\Rightarrow$ (i) Since $\lambda^c_\theta((\mathscr{D}(\Lambda^c_\theta)^* \cap \mathscr{D}(\Lambda^c_\theta))^2)$ is total in $\mathscr{H}_\theta$, it follows that Λ^{cc}_θ is well-defined and

$$\mathscr{D}(\Lambda^{cc}_\theta) = \left\{ A = \begin{pmatrix} U_1A_{11}U_1^* & U_1A_{12}U_2^* & 0 & 0 \\ U_2A_{21}U_1^* & U_2A_{22}U_2^* & 0 & 0 \\ 0 & 0 & U_3B_{11}U_3^* & U_3B_{12}U_4^* \\ 0 & 0 & U_4B_{21}U_3^* & U_4B_{22}U_4^* \end{pmatrix};\ \begin{array}{l} A_{ij} \in \mathscr{D}(\Lambda^{cc}_\varphi), \\ B_{ij} \in \mathscr{D}(\Lambda^{cc}_\psi)\ (i,j = 1,2)\ \text{s.t.} \\ B_{11}\Lambda^c_{\varphi,\psi}(C_3) = C_3\Lambda^{cc}_\varphi(A_{11}), \\ B_{21}\Lambda^c_{\varphi,\psi}(C_3) = C_3\Lambda^{cc}_\varphi(A_{21}), \\ A_{12}\Lambda^c_{\psi,\varphi}(C_2) = C_2\Lambda^{cc}_\psi(B_{12}), \\ A_{12}\Lambda^c_{\psi,\varphi}(C_2) = C_2\Lambda^{cc}_\psi(B_{22}) \\ \quad \text{for } {}^\forall C_2 \in \mathscr{D}(\Lambda^c_{\psi,\varphi}) \\ \quad \text{and } {}^\forall C_3 \in \mathscr{D}(\Lambda^c_{\varphi,\psi}) \end{array} \right\},$$

$$\Lambda^{cc}_\theta(A) = \begin{pmatrix} U_1\Lambda^{cc}_\varphi(A_{11}) \\ U_2\Lambda^{cc}_\varphi(A_{21}) \\ U_3\Lambda^{cc}_\psi(B_{12}) \\ U_4\Lambda^{cc}_\psi(B_{22}) \end{pmatrix}, \quad A \in \mathscr{D}(\Lambda^{cc}_\theta).$$

Then we have

$$S_{\Lambda_\theta^{cc}} = \begin{pmatrix} U_1 S_{11} U_1^* & 0 & 0 & 0 \\ 0 & 0 & U_2 S_{12} U_3^* & 0 \\ 0 & U_3 S_{21} U_2^* & 0 & 0 \\ 0 & 0 & 0 & U_4 S_{22} U_4^* \end{pmatrix},$$

where S_{ij} $(i, j = 1, 2)$ is a closed operator defined by

$$S_{11} : \Lambda_\varphi^{cc}(A_{11}) \mapsto \Lambda_\varphi^{cc}(A_{11}^*),$$

$$S_{22} : \Lambda_\psi^{cc}(B_{22}) \mapsto \Lambda_\psi^{cc}(B_{22}^*),$$

$$S_{12} : \Lambda_\psi^{cc}(B_{12}) \mapsto \Lambda_\varphi^{cc}(A_{12}^*),$$

$$S_{21} : \Lambda_\varphi^{cc}(A_{21}) \mapsto \Lambda_\psi^{cc}(B_{21}^*).$$

Let $S_{ij} = J_{ij} \Delta_{ij}^{1/2}$ be the polar decomposition of S_{ij} $(i, j = 1, 2)$. Then we have

$$\Delta_{\Lambda_\theta^{cc}} = \begin{pmatrix} U_1 \Delta_{11} U_1^* & 0 & 0 & 0 \\ 0 & U_2 \Delta_{21} U_2^* & 0 & 0 \\ 0 & 0 & U_3 \Delta_{12} U_3^* & 0 \\ 0 & 0 & 0 & U_4 \Delta_{22} U_4^* \end{pmatrix}, \tag{4.3}$$

$$J_{\Lambda_\theta^{cc}} = \begin{pmatrix} U_1 J_{11} U_1^* & 0 & 0 & 0 \\ 0 & 0 & U_2 J_{12} U_3^* & 0 \\ 0 & U_3 J_{21} U_2^* & 0 & 0 \\ 0 & 0 & 0 & U_4 J_{22} U_4^* \end{pmatrix}.$$

Then it follows from Lemma 4.5 that

$$\begin{aligned} C \equiv J_{\Lambda_\theta^{cc}} & \begin{pmatrix} 0 & U_1 U_2^* & 0 & 0 \\ U_2 U_1^* & 0 & 0 & 0 \\ 0 & 0 & 0 & U_3 U_4^* \\ 0 & 0 & U_4 U_3^* & 0 \end{pmatrix} J_{\Lambda_\theta^{cc}} \\ &= \begin{pmatrix} 0 & 0 & U_1 J_{11} J_{12} U_3^* & 0 \\ 0 & 0 & 0 & U_1 J_{12} J_{22} U_4^* \\ U_3 J_{21} J_{11} U_1^* & 0 & 0 & 0 \\ 0 & U_4 J_{22} J_{21} U_2^* & 0 & 0 \end{pmatrix} \\ &\in \pi_\theta(\mathscr{M} \otimes M_2(\boldsymbol{C}))'_{\mathrm{w}}. \end{aligned}$$

Hence we have

$$C\pi_\theta(X)C = \pi_\theta(X), \quad {}^\forall X \in \mathscr{M} \otimes M_2(\boldsymbol{C}),$$

which implies that $W \equiv J_{22}J_{21}$ is a unitary tranform of $\mathscr{H}_\varphi$ onto $\mathscr{H}_\psi$ such that

$$\pi_\varphi(X) = W^*\pi_\psi(X)W, \quad {}^\forall X \in \mathscr{M}.$$

This completes the proof.

PROPOSITION 4.7. *Let φ and ψ be faithful, σ-weakly continuous, semifinite* (*quasi-*) *weights on $\mathscr{P}(\mathscr{M})$ such that π_φ and π_ψ are self-adjoint. The following statements are equivalent*:

(i) *φ and ψ are quasi-standard* (*quasi-*)*weights which satisfy one of* (i)$\sim$(iv) *in Theorem* 4.6.

(ii) *θ is quasi-standard.*

PROOF. (i) $\Rightarrow$ (ii) By Theorem 4.6 there exists a unitary transform W of $\mathscr{H}_\varphi$ onto $\mathscr{H}_\psi$ such that $W\mathscr{D}(\pi_\varphi) = \mathscr{D}(\pi_\psi)$ and $\pi_\varphi(X) = W^*\pi_\psi(X)W$ for all $X \in \mathscr{M}$. Since $W^* \in \mathrm{II}(\pi_\psi, \pi_\varphi)$, it follows that

$$\mathfrak{A}_{\varphi,\psi} = \{(A, WAW^*);\ A \in (\pi_\varphi(\mathscr{M})'_{\mathrm{w}})'\},$$

which implies by Lemma 4.5 and (4.3) that for each $A_{ij} \in (\pi_\varphi(\mathscr{M})'_{\mathrm{w}})'$ $(i,j=1,2)$

$$\Delta_{A_\theta^{cc}}^{it}\begin{pmatrix} U_1A_{11}U_1^* & U_1A_{12}U_2^* & 0 & 0 \\ U_2A_{21}U_1^* & U_2A_{22}U_2^* & 0 & 0 \\ 0 & 0 & U_3WA_{11}W^*U_3^* & U_3WA_{12}W^*U_4^* \\ 0 & 0 & U_4WA_{21}W^*U_3^* & U_4WA_{22}W^*U_4^* \end{pmatrix}\Delta_{A_\theta^{cc}}^{-it}$$

$$= \begin{pmatrix} U_1\Delta_{11}^{it}A_{11}\Delta_{11}^{-it}U_1^* & U_1\Delta_{11}^{it}A_{12}\Delta_{21}^{-it}U_1^* & 0 & 0 \\ U_2\Delta_{21}^{it}A_{21}\Delta_{11}^{-it}U_1^* & U_2\Delta_{21}^{it}A_{22}\Delta_{21}^{-it}U_2^* & 0 & 0 \\ 0 & 0 & U_3\Delta_{12}^{it}WA_{11}W^*\Delta_{12}^{-it}U_3^* & U_3\Delta_{12}^{it}WA_{12}W^*\Delta_{22}^{-it}U_4^* \\ 0 & 0 & U_4\Delta_{22}^{it}WA_{21}W^*\Delta_{12}^{-it}U_3^* & U_4\Delta_{22}^{it}WA_{22}W^*\Delta_{22}^{-it}U_4^* \end{pmatrix}$$

$$\in (\pi_\theta(\mathscr{M} \otimes M_2(\boldsymbol{C}))'_{\mathrm{w}})'.$$

Hence we have by Lemma 4.5

$$W\Delta_{11}^{it}A_{11}\Delta_{11}^{-it}W^* = \Delta_{12}^{it}WA_{11}W^*\Delta_{12}^{-it}, \tag{4.4}$$

$$W\Delta_{21}^{it}A_{22}\Delta_{21}^{-it}W^* = \Delta_{22}^{it}WA_{22}W^*\Delta_{22}^{-it}, \tag{4.5}$$

$$W\Delta_{11}^{it}A_{12}\Delta_{21}^{-it}W^* = \Delta_{12}^{it}WA_{12}W^*\Delta_{22}^{-it}, \tag{4.6}$$

$$W\Delta_{21}^{it}A_{21}\Delta_{11}^{-it}W^* = \Delta_{22}^{it}WA_{21}W^*\Delta_{12}^{-it}. \tag{4.7}$$

It follows from (4.4) and (4.5) that $\Delta_{11}^{-it}W^*\Delta_{12}^{it}W$, $\Delta_{21}^{-it}W^*\Delta_{22}^{it}W \in \pi_\varphi(\mathscr{M})'_{\mathrm{w}}$ for all $t \in \boldsymbol{R}$, and hence

$$\Delta_{12}^{it}\mathscr{D}(\pi_\psi) = W\Delta_{11}^{it}(\Delta_{11}^{-it}W^*\Delta_{12}^{it}W)W^*\mathscr{D}(\pi_\psi) \subset \mathscr{D}(\pi_\psi)$$

for all $t \in \boldsymbol{R}$. Similarly,

$$\Delta_{21}^{it}\mathscr{D}(\pi_\varphi) \subset \mathscr{D}(\pi_\varphi), \quad {}^\forall t \in \boldsymbol{R}.$$

Hence we have

$$\Delta^{it}_{\Lambda^{cc}_{\theta}}\mathscr{D}(\pi_\theta) \subset \mathscr{D}(\pi_\theta), \quad {}^{\forall}t \in \boldsymbol{R}.$$

Therefore, θ is quasi-standard.

(ii) $\Rightarrow$ (i) Since $\Lambda^c_\theta((\mathscr{D}(\Lambda^c_\theta)^* \cap \mathscr{D}(\Lambda^c_\theta))^2)$ is total in $\mathscr{H}_\theta$, it follows from (4.2) that $\Lambda^c_\varphi((\mathscr{D}(\Lambda^c_\varphi)^* \cap \mathscr{D}(\Lambda^c_\varphi))^2)$ and $\Lambda^c_\psi((\mathscr{D}(\Lambda^c_\psi)^* \cap \mathscr{D}(\Lambda^c_\psi))^2)$ are total in $\mathscr{H}_\varphi$ and $\mathscr{H}_\psi$, respectively. Furthermore, since $\Delta^{it}_{\Lambda^{cc}_\theta}\mathscr{D}(\pi_\theta) \subset \mathscr{D}(\pi_\theta)$ for all $t \in \boldsymbol{R}$, it follows from (4.3) that $\Delta^{it}_{\Lambda^{cc}_\varphi}\mathscr{D}(\pi_\varphi) \subset \mathscr{D}(\pi_\varphi)$ and $\Delta^{it}_{\Lambda^{cc}_\psi}\mathscr{D}(\pi_\psi) \subset \mathscr{D}(\pi_\psi)$ for all $t \in \boldsymbol{R}$. Therefore, φ and ψ are quasi-standard. This completes the proof.

Theorem 4.8 (Generalized Connes cocycle theorem). *Suppose φ and ψ are faithful, σ-weakly continuous, semifinite, quasi-standard (quasi-)weights on $\mathscr{P}(\mathscr{M})$ which satisfy one of* (i)$\sim$(iv) *in Theorem* 4.6. *Then there exists a strongly continuous map $t \in \boldsymbol{R} \mapsto U_t \in \pi_\varphi(\mathscr{M})''_{\mathrm{wc}}$, uniquely determined, such that*

(i) *$\overline{U_t}$ is unitary for each $t \in \boldsymbol{R}$;*

(ii) *$U_{s+t} = U_t \sigma^{\Lambda^{cc}_\varphi}_t(U_s)$ for each $s, t \in \boldsymbol{R}$;*

(iii) *$\sigma^{\Lambda^{cc}_\psi}_t(WAW^\dagger) = WU_t\sigma^{\Lambda^{cc}_\varphi}_t(A)U^*_tW^*$ for each $A \in \pi_\varphi(\mathscr{M})''_{\mathrm{wc}}$ for each $t \in \boldsymbol{R}$, where W is a unitary transform of $\mathscr{H}_\varphi$ onto $\mathscr{H}_\psi$ such that $W\mathscr{D}(\pi_\varphi) = \mathscr{D}(\pi_\psi)$ and $\pi_\psi(X) = W\pi_\varphi(X)W^\dagger$ for all $X \in \mathscr{M}$;*

(iv) *for any $A \in (W^\dagger \mathfrak{N}_{\bar\psi} W) \cap \mathfrak{N}^\dagger_{\bar\varphi}$ and $B \in \mathfrak{N}_{\bar\varphi} \cap (W^\dagger \mathfrak{N}^\dagger_{\bar\psi} W)$ there exists an element $F_{A,B}$ of $A(0,1)$ such that*

$$F_{A,B}(t) = \dot{\bar\varphi}(AU_t\sigma^{\Lambda^{cc}_\varphi}_t(B)),$$

$$F_{A,B}(t+i) = \dot{\bar\psi}(\sigma^{\Lambda^{cc}_\psi}_t(WBW^\dagger)WU_tAW^\dagger),$$

for all $t \in \boldsymbol{R}$, where $\bar\varphi$ and $\bar\psi$ are the quasi-weights induced by φ and ψ, respectively.

Proof. We put

$$\begin{cases} \mathscr{D}(\Lambda^\psi_\varphi) = \{\pi_\varphi(X); X \in \mathfrak{N}_\psi\}, \\ \Lambda^\psi_\varphi(\pi_\varphi(X)) = W^*\lambda_\psi(X), \quad X \in \mathfrak{N}_\psi. \end{cases}$$

Then it is easily shown that Λ^ψ_φ is a generalized vector for $\pi_\varphi(\mathscr{M})$ such that

$$\begin{cases} \mathscr{D}((\Lambda^\psi_\varphi)^c) = \{W^*KW; K \in \mathscr{D}(\Lambda^c_\psi)\}, \\ (\Lambda^\psi_\varphi)^c(W^*KW) = W^*\Lambda^c_\psi(K), \quad K \in \mathscr{D}(\Lambda^c_\psi); \end{cases}$$

$$\begin{cases} \mathscr{D}((\Lambda^\psi_\varphi)^{cc}) = \{W^*AW; A \in \mathscr{D}(\Lambda^{cc}_\psi)\}, \\ (\Lambda^\psi_\varphi)^{cc}(W^*AW) = W^*\Lambda^{cc}_\psi(A), \quad A \in \mathscr{D}(\Lambda^{cc}_\psi); \end{cases}$$

$$S_{(\Lambda^\psi_\varphi)^{cc}} = W^*S_{\Lambda^{cc}_\psi}W.$$

Hence we have

$$\Delta^{it}_{(\Lambda^\psi_\varphi)^{cc}}\mathscr{D}(\pi_\varphi) = W^*\Delta^{it}_{\Lambda^{cc}_\psi}W\mathscr{D}(\pi_\varphi) \subset \mathscr{D}(\pi_\varphi)$$

for all $t \in \boldsymbol{R}$, and so Λ_φ^ψ is quasi-standard. By Theorem 3.4 $\overline{\Lambda_\varphi}$ and $\overline{\Lambda_\varphi^\psi}$ are standard generalized vectors for the generalized von Neumann algebra $\pi_\varphi(\mathcal{M})''_{\mathrm{wc}}$, and so it follows from ([**10**] Theorem 3.3) that there exists a strongly continuous map $t \in \boldsymbol{R} \mapsto U_t \in \pi_\varphi(\mathcal{M})''_{\mathrm{wc}}$ satisfying the conditions (i) $\sim$ (iv) and it is identical with the Connes cocycle $[D\overline{\Lambda_\varphi^\psi} : D\overline{\Lambda_\varphi}]_t$ $(= \Delta_{21}^{it}\Delta_{11}^{-it})$ associated with $\overline{\Lambda_\varphi^\psi}$ with respect to $\overline{\Lambda_\varphi}$. This completes the proof.

The map $t \in \boldsymbol{R} \mapsto U_t \in \pi_\varphi(\mathcal{M})''_{\mathrm{wc}}$, uniquely determined by the above theorem, is called *the cocycle associated with the quasi-weight* $\bar{\psi}$ *with respect to the quasi-weight* $\bar{\varphi}$, and denoted by $[D\bar{\psi} : D\bar{\varphi}]$. It follows from (4.6) that the cocycle $[D\bar{\varphi} : D\bar{\psi}]_t$ associated with $\bar{\varphi}$ with respect to $\bar{\psi}$ equals $W[D\bar{\psi} : D\bar{\varphi}]_t^* W^*$. By (iii) and (iv) in Theorem 4.8 we have

(iii)$'$ $\sigma_t^{\Lambda_\psi^{cc}}(\pi_\psi(X)) = W[D\bar{\psi} : D\bar{\varphi}]_t \sigma_t^{\Lambda_\varphi^{cc}}(\pi_\varphi(X))[D\bar{\psi} : D\bar{\varphi}]_t^* W^*$, $^\forall X \in \mathcal{M}$, $^\forall t \in \boldsymbol{R}$;

(iv)$'$ for any $X \in \mathfrak{N}_\varphi^\dagger \cap \mathfrak{N}_\psi$ and $Y \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\psi^\dagger$ there exists an element $F_{X,Y}$ of $A(0,1)$ such that

$$F_{X,Y}(t) = \dot{\bar{\varphi}}(\pi_\varphi(X)[D\bar{\psi} : D\bar{\varphi}]_t \sigma_t^{\Lambda_\varphi^{cc}}(\pi_\varphi(Y))),$$

$$F_{X,Y}(t+i) = \dot{\bar{\psi}}(\sigma_t^{\Lambda_\psi^{cc}}(\pi_\psi(Y))[D\bar{\varphi} : D\bar{\psi}]_t^* \pi_\psi(X)),$$

for all $t \in \boldsymbol{R}$.

Corollary 4.9. *Suppose* φ and ψ *are faithful*, σ*-weakly continuous, semifinite, standard* (*quasi-*)*weights which satisfy one of* (i) $\sim$ (iv) *in Theorem* 4.6 *and* $\pi_\varphi(\mathcal{M})$ *is a generalized von Neumann algebra. Then there exists a strongly continuous map* $t \in \boldsymbol{R} \mapsto [D\psi : D\varphi]_t \in \mathcal{M}$, *uniquely determined, such that*

(i) $\overline{[D\psi : D\varphi]_t}$ *is unitary for each* $t \in R$;

(ii) $[D\psi : D\varphi]_{s+t} = [D\psi : D\varphi]_t \sigma_t^\varphi([D\psi : D\varphi]_s)$;

(iii) $\sigma_t^\psi(X) = [D\psi : D\varphi]_t \sigma_t^\varphi(X)[D\psi : D\varphi]_t^*$, $^\forall X \in \mathcal{M}$ *for each* $t \in \boldsymbol{R}$;

(iv) *for any* $X \in \mathfrak{N}_\varphi^\dagger \cap \mathfrak{N}_\psi$ *and* $Y \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\psi^\dagger$ *there exists an element* $F_{X,Y}$ *of* $A(0,1)$ *such that*

$$F_{X,Y}(t) = \dot{\varphi}(X[D\psi : D\varphi]_t \sigma_t^\varphi(Y)),$$

$$F_{X,Y}(t+i) = \dot{\psi}(\sigma_t^\psi(Y)[D\psi : D\varphi]_t X)$$

for all $t \in \boldsymbol{R}$.

This $[D\psi : D\varphi]$ is called *the cocycle associated with the* (*quasi-*)*weight* ψ *with respect to the* (*quasi-*)*weight* φ.

5. Standard weights on generalized von Neumann algebras with strongly dense bounded part.

An seen in Corollary 4.9, if $\pi_\varphi(\mathcal{M})$ is a generalized von Neumann algebra, then the generalized Connes cocycle theorem for weights on O*-algebras becomes the best form. In this section we show that if $\mathcal{M}$ is a generalized von Neumann algebra with strongly dense bounded part and φ is a strongly faithful, σ-weakly continuous (quasi-) weight on $\mathscr{P}(\mathscr{M})$, then $\pi_\varphi(\mathscr{M})$ is spatially isomorphic to $\mathscr{M}$, and so it is a generalized von Neumann algebra.

Lemma 5.1. *Let $\mathscr{M}$ be a self-adjoint* O*-*algebra on $\mathscr{D}$ in $\mathscr{H}$ such that $\mathscr{M}_b'' = (\mathscr{M}_{\mathrm{w}}')'$ and φ a σ-weakly continuous (quasi-)weight on $\mathscr{P}(\mathscr{M})$. Then there exists a normal $*$-homomorphism $\overline{\pi_\varphi}$ of $(\mathscr{M}_{\mathrm{w}}')'$ onto $(\pi_\varphi(\mathscr{M})_{\mathrm{w}}')'$ such that $\overline{\pi_\varphi}(A) = \overline{\pi_\varphi(A)}$ for all $A \in \mathscr{M}_b$.*

Proof. Since φ_X can be extended to a σ-weakly continuous positive linear functional on $\mathscr{M}_b''$ for each $X \in \mathfrak{N}_\varphi$, it follows that

$$\varphi_X(A^\dagger A) \leq \|\bar{A}\|^2 \varphi_X(I), \quad {}^\forall A \in \mathscr{M}_b,$$

which implies

$$\pi_\varphi(\mathscr{M}_b) \subset \pi_\varphi(\mathscr{M})_b,$$

$$\|\overline{\pi_\varphi(A)}\| \leq \|\bar{A}\|, \quad {}^\forall A \in \mathscr{M}_b. \tag{5.1}$$

We now have the following:
If $\{A_\alpha\}$ is any uniformly bounded net in $\mathscr{M}_b$ such that $\overline{A_\alpha} \to A \in \mathscr{B}(\mathscr{H})$ weakly (resp. strongly, strongly*), then $\{\overline{\pi_\varphi(A_\alpha)}\}$ converges weakly (resp. strongly, strongly*) to an element of $\mathscr{B}(\mathscr{H}_\varphi)$. (5.2)

In fact, for each $X, Y \in \mathfrak{N}_\varphi$ we have

$$\begin{aligned} \lim_{\alpha,\beta}((\pi_\varphi(A_\alpha) - \pi_\varphi(A_\beta))\lambda_\varphi(X)|\lambda_\varphi(Y)) &= \lim_{\alpha,\beta} \varphi_{X,Y}(A_\alpha - A_\beta) \\ &= 0, \end{aligned}$$

and so we put

$$B(\lambda_\varphi(X), \lambda_\varphi(Y)) = \lim_\alpha(\pi_\varphi(A_\alpha)\lambda_\varphi(X)|\lambda_\varphi(Y)) \ \ X, Y \in \mathfrak{N}_\varphi.$$

By (5.1) B is a bounded sesquilinear form on $\lambda_\varphi(\mathfrak{N}_\varphi) \times \lambda_\varphi(\mathfrak{N}_\varphi)$, and so it can be extended to a bounded sesquilinear form on $\mathscr{H}_\varphi \times \mathscr{H}_\varphi$. It hence follows from the Riesz theorem that $\{\overline{\pi_\varphi(A_\alpha)}\}$ converges weakly (resp. strongly, strongly*) to an element of $\mathscr{B}(\mathscr{H}_\varphi)$. Since $\mathscr{M}_b'' = (\mathscr{M}_{\mathrm{w}}')'$, it follows from the Kaplansky density theorem that for each $A \in (\mathscr{M}_{\mathrm{w}}')'$ there exists a net $\{A_\alpha\}$ in $\mathscr{M}_b$ such that $\|\overline{A_\alpha}\| \leq \|A\|$ for all α and $\overline{A_\alpha} \to A$ strongly*, and so we put

$$\overline{\pi_\varphi}(A) = s^* - \lim_\alpha \overline{\pi_\varphi(A_\alpha)}, \quad A \in (\mathscr{M}_{\mathrm{w}}')'.$$

By (5.2) $\overline{\pi_\varphi}(A)$ is well-defined, i.e., it is independent for taking a net $\{A_\alpha\}$ in $\mathscr{M}_b$, and $\overline{\pi_\varphi}$ is a normal $*$-homomorphism of $(\mathscr{M}_{\mathrm{w}}')'$ to $\mathscr{B}(\mathscr{H}_\varphi)$. Hence, it follows that

$$\overline{\pi_\varphi}((\mathscr{M}_{\mathrm{w}}')') \text{ is a von Neumann algebra.} \tag{5.3}$$

We finally show that

$$\pi_\varphi(\mathscr{M}_b)'' = \overline{\pi_\varphi}((\mathscr{M}_{\mathrm{w}}')') = (\pi_\varphi(\mathscr{M})_{\mathrm{w}}')'. \tag{5.4}$$

In fact, take an arbitrary $C \in \pi_\varphi(\mathscr{M}_b)'$. Since $\bar{X}$ is affiliated with $(\mathscr{M}'_{\mathrm{w}})' = \mathscr{M}''_b$ for each $X \in \mathscr{M}$, there exists a net $\{A_\alpha\}$ in $\mathscr{M}_b$ which converges σ-strongly* to X. Hence we have

$$\lim_\alpha \|\pi_\varphi(A_\alpha)\lambda_\varphi(Y) - \pi_\varphi(X)\lambda_\varphi(Y)\|^2 = \lim_\alpha \varphi_Y((A_\alpha - X)^\dagger(A_\alpha - X)) = 0$$

and

$$\lim_\alpha \|\pi_\varphi(A_\alpha^\dagger)\lambda_\varphi(Y) - \pi_\varphi(X^\dagger)\lambda_\varphi(Y)\| = 0$$

for each $Y \in \mathfrak{N}_\varphi$, and so

$$\begin{aligned}(C\pi_\varphi(X)\lambda_\varphi(Y)|\lambda_\varphi(Z)) &= \lim_\alpha (C\pi_\varphi(A_\alpha)\lambda_\varphi(Y) \,|\, \lambda_\varphi(Z)) \\ &= \lim_\alpha (C\lambda_\varphi(Y) \,|\, \pi_\varphi(A_\alpha^\dagger)\lambda_\varphi(Z)) \\ &= (C\lambda_\varphi(Y) \,|\, \pi_\varphi(X^\dagger)\lambda_\varphi(Z))\end{aligned}$$

for all $Y, Z \in \mathfrak{N}_\varphi$. Hence, $C \in \pi_\varphi(\mathscr{M})'_{\mathrm{w}}$. Thus we have $\pi_\varphi(\mathscr{M}_b)' \subset \pi_\varphi(\mathscr{M})'_{\mathrm{w}}$, which implies by (5.3) that

$$\pi_\varphi(\mathscr{M}_b)'' \subset \overline{\pi_\varphi}((\mathscr{M}'_{\mathrm{w}})') \subset (\pi_\varphi(\mathscr{M})'_{\mathrm{w}})' \subset \pi_\varphi(\mathscr{M}_b)''.$$

Therefore, the statement (5.4) holds. This completes the proof.

As shown in Lemma 4.2, if φ is a faithful, semifinite (quasi-)weight on $\mathscr{P}(\mathscr{M})$, then π_φ is a $*$-isomorphism, but we don't know whether $\overline{\pi_\varphi}$ is a $*$-isomorphism in general. For this we have the following

Lemma 5.2. *Suppose φ is a faithful, σ-weakly continuous (quasi-)weight on $\mathscr{P}(\mathscr{M})$. Then the following statements are equivalent*:

(i) *$\overline{\pi_\varphi}$ is a $*$-isomorphism.*

(ii) *The map π_φ^{-1} from $\pi_\varphi(\mathscr{M}_b)[\tau_{\sigma s}]$ to $(\mathscr{M}'_{\mathrm{w}})'\lceil_{\mathscr{D}}[\tau_{\sigma s}]$ is closable.*

Proof. (i) $\Rightarrow$ (ii) Let $\{A_\alpha\}$ be any net in $\mathscr{M}_b$ such that $\tau_{\sigma s} - \lim_\alpha \pi_\varphi(A_\alpha) = 0$ and $\tau_{\sigma s} - \lim_\alpha A_\alpha = A \in (\mathscr{M}'_{\mathrm{w}})'\lceil_{\mathscr{D}}$. By Lemma 5.1 we have

$$\overline{\pi_\varphi}(\bar{A}) = \tau_s - \lim_\beta \overline{\pi_\varphi(B_\beta)},$$

where $\{B_\beta\}$ is a uniformly bounded net in $\mathscr{M}_b$ which converges σ-strongly* to A. And we have

$$\lim_{\alpha,\beta} \|\pi_\varphi(A_\alpha)\lambda_\varphi(X) - \pi_\varphi(B_\beta)\lambda_\varphi(X)\|^2 = \lim_{\alpha,\beta} \varphi_X((A_\alpha - B_\beta)^\dagger(A_\alpha - B_\beta)) = 0$$

for all $X \in \mathfrak{N}_\varphi$. Hence we have

$$\pi_\varphi(\bar{A})\lambda_\varphi(X) = \lim_\alpha \pi_\varphi(B_\beta)\lambda_\varphi(X)$$
$$= \lim_\alpha \pi_\varphi(A_\alpha)\lambda_\varphi(X)$$
$$= 0$$

for all $X \in \mathfrak{N}_\varphi$, and so $\overline{\pi_\varphi}(\bar{A}) = 0$. Since $\overline{\pi_\varphi}$ is a $*$-isomorphism, we have $\bar{A} = 0$.

(ii) $\Rightarrow$ (i) Suppose $\overline{\pi_\varphi}(A) = 0$, $A \in (\mathcal{M}'_{\mathrm{w}})'$. Then there exists a net $\{A_\alpha\}$ in $\mathcal{M}_b$ such that $\|\overline{A_\alpha}\| \leq r$ for all α and $\tau_{\sigma s} - \lim_\alpha \overline{A_\alpha} = A$. By Lemma 5.1 we have

$$\tau_{\sigma s} - \lim_\alpha \overline{\pi_\varphi}(A_\alpha) = \overline{\pi_\varphi}(A) = 0.$$

Hence, $A = 0$. This completes the proof.

DEFINITION 5.3. A σ-weakly continuous (quasi-)weight φ on $\mathscr{P}(\mathcal{M})$ is said to be *strongly faithful* if φ is faithful and one of the conditions (i) and (ii) of Lemma 5.2 holds.

PROPOSITION 5.4. *Let $\mathcal{M}$ be a self-adjoint* O**-algebra on $\mathscr{D}$ in $\mathscr{H}$ such that $\mathcal{M}''_b = (\mathcal{M}'_{\mathrm{w}})'$, and φ a strongly faithful, σ-weakly continuous* (*quasi-*)*weight on $\mathscr{P}(\mathcal{M})$ such that π_φ is self-adjoint. Suppose $(\mathcal{M}'_{\mathrm{w}})'$ and $(\pi_\varphi(\mathcal{M})'_{\mathrm{w}})'$ satisfy one of the following statements*:

(i) *they are standard von Neumann algebras.*

(ii) *$\mathcal{M}'_{\mathrm{w}}$ and $\pi_\varphi(\mathcal{M})'_{\mathrm{w}}$ are properly infinite and of countable type.*

(iii) *$\mathscr{H}$ and $\mathscr{H}_\varphi$ are separable and $(\mathcal{M}'_{\mathrm{w}})'$ and $(\pi_\varphi(\mathcal{M})'_{\mathrm{w}})'$ are von Neumann algebras of type III.*

Then the O**-algebras $\mathcal{M}$ and $\pi_\varphi(\mathcal{M})$ are spatially isomorphic.*

PROOF. It follows from Lemma 5.1, 5.2 and ([**23**] Corollary 8.12, 8.13, 10.15) that $\overline{\pi_\varphi}$ is spatial, that is, there exists a unitary transform U of $\mathscr{H}$ onto $\mathscr{H}_\varphi$ such that $\overline{\pi_\varphi}(A) = UAU^*$ for all $A \in (\mathcal{M}'_{\mathrm{w}})'$. This implies that

$$U\mathscr{D} = \mathscr{D}(\pi_\varphi) \quad \text{and} \quad \pi_\varphi(X) = UXU^* \tag{5.5}$$

for all $X \in \mathcal{M}$. Take an arbitrary $X \in \mathcal{M}$. For each $\xi \in \mathscr{D}$ and $Y \in \mathfrak{N}_\varphi$ we have

$$(\pi_\varphi(X^\dagger)\lambda_\varphi(Y)|U\xi) = \lim_\alpha(\pi_\varphi(A_\alpha^\dagger)\lambda_\varphi(Y)|U\xi)$$
$$= \lim_\alpha(UA_\alpha^\dagger U^*\lambda_\varphi(Y)|U\xi)$$
$$= (\lambda_\varphi(Y)|UX\xi),$$

where $\{A_\alpha\}$ is a net in $\mathcal{M}_b$ which converges σ-strongly* to X. By the self-adjointness of π_φ we have

$$U\xi \in \mathscr{D}(\pi_\varphi) \quad \text{and} \quad \pi_\varphi(X)U\xi = UX\xi, \tag{5.6}$$

and further

$$
\begin{aligned}
(X^{\dagger}\xi \mid U^{*}\eta) &= (UX^{\dagger}\xi|\eta) \\
&= (\pi_{\varphi}(X^{\dagger})U\xi|\eta) \\
&= (\xi|U^{*}\pi_{\varphi}(X)\eta)
\end{aligned}
$$

for all $\xi \in \mathscr{D}$ and $\eta \in \mathscr{D}(\pi_{\varphi})$. Hence it follows from the self-adjointness of $\mathscr{M}$ that $U^{*}\mathscr{D}(\pi_{\varphi}) \subset \mathscr{D}$, which implies that the statement (5.5) holds. This completes the proof.

Throughout the rest of this section let $\mathscr{M}$ be a self-adjoint generalized von Neumann algebra on $\mathscr{D}$ in $\mathscr{H}$ such that $\mathscr{M}_b'' = (\mathscr{M}_{\mathrm{w}}')'$ and $(\mathscr{M}_{\mathrm{w}}')'$ is a standard von Neumann algebra. We denote by $W_s(\mathscr{M})$ the set of all strongly faithful, σ-weakly continuous, semifinite, quasi-standard quasi-weights φ on $\mathscr{P}(\mathscr{M})$ such that π_{φ} are self-adjoint. Suppose $\varphi \in W_s(\mathscr{M})$. By Proposition 5.4 $\pi_{\varphi}(\mathscr{M})$ is a generalized von Neumann algebra on $\mathscr{D}(\pi_{\varphi})$ in $\mathscr{H}_{\varphi}$, and so φ is standard. By Theorem 3.3 we have the following

Corollary 5.5. *For every $\varphi \in W_s(\mathscr{M})$ there exists a one-parameter group $\{\sigma_t^{\varphi}\}_{t\in \boldsymbol{R}}$ of $*$-automorphisms of $\mathscr{M}$ such that*
(i) $\pi_{\varphi}(\sigma_t^{\varphi}(X)) = \Delta_{\varphi}^{it}\pi_{\varphi}(X)\Delta_{\varphi}^{-it}$, $X \in \mathscr{M}, t \in \boldsymbol{R}$;
(ii) *φ is a $\{\sigma_t^{\varphi}\}$-KMS quasi-weight on $\mathscr{P}(\mathscr{M})$.*

Suppose $\varphi, \psi \in W_s(\mathscr{M})$. By Proposition 5.4 $\pi_{\varphi}(\mathscr{M})$ and $\pi_{\psi}(\mathscr{M})$ are generalized von Neumann algebras, and φ and ψ are standard. Hence, by Corollary 4.7 we have the following

Corollary 5.6. *Suppose $\varphi, \psi \in W_s(\mathscr{M})$. Then, the cocycle $[D\psi : D\varphi]$ associated with the quasi-weight ψ with respect to the quasi-weight φ is well-defined in $\mathscr{M}$, that is, $t \mapsto [D\psi : D\varphi]_t$ is a strongly continuous map of $\boldsymbol{R}$ into $\mathscr{M}$ satisfying the conditions* (i) $\sim$ (iv) *in Corollary* 4.9.

We generalize the Pedersen-Takesaki theorem [**24**] for standard weights on von Neumann algebras to those on O*-algebras. Let $\varphi \in W_s(\mathscr{M})$. Since $\mathscr{M}$ is a generalized von Neumann algebra, the quasi-weight $\bar{\varphi}$ on $\mathscr{P}(\mathscr{M})$ in Theorem 3.4 is defined by

$$
\begin{cases}
\mathfrak{N}_{\bar{\varphi}} = \{X \in \mathscr{M}; {}^{\exists}\xi_X \in \mathscr{D}(\pi_{\varphi}) \text{ s.t. } \pi_{\varphi}(X)\Lambda_{\varphi}^{c}(K) = K\xi_X, {}^{\forall}K \in \mathscr{D}(\Lambda_{\varphi}^{c})\}, \\
\bar{\varphi}\left(\sum_k X_k^{\dagger}X_k\right) = \sum_k \|\xi_{X_k}\|^2, \quad \sum_k X_k^{\dagger}X_k \in \mathscr{P}(\mathfrak{N}_{\bar{\varphi}}).
\end{cases}
$$

Using ([**10**] Theorem 4.2, 4.7), we can show the following results:

Corollary 5.7. *Suppose $\varphi, \psi \in W_s(\mathscr{M})$. The following statements are equivalent*:
(i) $\bar{\psi} \circ \sigma_t^{\varphi} = \bar{\psi}$ *for each* $t \in \boldsymbol{R}$.
(ii) $\bar{\varphi} \circ \sigma_t^{\psi} = \bar{\varphi}$ *for each* $t \in \boldsymbol{R}$.
(iii) $[D\psi : D\varphi]_t \in \mathscr{M}^{\sigma^{\psi}}$ *for each* $t \in \boldsymbol{R}$, *where* $\mathscr{M}^{\sigma^{\psi}} \equiv \{X \in \mathscr{M}; \sigma_t^{\psi}(X) = X, {}^{\forall}t \in \boldsymbol{R}\}$.
(iv) $[D\psi : D\varphi]_t \in \mathscr{M}^{\sigma^{\varphi}}$ *for each* $t \in \boldsymbol{R}$, *where* $\mathscr{M}^{\sigma^{\varphi}} \equiv \{X \in \mathscr{M}; \sigma_t^{\varphi}(X) = X, {}^{\forall}t \in \boldsymbol{R}\}$.
(v) $\{[D\psi : D\varphi]_t\}_{t\in\boldsymbol{R}}$ *is strongly continuous group of unitary elements of* $\mathscr{M}$.

Corollary 5.8. *Suppose* $\varphi, \psi \in W_s(\mathscr{M})$. *The following statements are equivalent*:

(i) ψ *satisfies the* KMS-*condition with respect to* $\{\sigma_t^{\varphi}\}_{t\in\boldsymbol{R}}$.

(ii) $\sigma_t^{\psi} = \sigma_t^{\varphi}$ *for all* $t \in \boldsymbol{R}$.

(iii) *There exists a positive self-adjoint operator* A *in* $\mathscr{H}$ *affiliated with* $(\pi_\varphi(\mathscr{M})'_{\mathrm{w}})' \cap \pi_\varphi(\mathscr{M})'_{\mathrm{w}}$ *such that* $\bar{\psi} = \overline{\overline{\varphi_A}}$, *where* $\overline{\overline{\varphi_A}}$ *is the quasi-weight on* $\mathscr{P}(\pi_\varphi(\mathscr{M})''_{\mathrm{wc}})$ *induced by the quasi-weight on* $\mathscr{P}(\mathscr{M})$ *defined by*

$$\begin{cases} \mathfrak{N}_{\overline{\varphi_A}} = \{X \in \mathscr{M};\quad \overline{\Lambda_\varphi}(X) \in \mathscr{D}(A)\}, \\ \overline{\varphi_A}(X^\dagger X) = \|A\overline{\Lambda_\varphi}(X)\|^2, \quad X \in \mathfrak{N}_{\overline{\varphi_A}}. \end{cases}$$

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