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Convergence of formal morphisms of completions of complex spaces

Dedicated to the memory of Professor Nobuo Sasakura

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Abstract. We call a formal morphism between completions of complex spaces *convergent* if it comes from a holomorphic mapping between the complex spaces. We assume always that the source space is compact. Then a formal morphism is either convergent everywhere or nowhere, under very general conditions.

0. Introduction.

We are concerned with the claim that a formal morphism defined on the completion of a complex space along a compact subspace converges either everywhere or nowhere. As to the formal function case, such a claim for convergence domains first emerged in Gabrièlov's paper in an analytic form ([**G2**], (1.6)). This property of formal functions is extracted from those of formal functions along the exceptional set of a blowing up obtained as pullbacks. Further this property has turned out to be shared by all the formal functions on the completion of a complex space along more general space as follows.

($*$) *Let $\hat{X}_{|S}$ denote the completion of a complex space X along a Moishezon subspace S. If $\hat{X}_{|S}$ is globally integral* (*i.e. X is globally integral along S*). *Then a global formal function on $\hat{X}_{|S}$ converges either everywhere or nowhere* ([**I3**], (C)).

By the medium of this, two important theorems, Gabrièlov's on convergence of pullback of formal function ([**G2**], (5.6), cf. [**T2**]) and Sadullaev's criterion for algebraicity of analytic subsets of $\boldsymbol{C}^n$ ([**S**], (2.2)), turned out to be equivalent ([**I3**]).

Ohsawa posed the author the problem whether ($*$) can be generalized to morphisms or not. In the case of the famous example of Arnold [**Arn**], the formal isomorphism is known to be divergent everywhere. In this paper we show two affirmative answers to Ohsawa's problem, which cover a very general class of formal morphisms.

First we clarify the notion of convergence of formal morphisms in §1, which are concerned with formal morphisms between the special formal complex spaces obtained as the completions of complex spaces along analytic subsets. In this connection, we remember Whitney-Shiota function in Appendix. This function assures the existence

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of a formal complex space which is topologically 0-dimensional and not obtained as a completion of a complex space.

Now let $\hat{X}_{|S}$ be the same as in $(*)$ and Y a reduced complex space. Our first answer to Ohsawa's problem is Theorem I in §2, which asserts the following.

If $\boldsymbol{\Phi} : \hat{X}_{|S} \to Y$ is a formal morphism into a complex space Y such that none of the images of local irreducible components are locally thin (see §2) and if $|\boldsymbol{\Phi}|(S)$ can be contracted to one point by a generically finite morphism of Y, then $\boldsymbol{\Phi}$ converges either everywhere or nowhere.

The second is Theorem II in §7, which asserts the following.

Suppose that $\hat{Y}_{|T}$ is the completion of a reduced complex space Y along T such that Y is smooth outside T. Then, for the same $\hat{X}_{|S}$ as in $(*)$, an adic formal morphism (cf. §1) $\boldsymbol{\Phi} : \hat{X}_{|S} \to \hat{Y}_{|T}$ with the full generic rank (see §3) is convergent either everywhere or nowhere.

Both theorems are generalizations of $(*)$ above. Since our proofs of the main theorems are based upon $(*)$, they inherit the condition that S is a Moishezon subspace. It is left open whether these theorems hold or not for more general compact subspaces S.

The proof of Theorem I is a combination of Gabrièlov's theorem on convergence of pullback and other famous results in local analytic geometry.

The proof of Theorem II requires patience to develop the general theory of formal complex spaces (§3, 4, 5, 6). Formal complex spaces are introduced by Krasnov [**Kr**] and studied by Bingener [**B3**] systematically. Let us remember the definition. Let $\boldsymbol{X} := (S, \mathcal{O}_{\boldsymbol{X}})$ $(S = |\boldsymbol{X}|)$ be a $\boldsymbol{C}$-local ringed space and $\mathfrak{m}_\xi$ the maximal ideal of the local ring $\mathcal{O}_{\boldsymbol{X},\xi}$. Let $\mathscr{I}_T \subset \mathcal{O}_{\boldsymbol{X}}$ $(T \subset S)$ denote the ideal sheaf associated to the presheaf of sections $f \in \mathcal{O}_{\boldsymbol{X}}(U)$ over open sets $U \subset S$ such that $f_\xi \in \mathfrak{m}_\xi$ for all $\xi \in T \cap U$. Then, $\boldsymbol{X} := (S, \mathcal{O}_{\boldsymbol{X}})$ is by definition a *formal complex space* if $(S, \mathcal{O}_{\boldsymbol{X}}/\mathscr{I}_S^n)$ $(n \in \boldsymbol{N})$ are complex spaces and if the canonical homomorphism $\mathcal{O}_{\boldsymbol{X}} \to \varprojlim_n \mathcal{O}_{\boldsymbol{X}}/\mathscr{I}_S^n$ is bijective. Since the structure sheaves of formal complex spaces are coherent ([**B3**], (1.1)), many notions and constructions for complex spaces can be transferred to them. We sometimes neglect to write them up. In particular formal blowings up are important for the proof of Theorem II. There we need universal property of formal blowings up with not necessarily analytic center (in contrast to usual modification theory). Although these are covered by [**B3**] in principle, we need a few detailed properties. The difficulty lies in the fact that topology works little in formal geometry. Fortunately the spaces of Pfaffian forms often substitute for topology. It enables us to generalize Gabrièlov's generic rank for holomorphic maps to formal morphisms (cf. (3.4)). This generalized rank is used to measure the local dimensions of images of formal morphisms. Indeed, it is needed to deduce Theorem II from Theorem I in §7.

The category of complex spaces (resp. formal complex spaces) is expressed as **cs** (resp. **fcs**). Thus $\Phi \in \mathbf{cs} : X \to Y$ (resp. $\boldsymbol{\Phi} \in \mathbf{fcs} : \boldsymbol{X} \to \boldsymbol{Y}$) implies a morphism between complex spaces (resp. formal complex spaces). Let $\Phi_\xi : X_\xi \to Y_\eta$ $(\eta = |\Phi|(\xi))$ denote the germ of morphism $\Phi \in \mathbf{cs}$ (resp. **fcs**) at ξ. Then the induced homomorphism between local rings is expressed by the corresponding small letter as $\varphi_\xi : \mathcal{O}_{Y,\eta} \to \mathcal{O}_{X,\xi}$ (see §1). A *subspace* means a closed subspace defined by a coherent ideal sheaf unless stated otherwise. When we mention *Modifications* or *finite morphisms* between complex spaces, the underlying topological mappings are always assumed to be proper. Let P be a property of a complex space. We say that a complex space X satisfies P along a closed subset S if the germ of X along S has an arbitrarily small representative which satisfies P. A (formal) complex space X is called (*locally*) *integral* at $\xi \in |X|$ if the local ring $\mathcal{O}_{X,\xi}$ is an integral domain. A complex space is called (*globally*) *integral* if it is reduced and its smooth points form a connected subset. A formal complex space is (*globally*) *integral* if it is reduced and if its normalization is connected. These definitions are consistent (cf. (6.5)).

The author has made many mistakes in the preliminary stage of this paper. Many facts in formal geometry are seemingly apparent by the analogy to analytic geometry but they require careful algebraic verification. Finally, he wishes to express thanks to Professors A. Fujiki and M. Shiota for helpful discussions and to Professor T. Ohsawa for this not quite light task.

1. Convergence of formal morphisms.

Let $\Phi : X \to Y$ be a morphism between locally $\boldsymbol{C}$-ringed spaces. By definition, Φ consists of a continuous map $|\Phi| : |X| \to |Y|$ between underlying topological spaces and a homomorphism $\Phi^\sharp : \mathcal{O}_Y \to \Phi_*\mathcal{O}_X$ of sheaves of rings, where $\Phi_*\mathcal{O}_X$ denotes the *direct image sheaf*. The homomorphism $\Phi^\sharp$ canonically induces a local homomorphism $\varphi_\xi : \mathcal{O}_{Y,\eta} \to \mathcal{O}_{X,\xi}$ $(\eta = |\Phi|(\xi))$. For an $\mathcal{O}_Y$-module $\mathscr{F}$ on Y, let $\Phi^{-1}\mathscr{F} \equiv |X| \times_{|Y|} \mathscr{F}$ denote the *topological inverse image sheaf*. $\Phi^\sharp$ also induces canonically the adjoint homomorphism $\Phi^\sharp : \Phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ (the same notation $\Phi^\sharp$ is used). Let $\mathscr{I} \subset \mathcal{O}_Y$ be an ideal sheaf and let $(\Phi^{-1}\mathscr{I})\mathcal{O}_X \subset \mathcal{O}_X$ denote the ideal sheaf generated by its image by $\Phi^\sharp$. This is just the ideal sheaf that defines the inverse image of the subspace of Y defined by $\mathscr{I}$ and called the *inverse image ideal sheaf*.

Let X be a complex space with the structure sheaf $\mathcal{O}_X$ and S its analytic subset defined by a coherent ideal sheaf $\mathscr{I} \subset \mathcal{O}_X$ i.e. $S = \mathrm{spt}\mathcal{O}_X/\mathscr{I}$. Then $(S, \hat{\mathcal{O}}_{X|S})$ with $\hat{\mathcal{O}}_{X|S} \equiv (\varprojlim_k \mathcal{O}_X/\mathscr{I}^k)|S$ is a formal complex space. We call this the (*formal*) *completion* of X along S and express it as $\hat{X} = \hat{X}_{|S}$ and thus $\mathcal{O}_{\hat{X}} = \hat{\mathcal{O}}_{X|S}$ (cf. [**BS**], VI). The completion is determined by the *core* S and independent of the choice of $\mathscr{I}$. In this paper, the long vertical lines indicate

(i) the underlying topological spaces (mappings) of ringed spaces or

(ii) the restrictions of morphisms.

The short vertical lines indicate

(iii) the subscript case of (i) or

(iv) the cores of completions.

If $S = |X|$, then $\hat{\mathcal{O}}_{X|S} = \mathcal{O}_X$ and $\hat{X}_{|S} = X$. Hence **cs** is a full subcategory of **fcs**. If $S = \{\xi\}$, then $\mathcal{O}^*_{X,\xi} \equiv \hat{\mathcal{O}}_{X|\xi,\xi}$ is the $\mathfrak{m}_\xi$-adic completion of the local ring $\mathcal{O}_{X,\xi}$, where $\mathfrak{m}_\xi$ denotes the maximal ideal of $\mathcal{O}_{X,\xi}$. Let $S \subset |X|$ and $T \subset |Y|$ be analytic subsets of complex spaces and $\Phi \in \mathbf{cs} : X \to Y$ a morphism such that $|\Phi|(S) \subset T$. Then Φ induces a morphism $\hat{\Phi}_{S;T} \in \mathbf{fcs} : \hat{X}_{|S} \to \hat{Y}_{|T}$, which we call the *completion* of Φ along S and T (cf. [**BS**], VI, §2). Completion of a complex space is functorial: i.e. $(\Psi \circ \Phi)\hat{}_{S;U} = \hat{\Psi}_{T;U} \circ \hat{\Phi}_{S;T}$ for $\Phi \in \mathbf{cs} : X \to Y$ with $|\Phi|(S) \subset T$ and $\Psi \in \mathbf{cs} : Y \to Z$ with $|\Phi|(T) \subset U$. This implies that completion transforms a commutative diagram into a commutative diagram. The completions of the identity yield canonical homomorphisms $\mathcal{O}_{X,\xi} \to \hat{\mathcal{O}}_{X|S,\xi} \to \mathcal{O}^*_{X,\xi}$ for $\xi \in S$. These are faithfully flat morphisms (cf. [**M**], §7, §8). We may consider these homomorphisms inclusions: $\mathcal{O}_{X,\xi} \subset \hat{\mathcal{O}}_{X|S,\xi} \subset \mathcal{O}^*_{X,\xi}$. A section of $\hat{\mathcal{O}}_{X|S}$ over $U \subset S$ is called a *formal function* on U and it is identified with a morphism of $\hat{X}_{|S}|U$ into $\boldsymbol{C} \equiv \hat{\boldsymbol{C}}_{|\boldsymbol{C}}$. We say that $\boldsymbol{\Phi} \in \mathbf{fcs} : \hat{X}_{|S} \to \hat{Y}_{|T}$ is *convergent* at $\xi \in S$ if there exists a neighborhood $U \subset |X|$ of ξ such that $\Phi|S \cap U$ is the completion, along $S \cap U$ and T, of an analytic morphism $X|U \to Y$. A coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ is called an *ideal sheaf of definition* of $\boldsymbol{X} \in \mathbf{fcs}$ if $(|\boldsymbol{X}|, \mathcal{O}_{\boldsymbol{X}}/\mathcal{I}) \in \mathbf{cs}$ (i.e. $\mathcal{I}_{|\boldsymbol{X}|} = \sqrt{\mathcal{I}}$ by Ruckert's Nullstellensatz, cf. [**GR**], [**Moo**]). (See Introduction for the definition of $\mathcal{I}_{|\boldsymbol{X}|}$. The radicals of coherent ideal sheaves are coherent because so are the nilradicals of the structure sheaves (cf. (4.3)).) Thus $\mathcal{I}_{|\boldsymbol{X}|}$ is the maximal among ideals of definition of $|\boldsymbol{X}|$. A morphism $\boldsymbol{\Phi} \in \mathbf{fcs} : \boldsymbol{X} \to \boldsymbol{Y}$ is called *adic* if and only if every (or some) ideal of definition of $\boldsymbol{Y}$ generates that of $\boldsymbol{X}$. Take $\boldsymbol{\Phi} \in \mathbf{fcs} : \boldsymbol{X} \to \boldsymbol{Y}$, neighborhood U of $\xi \in |\boldsymbol{X}|$ and Stein neighborhood V of $\eta = |\boldsymbol{\Phi}|(\xi) \in |\boldsymbol{Y}|$ such that $|\boldsymbol{\Phi}|(U) \subset V$. By the embedding theorem of Bingener [**B3**], §1, there exist an open subset $W \subset \boldsymbol{C}^n$, closed analytic subset $S \subset W$ and an embedding $\boldsymbol{\Theta} \in \mathbf{fcs} : \boldsymbol{Y}|V \to \hat{W}_{|S}$ (i.e. $\boldsymbol{\Theta}$ is adic on V, $|\boldsymbol{\Theta}| : V \to S$ is an homeomorphism and the associated homomorphism $\boldsymbol{\theta}_\eta : \hat{\mathcal{O}}_{\Omega|S,\zeta} \to \mathcal{O}_{\boldsymbol{Y},\eta}$ ($\zeta = |\boldsymbol{\Theta}|(\eta)$) are surjective for all $\eta \in V$.) We call the pullbacks $\boldsymbol{\theta}(z_1), \dots, \boldsymbol{\theta}(z_n) \in \mathcal{O}_{\boldsymbol{Y}}(V)$ of affine coordinates $z_1, \dots, z_n$ of $\boldsymbol{C}^n$ the *ambient coordinates* around η and their pullbacks $\boldsymbol{\varphi} \circ \boldsymbol{\theta}(z_1), \dots, \boldsymbol{\varphi} \circ \boldsymbol{\theta}(z_n) \in \mathcal{O}_{\boldsymbol{X}}(U)$ *components* of $\boldsymbol{\Phi}$ around ξ. If W, $\boldsymbol{\Theta}$ and S are fixed, the components determine morphism $\boldsymbol{\Phi}|U$ ([**Kr**], §0). It is obvious that $\boldsymbol{\Phi} : \hat{X}_{|S} \to \hat{Y}_{|T}$ is convergent if and only if its components are all so. Convergence of $\boldsymbol{\Phi} \in \mathbf{fcs} : \hat{X}_{|S} \to \hat{Y}_{|T}$ between completions of complex spaces is not affected by the extension of $\hat{Y}_{|T}$ as follows.

Lemma 1.1. *Take $\boldsymbol{\Phi} \in \mathbf{fcs} : \hat{X}_{|S} \to \hat{Y}_{|T}$ between completions of complex spaces X and Y. Suppose that Y is a locally closed subspace of $Y' \in \mathbf{cs}$ and $T \subset T' \subset |Y'|$ and that $\hat{I}_{T;T'} : \hat{Y}_{|T} \to \hat{Y}'_{|T'}$ be the completion of the inclusion. Then $\boldsymbol{\Phi}$ is convergent if and only if $\hat{I}_{T;T'} \circ \boldsymbol{\Phi}$ is so.*

The proof is easy. If a formal complex space $\boldsymbol{X}$ is isomorphic to the completion $\hat{Y}_{|S}$ of a complex space Y, we call (Y, S) a *complex structure* of $\boldsymbol{X}$. A formal complex space may not have a complex structure even if its underlying topological space is 0-dimensional (see Appendix). If $\boldsymbol{X}$ and S are given, the germ of Y at a point of S is unique up to isomorphism ([**Art**], (1.6)). It is a difficult problem to discuss existence and uniqueness of the germ of Y up to isomorphism along entire S (cf. [**Arn**], [**K**]). At any rate, we can refer to convergence of a morphism between formal complex spaces when their complex structures are specified.

Example 1.2. The substitution $t = \sum_{n \ge 1} n! s^n$ defines a formal isomorphism $\boldsymbol{\Phi}$ of $\hat{\boldsymbol{C}}_{|0} \equiv (\{0\}, \boldsymbol{C}[[s]])$ into $\hat{\boldsymbol{C}}_{|0} \equiv (\{0\}, \boldsymbol{C}[[t]])$. If the complex structures are defined by s and t respectively, $\boldsymbol{\Phi}$ is not convergent.

An algebra isomorphic to a local ring of a formal complex space is called a *formal analytic algebra* (cf. (5.4)). Formal analytic algebras and $\boldsymbol{C}$-algebra homomorphisms form the category of formal analytic algebras, which is dual to the category of germs at points of formal complex spaces (cf. the analytic case [**Moo**], I).

LEMMA 1.3. *Let $\boldsymbol{\Phi} \in \mathbf{fcs} : \hat{X}_{|S} \to \hat{Y}_{|T}$ be a formal morphism between completions of complex spaces.*

(i) *For $\xi \in S$, the composition $\boldsymbol{\Phi} \circ \hat{I}_{\xi;S} : \hat{X}_{|\xi} \to \hat{Y}_{|T}$ is convergent if and only if $\boldsymbol{\Phi}$ is convergent at ξ.*

(ii) *If $\boldsymbol{\Phi}$ is convergent at all $\xi \in S$, there exist a neighborhood U of S and an analytic morphism $\Psi : X|U \to Y$ such that $\boldsymbol{\Phi} = \hat{\Psi}_{S;T}$.*

PROOF. (i) Suppose that $\boldsymbol{\Phi} \circ \hat{I}_{\xi;S}$ is convergent for $\xi \in S$. Then there exist a neighborhood U of ξ and an analytic morphism $\Psi \equiv \Psi^U \in \mathbf{cs} : X|U \to Y$ whose completion $\hat{\Psi}_{\xi;T}$ coincides with $\boldsymbol{\Phi} \circ \hat{I}_{\xi;S}$. Hence the left lower rectangle in the diagram is commutative. Since completion is functorial, the outer rectangle and upper triangle are also commutative. Hence we have $\hat{\Psi}_{S\cap U;T} \circ \hat{I}_{\xi;S\cap U} = \boldsymbol{\Phi} \circ \hat{I}_{S\cap U;S} \circ \hat{I}_{\xi;S\cap U}$. Since the homomorphism of local rings induced by $\hat{I}_{\xi;S\cap U}$ is injective, $\hat{I}_{\xi;S\cap U}$ is an epimorphism (a right cancelable morphism) in the category of germs of formal complex spaces. Hence the germ $(\hat{\Psi}_{S\cap U;T})_\xi$ coincides with $(\boldsymbol{\Phi} \circ \hat{I}_{S\cap U;S})_\xi = \boldsymbol{\Phi}_\xi$. Thus $\boldsymbol{\Phi}$ and $\hat{\Psi}_{S\cap U;T}$ coincides in a neighborhood of ξ and $\boldsymbol{\Phi}$ is convergent there. The converse is trivial.

$$
\begin{tikzcd}
(X|U)\hat{}_{|\xi} \arrow[rr, "\hat{I}_{\xi;S\cap U}"] \arrow[dd, "\cong"'] \arrow[dr, "\hat{\Psi}_{\xi;T}"] & & (X|U)\hat{}_{|S\cap U} \arrow[dl, "\hat{\Psi}_{S\cap U;T}"'] \arrow[dd, "\hat{I}_{S\cap U;S}"] \\
& \hat{Y}_{|T} & \\
\hat{X}_{|\xi} \arrow[rr, "\hat{I}_{\xi;S}"'] & & \hat{X}_{|S} \arrow[ul, "\boldsymbol{\Phi}"]
\end{tikzcd}
$$

(ii) Suppose that there exist analytic morphisms $\Psi^U : X|U \to Y$ and $\Psi^V : X|V \to Y$ defined on neighborhoods U and V of ξ such that $\boldsymbol{\Phi}|S\cap U = (\Psi^U)\hat{}_{S\cap U;T}$ and $\boldsymbol{\Phi}|S\cap V = (\Psi^V)\hat{}_{S\cup V;T}$. We have only to prove that each component of Ψ^U coincides with corresponding one of Ψ^V in a neighborhood of ξ. This is trivial because they coincide in $\hat{\mathcal{O}}_{X|S,\xi} (\supset \mathcal{O}_{X,\xi})$. □

LEMMA 1.4. *Let $X = X_1 \cup \cdots \cup X_k$ be the decomposition into irreducible components of reduced $X \in \mathbf{cs}$ and $S_i \subset |X_i|$ analytic subsets such that $S = S_1 \cup \cdots \cup S_k$. If $\boldsymbol{\Phi} \in \mathbf{fcs} : \hat{X}_{|S} \to \hat{Y}_{|T}$ is convergent on all $\hat{X}_{i|S_i}$ (i.e. if $\boldsymbol{\Phi} \circ \hat{I}_{S_i;S}$ are all convergent), then so is on entire $\hat{X}_{|S}$.*

PROOF. By (1.3) we have only to consider a formal function germ. Then the assertion follows from Artin's theorem on analytic equations [**Art**] (see Proof of [**I2**], (9.7) for details). □

2. The first main theorem.

We call an element a of a local ring A *active* if $a \bmod N_A$ is a nonzerodivisor in $A_{\mathrm{red}} \equiv A/N_A$, where $N_A \equiv \sqrt{0}$ denotes the nilradical of A. We call a subspace S of a locally ringed space X *thin* at ξ if the ideal $\mathscr{I}_{S,\xi} \subset \mathcal{O}_{X,\xi}$ of S_ξ contains an active element. Note that the underlying topological space of a thin subspace of a formal complex space is not always topologically thin. We call a homomorphism $\pi : A \to B$ between local rings a *crush* if the image of some active element by π is not active. If neither π nor θ is a crush, then neither is $\pi \circ \theta$. A morphism $\Pi : Y \to Z$ between locally ringed spaces is a *crush* at $\eta \in Y$ if the corresponding homomorphism $\mathcal{O}_{Z,\zeta} \to \mathcal{O}_{Y,\eta} (\zeta = |\Pi|(\eta))$ is so. Let $\Pi : Y \to Z$ be a morphism between complex or formal complex spaces. It is easy to see that Π is a crush at $\eta \in Y$ if and only if some irreducible component of $(Y_{\mathrm{red}})_\eta$ is mapped into a thin subspace by the canonically induced morphism $\Pi_{\mathrm{red}} : (Y_{\mathrm{red}})_\eta \to (Z_{\mathrm{red}})_\zeta$. A morphism $\Pi \in \mathbf{cs} : Y \to Z$ is called *generically finite* if $|\Pi|$ is proper and there exists a thin analytic subset $E \subset |Y|$ such that $|\Pi|^{-1}(|\Pi|(\eta))$ is a finite set for any $\eta \in |Y| \backslash E$. The *dimension* of $X \in \mathbf{cs}$ (or $\mathbf{fcs}$) at ξ (or the *dimension* of the germ X_ξ) is defined by the Krull dimension of the local ring: $\dim X_\xi \equiv \dim \mathcal{O}_{X,\xi}$. We say that $X \in \mathbf{cs}$ (or $\mathbf{fcs}$) is *equidimensional* at ξ (or X_ξ is *equidimensional*) if all the local irreducible components of the germ $(X_{\mathrm{red}})_\xi$ of the reduction have a same dimension and that X is *equidimensional* if X is equidimensional everywhere and its local dimensions are constant.

The following is our first main theorem.

Theorem I. *Suppose that X and Y are reduced equidimensional complex spaces, $S \subset |X|$ an analytic subset and $\boldsymbol{\Phi} : \hat{X}_{|S} \to Y$ a formal morphism satisfying the following*:

(i) *S is a thin Moishezon subspace* (*with respect to the reduced* (*or any*) *complex structure*);

(ii) *X is globally integral along S*;

(iii) *$\boldsymbol{\Phi}$ is nowhere a crush on S*;

(iv) *There exists a generically finite morphism $\Pi : Y \to Z$ such that $|\Pi \circ \boldsymbol{\Phi}|(S) =$* (*one point*).

If $\boldsymbol{\Phi}$ is convergent at some $\xi \in S$ along some irreducible component of $\hat{X}_{|S,\xi}$, then it is convergent everywhere on S.

Note that the condition (ii) implies that S is connected. The author has announced a claim similar to above in [**I1**], Theorem 3. In this occasion he would like to note that the assertion is incorrect. In fact his proof needs the condition of non-crush as stated above. If $\boldsymbol{\Phi}$ is a crush, its "image" may be contracted by Π and hence Π may bear no information on $\boldsymbol{\Phi}$.

Remark 2.1. Let $T \subset |Y|$ be an analytic subset in the theorem. Then, in view of (1.1), we may replace $\boldsymbol{\Phi} : \hat{X}_{|S} \to Y$ by $\boldsymbol{\Phi} : \hat{X}_{|S} \to \hat{Y}_{|T}$ and (iii) by the following:

(iii′) *$\hat{I}_{T;|Y|} \circ \boldsymbol{\Phi} : \hat{X}_{|S} \to Y$ is nowhere a crush, where I denotes the identity of Y.*

This is really weaker than (iii) by the following reasons. Since $\hat{I}_{T;|Y|}$ is flat, it is nowhere a crush. Then, if $\boldsymbol{\Phi}$ is nowhere a crush, neither is $\hat{I}_{T;|Y|} \circ \boldsymbol{\Phi}$. On the other hand, Gabrièlov [**G1**] has shown existence of $\Phi \in \mathbf{cs} : X \to Y$ such that $\hat{\Phi}_{\xi;\eta} : \hat{X}_{|\xi} \to \hat{Y}_{|\eta}$ is a crush and $\hat{I}_{\eta;|Y|} \circ \hat{\Phi}_{\xi;\eta} : \hat{X}_{|\xi} \to Y$ is not a crush.

If Y is Stein in the theorem, then $|\boldsymbol{\Phi}|(S)$ is one point and the condition (iv) is unnecessary. In particular, putting $Y = \boldsymbol{C}$, we see that this theorem is a generalization of the function case [**I3**], (C), the explicit form of which is found in (2.5), (C)$^\sharp$ below in an improved form.

To prove the theorem above, we prepare a few lemmas. A *formal curve* $\boldsymbol{K}$ on $\boldsymbol{X} \in \mathbf{fcs}$ (expressed as $\boldsymbol{K} \subset \boldsymbol{X}$) is simply a morphism $\boldsymbol{K} \in \mathbf{fcs} : \hat{\boldsymbol{C}}_{|0} \to \boldsymbol{X}$. We do not assume that $\boldsymbol{K}$ is adic, so that the image of $\boldsymbol{K}$ does not bear a canonical structure of a closed subspace. We call $|\boldsymbol{K}|(0) \in |\boldsymbol{X}|$ the *initial point* of $\boldsymbol{K}$.

Lemma 2.2. *Let $\Pi : Y \to Z$ denote a modification between reduced complex spaces. Then there exists a thin subspace $E \subset Z$ defined by an ideal sheaf $\mathscr{I}_E$ such that the following holds: if $\boldsymbol{K} \subset Z$ is a formal curve with initial point ζ such that $(\boldsymbol{K}^{-1}\mathscr{I}_{E,\zeta})\boldsymbol{C}[[t]] \neq 0$ (i.e. $\boldsymbol{K} \not\subset E$), then there exists a unique lifting $\boldsymbol{\Lambda} \subset Y$ of $\boldsymbol{K}$.*

Proof. (i) *The case of a blowing up*. Suppose that Π is the blowing up with centre E. By the condition on $\boldsymbol{K}, (\boldsymbol{K}^{-1}\mathscr{I}_{E,\zeta})\boldsymbol{C}[[t]]$ is invertible. Since Π is a blowing up in the category **fcs** also by the construction of projective spectra, there exists a unique lifting $\boldsymbol{\Lambda} \subset Y$ of $\boldsymbol{K}$ by the universality of a formal blowing up.

(ii) *The general case*. By Hironaka's version of Chow's lemma [**Hi**], Cor.2, there exist a locally finite succession $\Pi' : Y' \to Z$ of blowings up and a morphism $\Phi : Y' \to Y$ such that $\Pi' = \Pi \circ \Phi$. Since the problem is local on Z and since a composition of a finite number of blowings up is a blowing up locally (cf. [**HR**]), we may assume that Π' is a blowing up with centre E. Then a lifting of $\boldsymbol{K}$ to Y' exists by (i). Its image in Y is the lifting of $\boldsymbol{K}$ we seek for. Suppose that there exist two liftings $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}'$ to Y. By the universality of blowing up, Φ is also the blowing up of Y with centre $\Pi^{-1}(E)$ and liftings of $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}'$ to Y' exist by (i). By uniqueness (i) of lifting for the blowing up Π', they coincide. Hence $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}'$. □

Lemma 2.3. *Let $\Pi : Y \to Z$ denote a finite morphism between reduced complex spaces. Then, for any formal curve $\boldsymbol{K} : \hat{\boldsymbol{C}}_{|0} \to Z$, there exist only a finite number of liftings $\boldsymbol{\Lambda} : \hat{\boldsymbol{C}}_{|0} \to Y$ of $\boldsymbol{K}$.*

Proof. Let ζ be the initial point of $\boldsymbol{K}$. Since Π is proper, we have only to prove that there exist only a finite number of liftings through each point of $|\Pi|^{-1}(\zeta)$. Hence we may assume that $|\Pi|^{-1}(\zeta) = \eta$ (one point) and Y is integral at η. If $\boldsymbol{K}$ is not included in the image of Y, there exists no lifting. Otherwise we may assume that Z is the image of Y and hence Z is integral. These imply that $\mathcal{O}_{Y,\eta}$ and $\mathcal{O}_{Z,\zeta}$ are integral domains and $\mathcal{O}_{Y,\eta}$ is integral over $\mathcal{O}_{Z,\zeta}$.

Let $x_1, \dots, x_n$ be the ambient coordinates around η (cf. §1). There exist monic polynomials $A_p(s) \in \mathcal{O}_{Z,\zeta}[s]$ $(p = 1, \dots, n)$ such that $A_p(x_p) = 0$. Let $\boldsymbol{\Lambda}$ be a lifting of $\boldsymbol{K}$. If we apply $\boldsymbol{\kappa} : \mathcal{O}_{Z,\zeta} \to \boldsymbol{C}[[t]]$ to the coefficients of $A_p(s)$, we obtain a monic polynomial $B_p(s) \in \boldsymbol{C}[[t]][s]$. Then the components $\lambda_p = \boldsymbol{\lambda}(x_p) \in \boldsymbol{C}[[t]]$ of $\boldsymbol{\Lambda}$ satisfy $B_p(\lambda_p) = \boldsymbol{\lambda}(A_p(x_p)) = 0$. The coefficients of $B_p(s)$ depend upon $\boldsymbol{K}$ but not upon $\boldsymbol{\Lambda}$. Since $\boldsymbol{C}[[t]]$ is an integral domain, only a finite number of choices are permitted for the root $\lambda_p \in \boldsymbol{C}[[t]]$ of the equation $B_p(s) = 0$. Since Λ is determined by $\lambda_1, \dots, \lambda_n$, the number of liftings Λ is also finite. $\square$

The key lemma to Theorem I is the following.

Lemma 2.4. *Let X, Y be reduced complex spaces, $S \subset |X|$ an analytic subset and $\boldsymbol{\Phi} : \hat{X}_{|S} \to Y$ a formal morphism which is nowhere a crush on S. Suppose that there exists a generically finite morphism $\Pi \in \mathbf{cs} : Y \to Z$ such that $\Pi \circ \boldsymbol{\Phi} \in \mathbf{fcs} : \hat{X}_{|S} \to \hat{Z}_{||Z|} = Z$ is convergent at $\xi \in S$, then $\boldsymbol{\Phi}$ is convergent at ξ.*

Proof. By (1.4), we may assume that X is integral at ξ. Then by the Hironaka desingularization, we have only to treat the case X is smooth. Indeed, since the desingularization morphism has a full generic rank, convergence of the pullbacks of the components of $\boldsymbol{\Phi}$ implies convergence of the components themselves by Gabrièlov's theorem [**G2**], (5.6). Thus we may assume that X is a neighborhood of $\xi \in \boldsymbol{C}^n$. We may replace Z by the ambient space of its local model. Hence we have only to treat the case Z is an open subset of $\boldsymbol{C}^m$.

By Stein factorization theorem (cf. [**GR**], (10.6.1), (8.1.1)), there exist a reduced complex space W, a modification $\Theta : Y \to W$ and a finite morphism $\Sigma : W \to Z$ such that $\Pi = \Sigma \circ \Theta$. Let $E \subset |W|$ be the bad thin subspace for Θ described in (2.2). Let e be an active element in $\mathscr{I}_{|E|,\tau}$ ($\eta = |\boldsymbol{\Phi}|(\xi)$, $\tau = |\Theta|(\eta)$). The composition $\Theta \circ \boldsymbol{\Phi}$ is not a crush at ξ by the assumption on $\boldsymbol{\Phi}$. Hence $\boldsymbol{\varphi}_\xi \circ \theta_\eta(e)$ determine the germ of a hypersurface at ξ and its tangent cone H is also a hypersurface. Let L be a complex line through $\xi \in \boldsymbol{C}^n$ which is not contained in H. Parameterizing L by $t \in \boldsymbol{C}$ and composing it with $\boldsymbol{\Phi}$ and $\Pi \circ \boldsymbol{\Phi}$, we have a formal curve $\boldsymbol{K}$ on Y and an analytic curve K' on Z such that $\boldsymbol{K}$ is a lifting of $\hat{K}' \equiv \hat{K}'_{0;|Z|} : \hat{\boldsymbol{C}}_{|0} \to Z = \hat{Z}_{||Z|}$.

$$
\begin{tikzcd}
\xi \in H \subset \hat{X}_{|S} \arrow[d, "\boldsymbol{\Phi}"'] & \hat{\boldsymbol{C}}_{|0} \arrow[l, "L"'] \arrow[dl, "\boldsymbol{K}"] \arrow[dr, "\hat{K}'"] & \\
Y \arrow[r, "\Theta"'] & W \arrow[r, "\Sigma"'] & Z \\
\rotatebox{90}{$\in$} & \cup & \\
\eta & E & \\
 & \rotatebox{90}{$\in$} & \\
 & \tau &
\end{tikzcd}
$$

Let us take an embedding $Y \subset U$ where U is an open subset of $\boldsymbol{C}^p$. Let $\Lambda \subset U$ be a formal curve with initial point η. The condition for Λ to be a lifting to Y of $\hat{K}'$ is expressed by a system of simultaneous analytic equations for components of Λ around η (the constraint to Y and the lifting condition). Then the system has only a finite number of solutions in $\boldsymbol{C}[[t]]$ by (2.2) and (2.3). Since these solutions are arbitrarily approximated by analytic solutions with respect to the maximal-ideal-adic topology by Artin's theorem [**Art**], all solutions are analytic. Thus components of $\boldsymbol{K}$ are analytic. Since our choice of L is generic, the components of $\boldsymbol{\Phi}$ are convergent by a theorem of Tougeron [**T1**], (1.3). This proves the convergence of $\boldsymbol{\Phi}$ itself. $\square$

Finally, to complete the proof of Theorem I, we need the following supplement to a former paper.

REMARK 2.5. In view of the proofs of [**I3**], (C), (C*), we can improve them as follows:

Let S be a positive dimensional Moishezon subspace of a complex space X such that X is integral along S. Suppose that Y is an irreducible component of $X|U$ for an open set $U \subset |X|$. Then we have the following.

(C)$^\sharp$ *If $f \in \hat{\mathcal{O}}_{X|S}(S)$ is convergent along Y at $\xi \in S \cap |Y|$ (i.e. $(f|Y)_\xi \in \mathcal{O}_{Y,\xi}$), f is convergent everywhere (i.e. $f \in \mathcal{O}_X(S)$).*

(C*)$^\sharp$ *For any $\xi \in |Y|$ there exists $a \in \boldsymbol{R}$ such that*

$$a \cdot \bar{v}^S_{X,\mathscr{I}}(f) \geq \bar{v}_{Y,\xi}(f) \ (\geq \bar{v}^S_{X,\mathscr{I}}(f))$$

*for any $f \in \mathcal{O}_X(S)$, where $\mathscr{I} \subset \mathcal{O}_X(S)$ denotes the ideal sheaf of S. Such an a must satisfy $a > 1$. (cf. [**I3**] for notation.)*

In a similar way, we can improve [**I3**], (A), (A*), (B), (B*), obtaining assertions that the behaviors of polynomials (or sections) on a local irreducible component of S regulate the corresponding global properties. The improvement of (A*) is used in [**I4**].

PROOF OF THEOREM I. If we apply (2.5), (C)$^\sharp$ to the components of $\Pi \circ \boldsymbol{\Phi}$, they are convergent everywhere or nowhere. Hence $\Pi \circ \boldsymbol{\Phi}$ itself has the same property. The assertion follows from (2.4). □

3. Dimension of formal complex space.

This section, along with next three sections, is devoted to general theory of formal complex spaces as preliminary to the proof of the second main theorem in the last section. First we list ring theoretic properties of formal analytic algebras which are well-known for analytic algebras and then we generalize the definition of Gabrièlov's generic rank for analytic morphisms to one for formal morphisms.

LEMMA 3.1 ([**B3**], (1.10)). *If K is a Stein semianalytic compact subset of a formal complex space $\boldsymbol{X}$, then $\mathcal{O}_{\boldsymbol{X}}(K)$ is an excellent ring.*

Let us call a ring *normal* if its localizations at prime ideals are integrally closed in their respective total rings of fractions.

LEMMA 3.2. *Let A be a formal analytic algebra.*

(i) ([**AT1**]) *A is a Henselian local ring.*

(ii) *A has the approximation property for algebraic equations.*

(iii) *A is regular (or integral, normal, reduced,* CM, *Gorenstein) if and only if the maximal-ideal-adic completion A^* is so.*

PROOF. (i) This is proved by Ancona and Tomassini [**AT1**], (I.1.2).

(ii) It is known that a Henselian excellent local ring has the approximation property (Popescu [**P**], cf. [**Ro**], [**A**], [**O**], [**Sp**]).

(iii) By (ii) we have the implication

$$\exists f,\ \exists g \in A^* \text{ s.t. } f \neq 0,\ g \neq 0,\ fg = 0 \Rightarrow \exists f,\ \exists g \in A \text{ s.t. } f \neq 0,\ g \neq 0,\ fg = 0.$$

Hence, if A is integral, A^* is also so. Since A is excellent, it is a G-ring and the natural monomorphism $A \to A^*$ is regular and faithfully flat. Then A is regular (normal, reduced, CM or Gorenstein) if and only if A^* is so ([**M**], (32.2)). □

Suppose that A is an integral domain. We define the *rank* of an A-module M as the maximal number r such that there exists an injective A-homomorphism $A^r \to M$. If $Q(A)$ denote the fields of fractions, the rank is equal to the dimension of the $Q(A)$-vector space $Q(A) \otimes_A M$. This rank is denoted by $\mathrm{rank}_A M$.

Let A be a formal analytic algebra and $\Omega(A)$ the space of Pfaffian forms on A (= the universal finite differential module). This can be defined in the same way as the case of an analytic algebra and has similar properties ([**B3**], (1.8)).

Theorem 3.3 (cf. [**SS**], (4.1)). *If A is an integral formal analytic algebra, we have* $\mathrm{rank}_A \Omega(A) = \dim A$.

Proof. It is known that $\Omega(A^*) \cong A^* \otimes_A \Omega(A)$ ([**SS**], (1.6)). Since A^* is A-flat and integral by (3.2), $\mathrm{rank}_A \Omega(A) = \mathrm{rank}_{A^*} \Omega(A^*)$. Completion preserves Krull dimension also (cf. [**N**], (17.12)). Therefore we may assume that $A \equiv A^*$ i.e. A is a residue class algebra of $\boldsymbol{C}[[x_1, \dots, x_n]]$. Then it is well-known that, if $r = \dim A$, there exists an injective finite monomorphism $B \equiv \boldsymbol{C}[[x_1, \dots, x_r]] \to A$ (an application of the formal Weierstrass preparation theorem). Since $x_1, \dots, x_r$ form a system of parameters of $A, dx_1, \dots, dx_r$ are linearly independent over A ([**SS**], (8.12)). Any element $f \in A$ satisfies a monic polynomial relation over B. Taking the total derivative of this relation, we see that df is linearly dependent on $dx_1, \dots, dx_r$ over A. Hence $\mathrm{rank}_A \Omega(A) = r = \dim A$. □

Next we generalize the notion of Gabrièlov's generic rank $\mathrm{grk}\, \boldsymbol{\Phi}_\xi$ in analytic geometry (cf. [**G2**]). Let $\boldsymbol{\varphi} : B \to A$ be a homomorphism between integral formal analytic algebras. By the universality of $\Omega(B)$, $\boldsymbol{\varphi}$ naturally induces a homomorphism $\boldsymbol{\varphi}^1 : \Omega(B) \to \Omega(A)$ compatible with $\boldsymbol{\varphi}$ (cf. [**SS**], (1.1), (1.2), (1.3)). We define the *generic rank* of $\boldsymbol{\varphi}$ by $\mathrm{grk}\, \boldsymbol{\varphi} \equiv \mathrm{rank}_A A\boldsymbol{\varphi}^1(\Omega(B))$. Since $\boldsymbol{\varphi}$ splits through $B/\mathrm{Ker}\, \boldsymbol{\varphi}, \mathrm{grk}\, \boldsymbol{\varphi} \le \min\{\dim A, \dim B/\mathrm{Ker}\, \boldsymbol{\varphi}\}$. If $\boldsymbol{\Phi} : \boldsymbol{X} \to \boldsymbol{Y}$ is a morphism between formal complex spaces and if $\boldsymbol{X}_\xi$ is integral, the *generic rank* $\mathrm{grk}\, \boldsymbol{\Phi}_\xi$ of $\boldsymbol{\Phi}$ at ξ is defined to be that of the induced homomorphism $\boldsymbol{\varphi}_\xi : \mathcal{O}_{\boldsymbol{Y},\eta} \to \mathcal{O}_{\boldsymbol{X},\xi}$ $(\eta \equiv |\boldsymbol{\Phi}|(\xi))$.

Remark 3.4. Let us confirm that our definitions of generic rank coincides with Gabrièlov's *in analytic case*. Let X_ξ be an integral germ of a complex subspace of an open subset of $\boldsymbol{C}^m$. Then $A \equiv \mathcal{O}_{X,\xi}$ is an integral domain. Let $x_1, \dots, x_n$ be the ambient coordinates. We may assume that $x_1, \dots, x_k$ is a system of parameters of $\mathcal{O}_{X,\xi}$ $(k := \dim X)$. Then dx_i $(i = 1, \dots, k)$ are linearly independent over A and $\Omega(A) \subset \sum_{i=1}^k Q(A)\, dx_i$. Let $\Phi_\xi : X_\xi \to Y_\eta$ be a morphism between germs of complex spaces. Let $y_1, \dots, y_n$ be the ambient coordinates of Y around η. Then there exist $\beta \in A \backslash \{0\}$ and $\alpha_{ij} \in A$ such that $\beta d\varphi(y_j) = \sum_{i=1}^k \alpha_{ij} dx_i$. We may assume that Φ, α_{ij} and β have respective representatives defined on a neighborhood U of ξ. The representatives are indicated by $\tilde{}$. Let M be a minor of (α_{ij}). Since X_ξ is integral, the theorem of identity holds in a neighborhood of ξ, i.e. $M = 0$ if and only if $\tilde{M}$ vanishes on an open set arbitrarily near to ξ. Hence $\mathrm{grk}\, \Phi = r$ in our sense if and only if the maximum of the rank of the jacobian matrix $(\tilde{\alpha}_{ij}/\tilde{\beta})$ of $\Phi | X_\xi$ at a smooth point arbitrarily near to ξ is r. The latter is nothing but Gabrièlov's original definition.

By our new definition of generic rank, the troublesome inequality $\mathrm{grk}\, \varphi_\xi \le \dim B^*/\mathrm{Ker}\, \hat{\varphi}_{\xi;\eta}$ becomes trivial, cf. [**I2**], (1.5).

Lemma 3.5. *Suppose that $\boldsymbol{X} \in \mathbf{fcs}$, $\boldsymbol{X}_\xi$ is integral, $\boldsymbol{\Phi} \in \mathbf{fcs} : \boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{\Psi} \in \mathbf{fcs} : \boldsymbol{Y} \to \boldsymbol{Z}$. If we put $\eta = |\boldsymbol{\Phi}|(\xi)$ and $\zeta = |\boldsymbol{\Psi}|(\eta)$. Then we have the following.*

(i) *If there exist $f_1, \dots, f_n \in \mathcal{O}_{\boldsymbol{Y},\eta}$ such that $\boldsymbol{\varphi}_\xi(f_1), \dots, \boldsymbol{\varphi}_\xi(f_n)$ form a part of a system of parameters of $\mathcal{O}_{\boldsymbol{X},\xi}$, then $\operatorname{grk} \boldsymbol{\Phi}_\xi \geq n$.*

(ii) *If $\boldsymbol{\varphi}_\xi$ is injective and finite, then $\boldsymbol{Y}_\eta$ is integral and $\operatorname{grk} \boldsymbol{\Psi}_\eta \circ \boldsymbol{\Phi}_\xi = \operatorname{grk} \boldsymbol{\Psi}_\eta$. In particular $\operatorname{grk} \boldsymbol{\Phi}_\xi = \dim \boldsymbol{Y}_\eta$.*

Proof. (i) Obvious from the fact that $d(\boldsymbol{\varphi}_\xi(f_1)), \dots, d(\boldsymbol{\varphi}_\xi(f_n))$ generate a free submodule of $\Omega(\mathcal{O}_{\boldsymbol{X},\xi})$ of rank n ([**SS**], (8.12)).

(ii) The first assertion is trivial. If $\operatorname{grk} \boldsymbol{\Psi}_\eta = r$, there exist $g_1, \dots, g_r \in \mathcal{O}_{\boldsymbol{Z},\zeta}$ such that $d\boldsymbol{\psi}_\eta(g_i)$ are linearly independent over $\mathcal{O}_{\boldsymbol{Y},\eta}$. Let $f_1, \dots, f_m \in \mathcal{O}_{\boldsymbol{Y},\eta}$ be a system of parameters. Then $\Omega(\mathcal{O}_{\boldsymbol{Y},\eta}) / \sum \mathcal{O}_{\boldsymbol{Y},\eta} df_j$ is an $\mathcal{O}_{\boldsymbol{Y},\eta}$-torsion module ([**SS**], (4.1)). Hence there exist $h \in \mathcal{O}_{\boldsymbol{Y},\eta}$ and $\alpha_{ij} \in \mathcal{O}_{\boldsymbol{Y},\eta} \backslash \{0\}$ such that

$$h \cdot d\boldsymbol{\psi}_\eta(g_j) \in \sum_{i=1}^{m} \alpha_{ij} df_i, \quad \operatorname{rank}(\alpha_{ij}) = r.$$

Then

$$\boldsymbol{\varphi}_\xi(h) \cdot d(\boldsymbol{\varphi}_\xi \circ \boldsymbol{\psi}_\eta(g_j)) \in \sum_{i=1}^{m} \boldsymbol{\varphi}_\xi(\alpha_{ij}) d\boldsymbol{\varphi}_\xi(f_i), \quad \operatorname{rank}(\boldsymbol{\varphi}_\xi(\alpha_{ij})) = r.$$

Since a finite extension of a commutative ring preserves Krull dimension (cf. [**N**], (10.10)),

$$\dim \mathcal{O}_{\boldsymbol{X},\xi} / (\boldsymbol{\varphi}_\xi(f_1), \dots, \boldsymbol{\varphi}_\xi(f_m)) = \dim \mathcal{O}_{\boldsymbol{Y},\eta} / (f_1, \dots, f_m) = 0.$$

Then $\boldsymbol{\varphi}_\xi(f_1), \dots, \boldsymbol{\varphi}_\xi(f_n)$ form a system of parameters of $\mathcal{O}_{\boldsymbol{X},\xi}$ and $d\boldsymbol{\varphi}_\xi(f_i)$ are independent. This proves that $d(\boldsymbol{\varphi}_\xi \circ \boldsymbol{\psi}_\eta(g_j)) = \boldsymbol{\varphi}_\xi^1 \circ \boldsymbol{\psi}_\eta^1(dg_j)$ $(j = 1, \dots, r)$ generate a submodule of $\Omega(\mathcal{O}_{\boldsymbol{X},\xi})$ of rank r and that $\operatorname{grk} \boldsymbol{\Psi}_\eta \circ \boldsymbol{\Phi}_\xi \geq \operatorname{grk} \boldsymbol{\Psi}_\eta$. The converse inequality is trivial. If we take $\boldsymbol{\Psi}$ for the identity $\boldsymbol{I} : \boldsymbol{Y} \to \boldsymbol{Y}$, the last assertion follows. $\square$

Theorem 3.6. *Let $\mathcal{F}$ be a coherent $\mathcal{O}_{\boldsymbol{X}}$-module on $\boldsymbol{X} \in \mathbf{fcs}$. If $\boldsymbol{X}$ is locally and globally integral, then $\operatorname{rank}_{\mathcal{O}_{\boldsymbol{X},\xi}} \mathcal{F}_\xi$ is independent of $\xi \in |\boldsymbol{X}|$.*

Proof. Since $\mathcal{F}$ is coherent, there exists a local exact sequence:

$$(\mathcal{O}_{\boldsymbol{X}} | U)^p \xrightarrow{\lambda} (\mathcal{O}_{\boldsymbol{X}} | U)^q \to (\mathcal{F} | U) \to 0.$$

We may assume that $\mathcal{O}_{\boldsymbol{X}} | U$ is generated by global sections. Let $\mathcal{G}_n^U \subset \mathcal{O}_{\boldsymbol{X}} | U$ denote the ideal sheaf generated by the $(q-n) \times (q-n)$ minors of the matrix representing λ. This is known to be independent of the choice of the exact sequence. Gluing $\mathcal{G}_n^U$, we obtain a coherent ideal sheaf $\mathcal{G}_n \subset \mathcal{O}_{\boldsymbol{X}}$ (Fitting ideal sheaf). Let $\boldsymbol{Y}_n \subset \boldsymbol{X}$ denote the subspace defined by $\mathcal{G}_n$. Then we can easily verify that

$$\boldsymbol{Y}_{q-1} \subset \boldsymbol{Y}_{q-2} \subset \cdots \subset \boldsymbol{Y}_0 \subset \boldsymbol{X}$$

and that

$$\operatorname{rank}_{\mathcal{O}_{\boldsymbol{X},\xi}} \mathcal{F}_\xi \geq r+1 \iff \operatorname{rank}_{\mathcal{O}_{\boldsymbol{X},\xi}} \lambda_\xi \leq q-r-1 \iff (\mathcal{G}_r)_\xi = 0 \iff (\boldsymbol{Y}_r)_\xi = \boldsymbol{X}_\xi.$$

Since $\boldsymbol{X}$ is integral, putting

$$s \equiv \max\{\operatorname{rank}_{\mathcal{O}_{\boldsymbol{X},\xi}} \mathcal{F}_\xi : \xi \in |\boldsymbol{X}|\} - 1,$$

we have $\boldsymbol{X} = \boldsymbol{Y}_r$ for $r \leq s$. Then $\operatorname{rank}_{\mathcal{O}_{\boldsymbol{X},\xi}} \mathcal{F}_\xi = s + 1$ everywhere. □

The disjoint union $\Omega(\boldsymbol{X}) \equiv \coprod_{\xi \in \boldsymbol{X}} \Omega(\mathcal{O}_{\boldsymbol{X},\xi})$ has a natural structure of a coherent $\mathcal{O}_{\boldsymbol{X}}$-module on $|\boldsymbol{X}|$ and we call it the *sheaf of Pfaffian forms* ([**B3**], §1).

Corollary 3.7.

(i) *Suppose that* $\boldsymbol{X} \in \mathbf{fcs}$ *is globally integral and let* $\boldsymbol{Z}_\xi$ *be a local irreducible component of* $\boldsymbol{X}_\xi$. *Then, for* $\boldsymbol{\Phi} \in \mathbf{fcs} : \boldsymbol{X} \to \boldsymbol{Y}$, $\operatorname{grk} \boldsymbol{\Phi}_\xi \,|\, \boldsymbol{Z}_\xi$ *is independent of the choice of point* $\xi \in |\boldsymbol{X}|$ *and the component* $\boldsymbol{Z}_\xi$. *In particular* $\dim \boldsymbol{Z}_\xi = \operatorname{rank}_{\mathcal{O}_{\boldsymbol{Z},\xi}} \Omega(\mathcal{O}_{\boldsymbol{Z},\xi})$ *is constant and* $\boldsymbol{X}$ *is equidimensional.*

(ii) *For any* $\boldsymbol{X} \in \mathbf{fcs}$, $\dim \boldsymbol{X}_\xi$ *is upper semicontinuous with respect to* ξ.

Proof. By (3.5), (ii), we may assume that $\boldsymbol{X}$ is normal (cf. §3 of [**B3**] and §5 below for normalization) and $\boldsymbol{Z}_\xi = \boldsymbol{X}_\xi$. Then, since $\mathcal{O}_{\boldsymbol{X}} \boldsymbol{\varphi}^1(\Omega(\boldsymbol{Y})) \subset \Omega(\boldsymbol{X})$ is $\mathcal{O}_{\boldsymbol{X}}$-coherent, $\operatorname{grk} \boldsymbol{\Phi}_\xi$ is constant by (3.6). If we take $\boldsymbol{\Phi}$ for the identity of $\boldsymbol{X}$, the second assertion follows. To prove (ii), we may assume that $\boldsymbol{X}$ is reduced. Then it admits the normalization. Hence $\boldsymbol{X}$ admits the global Lasker-Noether decomposition by coherence of direct image sheaf ([**B3**], (3.1)) and (ii) follows from (i). □

4. Blowing up and dimension.

Using the basic works of Bingener [**B3**] on formal complex spaces, we can define the blowing up of a formal complex space $\boldsymbol{X}$ with center $\mathcal{I} \subset \mathcal{O}_{\boldsymbol{X}}$, a coherent ideal sheaf, in the same way as the complex analytic case. Since (formal) analytic spectra and projective (formal) analytic spectra are adic, blowings up are also so. Most of the properties of analytic blowings up hold in formal case too. In particular, the local dimension is invariant through formal blowings up. But this can not be obtained by an easy topological argument as in the analytic case (cf. [**Moo**], III, (1.4.6)). Instead we certify it algebraically, using the generalized generic rank defined in the previous section. This invariance is used to reduce the second main theorem to the first in §7 (cf. the first paragraph of §7).

Lemma 4.1. *Let* A *be an integral formal analytic algebra and* $h_1, \ldots, h_n$ $(n = \dim A)$ *a system of parameters of* A. *If* g *is a nonzero element of* A,

$$1 \otimes d(g^q h_1), \ldots, 1 \otimes d(g^q h_n)$$

form a basis for the $Q(A)$*-vector space* $Q(A) \otimes_A \Omega(A)$ *for any* $q \in \boldsymbol{Z}$ *possibly except one value.*

Proof. Since A is an integral domain, $dh_1, \ldots, dh_n$ generate a submodule of $\Omega(A)$ with the maximal rank ([**SS**], (8.12)). Hence there exists an expression

$$1 \otimes dg = g_1 \otimes dh_1 + \cdots + g_n \otimes dh_n$$

in $Q(A) \otimes_A \Omega(A)$ and we have

$$1 \otimes d(g^q h_\alpha) = q g^{q-1} h_\alpha (g_1 \otimes dh_1 + \cdots + g_n \otimes dh_n) + g^q \otimes dh_\alpha.$$

By a calculation of the determinant, we see that these are linearly independent unless

$$g + q(g_1 h_1 + \cdots + g_n h_n) = 0. \qquad \square$$

Remark 4.2. Let X be a ringed space with the structure sheaf $\mathcal{O}_X$, $\mathbf{coh}(X)$ the category of sheaves of $\mathcal{O}_X$-module locally of finite presentation whose morphisms are defined to be $\mathcal{O}_X$-homomorphisms and $\mathbf{cohi}(X)$ the category of ideal sheaves of $\mathcal{O}_X$ locally of finite presentation whose morphisms are inclusions. $\mathbf{cohi}(X)$ can be identified with the category of subobjects of $\mathcal{O}_X \in \mathbf{coh}(X)$. Namely an object of $\mathbf{cohi}(X)$ is a monomorphism $\theta \in \mathbf{coh}(X) : \mathscr{F} \to \mathcal{O}_X$. A morphism of $\mathbf{cohi}(X)$ comes from a monomorphism in $\mathbf{coh}(X)$.

Bingener [**B3**] called a formal complex space $\boldsymbol{X}$ *Stein* if $|\boldsymbol{X}|$ is a Stein space with respect to the reduced complex structure. Let $\boldsymbol{X}$ be a Stein formal complex space and put $A \equiv \mathcal{O}_{\boldsymbol{X}}(|\boldsymbol{X}|)$ (the Stein algebra of $\boldsymbol{X}$). Bingener has proved the following for an A-scheme $\boldsymbol{Y}$ which is locally of finite presentation ([**B3**], (4.5): the *existence theorem*):

If $\boldsymbol{Y}$ is proper over $\operatorname{Spec} A$ *and if $S \subset |\boldsymbol{X}|$ is a semianalytic Stein compact subset, there exists an equivalence*

$$\boldsymbol{E} : \mathbf{coh}(\boldsymbol{Y} \times_{\operatorname{Spec} A} \operatorname{Spec} \mathcal{O}_{\boldsymbol{X}}(S)) \to \varinjlim \mathbf{coh}(\boldsymbol{Y}^{\mathrm{an}} | U),$$

of categories, where the limit is taken over the directed system of all Stein neighborhoods U of S.

Since monomorphisms are characterized by left cancelability, a property defined only by the terms of the categories, $f \in \operatorname{Hom}(\mathscr{F}, \mathscr{F}')$ is a monomorphism if and only if $\boldsymbol{E}(f)$ is so. By exactness of inductive limit functor, $\boldsymbol{E}$ induces an isomorphism

$$\boldsymbol{E}' : \mathbf{cohi}(\boldsymbol{Y} \times_{\operatorname{Spec} A} \operatorname{Spec} \mathcal{O}_{\boldsymbol{X}}(S)) \to \varinjlim \mathbf{cohi}(\boldsymbol{Y}^{\mathrm{an}} | U).$$

Let $\boldsymbol{\Pi} \in \mathbf{fcs} : \boldsymbol{X}' \to \boldsymbol{X}$ be a blowing up with centre $\mathscr{I}$ and $S \equiv \{\xi\}$ ($\xi \in |\boldsymbol{X}|$). Then, $\boldsymbol{E}'$ implies the isomorphism between the lattice structures of the following

(i) the family of subspaces of $\operatorname{Proj} \oplus \mathscr{I}_\xi^n$;

(ii) the family of germs of subspaces of $|\boldsymbol{\Pi}|^{-1}(U) = \operatorname{Proj} \oplus (\mathscr{I}|U)^n$ around $|\boldsymbol{\Pi}|^{-1}(\xi)$, where U runs over the system of all Stein neighborhoods of ξ.

Remark 4.3. We can define *reduction* of formal complex spaces via those of Stein algebras using the *correspondence theorem* [**B2**] and [**B1**], (2.7). In other words, the nilradical of the structure sheaf of a formal complex space is coherent. Further, reduction is a covariant functor of $\mathbf{fcs}$ into itself. The reduction of $\boldsymbol{X}$ is denoted by $\boldsymbol{X}_{\mathrm{red}}$.

Theorem 4.4. *Let $\boldsymbol{\Pi} \in \mathbf{fcs} : \boldsymbol{X}' \to \boldsymbol{X}$ be a blowing up of a formal complex space and $\boldsymbol{Y}_\eta$ an irreducible component of $(\boldsymbol{X}'_{\mathrm{red}})_\eta$. Then we have the following:*

(i) *If $|\boldsymbol{\Pi}|(\eta) = \xi$, then* $\dim \boldsymbol{Y}_\eta = \operatorname{grk} \boldsymbol{\Pi} | \boldsymbol{Y}_\eta \le \dim \boldsymbol{X}_\xi$.

(ii) *If $\boldsymbol{X}_\xi$ is equidimensional, then*

$$\dim \boldsymbol{Y}_\eta = \operatorname{grk} \boldsymbol{\Pi} | \boldsymbol{Y}_\eta = \dim \boldsymbol{X}_\xi$$

and hence $\boldsymbol{X}'_\eta$ is equidimensional as well as $\boldsymbol{X}_\xi$.

Proof. (i) The inequality $\operatorname{grk} \boldsymbol{\Pi}|\boldsymbol{Y}_\eta \leq \dim \boldsymbol{X}_\xi$ is obvious. Let $\mathscr{I} \subset \mathcal{O}_{\boldsymbol{X}}$ be the centre of $\boldsymbol{\Pi}$. There exist a small neighborhood U of ξ and a degree-preserving $(\mathcal{O}_{\boldsymbol{X}}|U)$-epimorphism

$$\iota_U : (\mathcal{O}_{\boldsymbol{X}}|U)[T_0, \ldots, T_p] \to \bigoplus_{n \geq 0} (\mathscr{I}|U)^n,$$

whose kernel $\mathscr{K}_U$ is generated by a finite number of sections $k_1, \ldots, k_q \in \mathscr{K}_U(U)$ (cf. [**Moo**], III, (1.4.1)). Then $\boldsymbol{X}'|V$ $(V \equiv |\boldsymbol{\Pi}|^{-1}(U))$ is the projectivization of the formal complex subspace of $(\boldsymbol{X}|U) \times \boldsymbol{C}^{p+1}$ defined by $k_1, \ldots, k_q$. The images $f_\alpha \in \mathscr{I}(U)$ and $F_\alpha \equiv \boldsymbol{\pi}(f_\alpha) \in \mathcal{O}_{\mathbf{X}'}(V)$ of T_α are global sections over U and V respectively. Note that $\mathscr{K}_U$ is the ideal sheaf generated by the homogeneous polynomial relations among germs of f_α over $\mathcal{O}_{\boldsymbol{X}}|U$. We may assume that $T_0 \neq 0$ at η. Then we can find linear combinations

$$H_1 \equiv \sum_{\alpha=0}^{p} a_\alpha^1 T_\alpha / T_0, \ldots, H_d \equiv \sum_{\alpha=0}^{p} a_\alpha^d T_\alpha / T_0 \quad (a_\alpha^1, \ldots, a_\alpha^d \in \boldsymbol{C})$$

$(d = \dim \boldsymbol{Y}_\eta)$ which induce a system of parameters of the local ring $\mathcal{O}_{\boldsymbol{Y},\eta}$. For any $q \geq 1$, the restrictions of $T_0^q H_1, \ldots, T_0^q H_d$ to $\boldsymbol{Y}_\eta$ belong to the canonical image of $\mathcal{O}_{\boldsymbol{X},\xi}$ in $\mathcal{O}_{\boldsymbol{Y},\eta}$. By (4.1), $1 \otimes d(T_0^q H_1), \ldots, 1 \otimes d(T_0^q H_d)$ form a basis of $Q(\mathcal{O}_{\boldsymbol{Y},\eta}) \otimes_{\mathcal{O}_{\boldsymbol{Y},\eta}} \Omega(\mathcal{O}_{\boldsymbol{Y},\eta})$ for general $q \geq 1$. Let $\boldsymbol{\pi}' : \mathcal{O}_{\boldsymbol{X},\xi} \to \mathcal{O}_{\boldsymbol{Y},\eta}$ denote the canonical homomorphism induced by $\boldsymbol{\Pi}|\boldsymbol{Y}_\eta$. Since

$$d(T_0^q H_i)|\boldsymbol{Y}_\eta \in d(\pi'(\mathcal{O}_{\boldsymbol{X},\xi})) = \boldsymbol{\pi}'^1(d\mathcal{O}_{\boldsymbol{X},\xi}),$$

we have $\dim \boldsymbol{Y}_\eta = \operatorname{grk} \boldsymbol{\Pi}|\boldsymbol{Y}_\eta$.

(ii) As in analytic case, blowing up commutes with reduction and the irreducible components included in some multiple of the centre are lost. Then we may assume that $\boldsymbol{X}$ is reduced and $\mathscr{I}_\xi$ is not included in a minimal prime ideal of $\mathcal{O}_{\boldsymbol{X},\xi}$. The completion of a local ring preserves the dimension (cf. [**N**], (17.12)) and the decomposition into the irreducible components by (3.2), (iii). Hence, by the assumption that $\boldsymbol{X}_\xi$ (or $\mathcal{O}_{\boldsymbol{X},\xi}$) is equidimensional, it is formally so (= quasi-unmixed i.e. the maximal-ideal-adic completion is equidimensional). Further, since $\mathscr{I}_\xi$ is included in no minimal prime ideal, $A \equiv \bigoplus_{n \geq 0} \mathscr{I}_\xi^n$ is also formally, and hence plainly, equidimensional of dimension $r+1$ $(r \equiv \dim \boldsymbol{X}_\xi)$ ([**HIO**], (18.23), (9.7)). Then there exists a homogeneous prime chain of length $r+1$ in A which begins with the prime ideal corresponding to $\boldsymbol{Y}_\eta$ and ends with one corresponding to η (cf. [**M**], (13.7) and its proof). This proves that $\boldsymbol{X}'_\eta$ is equidimensional of dimension r by (4.2). $\square$

5. Completion.

In this section we show that the completion of a formal complex space is again a formal complex space.

Suppose that $\boldsymbol{X} \in \mathbf{fcs}$, $\mathscr{F} \in \mathbf{coh}(\boldsymbol{X})$ and $\mathscr{I} \in \mathbf{cohi}(\boldsymbol{X})$. Let $\hat{\mathscr{F}}_{|\mathscr{I}}$ denote the $\mathscr{I}$-adic completion: $\hat{\mathscr{F}}_{|\mathscr{I}} \equiv \varprojlim_p (\mathscr{F}/\mathscr{I}^p\mathscr{F})$ defined by the canonical presheaf which consists of inverse limits of sections of $\mathscr{F}$ over open sets of $|\boldsymbol{X}|$. Obviously $\hat{\mathscr{F}}_{|\mathscr{I}} \cong \hat{\mathscr{F}}_{|\sqrt{\mathscr{I}}}$.

LEMMA 5.1. *Suppose that $X \in \mathbf{fcs}$, and $\mathscr{I}, \mathscr{J} \in \mathbf{cohi}(X)$. If we put $\mathscr{J}' \equiv \mathscr{J}\hat{\mathcal{O}}_{X|\mathscr{I}} \subset \hat{\mathcal{O}}_{X|\mathscr{I}}$, then there exists a canonical isomorphism between $(\hat{\mathcal{O}}_{|\mathscr{I}})\hat{}_{|\mathscr{J}'}$ and $\hat{\mathcal{O}}_{X|\mathscr{I}+\mathscr{J}}$ as $\mathcal{O}_X$-algebras.*

PROOF. The canonical commutative diagram among $\mathcal{O}$-algebras ($\mathcal{O} \equiv \mathcal{O}_X$)

$$\begin{tikzcd}
\varprojlim_p (\mathcal{O}/\mathscr{I}^p)/\mathscr{J}(\mathcal{O}/\mathscr{I}^p) \arrow[d] & \varprojlim_p (\mathcal{O}/\mathscr{I}^p)/\mathscr{J}^2(\mathcal{O}/\mathscr{I}^p) \arrow[l] \arrow[d] & \varprojlim_p (\mathcal{O}/\mathscr{I}^p)/\mathscr{J}^3(\mathcal{O}/\mathscr{I}^p) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
(\mathcal{O}/\mathscr{I})/\mathscr{J}(\mathcal{O}/\mathscr{I}) & (\mathcal{O}/\mathscr{I}^2)/\mathscr{J}^2(\mathcal{O}/\mathscr{I}^2) \arrow[l] & (\mathcal{O}/\mathscr{I}^3)/\mathscr{J}^3(\mathcal{O}/\mathscr{I}^3) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

yields a homomorphism

$$\varprojlim_q \varprojlim_p (\mathcal{O}/\mathscr{I}^p)/\mathscr{J}^q(\mathcal{O}/\mathscr{I}^p) \longrightarrow \varprojlim_p (\mathcal{O}/\mathscr{I}^p)/\mathscr{J}^p(\mathcal{O}/\mathscr{I}^p).$$

Since all the positive dimensional cohomology groups with coefficients in coherent sheaves vanish on Stein open sets ([**B3**], (1.3)), they vanish on Stein compact sets also. Then by the argument [**BS**], VI, (2.2), (2.3) or [**H**], I, §4, §5, we see that

$$\varprojlim_p (\mathcal{O}/\mathscr{I}^p)/\mathscr{J}^q(\mathcal{O}/\mathscr{I}^p) \cong \varprojlim_p (\mathcal{O}/\mathscr{I}^p)/\varprojlim_p \mathscr{J}^q(\mathcal{O}/\mathscr{I}^p).$$

Thus we have

$$\varprojlim_q \varprojlim_p (\mathcal{O}/\mathscr{I}^p)/\mathscr{J}^q(\mathcal{O}/\mathscr{I}^p) \cong \varprojlim_q \hat{\mathcal{O}}_{|\mathscr{I}}/\mathscr{J}^q\hat{\mathcal{O}}_{|\mathscr{I}} \cong (\hat{\mathcal{O}}_{|\mathscr{I}})\hat{}_{|\mathscr{J}'}.$$

On the other hand, since $(\mathcal{O}/\mathscr{I}^p)/\mathscr{J}^p(\mathcal{O}/\mathscr{I}^p)$ is canonically isomorphic to $\mathcal{O}/(\mathscr{I}^p + \mathscr{J}^p)$ and since

$$(\mathscr{I}+\mathscr{J})^{2p} \subset \mathscr{I}^p + \mathscr{J}^p \subset (\mathscr{I}+\mathscr{J})^p,$$

we have an isomorphism

$$\varprojlim_p (\mathcal{O}/\mathscr{I}^p)/\mathscr{J}^p(\mathcal{O}/\mathscr{I}^p) \cong \hat{\mathcal{O}}_{|\mathscr{I}+\mathscr{J}}.$$

Thus we have a canonical homomorphism

$$(*) \qquad (\hat{\mathcal{O}}_{|\mathscr{I}})\hat{}_{|\mathscr{J}'} \to \hat{\mathcal{O}}_{|\mathscr{I}+\mathscr{J}}.$$

Similarly the commutative diagram among $\mathcal{O}$-algebras

$$\begin{tikzcd}
(\mathcal{O}/\mathscr{I}^p)/\mathscr{J}^p(\mathcal{O}/\mathscr{I}^p) \arrow[d] & (\mathcal{O}/\mathscr{I}^{p+1})/\mathscr{J}^{p+1}(\mathcal{O}/\mathscr{I}^{p+1}) \arrow[l] \arrow[d] & (\mathcal{O}/\mathscr{I}^{p+2})/\mathscr{J}^{p+2}(\mathcal{O}/\mathscr{I}^{p+2}) \arrow[l] \arrow[d] & \cdots \arrow[l] \\
(\hat{\mathcal{O}}/\mathscr{I}^p\hat{\mathcal{O}})/\mathscr{J}^p(\hat{\mathcal{O}}/\mathscr{I}^p\hat{\mathcal{O}}) & (\hat{\mathcal{O}}/\mathscr{I}^{p+1}\hat{\mathcal{O}})/\mathscr{J}^p(\hat{\mathcal{O}}/\mathscr{I}^{p+1}\hat{\mathcal{O}}) \arrow[l] & (\hat{\mathcal{O}}/\mathscr{I}^{p+2}\hat{\mathcal{O}})/\mathscr{J}^p(\hat{\mathcal{O}}/\mathscr{I}^{p+2}\hat{\mathcal{O}}) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

yields a canonical homomorphism

$$(**) \qquad \hat{\mathcal{O}}_{|\mathscr{I}+\mathscr{J}} \to (\hat{\mathcal{O}}_{|\mathscr{I}})\hat{}_{|\mathscr{J}'}.$$

The homomorphisms $(*)$ and $(**)$ are mutual inverses on the subalgebra $\mathcal{O}$, so on entire $(\hat{\mathcal{O}}_{|\mathscr{I}})\hat{}_{|\mathscr{J}'}$ and $\hat{\mathcal{O}}_{|\mathscr{I}+\mathscr{J}}$. □

Let $\mathscr{I}$ be an ideal of definition of $\boldsymbol{X} \in \mathbf{fcs}$. Then $\mathcal{O}_{\boldsymbol{X}} \cong \hat{\mathcal{O}}_{\boldsymbol{X}|\mathscr{I}}$ and $(|\boldsymbol{X}|, \mathcal{O}_{\boldsymbol{X}}/\mathscr{I}^p)$ are complex spaces. Take another $\mathscr{J} \in \mathbf{cohi}(\boldsymbol{X})$. If we put $\mathscr{K} \equiv \mathscr{I} + \mathscr{J}$, $\mathscr{K}' \equiv \mathscr{K}\hat{\mathcal{O}}_{\boldsymbol{X}|\mathscr{I}}$ and $\mathscr{J}' \equiv \mathscr{J}\hat{\mathcal{O}}_{\boldsymbol{X}|\mathscr{I}}$ then the analytic sets $S \equiv \operatorname{spt} \mathcal{O}_{\boldsymbol{X}}/\mathscr{K}^p \subset |\boldsymbol{X}|$ are independent of $p \in \boldsymbol{N}$ and $(S, \mathcal{O}_{\boldsymbol{X}}/\mathscr{K}^p) \cong (S, \hat{\mathcal{O}}_{\boldsymbol{X}|\mathscr{I}}/\mathscr{K}'^p) \in \mathbf{cs}$. Since

$$(\hat{\mathcal{O}}_{\boldsymbol{X}|\mathscr{I}})\hat{}_{|\mathscr{K}'} \cong \hat{\mathcal{O}}_{\boldsymbol{X}|\mathscr{I}+\mathscr{K}} = \hat{\mathcal{O}}_{\boldsymbol{X}|\mathscr{K}} \cong (\hat{\mathcal{O}}_{\boldsymbol{X}|\mathscr{I}})\hat{}_{|\mathscr{J}'} \cong \hat{\mathcal{O}}_{\boldsymbol{X}|\mathscr{J}}$$

by (5.1), we see that $(S, \hat{\mathcal{O}}_{\boldsymbol{X}|\mathscr{J}}) \in \mathbf{fcs}$ (cf. [**B3**]). Since $\hat{\mathcal{O}}_{\boldsymbol{X}|\mathscr{J}}$ is determined by the analytic set S, we may express it as $\hat{\mathcal{O}}_{\boldsymbol{X}|S}$. We call $(S, \hat{\mathcal{O}}_{\boldsymbol{X}|S})$ the *completion* of $\boldsymbol{X}$ along S and express it as $\hat{\boldsymbol{X}}_{|S}$ or as $\hat{\boldsymbol{X}}_{|\mathscr{K}}$.

Lemma 5.2. *Let $\boldsymbol{X} \in \mathbf{fcs}$ be a formal complex space and $S \subset |\boldsymbol{X}|$ an analytic subset. Then $\mathcal{O}_{\mathbf{X},\xi}$ is regular (or integral, normal, reduced,* CM, *Gorenstein) if and only if the completion $\hat{\mathcal{O}}_{\boldsymbol{X}|S,\xi}$ is so.*

Proof. Since the maximal-ideal-adic completions of the two algebras coincide, this follows from (3.2). □

Let $\mathscr{K}$ be an ideal which includes an ideal of definition of $\boldsymbol{X}$. If $\mathscr{F} \in \mathbf{coh}(\boldsymbol{X})$, $\hat{\mathscr{F}} \equiv \hat{\mathscr{F}}_{|S}$ denote the sheaf $\varprojlim_p \mathscr{F}(\mathcal{O}_{\boldsymbol{X}}/\mathscr{K}^p)$ or its restriction to $S \equiv \operatorname{spt} \mathcal{O}_{\boldsymbol{X}}/\mathscr{K}$. It has the canonical structure of an $\hat{\mathcal{O}}_{\boldsymbol{X}|S}$-modules. For $\boldsymbol{\Phi} \in \mathbf{fcs} : \boldsymbol{X} \to \boldsymbol{Y}$ and an $\mathcal{O}_{\boldsymbol{Y}}$-module $\mathscr{F}$, the *analytic inverse image sheaf* $\boldsymbol{\Phi}^*\mathscr{F}$ is defined by $\boldsymbol{\Phi}^*\mathscr{F} \equiv \boldsymbol{\Phi}^{-1}\mathscr{F} \otimes_{\boldsymbol{\Phi}^{-1}\mathcal{O}_{\boldsymbol{Y}}} \mathcal{O}_{\boldsymbol{X}}$. We can prove the following in the same way as analytic case (cf. [**BS**], VI, (2.3), [**H**], §5)

Lemma 5.3.

(i) *The correspondence $\mathscr{F} \to \hat{\mathscr{F}}$ defines an exact functor from $\mathbf{coh}(\boldsymbol{X})$ to $\mathbf{coh}(\hat{\boldsymbol{X}}_{|S})$.*

(ii) *(lifting to completion) Let $\boldsymbol{\Phi} \in \mathbf{fcs} : \boldsymbol{X} \to \boldsymbol{Y}$ be a morphism and $S \subset |\boldsymbol{X}|, T \subset |\boldsymbol{Y}|$ subsets such that $|\boldsymbol{\Phi}|(S) \subset T$. Then there is a unique lifting*

$$\hat{\boldsymbol{\Phi}}_{S;T} \in \mathbf{fcs} : \hat{\boldsymbol{X}}_{|S} \to \hat{\boldsymbol{Y}}_{|T}$$

of $\boldsymbol{\Phi}$.

(iii) *Let*

$$\hat{\boldsymbol{I}} \equiv \hat{\boldsymbol{I}}_{S;|\boldsymbol{X}|} \in \mathbf{fcs} : \hat{\boldsymbol{X}}_{|S} \to \boldsymbol{X}$$

be the lifting of the identity. Then there exists a canonical isomorphism

$$\hat{\boldsymbol{I}}^*\mathscr{F} \equiv \hat{\boldsymbol{I}}^{-1}\mathscr{F} \otimes_{\hat{\boldsymbol{I}}^{-1}\mathcal{O}_{\boldsymbol{X}}} \hat{\mathcal{O}}_{\boldsymbol{X}|S} \to \hat{\mathscr{F}}.$$

Remark 5.4. We must be careful to the following fact. Let $\mathscr{F}$ be an $\mathcal{O}_{\boldsymbol{X}}$-module on $\boldsymbol{X} \in \mathbf{fcs}$ and $S \subset |\boldsymbol{X}|$ an analytic subset defined by $\mathscr{K}$. Then the $\mathcal{O}_{\boldsymbol{X}}(T)$-module $\hat{\mathscr{F}}_{|S}(T)$ of sections over $T \subset S$ does not necessarily coincide with $\varprojlim_p \{(\mathscr{F}/\mathscr{K}^p)(T)\}$, unless T is open. In particular, $\hat{\mathcal{O}}_{\boldsymbol{X}|\mathscr{K},\xi}$ is a proper subalgebra of the $\mathscr{K}_\xi$-adic completion of $\mathcal{O}_{\boldsymbol{X},\xi}$ in general.

6. Normalization and completion.

Here we consider the notion of integrality of a completion of a complex space using normalization of formal complex spaces. This integrality assures constancy of the generalized generic rank in the proof of the second main theorem. On the way, we need the formal version of Forster's duality between Stein algebras and Stein spaces.

Let $\boldsymbol{X}$ be a reduced formal complex space and $\tilde{\mathcal{O}}_{\boldsymbol{X}}$ the sheaf of integral closures of the structure sheaf $\mathcal{O}_{\boldsymbol{X}}$ in the sheaf of the total rings of fractions of $\mathcal{O}_{\boldsymbol{X}}$. Bingener has proven that $\tilde{\mathcal{O}}_{\boldsymbol{X}}$ is an $\mathcal{O}_{\boldsymbol{X}}$-coherent algebra ([**B2**], §1, Bemerkung, [**B3**], §3). The formal complex space $\tilde{\boldsymbol{X}} \equiv \operatorname{Specan} \tilde{\mathcal{O}}_{\boldsymbol{X}}$ with the canonical epimorphism $\boldsymbol{N} \in \mathbf{fcs} : \tilde{\boldsymbol{X}} \to \boldsymbol{X}$ is called the *normalization* of $\boldsymbol{X}$. By coherence of $\tilde{\mathcal{O}}_{\boldsymbol{X}}$, we have $\boldsymbol{N}_* \mathcal{O}_{\tilde{\boldsymbol{X}}} \cong \tilde{\mathcal{O}}_{\boldsymbol{X}}$.

THEOREM 6.1 (lifting to normalization). *Let $\boldsymbol{\Phi} \in \mathbf{fcs} : \boldsymbol{X} \to \boldsymbol{Y}$ be a morphism between reduced formal complex spaces. If $\boldsymbol{\Phi}$ is nowhere a crush, then there exists a unique lifting $\tilde{\boldsymbol{\Phi}} \in \mathbf{fcs} : \tilde{\boldsymbol{X}} \to \tilde{\boldsymbol{Y}}$.*

PROOF. By the assumption on $\boldsymbol{\Phi}$, the canonical homomorphism $\boldsymbol{\Phi}^\sharp : \boldsymbol{\Phi}^{-1}\mathcal{O}_{\boldsymbol{Y}} \to \mathcal{O}_{\boldsymbol{X}}$ can be extended to $\boldsymbol{\Phi}^{-1}\mathcal{M}_{\boldsymbol{Y}} \to \mathcal{M}_{\boldsymbol{X}}$, where $\mathcal{M}_{\boldsymbol{X}}$ and $\mathcal{M}_{\boldsymbol{Y}}$ denote the sheaves of meromorphic function germs on $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively. Writing up the relation of integral dependence, we see that this extension maps $\boldsymbol{\Phi}^{-1}\tilde{\mathcal{O}}_{\boldsymbol{Y}} \subset (\boldsymbol{\Phi}^{-1}\mathcal{M}_{\boldsymbol{Y}})$ into $\tilde{\mathcal{O}}_{\boldsymbol{X}} \subset (\mathcal{M}_{\boldsymbol{X}})$. Then the composition $\boldsymbol{\theta}$ of

$$\boldsymbol{\Phi}^*\tilde{\mathcal{O}}_{\boldsymbol{Y}} \equiv \boldsymbol{\Phi}^{-1}\tilde{\mathcal{O}}_{\boldsymbol{Y}} \otimes_{\boldsymbol{\Phi}^{-1}\mathcal{O}_{\boldsymbol{Y}}} \mathcal{O}_{\boldsymbol{X}} \to \tilde{\mathcal{O}}_{\boldsymbol{X}} \otimes_{\boldsymbol{\Phi}^{-1}\mathcal{O}_{\boldsymbol{Y}}} \mathcal{O}_{\boldsymbol{X}} \to \tilde{\mathcal{O}}_{\boldsymbol{X}} \otimes_{\mathcal{O}_{\boldsymbol{X}}} \mathcal{O}_{\boldsymbol{X}} \equiv \tilde{\mathcal{O}}_{\boldsymbol{X}}$$

of canonical homomorphisms of $\mathcal{O}_{\boldsymbol{X}}$-algebras yields an extension $\tilde{\boldsymbol{X}} \equiv \operatorname{Specan}_{\boldsymbol{X}} \tilde{\mathcal{O}}_{\boldsymbol{X}} \to \operatorname{Specan}_{\boldsymbol{X}} \boldsymbol{\Phi}^*\tilde{\mathcal{O}}_{\boldsymbol{Y}}$. By the base change theorem for formal analytic spectra (cf. [**B3**], p. 33, [**B1**]):

$$\operatorname{Specan}_{\boldsymbol{X}} \boldsymbol{\Phi}^*\tilde{\mathcal{O}}_{\boldsymbol{Y}} \equiv \operatorname{Specan}_{\boldsymbol{Y}} \tilde{\boldsymbol{Y}} \times_{\boldsymbol{Y}} \boldsymbol{X},$$

we have the projection

$$\operatorname{Specan}_{\boldsymbol{X}} \boldsymbol{\Phi}^*\tilde{\mathcal{O}}_{\boldsymbol{Y}} \to \operatorname{Specan}_{\boldsymbol{Y}} \tilde{\mathcal{O}}_{\boldsymbol{Y}} \equiv \tilde{\boldsymbol{Y}}$$

over $\boldsymbol{\Phi}$. Composing these morphisms we obtain a lifting of $\boldsymbol{\Phi}$. By the universality of the fiber product, any lifting splits through

$$\operatorname{Specan}_{\boldsymbol{X}} \boldsymbol{\Phi}^*\tilde{\mathcal{O}}_{\boldsymbol{Y}} \cong (\operatorname{Specan}_{\boldsymbol{Y}} \tilde{\mathcal{O}}_{\boldsymbol{Y}}) \times_{\boldsymbol{Y}} \boldsymbol{X}.$$

An element of $(\boldsymbol{\Phi}^*\tilde{\mathcal{O}}_{\boldsymbol{Y}})_\xi$ is expressed as $(f/g) \otimes h$ $(f, g, h \in \mathcal{O}_{\boldsymbol{X},\xi})$ such that g is not a zerodivisor. Since the image $\boldsymbol{\theta}_\xi(g \otimes 1) = \boldsymbol{\varphi}_\xi(g)$ is not a zerodivisor, the image of $(f/g) \otimes h$ is uniquely determined by $\boldsymbol{\varphi}_\xi(fh)$ and $\boldsymbol{\varphi}_\xi(g)$ and the extension $\boldsymbol{\Phi}^*\tilde{\mathcal{O}}_{\boldsymbol{Y}} \to \tilde{\mathcal{O}}_{\boldsymbol{X}}$ is unique. This proves the uniqueness. $\square$

LEMMA 6.2. *Let $S \subset |\boldsymbol{X}|$ be an analytic subset of $\boldsymbol{X} \in \mathbf{fcs}$. Then $(\boldsymbol{X}_{\mathrm{red}})\hat{}_{|S}$ and $(\hat{\boldsymbol{X}}_{|S})_{\mathrm{red}}$ are canonically isomorphic.*

PROOF. Let us put $A \equiv \mathcal{O}_{\boldsymbol{X},\xi}$ and $\hat{A} \equiv \hat{\mathcal{O}}_{\boldsymbol{X}|S,\xi}$. Since $((A_{\mathrm{red}})\hat{}\,)_{\mathrm{red}} \cong (A_{\mathrm{red}})\hat{}$ by (5.2), we have a homomorphism

$$\hat{A}/N_{\hat{A}} \cong (\hat{A})_{\mathrm{red}} \to (A_{\mathrm{red}})\hat{} \cong (A/N_A)\hat{} \cong \hat{A}/N_A\hat{A}$$

by (5.3). Then we have $N_{\hat{A}} \subset N_A \hat{A}$. Since $N_A \hat{A} \subset N_{\hat{A}}$ is obvious, the equality holds and the local rings $(A_{\mathrm{red}})^{\wedge}$ and $\hat{A}_{\mathrm{red}}$ are canonically isomorphic. This proves the assertion. $\square$

We call a $\boldsymbol{C}$-algebra a *Stein formal algebra* if it is isomorphic to the algebra of global sections $\mathcal{O}_{\boldsymbol{X}}(|\boldsymbol{X}|)$ for some Stein formal complex space $\boldsymbol{X}$.

The following is the formal version of Forster's theorem ([**F**], Satz 1).

Theorem 6.3. *The category of Stein formal complex spaces is isomorphic to the dual of the category of Stein formal algebras.*

Proof. Let $\boldsymbol{\varphi}(|\boldsymbol{X}|) : \mathcal{O}_{\boldsymbol{Y}}(|\boldsymbol{Y}|) \to \mathcal{O}_{\boldsymbol{X}}(|\boldsymbol{X}|)$ be a homomorphism between Stein formal algebras. Since $\boldsymbol{\varphi}(\mathcal{I}_{|\boldsymbol{Y}|}(|\boldsymbol{Y}|)) \subset \mathcal{I}_{|\boldsymbol{X}|}(|\boldsymbol{X}|)$, this induces a homomorphism

$$\varphi_n(|\boldsymbol{X}|) : \mathcal{O}_{\boldsymbol{Y}}(|\boldsymbol{Y}|)/\mathcal{I}^n_{|\boldsymbol{Y}|}(|\boldsymbol{Y}|) \to \mathcal{O}_{\boldsymbol{X}}(|\boldsymbol{X}|)/\mathcal{I}^n_{|\boldsymbol{X}|}(|\boldsymbol{X}|)$$

between Stein algebras. Let us put

$$X_n \equiv (|\boldsymbol{X}|, \mathcal{O}_{\boldsymbol{X}}/\mathcal{I}^n_{|\boldsymbol{X}|}), \quad Y_n \equiv (|\boldsymbol{Y}|, \mathcal{O}_{\boldsymbol{Y}}/\mathcal{I}^n_{|\boldsymbol{Y}|}).$$

By the classical Forster's theorem, there exists a morphism $\Phi_n \in \mathbf{cs} : X_n \to Y_n$ which induces $\varphi_n(|\boldsymbol{X}|)$. Taking the inverse limit of Φ_n, we have a morphism $\boldsymbol{\Phi} \in \mathbf{fcs} : \boldsymbol{X} \to \boldsymbol{Y}$ which induces $\boldsymbol{\varphi}(|\boldsymbol{X}|)$. The rest are easy to see. $\square$

Theorem 6.4. *A complex space $X \in \mathbf{cs}$ is reduced along $S \subset X$ if and only if $\hat{X} \equiv \hat{X}_{|S}$ is reduced. If X is reduced along S, then $(\tilde{X})^{\wedge}_{|S'}$, $(\hat{X}_{|S})^{\sim}$ and $\hat{X} \times_X \tilde{X}$ are canonically isomorphic, where $S' \subset |\tilde{X}|$ is the inverse image of S.*

Proof. If we apply [**BS**], VI, (2.8) to the sheaves of nilradicals, we obtain the first assertion. In a) and b) below, we consider the sheaves on X.

a) First we claim that $\hat{\mathcal{O}}_X \to \mathcal{O}_X{}^{\sim\wedge}$ is nowhere a crush. To see this, suppose the contrary: the injective homomorphism

$$\hat{A} \cong A \otimes_A \hat{A} \to \tilde{A} \otimes_A \hat{A} \cong A^{\sim\wedge} \quad (A \equiv \mathcal{O}_{X,\xi}, \hat{A} \equiv \hat{\mathcal{O}}_{X|S,\xi})$$

is a crush for some $\xi \in S$. Then there exist a non-zerodivisor $1 \otimes f \in A \bigotimes_A \hat{A}$ and a nonzero $g \otimes h \in \tilde{A} \otimes \hat{A}$ such that $g \otimes fh = 0$ in $\tilde{A} \otimes_A \hat{A}$. We can express g as $g = g'/k$ for some $g' \in A$ and some non-zerodivisor $k \in A$. Therefore $g' \otimes fh = 0$ in $A \otimes_A \hat{A}$. This contradicts the assumption on $1 \otimes f$.

b) By a) there exists a lifting $\mathcal{O}_X{}^{\wedge\sim} \to \mathcal{O}_X{}^{\sim\wedge\sim}$ (cf. Proof of (6.1)). Fibers of the sheaf $\mathcal{O}_X{}^{\sim\wedge}$ are normal i.e. $\mathcal{O}_X{}^{\sim\wedge\sim} \cong \mathcal{O}_X{}^{\sim\wedge}$ by (5.2). Therefore we have a canonical homomorphism $\boldsymbol{\sigma} : \mathcal{O}_X{}^{\wedge\sim} \to \mathcal{O}_X{}^{\sim\wedge}$. Since $\mathcal{O}_X{}^{\wedge\sim}$ is $\hat{\mathcal{O}}_X$-coherent, $\mathcal{O}_X{}^{\wedge\sim\wedge} \cong \mathcal{O}_X{}^{\wedge\sim}$. By the flatness of $\mathcal{O}_{X,\xi} \to \hat{\mathcal{O}}_{X,\xi}$ and by (5.3), (6.1), we have a canonical homomorphism $\mathcal{O}_X{}^{\sim\wedge} \to \mathcal{O}_X{}^{\wedge\sim\wedge}$. Thus we have a homomorphism $\boldsymbol{\tau} : \mathcal{O}_X{}^{\sim\wedge} \to \mathcal{O}_X{}^{\wedge\sim}$. The homomorphism $\boldsymbol{\sigma}$ is the inverse of $\boldsymbol{\tau}$ on their subsheaf $\mathcal{O}_X$ and hence on $\mathcal{O}_X{}^{\wedge\sim}$ and $\mathcal{O}_X{}^{\sim\wedge}$. Therefore they are canonically isomorphic.

c) Let $U \subset |X|$ be a Stein open set. Since the normalization morphism $N : \tilde{X} \to X$ is surjective and finite, $\tilde{U} \equiv |N|^{-1}(U) \subset |\tilde{X}|$ is also Stein (cf. [**KK**], (73.1)). Let us put $\hat{X}_{|S}$, $X^{\sim\wedge} \equiv (\tilde{X})^{\wedge}_{|S'}$ and let $\boldsymbol{N} \in \mathbf{fcs} : X^{\wedge\sim} \to \hat{X}$ denote the formal normalization morphism and

$$\boldsymbol{K} \equiv \hat{I}_{S;|X|} \in \mathbf{fcs} : \hat{X} \to X$$

the completion morphism. Thus we have the following diagram.

$$\begin{tikzcd}
 & (\tilde{S}_U \subset) X^{\tilde{}\,\hat{}} \arrow[r] \arrow[dl, dotted, "?"'] & \tilde{X}(\supset \tilde{U} \supset \tilde{S}_U) \arrow[dd, "N"] \\
(T_U \subset) X^{\hat{}\,\tilde{}} \arrow[d, "\boldsymbol{N}"'] & & \\
(S \cap U \subset) \hat{X} \arrow[rr, "\boldsymbol{K}"] & & X(\supset U \supset S \cap U)
\end{tikzcd}$$

Let $\mathscr{I}_U \subset \mathcal{O}_X(U)$ denote the ideal of sections vanishing on $S \cap U$. This generates the ideals of S_ξ at each $\xi \in U$ (Theorem A). Let $\tilde{\mathscr{I}}_U \subset \mathcal{O}_{\tilde{X}}(\tilde{U})$ and $\hat{\mathscr{I}}_U \subset \mathcal{O}_{\hat{X}}(S \cap U)$ denote the ideals generated by the images of $\mathscr{I}_U$. Their images generate ideals of definition of $X^{\tilde{}\,\hat{}}|\tilde{S}_U$ and $X^{\hat{}\,\tilde{}}|T_U$ respectively ($\tilde{S}_U \equiv S' \cap \tilde{U}$, $T_U \equiv |X^{\hat{}\,\tilde{}}| \cap |\boldsymbol{K} \circ \boldsymbol{N}|^{-1}(U)$). Then we have

$$\begin{aligned}
\mathcal{O}_{X^{\tilde{}\,\hat{}}}(\tilde{S}_U) &\cong \varprojlim_k (\mathcal{O}_{\tilde{X}}/\tilde{\mathscr{I}}_U^k \mathcal{O}_{\tilde{X}})(\tilde{U}) \quad (\text{cf. [}\mathbf{BS}\text{], VI, §2}) \\
&\cong \varprojlim_k \mathcal{O}_{\tilde{X}}(\tilde{U})/(\tilde{\mathscr{I}}_U^k \mathcal{O}_{\tilde{X}})(\tilde{U}) \quad (\text{since } \tilde{U} \text{ is Stein}) \\
&\cong \varprojlim_k \tilde{\mathcal{O}}_X(U)/\mathscr{I}_U^k(U)\tilde{\mathcal{O}}_X(U) \quad (\text{by a property of normalization}) \\
&\cong \varprojlim_k (\tilde{\mathcal{O}}_X/\mathscr{I}_U^k \tilde{\mathcal{O}}_X)(U) \quad (\text{since } U \text{ is Stein}) \\
&\cong \mathcal{O}_X{}^{\tilde{}\,\hat{}}(S \cap U) \quad (\text{cf. [}\mathbf{BS}\text{], VI, §2}).
\end{aligned}$$

On the other hand, we have

$$\mathcal{O}_{X^{\hat{}\,\tilde{}}}(T_U) \cong (\boldsymbol{N}_* \mathcal{O}_{X^{\hat{}\,\tilde{}}})(S \cap U) \cong (\mathcal{O}_{\hat{X}})^{\tilde{}}(S \cap U) \cong \mathcal{O}_X{}^{\hat{}\,\tilde{}}(S \cap U).$$

d) We have an isomorphism

$$\boldsymbol{\varphi}_U(\tilde{S}_U) : \mathcal{O}_{X^{\hat{}\,\tilde{}}}(T_U) \to \mathcal{O}_{X^{\tilde{}\,\hat{}}}(\tilde{S}_U)$$

by b) and c). Since $\tilde{S}_U$ and T_U are Stein, we have an isomorphism

$$\boldsymbol{\Phi}_U \in \mathbf{fcs} : X^{\tilde{}\,\hat{}}|\tilde{S}_U \to X^{\hat{}\,\tilde{}}|T_U$$

which induces $\boldsymbol{\varphi}_U(\tilde{S}_U)$ by (6.3).

e) Suppose that U and V are Stein open sets of $|X|$. We claim that $\boldsymbol{\Phi}_U$ and $\boldsymbol{\Phi}_V$ coincide on $T_{U \cap V}$. Since the intersection of two Stein open sets is Stein, we may assume that $V \subset U$. Then $\boldsymbol{\varphi}_U(\tilde{S}_U)$ and $\boldsymbol{\varphi}_V(\tilde{S}_V)$ commute with the restriction homomorphisms

$$\mathcal{O}_{X^{\hat{}\tilde{}}}(T_U) \to \mathcal{O}_{X^{\hat{}\tilde{}}}(T_V), \quad \mathcal{O}_{X^{\tilde{}\hat{}}}(\tilde{S}_U) \to \mathcal{O}_{X^{\tilde{}\hat{}}}(\tilde{S}_V),$$

because they are defined through the sheaf homomorphism $\mathcal{O}_{X^{\hat{}\tilde{}}} \to \mathcal{O}_{X^{\tilde{}\hat{}}}$ on $|X|$. This verifies the claim by (6.3). Thus $\boldsymbol{\Phi}_U$ are glued to a canonical isomorphism $\boldsymbol{\Phi} : X^{\tilde{}\hat{}} \to X^{\hat{}\tilde{}}$.

f) The $\mathcal{O}_{\hat{X}}$-modules $\tilde{\mathcal{O}}_{\hat{X}}$ and $\boldsymbol{K}^*\tilde{\mathcal{O}}_X$ are canonically isomorphic as $\mathcal{O}_X$-modules by (b). Then they are canonically isomorphic as $\mathcal{O}_{\hat{X}}$-modules also by continuity of multiplications by elements of $\mathcal{O}_{\hat{X}}$ in each stalk module with the maximal-ideal-adic topology. Then the claim $X^{\hat{}\tilde{}} \cong \hat{X} \times_X \tilde{X}$ is proved as follows.

$$\begin{aligned} X^{\tilde{}\hat{}} &\equiv \mathrm{Specan}_{\hat{X}}\, \tilde{\mathcal{O}}_{\hat{X}} \quad \text{(by definition of the normalization)} \\ &\cong \mathrm{Specan}_{\hat{X}}\, \boldsymbol{K}^*\tilde{\mathcal{O}}_X \\ &\cong \hat{X} \times_X \mathrm{Specan}_X\, \tilde{\mathcal{O}}_X \quad \text{(by the base change theorem [}\mathbf{B3}\text{], §4 and [}\mathbf{B1}\text{])} \\ &\equiv \hat{X} \times_X \tilde{X} \quad \text{(by definition of normalization).} \end{aligned}$$

□

We call a formal complex space (*globally*) *integral* if it is reduced and its normalization is connected (cf. [**GR**], (9.1.2)).

Theorem 6.5. *A complex space X is globally integral along an analytic set $S \subset |X|$ if and only if $\hat{X}_{|S}$ is globally integral.*

Proof. Since the equivalence of reducedness is stated in (6.4), we may assume that X and $\hat{X}$ are reduced. Let $N : \tilde{X} \to X$ denote the normalization morphism and put $S' \equiv |N|^{-1}(S)$. Suppose that X is integral along S. There exists an arbitrarily small neighborhood U of S such that $X|U$ is integral i.e. $U \backslash \mathrm{Sing}\, X$ is connected. Then $U' \equiv |N|^{-1}(U)$ is connected. Since N is proper, we can choose an arbitrarily small U' by the choice of U. This, together with (6.4), proves that $|X^{\hat{}\tilde{}}| \cong |X^{\tilde{}\hat{}}| \equiv S'$ is connected i.e. $\hat{X}$ is integral. Now the proof of the converse is easy. □

7. The second main theorem.

The purpose here is to take off the condition of contraction (iv) in the first main theorem. We can not get rid of the condition (iii) of non-crush. This condition is not preserved by the canonical lifting to blowings up. Indeed, the example of Osgood (cf. [**KK**], p. 188) is a non-crush morphism, whose canonical lifting induced by the point blowing up of the target space is a crush. So we adopt a more stable condition, the formal version of Gabrièlov's rank condition, which is stronger than the condition of non-crush. Furthermore, we are obliged to assume adicity of the morphism and smoothness of the target space outside the core of the completion to assure the existence of a lifting to desingularization.

Theorem II. *Suppose that $S \subset |X|$ and $T \subset |Y|$ be analytic subsets of reduced complex spaces and $\boldsymbol{\Phi} : \hat{X}_{|S} \to \hat{Y}_{|T}$ a formal morphism between the completions satisfying the following*:

(i) *S is a thin Moishezon subspace* (*with respect to the reduced complex structure*);

(ii) *X is globally integral along S*;

(iii) *There exist $\xi \in S$ and a local irreducible component $\boldsymbol{Z}_\xi$ of $\hat{X}_{|S,\xi}$ such that* $\operatorname{grk} \boldsymbol{\Phi}|\boldsymbol{Z}_\xi = \dim Y_\eta$ $(\eta = |\boldsymbol{\Phi}|(\xi))$;

(iv) *Y is equidimensional and Y is smooth outside T*;

(v) *$\boldsymbol{\Phi} : \hat{X}_{|S} \to \hat{Y}_{|T}$ is adic all over S*;

(vi) *$\boldsymbol{\Phi}$ is convergent at some $\xi_0 \in S$ along some irreducible component of $\hat{X}_{|S,\xi_o}$.*
Then $\boldsymbol{\Phi}$ is convergent everywhere on S.

By (3.7) and (6.5), the condition (ii) and (iii) imply that $\operatorname{grk} \boldsymbol{\Phi}|\boldsymbol{Z}_\xi = \dim Y_\eta$ $(\eta = |\boldsymbol{\Phi}|(\xi))$ for any $\xi \in S$ and any local irreducible component $\boldsymbol{Z}_\xi$ of $\hat{X}_{|S,\xi}$.

Proof. We equip analytic subsets with the reduced complex structures if necessary. In view of (1.1), convergence of $\boldsymbol{\Phi}$ and the condition (v) are not affected even if we replace T by the image of S: $T = |\Phi|(S)$. Then T is known to be also Moishezon (Theorem of Ueno-Iitaka, [**Ii**], §8, Remark 4).

Let us consider the set $\mathscr{A}$ of all the triples $\{S_\alpha, X_\alpha, V_\alpha\}$, where S_α is an irreducible component of S, V_α an open neighborhood of S_α and X_α an irreducible component of $X|V_\alpha$ with $S_\alpha \subset |X_\alpha|$ such that X_α is integral along S_α. The join of all X_α is a neighborhood of S in $|X|$.

By our assumption (vi), there exists $\{S_0, X_0, V_0\} \in \mathscr{A}$ such that $\boldsymbol{\Phi}$ is convergent at a point of S_0 along a local irreducible component of X_0. Let $I : X_0 \to X$ be the inclusion. The composition $\boldsymbol{\Phi}_0 \equiv \boldsymbol{\Phi} \circ \hat{I}_{S \cap |X_0|;S}$ is adic by (v). Now we apply the argument in [**I3**], §3. First by the theory of Moishezon spaces and Hironaka desingularization, shrinking Y if necessary, we have a blowing up $\Theta \in \mathbf{cs} : Y' \to Y$ with centre $D \subset Y$ such that Y' is smooth and all irreducible components of $T' \equiv |\Theta|^{-1}(T)$ are smooth and projective. By the assumption (iv) and by the nature of Hironaka desingularization ([**AHV**], cf. [**BM**]), D is a complex subspace of $\hat{Y}_{|T}$ (i.e. $\mathscr{I}_T \subset \sqrt{\mathscr{I}_D} \subset \hat{\mathcal{O}}_{Y|T}$). Since $\boldsymbol{\Phi}_0$ is adic, $C \equiv \boldsymbol{\Phi}_0^{-1}(D)$ is a complex subspace of $\hat{X}_{|S}$. Let $\Sigma \in \mathbf{cs} : X' \to X_0$ be the blowing up with centre $C \cap X_0$. It is known that the completion of an analytic blowing up along an analytic subspace whose reduction is included in the core is a formal blowing up ([**AT2**], III, Prop.6). If we put $S' \equiv |\Sigma|^{-1}(S \cap |X_0|)$, $\boldsymbol{\Phi}_0$ has the canonical lifting $\boldsymbol{\Phi}' \in \mathbf{fcs} : \hat{X}'_{|S'} \to \hat{Y}'_{|T'}$ by the universality of formal blowing up. Since $\boldsymbol{\Phi}_0$ and $\hat{\Sigma}_{S';S\cap|X_0|}$ are adic, $\boldsymbol{\Phi}'$ is also so. There exists an irreducible component S'_0 of $|\Sigma|^{-1}(S_0)$ such that $|\Sigma|(S'_0) = S_0$. We put $T_0 \equiv |\boldsymbol{\Phi}|(S_0)$ and $T'_0 \equiv |\boldsymbol{\Phi}'|(S'_0)$. Then $|\Theta|(T'_0) \subset T'$ is an irreducible projective variety.

Next, applying Grauert's theorem on weakly negative line bundle neighborhoods [**Gr**], we get a further blowing up $\Theta' : Y'' \to Y'$ whose centre D' is a thin complex subspace of T'_0 such that the strict transform T''_0 of T'_0 is exceptional in Y'' (cf. [**I3**], (3.2)). Since $\boldsymbol{\Phi}'$ is adic, $C' \equiv \boldsymbol{\Phi}'^{-1}(D')$ is an complex subspace of $\hat{X}'_{|S'}$. Let $\Sigma' \in \mathbf{cs} : X'' \to X'$ be the blowing up with centre C' and

$$\boldsymbol{\Phi}'' \in \mathbf{fcs} : \hat{X}''_{|S''} \to \hat{Y}''_{|T''} \quad (S'' \equiv |\Sigma'|^{-1}(S'), T'' \equiv |\Theta'|^{-1}(T'))$$

the canonical lifting of $\boldsymbol{\Phi}'$. We have the following commutative diagram.

$$\begin{array}{ccccc}
\hat{X}''_{|S''} & \xrightarrow{\hat{\Sigma}'_{S'';S'}} & \hat{X}'_{|S'} & \xrightarrow{\hat{\Sigma}_{S';S\cap|X_0|}} & \hat{X}_{0|S\cap|X_0|} \\
{\scriptstyle \boldsymbol{\Phi}''}\big\downarrow & & {\scriptstyle \boldsymbol{\Phi}'}\big\downarrow & & {\scriptstyle \boldsymbol{\Phi}_0}\big\downarrow \\
\hat{Y}''_{|T''} & \xrightarrow[\hat{\Theta}'_{T'';T'}]{} & \hat{Y}'_{|T'} & \xrightarrow[\hat{\Theta}_{T';T}]{} & \hat{Y}_{0|T}
\end{array}$$

Let $S''_0 \subset S''$ be the strict transformation of S'_0. It is easy to see that $|\boldsymbol{\Phi}''|(S''_0) \subset T''_0$. By (4.4) the generic rank of a formal blowing up on each local irreducible component is equal to the dimension of the target space so long as the target space is equidimensional. Since $\hat{X}_{0|S\cap|X_0|}$ is equidimensional by (ii) and (3.7), $\hat{\Sigma}_{S';S\cap|X_0|}$ and $\hat{\Sigma}'_{S'';S'}$ satisfy this condition. Similarly $\hat{\Theta}'_{T'';T'} \circ \hat{\Theta}_{T';T}$ preserves local dimension by the assumption (iv). $\boldsymbol{\Phi}$ has the full generic rank on all the local irreducible components by (iii) and (3.7). These prove that $\boldsymbol{\Phi}''$ has the full generic rank on all the local irreducible components as well as $\boldsymbol{\Phi}$ and hence $\boldsymbol{\Phi}''$ can not split through a thin subspace of $\hat{Y}''_{|T''}$ on any local irreducible component of $\hat{X}''_{|S''}$. Then $\boldsymbol{\Phi}''$ is nowhere a crush. Since

$$\Theta \circ \Theta' \circ \hat{I}_{T'';|Y''|} \circ \boldsymbol{\Phi}'' = \hat{I}_{T;|Y|} \circ \hat{\Theta}_{T';T} \circ \hat{\Theta}'_{T'';T'} \circ \boldsymbol{\Phi}'' = \hat{I}_{T;|Y|} \circ \boldsymbol{\Phi}_0 \circ \hat{\Sigma}_{S';S\cap|X_0|} \circ \hat{\Sigma}'_{S'';S'}$$

is convergent at a point $\xi'' \in |\Sigma \circ \Sigma'|^{-1}(\xi_0)$ along a local irreducible component, $\hat{I}_{T'';|Y''|} \circ \boldsymbol{\Phi}''$ is also so by (2.4). Then $\hat{I}_{T''_0;|Y''|} \circ \boldsymbol{\Phi}''$ is globally convergent on S''_0 by Remark (2.1). The composition $\boldsymbol{\Phi}_0 \circ \hat{\Sigma}_{S';S\cap|X_0|} \circ \hat{\Sigma}'_{S'';S'}$ is also so. By Gabrièlov's theorem, $\boldsymbol{\Phi}_0$ is convergent on S_0 as in the first part of the proof of (2.4), i.e. $\boldsymbol{\Phi}$ is convergent along X_0 at points of S_0 by (1.3) and (1.4).

We say that $\{S_\alpha, X_\alpha, V_\alpha\}$ and $\{S_\beta, X_\beta, V_\beta\}$ of $\mathscr{A}$ are *linked* if there exists a point $\zeta \in S_\alpha \cap S_\beta$ such that the germs $(X_\alpha)_\zeta$ and $(X_\beta)_\zeta$ share at least one local irreducible component. If this is the case, convergence of $\boldsymbol{\Phi}$ along X_α on S_α implies that on X_β on S_β by the same argument as above. Since X is integral along S, any two elements of $\mathscr{A}$ are joined by successively linked elements and the convergence on $\hat{X}_{0|S_0}$ influences all other $\hat{X}_{\alpha|S_\alpha}$ of $\{S_\alpha, X_\alpha, V_\alpha\} \in \mathscr{A}$. Then Theorem II follows from (1.3) and (1.4). $\square$

Appendix. Whitney-Shiota function.

Here we show an example of topologically 0-dimensional formal complex space which has no complex structure (cf. §1 for complex structure). Whitney [**W**] has constructed a convergent power series which cannot be transformed into a polynomial through analytic change of variables. Modifying this, Shiota has obtained the following. (There are a little improvement in the assertion and a little alteration of the form of the function.)

Example 8.1 ([**Sh**]). Let us put $O \equiv k\{x, y, z\}$ and $O^* \equiv k[[x, y, z]]$ ($k = \boldsymbol{R}$ or $\boldsymbol{C}$). If $\lambda(z)$ is a divergent power series in z with $\lambda(0) = 3$, then

$$f \equiv xy(y - x)\{y - (2 + z)x\}\{y - \lambda(z)x\} \in O^*$$

can never be transformed into a convergent power series by multiplication by an invertible element of O^* and by a formal change of variables.

This fact implies that $(\{0\}, A) \in \mathbf{fcs}$ with $A \equiv O^*/fO^*$ has no complex structure when $k = \boldsymbol{C}$. For, otherwise A would be the maximal-ideal-adic completion of an analytic algebra $B : A \equiv B^*$. The completion does not alter the dimension and the embedding dimension and the completion of a complete intersection ring is also so. Then B is expressed as $B \equiv k\{x', y', z'\}/f'k\{x', y', z'\}$ with $f' \in k\{x', y', z'\}$ with respect to new formal coordinate system (x', y', z'). Then there exist an automorphism $\boldsymbol{\varphi}$ of O^* and an invertible $g \in A$ such that $f \circ \boldsymbol{\varphi} = gf'$, a contradiction.

Since Shiota is concerned with real problems in [**Sh**], his geometric proof does not works in the complex case. Hence we note a proof which covers both cases.

Suppose that there exist an automorphism $\boldsymbol{\varphi}$ of O^* and an invertible element $g \in O^*$ such that $f \circ \boldsymbol{\varphi} \in gO$. Since O is normal, the formal decomposition of $(f \circ \boldsymbol{\varphi})g^{-1} \in O$ reduces to an analytic one by [**I3**], (E), that is to say, there exist $X, Y, U, V, W \in O$ such that

$$(f \circ \boldsymbol{\varphi})g^{-1} = XYUVW,$$

$$x \circ \boldsymbol{\varphi} = g_1 X, \quad y \circ \boldsymbol{\varphi} = g_2 Y, \quad (y - x) \circ \boldsymbol{\varphi} = g_3 U,$$

$$\{y - (2 + z)x\} \circ \boldsymbol{\varphi} = g_4 V, \quad \{y - \lambda(z)x\} \circ \boldsymbol{\varphi} = g_5 W.$$

(We may as well use the approximation theorem of Artin.)

We may assume that $g_i \in 1 + \mathfrak{m}^*$($\mathfrak{m}^*$: the maximal ideal of O^*). If $\boldsymbol{\psi}$ denote the linear part of $\boldsymbol{\varphi}$, then the linear part of $\boldsymbol{\varphi} \circ \boldsymbol{\psi}^{-1}$ is the identity. Hence we may assume that the linear part of $\boldsymbol{\varphi}$ is the identity from the first. Then by the inverse mapping theorem, we see that X, Y and $Z \equiv z$ form a regular system of parameters of O and O^*. Then every elements of O (resp. O^*) can be expressed as a convergent (resp. formal) power series in them. Eliminating x, y, z, we have

$$W(X, Y, Z) = \frac{g_2(X, Y, Z)}{g_5(X, Y, Z)} Y - \frac{g_1(X, Y, Z)}{g_5(X, Y, Z)} \lambda(Z - \mu(X, Y, Z))X$$

$$(Z - \mu(X, Y, Z) = z \circ \boldsymbol{\varphi}, \quad \mu(X, Y, Z) \in \mathfrak{m}^{*2}).$$

Then we see that

$$\frac{g_2(0,0,Z)}{g_5(0,0,Z)}, \quad -\frac{g_1(0,0,Z)}{g_5(0,0,Z)} \lambda(Z - \mu(0,0,Z)),$$

are convergent because they are respectively the coefficients of Y and X in W as an element of $k\{Z\}[[X, Y]]$. Similarly, from the convergence of U and V,

$$\frac{g_2(0,0,Z)}{g_3(0,0,Z)}, \quad \frac{g_1(0,0,Z)}{g_3(0,0,Z)}, \quad \frac{g_2(0,0,Z)}{g_4(0,0,Z)}, \quad \frac{g_4(0,0,Z)}{g_1(0,0,Z)}\{2 + Z - \mu(0,0,Z)\}$$

are convergent. Observing these six expressions, we see that $2 + Z - \mu(0,0,Z)$ and $\lambda(Z - \mu(0,0,Z))$ are convergent. This contradicts divergence of $\lambda(z)$ and Example is confirmed.

If we want to have an integral example, we have only to take the formal complex subspace of $(\{0\}, k[[x, y, z, u]])$ defined by $u^2 - f(x, y, z) \in k[[x, y, z, u]]$ with f used above.

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