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On strength of precipitousness of some ideals and towers

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Abstract. We show that the existence of a precipitous ideal over the successor of some limit cardinals implies the existence of some large cardinals, in the sense of consistency. Moreover we use the same technique to evaluate the consistency strength of precipitousness of Woodin's stationary tower.

§ 0. Introduction.

In this paper we mainly consider two kinds of objects: ideals over uncountable regular cardinals and stationary towers. The notion of precipitousness of ideals over uncountable regular cardinals was first introduced by Jech and Prikry in [**JP**].

Since then, the strength of the existence of precipitous ideals has been investigated. In particular, researchers are interested in the following two questions.

Question 1. How strong is *Con*(ZFC + "there is a precipitous ideal over κ"), for various uncountable regular κ's?

Question 2. How strong is *Con*(ZFC + "NS_κ is precipitous"), for various uncountable regular κ's?

For certain types of κ's, some results are already known.

Proposition 0.1 (Jech-Magidor-Mitchell-Prikry [**JMMP**]).

(1) *Con*(ZFC + "*There is a precipitous ideal*") *implies* *Con*(ZFC + "*There is a measurable cardinal*") *in* ZFC.

(2) *The following are equiconsistent*:

(a) ZFC + "*There is a measurable cardinal.*"

(b) ZFC + "*There is a precipitous ideal over* ω_1."

(c) ZFC + "NS_{ω_1} *is precipitous.*"

For $\kappa > \omega_1$, better lower bounds were given for the consistency strength of the precipitousness of NS_κ. These partially answer Question 2 above.

Proposition 0.2 (Gitik [**Gi**]). *The following are equiconsistent*:

(a) ZFC + $\exists\kappa(o(\kappa) \geq 2)$

(b) ZFC + "NS_{ω_2} *is precipitous.*"

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Proposition 0.3 (Jech [**Jec84**]).

(1) *If κ is a measurable cardinal and NS_κ is precipitous, then κ is at least of Mitchell order $\kappa^+ + 1$ in some inner model of* ZFC. *Consequently, Con*(ZFC $+ \exists\kappa$ ("*κ is measurable and NS_κ is precipitous*")) *is at least as strong as Con*(ZFC $+ \exists\kappa$ $(o(\kappa) \geq \kappa^+ + 1)$).

(2) *If $\kappa > \omega_2$ is a successor cardinal, and NS_κ is precipitous, then κ is at least of Mitchell order θ in some inner model of* ZFC (where θ is given by the following table.)

κ	θ
(a) λ^+ (λ is singular)	λ
(b) λ^+ (λ is weakly inaccessible)	$\lambda + 1$
(c) λ^{++} (λ is regular)	$\lambda + 1$
(d) λ^{++} (λ is regular uncountable)	$\lambda + 2$

Now we consider Question 1 above. It is not very interesting in the case that κ is a limit cardinal. For, if κ is measurable, then the dual ideal of a measure over κ is trivially a precipitous ideal, and therefore we cannot expect to improve the lower bound given in Proposition 0.1. We consider the case that κ is a successor cardinal. In the case $\kappa = \lambda^+$ where λ is a regular cardinal, the following is already known.

Proposition 0.4 (Jech-Magidor-Mitchell-Prikry [**JMMP**]). *Let λ be a regular cardinal, κ a measurable cardinal above λ. Then*:

$$\Vdash_{Col(\lambda,<\kappa)} \text{“}\kappa = \lambda^+, \lambda \textit{ is regular, and } \kappa \textit{ carries a precipitous ideal”}.$$

Consequently, in ZFC, *Con*(ZFC + "*There is a measurable cardinal*") *is at least as strong as Con*(ZFC $+ \exists\kappa$ ("*$\kappa = \lambda^+$ for some regular cardinal λ and κ carries a precipitous ideal*")) (*by Proposition* 0.1 (1), *these two are equiconsistent*).

Thus the case when λ is singular is a problem. We are also interested in the case when λ itself is large. In these cases it is possible that some larger cardinals are implied in the sense of consistency. Our results on ideals are concerned with such cases. The main theorem of this paper is the following.

Theorem 0.5. *Suppose our back ground theory is* ZFC + "OR *is measurable*". *Then the following are equiconsistent*:

(1) *There exists a Woodin cardinal.*

(2) *NS_κ is precipitous for some κ which is the successor of a singular cardinal* (*or of a weakly compact cardinal*).

(3) *There exists an elementary embedding of **V** to some transitive class **M**, defined within some generic extension **N*** of ***V**, with critical point κ which is the successor of a singular cardinal* (*or of a weakly compact cardinal*).

In the middle of the 1980's, Martin, Steel, and Woodin established their celebrated work on the axiom of determinacy. One of the key tools in their work was the method of *stationary towers*, which was first introduced by Woodin. A stationary tower is a notion of forcing which gives a directed system of many ***V***-ultrafilters, and a direct limit of many ultrapowers of ***V***, like extenders, but in generic extensions. Thus precipitousness of stationary towers is also considered. The following was one of the breakthroughs in the proof of the theorem of Martin-Steel-Woodin.

PROPOSITION 0.6 (Woodin [**W**]). *If δ is Woodin, then the (full) stationary tower $\boldsymbol{P}_\delta$ of height δ is precipitous. Moreover,*

$$\Vdash_{\boldsymbol{P}_\delta} \text{“}{}^{<\delta}Ult_G(\boldsymbol{V}) \cap \boldsymbol{V}[G] \subseteq Ult_G(\boldsymbol{V})\text{”}.$$

Steel has proved the "closure" property above needs a Woodin cardinal in the sense of consistency, under the assumption "OR is measurable". In this paper we show that even the precipitousness of a stationary tower needs a Woodin cardinal, under the same assumption. Someday this technical assumption might be dropped, by the development of inner model theory.

This paper is organized as follows. In §1, we review some properties of the *core model*, which are key tools in the proof of our theorems. In §2, we prove our main theorem above and exhibit some related results. In §3, we use the technique in §2 to derive our result on stationary towers.

Throughout this paper, we assume that our background theory is ZFC, and let $\boldsymbol{V}$ denote the (real) universe.

§ 1. The core model.

The core model $\boldsymbol{K}$ we mention here is the one introduced by Steel, and therefore is usually called the Steel core model. It is uniformly defined in ZFC, and is a generalization of other core models which have been considered before.

Instead of stating the long definition of $\boldsymbol{K}$, we exhibit only some known results about $\boldsymbol{K}$.

In the following proposition, "OR is measurable" denotes the third-order sentence which says that there exists a fine ultrafilter over OR which is closed under intersections of set-many classes. Of course, if κ is a measurable cardinal, then $\boldsymbol{V}_\kappa$ (associated with $\boldsymbol{V}_{\kappa+1}$, $\boldsymbol{V}_{\kappa+2}$ as the 2nd-order and the 3rd-order domain, respectively) satisfies "OR is measurable".

PROPOSITION 1.1. *In* ZFC *with assumptions that there is no inner model of* ZFC *with a Woodin cardinal and that* OR *is measurable, the proper class $\boldsymbol{K}$ called the Steel core model is (uniformly) defined and has the following property.*

(1) (Steel [**St1**]) *If $\boldsymbol{N}$ is a set generic extension of $\boldsymbol{V}$, then $\boldsymbol{N}$ has no inner model of* ZFC *with a Woodin cardinal, and $\boldsymbol{K}^{\boldsymbol{N}} = \boldsymbol{K}$ holds.*

(2) (Mitchell-Schimmerling [**MS**] for singular cardinals, Schimmerling-Steel [**SS**] for weakly compact cardinals) *If λ is a singular cardinal or a weakly compact cardinal in $\boldsymbol{V}$, then $(\lambda^+)^{\boldsymbol{K}} = \lambda^+$.*

(3) (Steel [**St1**] [**St2**]) *Whenever $j : \boldsymbol{K} \to \boldsymbol{M}$ is an elementary embedding defined within $\boldsymbol{V}$, $\boldsymbol{M}$ is transitive, and $crit(j) = \kappa$, then $\mathscr{P}^{\boldsymbol{K}}(\kappa) = \mathscr{P}^{\boldsymbol{M}}(\kappa)$.*

REMARK. It is conjectured that the assumption "OR is measurable" can be dropped, but for the present, without this we must replace each Woodin cardinal in the above statement by a strong cardinal.

In fact, most of our theorems in this paper will be shown as corollaries of the following lemma, the proof of which heavily depends on Proposition 1.1.

Lemma 1.2. *Suppose that there is no inner model of* ZFC *with a Woodin cardinal, and that* OR *is measurable. Let j be any elementary embedding of $\boldsymbol{V}$ into some transitive $\boldsymbol{M}$, which is defined within some set generic extension of $\boldsymbol{V}$, with the critical point κ. Then $\boldsymbol{K} \models$ "κ is a limit cardinal."*

Proof of Lemma 1.2. Suppose j of our assumption is defined within a set generic extension $\boldsymbol{N}$ of $\boldsymbol{V}$. By the usual arguement, κ is a regular cardinal in $\boldsymbol{V}$, and so in $\boldsymbol{K}$. Thus it is enough to show that κ is not a successor cardinal in $\boldsymbol{K}$. Assume $\kappa = (\lambda^+)^{\boldsymbol{K}}$. By the elementarity of j, the image of every element of $\boldsymbol{K}$ by j belongs to $\boldsymbol{K}^{\boldsymbol{M}}$, and $j \lceil \boldsymbol{K} : \boldsymbol{K} \to \boldsymbol{K}^{\boldsymbol{M}}$ is an elementary embedding. On the other hand, we know by Proposition 1.1(1), the core model $\boldsymbol{K}$ is also defined within $\boldsymbol{N}$, and $\boldsymbol{K}^{\boldsymbol{N}} = \boldsymbol{K}$ holds. Since j is defined within $\boldsymbol{N}$ (and so is $j \lceil \boldsymbol{K}^{\boldsymbol{N}}$), we see that $\mathscr{P}^{\boldsymbol{K}}(\kappa) = \mathscr{P}^{\boldsymbol{K}^{\boldsymbol{M}}}(\kappa)$ by applying Proposition 1.1(3) within $\boldsymbol{N}$. Hence the set of all well-ordered relations on λ computed in $\boldsymbol{K}$ and $\boldsymbol{K}^{\boldsymbol{M}}$ coincide with each other. By this we have $(\lambda^+)^{\boldsymbol{K}} = (\lambda^+)^{\boldsymbol{K}^{\boldsymbol{M}}}$. But again by the elementarity of j we have $j(\kappa) = j((\lambda^+)^{\boldsymbol{K}}) = (\lambda^+)^{\boldsymbol{K}^{\boldsymbol{M}}} > \kappa = (\lambda^+)^{\boldsymbol{K}}$. Contradiction. □

Remark. In fact, by similar arguement we can show that κ is strong limit and thus inaccessible in $\boldsymbol{K}$ (or even more), but it is not necessary for our purpose.

§ 2. The strength of precipitous ideals.

In this section we show our results on the consistency strength of the existence of precipitous ideals over κ, when κ is the successor of a limit cardinal. The definition of precipitous ideals can be seen in [**JP**].

Proof of Theorem 0.5. The direction (1) implies (2) follows from the following theorem which is due to Goldring, an improvement of the former results of Foreman, Magidor, and Shelah [**FMS**]:

Proposition 2.1 (Goldring [**Go**]). *Let κ be a regular uncountable cardinal, and suppose that $\lambda > \kappa$ is a Woodin cardinal. Then,*

$$\Vdash_{Col(\kappa,<\lambda)} \text{“}NS_\kappa \text{ is precipitous”},$$

where $Col(\kappa, < \lambda)$ denotes the Levy collapse which makes λ be the successor of κ.

Using this we can argue as follows: Assume (1) in our ground model, and let λ be a Woodin cardinal, $\kappa = \mu^+$ where μ is a singular or of a weakly compact cardinal less than λ, and G a $Col(\kappa, < \lambda)$-generic over $\boldsymbol{V}$. Then by the above Proposition $\boldsymbol{V}[G]$ satisfies (2), since in $\boldsymbol{V}[G]$ $\kappa = \mu^+$ holds and μ remains singular (or weakly compact) as $Col(\kappa, < \lambda)$ is $< \kappa$-closed.

It is clear that (2) implies (3). Now Assume (3), and $\kappa = \mu^+$. By Lemma 1.2, κ is a limit cardinal in $\boldsymbol{K}$. Then $\mu^+ > (\mu^+)^{\boldsymbol{K}}$ holds and by Proposition 1.1(2), there must be an inner model of (1). □

THEOREM 2.2. *In* ZFC, *Con*(ZFC + $\exists\kappa$ ("$\kappa = \lambda^+$ *for some* λ *which is* $\mathscr{P}^3(\lambda)$-*hypermeasurable, and* κ *carries a precipitous ideal*")) *is strictly stronger than Con*(ZFC + $\exists\kappa$ ("κ *is Woodin*")).

PROOF OF THEOREM 2.2. Let λ be $\mathscr{P}^3(\lambda)$-hypermeasurable and suppose that $\kappa = \lambda^+$ carries a precipitous ideal I. Let $j : \boldsymbol{V} \to \boldsymbol{M}$ be an elementary embedding defined within $\boldsymbol{V}$ such that $\boldsymbol{M}$ is transitive, $crit(j) = \lambda$, and $\mathscr{P}^3(\lambda) \subset \boldsymbol{M}$.

Since $\mathscr{P}^3(\lambda) = (\mathscr{P}^3(\lambda))^{\boldsymbol{M}}$, we have $\kappa = \lambda^+ = (\lambda^+)^{\boldsymbol{M}}$, and $\mathscr{P}^2(\kappa) = (\mathscr{P}^2(\kappa))^{\boldsymbol{M}}$. Since $I \in \mathscr{P}^2(\kappa)$, we have $I \in \boldsymbol{M}$. We also have, by an absoluteness argument, $\boldsymbol{M} \models$ "I is precipitous". On the other hand, since $(\mathscr{P}^2(\lambda))^{\boldsymbol{M}} = \mathscr{P}^2(\lambda)$, then measures over λ in $\boldsymbol{V}$ belong to $\boldsymbol{M}$, and hence λ is measurable in $\boldsymbol{M}$. Thus we have

$$\boldsymbol{M} \models \exists\rho < j(\lambda)(\text{"}\rho \text{ is measurable and } \rho^+ \text{ carries a precipitous ideal"}).$$

Then by the elementarity of j,

$$\boldsymbol{V} \models \exists\rho < \lambda(\text{"}\rho \text{ is measurable and } \rho^+ \text{ carries a precipitous ideal"}).$$

Now, since λ is measurable in $\boldsymbol{V}$, $\boldsymbol{V}_\lambda$ is a model of ZFC + "OR is measurable", and

$$\boldsymbol{V}_\lambda \models \exists\rho < \lambda\ (\text{"}\rho \text{ is measurable and } \rho^+ \text{ carries a precipitous ideal"}).$$

Then by Theorem 0.5, $\boldsymbol{V}_\lambda$ has an inner model with a Woodin cardinal. Since this inner model is a set in our ground model $\boldsymbol{V}$, we have the conclusion. □

In the last theorem, we can reduce the large cardinal assumption if the precipitous ideal is specified.

THEOREM 2.3. *In* ZFC, *Con*(ZFC + $\exists\kappa$ ("$\kappa = \lambda^+$ *for some* $\mathscr{P}^2(\lambda)$-*hypermeasurable cardinal* λ, *and* NS_κ *is precipitous*")) *is strictly stronger than Con*(ZFC + $\exists\kappa$ ("κ *is Woodin*")).

PROOF OF THEOREM 2.3. Let λ be $\mathscr{P}^2(\lambda)$-hypermeasurable, $\kappa = \lambda^+$, and suppose that NS_κ is precipitous. Let $j : \boldsymbol{V} \to \boldsymbol{M}$ be an elementary embedding defined within $\boldsymbol{V}$ such that $\boldsymbol{M}$ is transitive, $crit(j) = \lambda$, and $\mathscr{P}^2(\lambda) \subset \boldsymbol{M}$.

Since $\mathscr{P}^2(\lambda) = (\mathscr{P}^2(\lambda))^{\boldsymbol{M}}$, we have $\kappa = (\lambda^+)^{\boldsymbol{M}} = \lambda^+$, and $\mathscr{P}^{\boldsymbol{M}}(\kappa) = \mathscr{P}(\kappa)$. Then, by the absoluteness of clubness, $NS_\kappa^{\boldsymbol{M}} = NS_\kappa$ holds. The rest of the proof is completely the same as that of Theorem 2.2. □

REMARK.

(1) In fact, the proof of Theorem 2.2 (or 2.3) works even if λ is measurable and Σ_1^3-indescribable (or Σ_1^2-indescribable, respectively) which is weaker (in the sense of consistency) than a measurable cardinal of Mitchell order 2.

(2) As for the upper bound, one can argue as follows. Let κ be a supercompact cardinal, and λ be a Woodin cardinal above κ. By [**L**] there exists a forcing notion $\boldsymbol{P}$ of size κ such that in $\boldsymbol{P}$-generic extensions κ remains supercompact and is never destructed by any $< \kappa$-directed closed forcing. Then apply Proposition 2.1 in such a generic extension to obtain a desired model.

QUESTION. What is the exact consistency strength of our assumption of Theorems 2.2 and 2.3?

§ 3. Stationary towers and their precipitousness.

The definition of stationary towers and their elementary properties can be seen in [**W**]. For a limit ordinal δ, let us denote $\boldsymbol{P}_\delta$ the (full) stationary tower of height δ.

Now we state our theorem on precipitousness of stationary towers.

Theorem 3.1. *In* ZFC, *Con*(ZFC + "OR *is measurable*" + $\exists\delta$ ("δ *is inaccessible and* $\boldsymbol{P}_\delta$ *is precipitous*")) *implies* *Con*(ZFC + $\exists\kappa$ ("κ *is Woodin*")).

Proof of Theorem 3.1. Let δ be an inaccessible and suppose that OR is measurable and $\boldsymbol{P}_\delta$ is precipitous. Let λ be a singular cardinal $< \delta$. Let $\kappa = \lambda^+$.

Claim. *κ is stationary in $\mathscr{P}(\kappa)$ (and thus $\kappa \in \boldsymbol{P}_\delta$), and $\kappa \Vdash_{\boldsymbol{P}_\delta} crit(j) = \kappa$ holds, where j denotes the canonical elementary embedding of $\boldsymbol{V}$.*

Proof of Claim. For any function $f : {}^{<\omega}\kappa \to \kappa$, there is an ordinal $\gamma < \kappa$ which is closed under f, since κ is regular and uncountable. This shows that $\kappa \in \boldsymbol{P}_\delta$.

Let G be any $\boldsymbol{P}_\delta$-generic filter over $\boldsymbol{V}$ such that $\kappa \in G$. Let *Ult* denote the generic ultrapower of $\boldsymbol{V}$ by G and $j : \boldsymbol{V} \to Ult$ the canonical elementary embedding. Note that each element of *Ult* is represented by some function h in $\boldsymbol{V}$ defined on $\mathscr{P}(X)$ where X is some set in $\boldsymbol{V}_\delta$. This element is denoted by $[h]_G$.

By a normality arguement, $[\mathrm{id} \restriction \mathscr{P}(\kappa)]_G = j''\kappa$ holds. Thus, the trivial statement $\forall\alpha \in \kappa(\alpha \in \kappa)$ shows that $\kappa \Vdash_{\boldsymbol{P}_\delta} j''\kappa \in j(\kappa)$. Suppose in *Ult*, $j''\kappa \in j(\kappa)$. Then $j''\kappa$ must be an ordinal, and $j \restriction \kappa$ cannot have a leap. Therefore $crit(j) \geq \kappa$, and $j''\kappa = \kappa$. But now we have $\kappa \in j(\kappa)$ and thus $crit(j) = \kappa$. □(Claim)

Now Theorem 3.1 turns out a corollary of Lemma 2.1, since $\kappa \in \boldsymbol{P}_\delta$ forces the existence of an elementary embedding of $\boldsymbol{V}$ whose critical point is the successor of a singular cardinal. □(Theorem 3.1)

Theorem 3.1 implies that the precipitousness of $\boldsymbol{P}_\delta$ and its "closure" property as in Proposition 0.6 are close in their consistency strength. We ask the following question.

Question: Does the precipitousness of $\boldsymbol{P}_\delta$ imply that

$$\Vdash_{\boldsymbol{P}_\delta} \text{"}{}^{<\delta}Ult_G(\boldsymbol{V}) \cap \boldsymbol{V}[G] \subseteq Ult_G(\boldsymbol{V})\text{"?}$$

For more general types of towers (argued in [**B**]), this is not true. The following is a counterexample (by Burke): Let δ be Woodin, and consider the tower which consists of all sets in $\boldsymbol{P}_\delta$ whose elements have size $< \aleph_\omega$. It turns out to be precipitous, but its generic ultrapowers are not even ω-closed.

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