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Asymptotics in time for nonlinear nonlocal Shrödinger equations with a source

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Abstract. We consider the asymptotic behavior of the solution of the Cauchy problem for the nonlinear nonlocal Shrödinger equation (NNS) with a source. The source in the NNS equation makes essential alterations to the asymptotic behavior. We study the cases of both small and large initial data.

1. Introduction

In this paper we study the asymptotic behavior for large time of solutions of the Cauchy problem for the nonlinear nonlocal Schrödinger (NNS) equation, proposed in [**1**], with a source:

$$\begin{cases} iu_t + |u|^2u + iKu = f(x,t) & t > 0, \quad x \in \boldsymbol{R}, \\ u(x,0) = \bar{u}(x). \end{cases} \tag{NNS}$$

Here the linear pseudo differential operator Ku is defined by

$$Ku = \frac{1}{2\pi}\int_R e^{ipx}K(p)\hat{u}(p,t)\,dp$$

where $K(p)$ is the symbol of the operator Ku and $\hat{u}(p,t)$ is the Fourier transform of the function $u(x,t)$

$$\hat{u}(p,t) = \int_R e^{-ipx}u(x,t)\,dx.$$

The (NNS) equation describes wave propagation in plasma physics, nonlinear optics, chemical kinetics, hydrodynamics [**2–5**]. The (NNS) equation is a very general nonlinear equation and due to the choice of the operator K it includes a number of well-known equations. For example, when $K(p) = (i + a^2)p^2$, $(a \in \boldsymbol{R})$, the (NNS) equation is the generalized Landau-Ginzburg equation [**6**]. If $K(p) = -iq(p) + ip^2$ where $q(\cdot)$ is a suitable real valued function, the (NNS) equation is the generalized nonlocal nonlinear Schrödinger equation and describes different processes, connected with the dissipation or pumping of energy [**7**].

Without a source (i.e. $f(x,t) = 0$) the Cauchy problem (NNS) was studied in papers [**7**], [**8**]. The local and global existence of solutions and the smoothing property of solutions were proved. In the case of the dissipative operator K and small initial data

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the asymptotics for large time of solutions of (NNS) without a source were studied in [**12**].

The aim of the present paper is to study a symptotics as $t \to +\infty$ of the solutions of the Cauchy problem (NNS) with a source. We use estimates in Sobolev spaces as in paper [**13**]. The source in (NNS) makes essential alterations to the asymptotic behavior.

Before stating our results, we give *notation and function spaces*. We denote

$$m_p = \min(1, |p|), \qquad M_p = \max(1, |p|), \qquad \partial_p u = \frac{\partial}{\partial p} u$$

and

$$\|\cdot\|_k^2 = \int_{\mathbf{R}_1} |x|^{2k} |\cdot|^2\, dx, \quad \|\cdot\|_{n,k}^2 = \int_{\mathbf{R}_1} m_p^{2n} M_p^{2k} |\cdot|^2\, dp.$$

We introduce some function spaces: $X = \{\phi(x) \in L^\infty(\boldsymbol{R}) \cap H^{0,1/2+\gamma}(\boldsymbol{R})\}$ with $\|\cdot\|_X = \|\cdot\|_{L^\infty(\boldsymbol{R})} + \|\cdot\|_{1/2+\gamma}$, here $H^{0,s} = \{f \in S'; \|(1+x^2)^{s/2} f\|_{L^2} < \infty\}$. And $Z = \{\phi \in L^\infty(\boldsymbol{R}) \cap L^2(\boldsymbol{R})\}$ with $\|\cdot\|_Z = \|\cdot\|_{L^\infty(\boldsymbol{R})} + \|\cdot\|_{L^2(\boldsymbol{R})}$.

Different positive constants might be denoted by the same letter C and sufficiently small positive constants by the letters γ, γ'.

We now state our results in this paper. There are three main cases of large time asymptotic behavior of solutions.

In the first case the asymptotics of solutions is determined by the source. The following statement is valid.

Theorem 1. *Assume that the symbol $K(p)$ is dissipative, that is*

$$\operatorname{Re} K(p) \geq \theta m_p^\delta, \tag{1.1}$$

for all $p \in \boldsymbol{R}$, where $\theta > 0$, $0 < \delta < 1$. Suppose that the source $f(x,t)$ satisfies the following condition for $p \in \boldsymbol{R}$, $t > 0$

$$\hat{f} = \frac{g(p)}{(1+t)^\alpha} + \psi(p,t), \quad \|g\|_X \leq \varepsilon, \quad \|\psi\|_X \leq \frac{\varepsilon}{(1+t)^{\alpha+\eta}}, \tag{1.2}$$

where $\varepsilon > 0$ is sufficiently small, $\eta > 0$, $\alpha \in (\max(0, 1 - 1/2\delta), 1/\delta)$.

Suppose that the initial data are small enough, that is the following estimate is valid

$$\|\hat{\bar{u}}\|_X \leq \varepsilon. \tag{1.3}$$

Then the solution $u(x,t)$ of the Cauchy problem (NNS) *has the asymptotics as $t \to \infty$ uniformly with respect to $x \in \boldsymbol{R}$*

$$u(x,t) = \frac{-i}{2\pi t^\alpha} \int_R \frac{g(p) e^{ipx}}{K(p)}\, dp + O(t^{-\alpha-\rho}), \tag{1.4}$$

where $\rho > 0$ is some constant.

In the second case the source decays in time more rapidly and interacts with the operator K. Therefore the asymptotic behavior of the solution is of intermediate character. The solution decays slower than the source and the response from the initial data.

The following theorem is proved.

THEOREM 2. *Let the operator K satisfy the condition* (1.1) *with* $\delta \in (1,2)$ *and*

$$K(p) = \omega|p|^{\delta} + O(|p|^{\delta+\sigma}), \quad \textit{for } |p| < 1, \tag{1.5}$$

where $\omega > 0$, $\sigma > 0$.

Suppose that the source $f(x,t)$ *satisfies estimates for all* $t > 0$

$$\|\hat{f}\|_X \le \frac{\varepsilon}{(1+t)^{\alpha}}, \quad \hat{f}(0,t) = \frac{\Theta}{t^{\alpha}} + O((1+t)^{-\alpha-\eta}), \tag{1.6}$$

where ε, $\Theta > 0$ *are sufficiently small,* $\eta > 0$ *and* $\alpha \in (3/2 - 1/\delta, 1)$, *and*

$$|\hat{f}(p,t) - \hat{f}(0,t)| \le \frac{C|p|^{\nu}}{(1+t)^{\alpha}} \quad \textit{for } |p| \le 1, \tag{1.7}$$

where $0 < \nu < \delta - 1$.

Assume that the initial data are small enough in the sense that the estimate (1.3) *is valid.*

Then the solution $u(x,t)$ *of the Cauchy problem* (NNS) *has the asymptotics as* $t \to +\infty$ *uniformly with respect to* $x \in \boldsymbol{R}$

$$u(x,t) = \frac{-i\Theta}{\pi\omega^{1/\delta}} \frac{1}{t^{\beta}} \int_0^1 \frac{dz}{z^{\alpha}(1-z)^{1/\delta}} \int_0^{+\infty} \cos\left(y \frac{x}{(\omega t)^{1/\delta}(1-z)^{1/\delta}}\right) e^{-y^{\delta}}\, dy + O(t^{-\beta-\rho}), \tag{1.8}$$

where $\beta = \alpha + 1/\delta - 1$ *and* $\rho > 0$.

In the third case the source decays sufficiently rapidly and does not play a role in the character of asymptotic behavior of solution. For this case we prove the following results.

THEOREM 3. *Let the symbol* $K(p)$ *satisfy conditions* (1.1), (1.5) *with* $\delta \in (0,2)$ *and*

$$|\partial_p K| \le \lambda m_p^{\delta-1}, \quad \textit{for all } p \in \boldsymbol{R}, \tag{1.9}$$

where $\lambda > 0$. *Assume that the right-hand-side* $f(x,t)$ *satisfies conditions with* $\alpha > \max(1, 1/\delta)$

$$\|\hat{f}\|_X \le \frac{\varepsilon}{M_t^{\alpha}}, \quad \hat{f}(0,t) = \frac{\Theta}{M_t^{\alpha}} + O(M_t^{-\alpha-\eta}), \quad \|\partial_p \hat{f}\|_{1/2-\nu,1/2+\gamma} \le C M_t^{\nu/\delta-\alpha}, \tag{1.10}$$

where ε, Θ, $\nu > 0$ *are sufficiently small,* $\eta > 0$. *Assume that initial data satisfy condition* (1.3) *and*

$$\partial_p \hat{\bar{u}} \in H^{0,1/2+\gamma} \tag{1.11}$$

Then the solution $u(x,t)$ *of the Cauchy problem* (NNS) *has the following asymptotics as* $t \to +\infty$ *uniformly with respect to* $x \in \boldsymbol{R}$

$$u(x,t) = A \frac{1}{t^{1/\delta}} \int_0^{+\infty} \cos\left(y \frac{x}{(\omega t)^{1/\delta}}\right) e^{-y^{\delta}}\, dy + O(t^{-(1/\delta)-\rho}), \tag{1.12}$$

where $\rho > 0$ *and*

$$A = \frac{1}{\omega^{1/\delta}\pi}\left(i\int_0^{+\infty}(w(0,\tau) - \hat{f}(0,\tau))\,d\tau + \hat{u}(0)\right),$$

with $w(0,t) = \int_{\mathrm{R}} |u(x,t)|^2 u(x,t)\,dx$.

In the previous theorems we consider the case of sufficiently small initial data. This smallness condition enables us to prove the global existence of solutions and the necessary time decay estimates. If the initial data are not small (we will call such initial data large data), the solution of the Cauchy problem can blow up in finite time. It is known [**7**], [**8**] that the global existence of the solution of the Cauchy problem can be obtained under the condition that the operator K is strongly dissipative. Also it is interesting to obtain asymptotics of solutions of the Cauchy problem with large initial data.

We consider the case of strongly dissipative operator K. As in the paper [**14**] we will use the basic estimate of the solution in L^2 norm. However, for the case of large initial data, we can not obtain the estimate of this L^2 norm decaying in time. Therefore, decay estimates of the solution in the case of large initial data can be obtained for more rough condition $\delta < 1$ (instead of $\delta < 2$) on the symbol of the operator K. We can not say that the condition is essential, or it is caused only by our approach.

We will prove the following theorems. For the first case we have

Theorem 4. *Suppose that the symbol* $K(p)$ *is strongly dissipative, so that for all* $p \in \boldsymbol{R}$

$$\operatorname{Re} K(p) \geq \theta m_p^{\delta} M_p^{\beta}, \tag{1.13}$$

where $\theta > 0$, $0 < \delta < 1$, $\beta > 1$, *and that the source* $f(x,t)$ *satisfies the condition* (1.2) *with* $1 < \alpha < 1/\delta$ *and*

$$g(p) \in Z, \quad \sup_{t>0}\,(1+t)^{\alpha+\eta}\|\psi(p,t)\|_Z \leq C.$$

Then the solution $u(x,t)$ *of the Cauchy problem* (NNS) *with any large initial data* $\hat{u} \in Z$ *has the asymptotics* (1.4) *as* $t \to \infty$.

For the third case we obtain (the second case does not appear since we put restriction on δ $0 < \delta < 1$):

Theorem 5. *Let symbol* $K(p)$ *satisfy conditions* (1.5), (1.13) *and*

$$|\partial_p K| \leq \lambda m_p^{\delta-1} M_p^{\beta-1}, \quad \textit{for all } p \in \boldsymbol{R}, \tag{1.14}$$

where $\lambda > 0$. *Suppose that the source* $f(x,t)$ *satisfies the following estimates with* $\alpha > 1/\delta$

$$\partial_p \hat{f} \in H^{0,1/2+\gamma}, \quad \sup_{t>0}(1+t)^{\alpha}\|\hat{f}\|_Z < C, \tag{1.15}$$

$$\hat{f}(0,t) = \frac{\Theta}{(1+t)^{\alpha}} + O((1+t)^{-\alpha-\eta}), \quad \textit{for all } t > 0, \tag{1.16}$$

where $\Theta > 0$, $\eta > 0$.

The initial data are large and such that

$$\hat{\bar{u}} \in Z, \quad \partial_p \hat{\bar{u}} \in H^{0,1/2+\gamma}. \tag{1.17}$$

Then the solution $u(x,t)$ *of the Cauchy problem* (NNS) *has the asymptotics* (1.12) *as* $t \to +\infty$.

We organize our paper as follows. In section 2 we give some preliminary results. First we mention a local existence result in Theorem 0 without giving a proof. Further we prove Lemma 1 which establish time decay estimates of the solution for small initial data and is necessary in section 3 in proving Theorems 1-3. Then, in Lemma 2 we prove time decay estimates of the solution for large initial data. In section 3 we give proofs of the theorems.

2. Preliminaries

By using the standard successive approximation method it is easy to prove the following theorem.

Theorem 0 (local existence in time). *Suppose that the symbol* $K(p)$ *satisfies condition*

$$\operatorname{Re} K(p) \geq 0 \quad \text{for all } p \in \boldsymbol{R}.$$

Then there exists the unique solution of the Cauchy problem (NNS) *for any* $\bar{u}$, $f \in H^k$, $k > 1/2$ *on some interval* $[0,T]$ *such that*

$$u(x,t) \in C^0([0,T], H^k(\boldsymbol{R})),$$

where $T > 0$ *depends on the sizes of the data* $\bar{u}$ and f.

We denote

$$w(p,\tau) = \frac{1}{4\pi^2} \iint_{\mathrm{R}} \hat{u}(p-q,\tau)\hat{u}^*(r-q,\tau)\hat{u}(r,\tau)\,dq\,dr.$$

Lemma 1. *Let the operator* K *be dissipative, that is*

$$\operatorname{Re} K(p) \geq \theta m_p^{\delta}, \quad \text{for all } p \in \boldsymbol{R}, \tag{2.1}$$

where $\delta \in (0,2)$, $\theta > 0$.

Suppose that the source $f(x,t)$ *satisfies the estimate for all* $t > 0$

$$\|\hat{f}(t)\|_X \leq \frac{\varepsilon}{(1+t)^{\alpha}}, \tag{2.2}$$

where $\varepsilon > 0$ *is sufficiently small and*

if $\delta \in (0,1)$, *then* $\alpha \in (\max(0, 1-1/2\delta), 1/\delta) \cup (1/\delta, +\infty)$

if $\delta \in (1,2)$, *then* $\alpha \in (3/2 - 1/\delta, 1) \cup (1, +\infty)$

if $\delta = 1$, *then* $\alpha \in (1, +\infty)$.

Assume that the initial data are small in the sense

$$\|\hat{\bar{u}}\|_X \leq \varepsilon. \tag{2.3}$$

Then for the solution $u(x,t)$ of the Cauchy problem (NNS) *and $w(p,t)$ the following estimates are valid for all $t > 0$*

$$\|\hat{u}(t)\|_k < \frac{\sqrt{\varepsilon}}{(1+t)^{\nu_k}}, \tag{2.4}$$

$$\|w(t)\|_X \leq \frac{\varepsilon}{(1+t)^{\alpha_1}}, \tag{2.4'}$$

where

$$\nu_k = \min\left(\frac{1+2k}{2\delta}, \alpha - 1 + \frac{1+2k}{2\delta}, \alpha - \mu\right),$$

$$\alpha_1 = 2\nu_0 + \nu_{1/2-\gamma},$$

$k \in [-1/2+\gamma, 1/2+\gamma]$ *and small* $\mu > 0$.

Proof. We prove (2.4) by the contradiction. By virtue of (2.3) the estimate (2.4) is valid at $t = 0$. Suppose that at some $T > 0$ the estimate (2.4) is violated. Then by continuity we have

$$\|\hat{u}(t)\|_k \leq \frac{\sqrt{\varepsilon}}{(1+t)^{\nu_k}}. \tag{2.5}$$

for $t \in [0, T]$.

We consider the (NNS) equation on $[0, T]$. Taking the Fourier transform, we have

$$\hat{u}(p,t) = e^{-K(p)t}\hat{\bar{u}}(p) + i\int_0^t e^{-K(p)(t-\tau)}(w(p,\tau) - \hat{f}(p,\tau))\,d\tau. \tag{2.6}$$

Denote $h(p,t) = w - \hat{f}$.

Multiplying (2.6) by $|p|^k$ and taking the L^2 norm, we obtain

$$\|\hat{u}\|_k \leq C\left(\|\hat{\bar{u}}e^{-K(p)t}\|_k + \int_0^t \|he^{-K(p)(t-\tau)}\|_k\,d\tau\right). \tag{2.7}$$

Making a change of the variable $y^\delta = |p|^\delta t$ and using (2.1), (2.3), we obtain the estimate of the first term in the right-hand side of (2.7)

$$\|\hat{\bar{u}}e^{-K(p)t}\|_k \leq \|\hat{\bar{u}}\|_{L^\infty}\sqrt{\int_{|p|\leq 1} e^{-2\theta|p|^\delta t}|p|^{2k}\,dp} + C\|\hat{\bar{u}}\|_l e^{-\theta t} \leq C\frac{\varepsilon}{(1+t)^{(1+2k)/2\delta}}, \tag{2.8}$$

where $l = \min(0,k)$. To estimate the second term in the right-hand side of (2.7) we need a number of preliminary estimates.

Using the following estimates

$$\left\| \int_{\mathbf{R}} \phi(p-q)\psi(q)\,dq \right\|_{L^2} \le \|\phi\|_{L_2}\|\psi\|_{L^1},$$

$$|p|^l \le C(|p-q|^l + |q-r|^l + |r|^l),$$

$$\|\hat{u}\|_{L^1} = \int_{\mathbf{R}} |\hat{u}| \sqrt{\frac{|p|^{1-2\gamma}+|p|^{1+2\gamma}}{|p|^{1-2\gamma}+|p|^{1+2\gamma}}}\,dp$$

$$\le \sqrt{\int_{\mathbf{R}} \frac{dp}{|p|^{1-2\gamma}+|p|^{1+2\gamma}}} \sqrt{\int_{\mathbf{R}} |\hat{u}|^2(|p|^{1-2\gamma}+|p|^{1+2\gamma})\,dp}$$

$$\le C(\|\hat{u}\|_{1/2-\gamma} + \|\hat{u}\|_{1/2+\gamma}),$$

we have

$$\|w(t)\|_{L^\infty(\mathbf{R})} \le C\|\hat{u}\|_0^2\|\hat{u}\|_{L^1} \le C\|\hat{u}\|_0^2(\|\hat{u}\|_{1/2-\gamma} + \|\hat{u}\|_{1/2+\gamma}), \tag{2.9}$$

and

$$\|w(t)\|_l \le C\|\hat{u}\|_l\|\hat{u}\|_{L^1}^2 \le C\|\hat{u}\|_l(\|\hat{u}\|_{1/2-\gamma} + \|\hat{u}\|_{1/2+\gamma})^2. \tag{2.10}$$

Since $v_{1/2-\gamma} \le v_{1/2+\gamma}$, substituting (2.5) in (2.9) and (2.10), we get

$$\|w(t)\|_{L^\infty} \le C\frac{\varepsilon^{3/2}}{(1+t)^{\alpha_1}} \tag{2.11}$$

and

$$\|w(t)\|_l \le C\frac{\varepsilon^{3/2}}{(1+t)^{\alpha_2}}, \tag{2.12}$$

where $\alpha_1 = 2v_0 + v_{1/2-\gamma}$ and $\alpha_2 = v_l + 2v_{1/2-\gamma}$. It is easy to see that $\alpha_1 \le \alpha_2$.
Denote $\alpha_3 = \min(\alpha_1, \alpha)$. Then we obtain from (2.2), (2.11), (2.12)

$$\|h(t)\|_{L^\infty} \le C(\|f\|_{L^\infty} + \|w\|_{L^\infty}) \le C\frac{\varepsilon^{3/2}}{(1+t)^{\alpha_3}}, \tag{2.13}$$

$$\|h(t)\|_l \le C(\|f\|_l + \|w\|_l) \le C\frac{\varepsilon^{3/2}}{(1+t)^{\alpha_3}}. \tag{2.14}$$

Using (2.1) (2.13) (2.14), we estimate the second term in the right-hand-side of (2.7)

$$\int_0^t \|e^{-K(p)(t-\tau)}h(p,\tau)\|_k\,d\tau \le C\Bigg(\int_0^{t/2} \|h(\tau)\|_{L^\infty}\sqrt{\int_{|p|\le 1} e^{-2\theta|p|^\delta(t-\tau)}|p|^{2k}\,d\tau}$$

$$+ \int_{t/2}^t \|h(\tau)\|_{L^\infty}\sqrt{\int_{|p|\le 1} e^{-2\theta|p|^\delta(t-\tau)}|p|^{2k}\,d\tau}$$

$$+ \int_0^t \|h(\tau)\|_l e^{-\theta(t-\tau)}\,d\tau\Bigg)$$

$$\leq C\varepsilon\left(\int_0^{t/2} \frac{1}{(t-\tau)^{(1+2k)/2\delta}(1+\tau)^{\alpha_3}}\,d\tau + \int_{t/2}^{t} \frac{1}{(t-\tau)^{\min((1+2k)/2\delta,\,1-\mu)}(1+\tau)^{\alpha_3}}\,d\tau\right)$$

$$\leq C\varepsilon \frac{1}{(1+t)^{\hat{\nu}_k}} \tag{2.15}$$

where $\hat{\nu}_k = \min((1+2k)/2\delta, \alpha_3 + (1+2k)/2\delta - 1, \alpha_3 - \mu)$, $\mu > 0$ is sufficiently small.

Easily see that

$$\hat{\nu}_k = \nu_k. \tag{2.15$'$}$$

Indeed, if $\alpha \in (\max(0, 1 - 1/2\delta), 1/\delta)$ and $\delta \in (0, 1)$, then $\alpha_1 = 2\min(1/2\delta, 1/2\delta + \alpha - 1, \alpha - \mu) + \alpha - \mu > \alpha$ and therefore $\alpha_3 = \alpha$ and so we have (2.15$'$). If $\alpha \in (3/2 - 1/\delta, 1)$ and $\delta \in (1, 2)$, then $\alpha_1 = 3\alpha + 2/\delta - 3 - \gamma > \alpha$ and so we get (2.15$'$). When $\alpha > \max(1, 1/\delta)$ and $\delta \in (0, 2)$, $\alpha_1 = 2/\delta - \gamma > \max(1, 1/2\delta)$ and therefore $\nu_k = \hat{\nu}_k = (1+2k)/2\delta$. Substitution of estimates (2.8), (2.15), (2.15$'$) in (2.7) yields for $t \in [0, T]$

$$\|\hat{u}\|_k \leq C\varepsilon\left(\frac{1}{(1+t)^{(1+2k)/2\delta}} + \frac{1}{(1+t)^{\hat{\nu}_k}}\right) < \frac{\sqrt{\varepsilon}}{(1+t)^{\nu_k}}$$

This contradiction proves estimate (2.4) for all $t > 0$. From (2.11), (2.12) and (2.4) we have (2.4$'$). This completes the proof. □

Remark. Theorem 0 gives us the solution of the Cauchy problem (NNS) on some interval $[0, T]$. By the standard prolongation argument we obtain easily the global solution from the estimates of the Lemma 1.

Lemma 2. *Suppose that the operator K is strongly dissipative, that is*

$$\operatorname{Re} K(p) \geq \theta m_p^{\delta} M_p^{\beta}, \quad \text{for all } p \in \mathbf{R}, \tag{2.16}$$

where $\delta \in (0, 1)$, $\theta > 0$, $\beta > 1$, and the right-hand-side $f(x, t)$ of the equation satisfies the estimate with $\alpha > 1$

$$\sup_{t>0}(1+t)^{\alpha}\|\hat{f}\|_Z < C. \tag{2.17}$$

Then for the solution $u(x, t)$ of the Cauchy problem (NNS) *with any initial data $\hat{u} \in Z$ we have*

$$\sup_{t>0} m_t^{1/2-\gamma'}\|\hat{u}(t)\|_X \leq C \tag{2.17$'$}$$

and for any sufficiently small $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that,

$$\|\hat{u}(t)\|_{1/2-\gamma, 1/2+\gamma} < \frac{\varepsilon}{(1+t-T_\varepsilon)^{\min(\alpha, 1/\delta)-\gamma'}}, \qquad t \geq T_\varepsilon \tag{2.18}$$

where $\gamma, \gamma' > 0$ are sufficiently small.

Proof. Arguing in the same way as in the paper [**8**] it is easy to prove that the Cauchy problem (NNS) has the unique solution $u(x,t)$ and

$$u(x,t) \in C^{\infty}((0,+\infty); H^{\infty}(\boldsymbol{R})) \cap C^0([0+\infty); L^2(\boldsymbol{R})).$$

Since

$$2\|f(t)\|_{L^2}\|u(t)\|_{L^2} \le \left(\|f(t)\|_{L^2}^2(1+t)^{1+\gamma} + \frac{\|u(t)\|_{L^2}^2}{(1+t)^{1+\gamma}}\right),$$

multiplying the (NNS) equation by u^* and taking real part we have

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 \le C\left(-\int_{\mathrm{R}} \operatorname{Re} K(p)|\hat{u}|^2\,dp + \operatorname{Re} i\int_{\mathrm{R}} \hat{f}(p,t)\hat{u}(p,t)\,dp\right)$$

$$\le -C\int_{\mathrm{R}} m_p^{\delta} M_p^{\beta}|\hat{u}|^2\,dp + C\|f(t)\|_{L^2}^2(1+t)^{1+\gamma} + \frac{C\|u(t)\|_{L^2}^2}{(1+t)^{1+\gamma}}. \tag{2.19}$$

By virtue of (2.16), (2.17) the integration of (2.19) yields

$$\|u(t)\|_{L^2}^2 + C\int_0^t d\tau \int_{\mathrm{R}} m_p^{\delta} M_p^{\beta}|\hat{u}|^2\,dp$$

$$\le C\left(\|\bar{u}\|_{L^2}^2 + \int_0^{+\infty} \|f\|_{L^2}^2(1+\tau)^{1+\gamma}\,d\tau\right) \le C \tag{2.20}$$

and since C is independent of t, for any $\varepsilon > 0$ there is an sufficiently large $T_\varepsilon > 0$ such that

$$\int_{\mathrm{R}} m_p^{\delta} M_p^{\beta}|\hat{u}|^2\,dp \le \varepsilon_1(\varepsilon), \quad \text{at } t = T_\varepsilon, \tag{2.21}$$

where $\varepsilon_1(\varepsilon) = \varepsilon^{2+4(1-\gamma')/\mu}$, $\mu > 0$. As in (2.9), (2.10) we have

$$\|w(t)\|_X \le C\|\hat{u}(t)\|_X \|\hat{u}(t)\|_{1/2-\gamma,1/2+\gamma}^2 \le C\|\hat{u}(t)\|_X \int_{\mathrm{R}} m_p^{\delta} M_p^{\beta}|\hat{u}|^2\,dp \tag{2.22}$$

using (2.16), (2.17), (2.20) and Gronwall's inequality, we have from (2.6)

$$\|\hat{u}(t)\|_X \le C\left(\|\hat{\bar{u}}e^{-K(p)t}\|_X + \int_0^t \|w(\tau)\|_X\,d\tau + \int_0^t \|\hat{f}(\tau)e^{-K(p)(t-\tau)}\|_X\,d\tau\right)$$

$$\le \frac{C}{m_t^{1/2-\gamma}} + C\int_0^t \|\hat{u}(\tau)\|_X \int_{\mathrm{R}} m_p^{\delta} M_p^{\beta}|\hat{u}(\tau)|^2\,dp\,d\tau$$

$$\le \frac{C}{m_t^{1/2-\gamma}} \exp\left(\int_0^{+\infty} d\tau \int_{\mathrm{R}} m_p^{\delta} M_p^{\beta}|\hat{u}|^2\,dp\right) \le \frac{C}{m_t^{1/2-\gamma}}. \tag{2.23}$$

We prove now (2.18). By virtue of (2.21) the estimate (2.18) is valid at $t = T_\varepsilon$. Suppose that at some $T_1 > T_\varepsilon$ the estimate (2.18) is violated. Then by continuity we have for $t \in [T_\varepsilon, T_1]$

$$\|\hat{u}(t)\|_{1/2-\gamma,1/2+\gamma} \le \frac{\varepsilon}{(1+t-T_\varepsilon)^{\min(\alpha,1/\delta)-\gamma'}}. \tag{2.24}$$

We consider (NNS) on $[T_\varepsilon, T_1]$. In the same way as in the proof of (2.7) in Lemma 1 we get

$$\|\hat{u}(t)\|_{1/2-\gamma,1/2+\gamma} \le C\biggl(\|\hat{u}_1 e^{-K(p)t}\|_{1/2-\gamma,1/2+\gamma} + \int_{T_\varepsilon}^{t} \|e^{-K(p)(t-\tau)}\hat{f}\|_{1/2-\gamma,1/2+\gamma}\,d\tau$$

$$+ \int_{T_\varepsilon}^{t} \|e^{-K(p)(t-\tau)}w\|_{1/2-\gamma,1/2+\gamma}\,d\tau\biggr) = I_1 + I_2 + I_3, \tag{2.25}$$

where $\hat{u}_1 = \hat{u}(p, T_\varepsilon)$. By virtue of the (2.23) $\|\hat{u}_1\|_{L^\infty} \le C$.

Using (2.21) and (2.16) we have

$$I_1 \le C\left(\|\hat{u}_1\|_{L^\infty}\varepsilon_2^{\mu}\sqrt{\int_{|p|\le\varepsilon_2} e^{-2\theta|p|^{\delta}t}|p|^{1-2\gamma-2\mu}\,dp} + \sqrt{\int_{|p|\ge\varepsilon_2} e^{-2\theta\varepsilon_2^{\delta}t} m_p^{\delta} M_p^{\beta}|\hat{u}_1|^2\,dp}\right)$$

$$\le \frac{C\varepsilon^2}{t^{1/\delta-\gamma'}} + \frac{C}{\varepsilon_2^{1-\gamma'}t^{1/\delta-\gamma'}}\sqrt{\int_{\mathrm{R}} m_p^{\delta} M_p^{\beta}|\hat{u}_1|^2\,dp} \le \frac{\varepsilon^2}{t^{1/\delta-\gamma'}} \le \frac{\varepsilon^2}{(1+t-T_\varepsilon)^{1/\delta-\gamma'}}, \tag{2.26}$$

where $\varepsilon_2 = \varepsilon^{2/\mu}$, small $\mu > 0$. Applying (2.16) and (2.17) and changing variables $y^{\delta} = |p|^{\delta}t$ we obtain for I_2

$$I_2 \le C\Biggl(\int_{T_\varepsilon}^{t} \|\hat{f}(\tau)\|_{L^\infty}\,d\tau\sqrt{\int_{|p|\le 1} |p|^{1-2\gamma}e^{-2\theta|p|^{\delta}(t-\tau)}\,dp}$$

$$+ \int_{T_\varepsilon}^{t} d\tau\sqrt{\int_{|p|\ge 1} |p|^{1+2\gamma e^{-2\theta|p|^{\beta}(t-\tau)}}|\hat{f}|^2\,dp}\Biggr)$$

$$\le C\left(\int_{T_\varepsilon}^{t/2} \frac{d\tau}{(1+\tau)^{\alpha}(t-\tau)^{1/\delta-\gamma}}\left(\sqrt{\int_0^{+\infty} y^{1-2\gamma}e^{-\theta y^{\delta}}\,dy} + \|\hat{f}\|_0\sqrt{\sup_{|p|\ge 1}|p|^{1+2\gamma-\beta(1/\delta-\gamma)}}\right)\right.$$

$$\left. + \int_{t/2}^{t} \frac{d\tau}{(1+\tau)^{\alpha}(t-\tau)^{1-\gamma}}\left(\sqrt{\int_{|p|\le 1}\frac{|p|^{1-2\gamma}dp}{|p|^{\delta(2-2\gamma)}}} + \|\hat{f}\|_0\sqrt{\sup_{|p|\ge 1}|p|^{1+2\gamma-\beta(2-2\gamma)}}\right)\right)$$

$$\le \frac{C}{(1+t)^{\min(1/\delta,\alpha)-\gamma}} \le \frac{C\varepsilon^2}{(1+t-T_\varepsilon)^{\min(1/\delta,\alpha)-\gamma'}}, \tag{2.27}$$

since $T_\varepsilon > 0$ is large enough. By virtue of (2.22)–(2.24) analogously to I_2 we have

$$I_3 \le C\int_{T_\varepsilon}^{t}\Biggl(\|w(\tau)\|_{L^\infty}\sqrt{\int_{|p|\le 1}|p|^{1-2\gamma}e^{-2\theta|p|^{\delta}(t-\tau)}\,dp}$$

$$+ \sqrt{\int_{|p|\ge 1}|p|^{1+2\gamma}e^{-2\theta|p|^{\beta}(t-\tau)}|w(\tau)|^2\,dp}\Biggr)\,d\tau \le \frac{C\varepsilon^2}{(1+t-T_\varepsilon)^{\min(1/\delta,\alpha)-\gamma'}} \tag{2.28}$$

Substitution of (2.26)–(2.28) in (2.25) yields for $t \in [T_\varepsilon, T_1]$

$$\|\hat{u}\|_{1/2-\gamma, 1/2+\gamma} \le \frac{C\varepsilon^2}{(1+t-T_\varepsilon)^{\min(1/\delta,\alpha)-\gamma'}} < \frac{\varepsilon}{(1+t-T_\varepsilon)^{\min(1/\delta,\alpha)-\gamma'}}.$$

This contradiction proves (2.18) for $t > T_\varepsilon$.

This completes the proof. □

3. Proofs of the theorems

Proof of Theorem 1. From Remark to Lemma 1 it follows that the Cauchy problem has the unique solution $u(x,t)$ such that

$$u(x,t) \in C^0([0,+\infty), H^{1/2+\gamma}(\boldsymbol{R})).$$

Integrating the (NNS) equation, we have

$$u(x,t) = \frac{1}{2\pi}\left(\int_{\mathrm{R}} e^{ipx} e^{-K(p)t}\hat{u}(p)\,dp + i\int_{\mathrm{R}} e^{ipx}\,dp \int_0^t e^{-K(p)(t-\tau)} w(p,\tau)\,d\tau \right.$$
$$\left. - i\int_{\mathbf{R}_1} e^{ipx}\,dp\int_0^t e^{-K(p)(t-\tau)}\hat{f}(p,\tau)\,d\tau\right) = I_1 + I_2 + I_3. \tag{3.1}$$

We estimate each integral in the formula (3.1). The first integral I_1 decays faster than $t^{-\alpha}$ as $t \to \infty$ and forms the remainder term. Indeed, using conditions (1.1) and (1.3), we have

$$|I_1| \le \|\hat{u}\|_{L^\infty}\int_{|p|\le 1} e^{-\theta|p|^\delta t}\,dp + e^{-\theta t}\|\hat{u}\|_{H^{0,1/2+\gamma}} \le C\frac{1}{t^{1/\delta}}. \tag{3.2}$$

The second integral I_2 in (3.1) also forms the remainder term. To prove this we use the results of Lemma 1. We have

$$\|w\|_X \le C\|\hat{u}\|_{L_2}^2(\|\hat{u}\|_{1/2+\gamma/2} + \|\hat{u}\|_{1/2+\gamma}) \le \frac{C}{(1+t)^{\alpha_1}} \tag{3.3}$$

where $\alpha_1 = 2\min(\alpha - 1 + 1/2\delta, \alpha - \mu) + \alpha - \mu > \alpha$, since $\max(0, 1-1/2\delta) < \alpha < 1/\delta$, here $\mu > 0$ being sufficiently small.

Then we obtain the following estimate

$$|I_2| \le C\left(\int_{|p|\le 1} dp\int_0^t e^{-\theta|p|^\delta(t-\tau)}\|w\|_{L_\infty}\,d\tau + \int_0^t e^{-\theta(t-\tau)}\|w\|_{1/2+\gamma}\,d\tau\right)$$
$$\le C\left(\int_0^{t/2}\frac{d\tau}{(1+\tau)^{\alpha_1}(t-\tau)^{1/\delta}} + \int_{t/2}^t \frac{d\tau}{(1+\tau)^{\alpha_1}(t-\tau)^{1-\mu}}\right) = O(t^{-\alpha-\rho}), \tag{3.4}$$

where $0 < \rho < \alpha_1 - \alpha - \mu$.

We now prove that the third integral in (3.1) gives the main term of the asymptotics (1.4), that is,

$$I_3 = \frac{-i}{2\pi t^\alpha} \int_{\mathbf{R}} \frac{g(p)e^{ipx}}{K(p)}\, dp + O(t^{-\alpha-\rho}). \tag{3.5}$$

Using (1.2) we have

$$I_3 = \frac{-i}{2\pi} \int_{t/2}^{t} d\tau \int_{\mathbf{R}} e^{ipx} e^{-K(p)(t-\tau)} \frac{g(p)}{(1+\tau)^\alpha}\, dp + R(t) = \hat{I} + R(t), \tag{3.6}$$

where

$$R(t) = \frac{-i}{2\pi} \left(\int_0^{t/2} d\tau \int_{\mathbf{R}} e^{ipx} e^{-K(p)(t-\tau)} \hat{f}(p,\tau)\, dp + \int_{t/2}^{t} d\tau \int_{\mathbf{R}} e^{ipx} e^{-K(p)(t-\tau)} \psi(p,\tau)\, dp \right).$$

Integrating by parts we get for the first term in (3.6)

$$\begin{aligned}\hat{I} &= \frac{-i}{2\pi} \left(\int_{\mathbf{R}} \frac{e^{ipx} g(p)}{K(p)} e^{-K(p)(t-\tau)} \frac{1}{(1+\tau)^\alpha} \bigg|_{t/2}^{t} dp \right.\\ &\quad \left. + (\alpha+1) \int_{\mathbf{R}} \frac{e^{ipx} g(p)}{K(p)}\, dp \int_{t/2}^{t} \frac{e^{-K(p)(t-\tau)}}{(1+\tau)^{\alpha+1}}\, d\tau \right)\\ &= \frac{-i}{2\pi (1+t)^\alpha} \int_{\mathbf{R}} \frac{e^{ipx} g(p)}{K(p)}\, dp + O(t^{-\alpha-\rho}). \end{aligned} \tag{3.7}$$

Indeed, since $e^{-K(p)t/2} < C/m_p^{\delta\rho} t^\rho$, from (1.1) and (1.3) we have for $\rho < (1-\delta)/\delta$

$$\begin{aligned}\left| \frac{1}{t^\alpha} \int_{\mathbf{R}_1} \frac{g(p)}{K(p)} e^{-K(p)t/2}\, dp \right| &\le \frac{C}{t^\alpha} \left(\int_{|p|\le 1} \frac{|g|}{|p|^{\delta(1+\rho)} t^\rho}\, dp + e^{-\theta t/2} \int_{|p|\ge 1} |g(p)|\, dp \right)\\ &\le \frac{C}{t^{\alpha+\rho}} \|g\|_X = O(t^{-\alpha-\rho})\end{aligned}$$

and

$$\begin{aligned}\left| \int_{t/2}^{t} \frac{1}{(1+\tau)^{\alpha+1}}\, d\tau \int_{\mathbf{R}} \frac{e^{ipx} e^{-K(p)(t-\tau)} g(p)}{K(p)}\, dp \right| &\le C \left(\int_{\mathbf{R}} \frac{|g|}{|p|^{\delta(1+\rho)}}\, dp \int_{t/2}^{t} \frac{1}{(1+\tau)^{\alpha+1}(t-\tau)^\rho}\, d\tau \right)\\ &\le \frac{C}{t^{\alpha+\rho}} \|g\|_X = O(t^{-\alpha-\rho}). \end{aligned} \tag{3.8}$$

Now we show that $R(t)$ in (3.6) decays faster than $t^{-\alpha}$ and forms the remainder term:

$$R(t) = O(t^{-\alpha-\rho}). \tag{3.9}$$

From (1.1)–(1.2), changing the variable $y^\delta = \theta |p|^\delta (t-\tau)$, we obtain

$$\left|\int_0^{t/2} d\tau \int_{\mathrm{R}} e^{ipx} e^{-K(p)(t-\tau)} \hat{f}(p,t)\,dp\right|$$

$$\le C\left(\int_0^{t/2} \|\hat{f}\|_{L_\infty} \frac{1}{(t-\tau)^{1/\delta}}\,d\tau \int_0^{+\infty} e^{-y^\delta}\,dy + \int_0^{t/2} e^{-\theta(t-\tau)} \|\hat{f}\|_{1/2+\gamma}\,d\tau\right)$$

$$\le \frac{C}{t^{1/\delta}} \int_0^{t/2} \frac{1}{(1+\tau)^\alpha}\,d\tau = O(t^{-\alpha-\rho})$$

where $\rho < 1/\delta - 1$, and by the analogy of (3.6) for $0 < \rho < \min(1, \eta/2)$

$$\left|\int_{t/2}^{t} d\tau \int_{\mathrm{R}} e^{ipx} e^{-K(p)(t-\tau)} \psi(p,\tau)\,dp\right|$$

$$\le C\left(\int_{t/2}^{t} \|\psi\|_{L_\infty}\,d\tau \int_{|p|\le 1} e^{-\theta|p|^\delta(t-\tau)}\,dp + \int_{t/2}^{t} e^{-\theta(t-\tau)} \|\psi\|_{1/2+\gamma}\,d\tau\right)$$

$$\le C\int_{t/2}^{t} \frac{1}{(1+\tau)^{\alpha+\eta}(t-\tau)^{1-\rho}}\,d\tau = O(t^{-\alpha-\rho}).$$

Thus the estimate (3.9) is proved. Substituting (3.7) and (3.9) in (3.6), we get (3.5).
Using (3.2), (3.4) and (3.5) from (3.1) we have

$$u(x,t) = \frac{-i}{2\pi t^\alpha} \int_{\mathrm{R}} \frac{g(p)e^{ipx}}{K(p)}\,dp + O(t^{-\alpha-\rho}),$$

as $t \to +\infty$ and $0 < \rho < \min((1-\delta)/\delta, \alpha_1 - \alpha, \beta, 1, \eta/2)$. □

Proof of Theorem 2. We estimate each of the integrals in (3.1).

For I_1 we again use the estimate (3.2). The integral I_2 decays faster than $t^{-\beta}$ and also forms the remainder.

$$I_2 = O(t^{-\beta-\rho}) \tag{3.10}$$

Indeed, since $\alpha \in (3/2 - 1/\delta, 1)$ and $\delta \in (1,2)$ using estimates of Lemma 1, we have (3.3) with $\alpha_1 = 2(\alpha + 1/2\delta - 1) + \alpha + 1/\delta - 1 - \gamma = 3\alpha - 3 + 2/\delta - \gamma > \alpha$.

Therefore as (3.4) we obtain

$$|I_2| \le C\int_0^t \frac{d\tau}{(1+\tau)^{\alpha_1}(t-\tau)^{1/\delta}} = O(t^{-\beta-\rho}) \tag{3.11}$$

where $\rho < \alpha_1 - \alpha$.

The third integral in (3.1) gives the main term of the asymptotics. Indeed, by virtue of (1.6), we have

$$I_3 = \frac{-i}{2\pi}\int_{\mathrm{R}} e^{ipx}\,dp \int_0^t e^{-K(p)(t-\tau)} \hat{f}(p,\tau)\,d\tau$$

$$= \frac{-i\Theta}{2\pi}\int_0^t \frac{d\tau}{\tau^\alpha} \int_{|p|\le 1} e^{ipx} e^{-\omega|p|^\delta(t-\tau)}\,dp - \frac{i}{2\pi}R(t) = I - \frac{i}{2\pi}R(t) \tag{3.12}$$

where

$$R(t) = \int_0^t \frac{d\tau}{\tau^\alpha} \int_{|p|\le 1} e^{ipx}(e^{-K(p)(t-\tau)} - e^{-\omega|p|^\delta(t-\tau)})\,dp$$
$$+ \int_0^t d\tau \int_{|p|\le 1} e^{ipx} e^{-K(p)(t-\tau)}(\hat{f}(p,\tau) - \hat{f}(0,\tau))\,dp$$
$$+ \int_0^t d\tau \int_{|p|\le 1} e^{ipx} e^{-K(p)(t-\tau)} O((1+\tau)^{-\alpha-\eta})\,dp$$
$$+ \int_0^t d\tau \int_{|p|\ge 1} e^{ipx} e^{-K(p)(t-\tau)} \hat{f}(p,\tau)\,dp = \sum_{i=1}^4 J_i.$$

Making the change of variables $y^\delta = \omega|p|^\delta(t-\tau)$ and $tz = \tau$ in the first summand of (3.12) we obtain

$$I = \frac{-i\Theta}{\pi\omega^{1/\delta}} \frac{1}{t^\beta} \int_0^1 \frac{dz}{z^\alpha(1-z)^{1/\delta}} \int_0^{+\infty} \cos\left(y \frac{x}{(\omega t)^{1/\delta}}(1-z)^{1/\delta}\right) e^{-y^\delta}\,dy. \tag{3.13}$$

Now we prove that

$$R(t) = O(t^{-\beta-\rho}). \tag{3.14}$$

Using the trivial inequality for $b, c \ge 0$: $|e^{-b} - e^{-c}| \le C(e^{-c} + e^{-b})|b-c|^\mu$, $\mu \in (0,1]$ and (1.1), (1.5), we get

$$|e^{-K(p)(t-\tau)} - e^{-\omega|p|^\delta(t-\tau)}| \le Ce^{-\theta|p|^\delta(t-\tau)}|p|^{(\delta+\sigma)\delta\rho/\sigma}(t-\tau)^{\delta\rho/\sigma}, \quad |p| \le 1,$$

and therefore we obtain the following estimate for the first integral in $\boldsymbol{R}(t)$

$$|J_1| \le \frac{C}{t^{\beta+\rho}} \int_0^1 \frac{dz}{z^\alpha(1-z)^{\rho+1/\delta}} \int_0^{+\infty} y^{(\delta+\sigma)\delta\rho/\sigma} e^{-y^\delta}\,dy = O(t^{-\beta-\rho}), \tag{3.15}$$

where $0 < \rho < 1 - 1/\delta$.

In view of (1.1), (1.7) we have for $0 < \rho < \nu/\delta$

$$|J_2| \le C \int_0^t \frac{d\tau}{\tau^\alpha} \int_{|p|\le 1} e^{-\theta|p|^\delta(t-\tau)}|p|^\nu\,dp$$
$$\le C \int_0^t \frac{d\tau}{\tau^\alpha(t-\tau)^{(1+\nu)/\delta}} \int_0^{+\infty} y^\nu e^{-y^\delta}\,dy = O(t^{-\beta-\rho}), \tag{3.16}$$

and

$$|J_3| \le C \int_0^t \frac{d\tau}{(1+\tau)^{\alpha+\eta}} \int_{|p|\le 1} e^{-\theta|p|^\delta(t-\tau)}\,dp$$
$$\le C \int_0^t \frac{d\tau}{(t-\tau)^{1/\delta}(1+\tau)^{\alpha+\eta}} = O(t^{-\beta-\rho}), \tag{3.17}$$

where $0 < \rho < \eta$.

Using (1.1) and (1.6), we get for $\rho > 0$

$$|J_4| \le C\int_0^t d\tau \int_{|p|\ge 1} e^{-\theta(t-\tau)}|\hat{f}|\,dp$$

$$\le C\int_{|p|\ge 1}\frac{dp}{|p|^{1+\gamma}}\left(e^{-\theta t/2}\int_0^{t/2}\|\hat{f}\|_{1/2+\gamma}\,d\tau + \int_{t/2}^t e^{-\theta(t-\tau)}\|\hat{f}\|_{1/2+\gamma}\,d\tau\right)$$

$$\le C\left(e^{-\theta t/2}\int_0^{t/2}\frac{d\tau}{\tau^\alpha} + \frac{1}{t^\alpha}\int_{t/2}^t e^{-\theta(t-\tau)}\,d\tau\right) = O(t^{-\beta-\rho}). \qquad (3.18)$$

From (3.15)–(3.18) follows (3.14) and from (3.12)–(3.14) we have

$$I_3 = \frac{-i\Theta}{\pi\omega^{1/\delta}}\frac{1}{t^\beta}\int_0^1 \frac{dz}{z^\alpha(1-z)^{1/\delta}}\int_0^{+\infty}\cos\left(y\frac{x}{(\omega t)^{1/\delta}(1-z)^{1/\delta}}\right)e^{-y^\delta}\,dy + O(t^{-\beta-\rho}). \qquad (3.19)$$

Substitution of (3.2), (3.10) and (3.19) in (3.1) yields the asymptotics (1.8). This completes the proof. □

Proof of Theorem 3. In this case each of the integrals in (3.1) forms the main term of asymptotics. Consider the first integral I_1. Making the change of variables $\omega|p|^\delta t = y^\delta$ and using (1.1) and (1.3) we have

$$I_1 = \frac{1}{2\pi}\int_{\mathbf{R}_1} e^{ipx}e^{-\omega|p|^\delta t}\hat{u}(0)\,dp + R_1(t)$$

$$= \frac{\hat{u}(0)}{\pi\omega^{1/\delta}t^{1/\delta}}\int_0^{+\infty}\cos\left(y\frac{x}{(\omega t)^{1/\delta}}\right)e^{-y^\delta}\,dy + \frac{1}{2\pi}R_1(t), \qquad (3.20)$$

where

$$R_1(t) = \int_{|p|\le 1} e^{ipx}\hat{u}(0)(e^{-K(p)t} - e^{-\omega|p|^\delta t})\,dp$$

$$+ \int_{|p|\le 1} e^{ipx}e^{-K(p)t}(\hat{u}(p) - \hat{u}(0))\,dp$$

$$+ \int_{|p|\ge 1} e^{ipx}e^{-K(p)t}\hat{u}(p)\,dp = \Sigma_{i=1}^3 J_i = O(t^{-1/\delta-\rho}). \qquad (3.21)$$

Indeed, by the analogy of (3.15), making the change of variables as in (3.20), we get for $\rho > 0$

$$|J_1| \le C\int_{|p|\le 1} e^{-\theta|p|^\delta t}|p|^{(\delta+\sigma)\rho\delta/\sigma}t^{\rho\delta/\sigma}\,dp \le \frac{C}{t^{1/\delta+\rho}}\int_0^{+\infty} e^{-y^\delta}y^{(\delta+\sigma)\rho\delta/\sigma}\,dy = O(t^{-1/\delta-\rho}).$$

By virtue of (1.11) we have for small $\nu > 0$

$$|\hat{u}(p) - \hat{u}(0)| \le C\int_0^p |\partial_r\hat{u}(r)|\frac{|r|^{1/2-\nu}}{|r|^{1/2-\nu}}\,dr \le C\sqrt{\int_0^p \frac{dr}{|r|^{1-2\nu}}}\,\|\partial_p\hat{u}\|_{H^{0,1/2+\gamma}} \le C|p|^\nu.$$

Hence, using (1.1), we get

$$|J_2| \le C\frac{1}{t^{1/\delta+\rho}}\int_0^{+\infty} e^{-y^\delta} y^\nu\, dy = O(t^{-1/\delta-\rho}),$$

where $0 < \rho < \nu/\delta$.

In view of (1.3) and (1.1) we easily obtain

$$|J_3| < Ce^{-\theta t}\|\hat{u}\|_X = O(t^{-1/\delta-\rho}).$$

To estimate the second and the third integrals in (3.1) we obtain a number of preliminary estimates. Denote $h(p,t) = w(p,t) - \hat{f}(p,t)$.

By virtue of Lemma 1 and (1.10), we have

$$\|h(t)\|_X \le C(\|w(t)\|_X + \|\hat{f}(t)\|_X) \le \frac{C}{(1+t)^{\alpha_3}}, \quad \alpha_3 = \min\left(\alpha, \frac{2}{\delta} - \gamma\right). \tag{3.22}$$

Now we prove for all $|p| < 1$

$$|h(p,t) - h(0,t)| \le C\frac{|p|^\nu (1+t)^{\nu/\delta}}{(1+t)^{\alpha_3}}, \qquad \nu > 0. \tag{3.23}$$

For this purpose we need the following inequality

$$\|\partial_p \hat{u}\|_{1/2-\nu,1/2+\gamma} < C(1+t)^{\nu/\delta}, \tag{3.25}$$

for small $\nu > 0$. Indeed in view of (1.9)–(1.11), Lemma 1 and Gronwall's inequality from (NNS) we get

$$\|\partial_p \hat{u}\|_{1/2-\nu,1/2+\gamma} \le C\Bigg(\|\partial_p \hat{\bar{u}}\|_{1/2-\nu,1/2+\gamma} + \int_0^t (\|\partial_p \hat{f}\|_{1/2-\nu,1/2+\gamma} + \|\hat{u}\|_{\delta-1/2-\nu,1/2+\gamma})\, d\tau$$

$$+ \int_0^t \|\partial_p \hat{u}\|_{1/2-\nu,1/2+\gamma}(\|\hat{u}\|_{1/2-\nu} + \|\hat{u}\|_{1/2+\gamma})^2\, d\tau\Bigg)$$

$$\le C(1+t)^{\nu/\delta} + \int_0^t \frac{\|\partial_p \hat{u}\|_{1/2-\nu,1/2+\gamma}}{(1+t)^{2/\delta-2\nu/\delta}}\, d\tau \le C(1+t)^{\nu/\delta}.$$

Then using (1.10), Lemma1 we get for all $|p| < 1$:

$$|h(p,t) - h(0,t)| \le C\Bigg(\sqrt{\int_0^p \frac{dr}{|r|^{1-2\nu}}}\|\partial_p \hat{f}\|_{1/2-\nu,1/2+\gamma}$$

$$+ \sup_{q\in \mathrm{R}} \sqrt{\int_{-q}^{p-q} \frac{dr}{|r|^{1-2\nu}}}\, \|\partial_p \hat{u}\|_{1/2-\nu,1/2+\gamma} \|\hat{u}\|^2_{1/2-\nu,1/2+\gamma}\Bigg)$$

$$\le C|p|^\nu (1+t)^{\nu/\delta-\alpha_3},$$

for all $|p| < 1$. Thus, the estimate (3.23) is proved.

Using (3.22) and (3.23) we can obtain the asymptotics of integrals I_2 and I_3 in (3.1).

We have

$$I_2 + I_3 = \frac{1}{2\pi}\int_{\mathrm{R}} e^{ipx}\,dp \int_0^t e^{-K(p)(t-\tau)} h(p,\tau)\,d\tau$$
$$= \frac{1}{2\pi}\int_{|p|\le 1} e^{ipx}\,dp \int_0^{t/2} e^{-\omega|p|^{\delta}t} h(0,\tau)\,d\tau + \frac{1}{2\pi}R_2(t)$$
$$= \frac{1}{\pi\omega^{1/\delta}t^{1/\delta}}\int_0^{+\infty} h(0,\tau)\,d\tau \int_0^{+\infty} e^{-y^{\delta}}\cos\left(y\frac{x}{(\omega t)^{1/\delta}}\right)dy$$
$$+ \frac{1}{2\pi}R_2(t) + O(t^{-1/\delta-\rho}), \tag{3.26}$$

where

$$R_2(t) = \int_0^{t/2} h(0,\tau)\,d\tau \int_{|p|\le 1} e^{ipx}(e^{-K(p)(t-\tau)} - e^{-\omega|p|^{\delta}t})\,dp$$
$$+ \int_0^{t/2} d\tau \int_{|p|\le 1} e^{ipx}e^{-K(p)(t-\tau)}(h(p,\tau) - h(0,\tau))\,dp$$
$$+ \int_{t/2}^{t} d\tau \int_{|p|\le 1} e^{ipx}e^{-K(p)(t-\tau)} h(p,\tau)\,dp$$
$$+ \int_0^{t} d\tau \int_{|p|\ge 1} e^{ipx}e^{-K(p)(t-\tau)} h(p,\tau)\,dp = \Sigma_{i=1}^4 J_i. \tag{3.27}$$

Now we prove

$$R_2(t) = O(t^{-1/\delta-\rho}). \tag{3.28}$$

Since $\alpha_3 > 1, \alpha_3 > 1/\delta$, in view of (3.22), by the analogy of estimate (3.15), we get

$$|J_1| \le C\int_0^{t/2} h(0,\tau)\,d\tau \int_{p\le 1} (|e^{-K(p)(t-\tau)} - e^{-\omega|p|^{\delta}(t-\tau)}| + |e^{-\omega|p|^{\delta}(t-\tau)} - e^{-\omega|p|^{\delta}t}|)\,dp$$
$$\le C\int_0^{t/2} h(0,\tau)\,d\tau \int_{|p|\le 1} e^{-C|p|^{\delta}(t-\tau)}(|p|^{(\delta+\sigma)\delta\rho/\sigma}(t-\tau)^{\delta\rho/\sigma} + |p|^{\rho}\tau^{\rho/\delta})\,dp$$
$$\le \frac{C}{t^{1/\delta+\rho}}\int_0^{t/2} \frac{d\tau}{(1+\tau)^{\alpha_3}} = O(t^{-1/\delta-\rho})$$

where $0 < \rho < (\alpha_1 - 1)\delta$.

Using (1.1), (3.23), we obtain

$$|J_2| \le C\int_{|p|\le 1} dp \int_0^{t/2} e^{-\operatorname{Re} K(p)(t-\tau)} |h(p,\tau) - h(0,\tau)|\,d\tau$$
$$\le C\int_0^{t/2} \frac{\tau^{\nu/\delta}}{(1+\tau)^{\alpha_3}(t-\tau)^{(1+\nu)/\delta}}\,d\tau = O(t^{-1/\delta-\rho}),$$

for $0 < \rho < \min(\nu/\delta, \alpha_3 - 1 - \nu/\delta)$. By virtue of (3.22) the following estimates for J_3 and J_4 are valid.

$$|J_3| \le C \int_{t/2}^{t} \|h(\tau)\|_{L^\infty} \frac{d\tau}{(t-\tau)^{1/\delta}} \int_0^{+\infty} e^{-y^\delta}\, dy$$

$$\le C \int_{t/2}^{t} \frac{d\tau}{(t-\tau)^{1/\delta}(1+\tau)^{\alpha_3}} = O(t^{-1/\delta-\rho})$$

and

$$|J_4| \le C \int_0^t \|h(\tau)\|_{1/2+\gamma} e^{-\theta(t-\tau)} \le C \int_0^t \frac{e^{-\theta(t-\tau)}}{(1+\tau)^{\alpha_3}}\, d\tau = O(t^{-1/\delta-\rho}),$$

where $0 < \rho < \min(\alpha_3 - 1/\delta, \alpha_3 - 1)$. Thus, the estimate (3.28) is proved. Substituting (3.20), (3.26) and (3.28) in (3.1), we have the asymptotics (1.12). This completes the proof. □

Proof of Theorem 4. It is sufficient to estimate I_2 in (3.1). By applying Lemma 2 we have, for $T_\varepsilon > 0$

$$\|w(t)\|_{L^\infty} \le C\|\hat{u}\|_{L^\infty}\|\hat{u}\|^2_{1/2-\gamma,1/2+\gamma} \le \frac{C}{(1+t-T_\varepsilon)^{2\alpha-\gamma}}, \quad t > T_\varepsilon, \tag{3.29}$$

$$\|\hat{w}(t)\|_{L^\infty} \le C\|\hat{u}\|_{L^\infty}\|\hat{u}\|^2_X \le \frac{C}{m_t^{1-\gamma}}, \quad t > 0. \tag{3.30}$$

Then by the analogy of (3.4), using (3.29) and (3.30), we get for the second integral in (3.1)

$$\begin{aligned}
|I_2| &\le C \int_0^t d\tau \int_{\mathrm{R}} e^{-K(p)(t-\tau)} w(p,\tau)\, dp \\
&\le C\left(\int_{|p|\le 1} dp \left(\int_0^{T_\varepsilon} d\tau + \int_{T_\varepsilon}^t d\tau\right)\|w\|_{L^\infty} e^{-\theta|p|^\delta(t-\tau)}\right) \\
&\quad + \int_{|p|\ge 1} dp \left(\int_0^{T_\varepsilon} d\tau + \int_{T_\varepsilon}^t d\tau\right) e^{-\theta|p|^\beta(t-\tau)}\|w\|_{L^\infty}) \\
&\le C\left(\int_0^{+\infty} e^{-y^\delta}\, dy + \int_{|p|\ge 1} \frac{dp}{|p|^{\beta/\delta}}\right)\left(\frac{CT_\varepsilon}{(t-T_\varepsilon)^{1/\delta}} + \int_{T_\varepsilon}^{t/2} \frac{d\tau}{(1+\tau-T_\varepsilon)^{2\alpha-\gamma}(t-\tau)^{1/\delta}}\right) \\
&\quad + \int_{t/2}^t \frac{d\tau}{(1+\tau-T_\varepsilon)^{2\alpha-\gamma}(t-\tau)^{1-\mu}}\left(\int_{|p|\le 1} \frac{dp}{|p|^{\delta(1-\mu)}} + \int_{|p|\ge 1} \frac{dp}{|p|^{\beta(1-\mu)}}\right) \\
&= O(t^{-\alpha-\rho})
\end{aligned} \tag{3.31}$$

where $0 < \rho < \min(1/\delta - \alpha, \alpha)$.

In the same way as the proof of Theorem 1, using the norm L^2 instead of $\|\cdot\|_{1/2+\gamma}$ and condition (1.13) of strong dissipativity, we get estimates (3.2) and (3.5). This completes the proof. □

Proof of Theorem 5. To estimate I_2 in (3.1) requires a number of preliminary estimates. From Lemma 2 we have

$$\|\hat{w}\|_{L^\infty} \le \frac{C}{m_t^{1-\gamma}}, \qquad t > 0 \tag{3.32}$$

and for small enough $\varepsilon > 0$

$$\|\hat{w}\|_{L^\infty} \le \frac{\varepsilon}{(1+t)^{1/\delta-\gamma}}, \qquad t > T_\varepsilon, \tag{3.33}$$

since $\alpha > 1/\delta$. From (2.20) we get

$$\int_0^{+\infty} d\tau \int_{\mathbf{R}} m_p^\delta M_p^\beta |\hat{u}|^2\, dp \le C. \tag{3.34}$$

Now we prove that for $0 < \gamma' < 1$

$$\|\hat{u}\|_k \le \frac{C}{m_t^{1-\gamma'}}, \quad t > 0, \quad k \in \left[0, \beta - \frac{1}{2} + \gamma\right]. \tag{3.35}$$

Indeed, since for $k \ge 0$

$$\|w\|_k \le C\|\hat{u}\|_k \int_{\mathbf{R}} m_p^\delta M_p^\beta |\hat{u}|^2\, dp,$$

by applying Gronwall's inequality and (1.15), (1.17) and (3.34), we get

$$\begin{aligned}\|\hat{u}\|_k &\le C\left(\|e^{-K(p)t}\hat{\bar{u}}\|_k + \int_0^t \|e^{-K(p)(t-\tau)}\hat{f}\|_k\, d\tau + \int_0^t \|\hat{u}\|_k \int_{\mathbf{R}} m_p^\delta M_p^\beta |\hat{u}|^2\, dp\, d\tau\right)\\ &\le \frac{C}{t^{1-\gamma'}} + \int_0^t \|\hat{u}\|_k \int_{\mathbf{R}} m_p^\delta M_p^\beta |\hat{u}|^2\, dp\, d\tau \le \frac{C}{m_t^{1-\gamma'}}.\end{aligned}$$

For $k \in [0, 1/2 - \gamma)$ we can prove

$$\|\hat{u}\|_{-k} \le C. \tag{3.36}$$

Indeed, since as in (2.9) we have

$$\|w\|_{L^\infty} \le C\|\hat{u}\|_{L^\infty}\|\hat{u}\|^2_{1/2-\gamma,\,1/2+\gamma} \le C\|\hat{u}\|_{L^\infty} \int_{\mathbf{R}} m_p^\delta M_p^\beta |\hat{u}|^2\, dp. \tag{3.37}$$

Using (3.32), (3.33), (3.37) we obtain

$$\begin{aligned}\|\hat{u}\|_{-k} &\le \|\hat{\bar{u}}e^{-K(p)t}\|_{-k} + \int_0^t \|\hat{f}e^{-K(p)(t-\tau)}\|_{-k}\, d\tau\\ &\quad + \int_0^t \|w(\tau)\|_{L^\infty}\, d\tau \sqrt{\left(\int_{|p|\le 1} + \int_{|p|\ge 1}\right) e^{-2K(p)(t-\tau)} \frac{dp}{|p|^{2k}}}\\ &\le C + C\int_0^t d\tau \int_{\mathbf{R}} m_p^\delta M_p^\beta |\hat{u}|^2\, dp + \int_0^t \frac{\|\hat{u}\|^2_{1/2-\gamma,\,1/2+\gamma}}{(t-\tau)^{1-\gamma}}\, d\tau \le C.\end{aligned}$$

Applying (3.35) and (3.36), we can prove the following estimate for small $\nu > 0$:

$$\|\partial_p \hat{u}\|_{1/2-\nu,1/2+\gamma} \leq CM_t, \qquad t > 0. \tag{3.38}$$

Indeed, as in proof of Theorem 3 by virtue of (1.14), (3.35), (3.36), applying Gronwall's inequality, we obtain

$$\begin{aligned}\|\partial_p \hat{u}\|_{1/2-\nu,1/2+\gamma} &\leq \|\hat{u}\|_{1/2-\gamma,1/2+\gamma} + \int_0^t (\|\partial_p \hat{f}\|_{1/2-\nu,1/2+\gamma} + \|\hat{u}\|_{\delta-1/2-\nu} + \|\hat{u}\|_{\beta-1/2+\gamma})\, d\tau \\ &\quad + \int_0^t \|\hat{u}\|^2_{1/2-\nu,1/2+\gamma} \|\partial_p \hat{u}\|_{1/2-\nu,1/2+\gamma}\, d\tau \\ &\leq CM_t \exp\left(\int_0^t d\tau \int_{\mathrm{R}} m_p^\delta M_p^\beta |\hat{u}|^2\, dp\right).\end{aligned}$$

Then, since

$$|w(p,t) - w(0,t)| \leq C \sup_{q \in \mathbf{R}} \sqrt{\int_{-q}^{p-q} \frac{dr}{|r|^{1-2\nu}}}\, \|\partial_p \hat{u}\|_{1/2-\nu,1/2+\gamma} \|\hat{u}\|^2_{1/2-\gamma,1/2+\gamma},$$

using Lemma 2 and (3.37), we get for $t > 0$

$$|w(p,t) - w(0,t)| \leq C \frac{|p|^\nu}{M_t^{2/\delta-1-\gamma} m_t^{1-\gamma'}}.$$

From this estimate and (3.32), (3.33), as in Theorem 3 we obtain the asymptotics (1.12). This completes the proof. □

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References

[1] P. I. Naumkin and I. A. Shishmarev, Asymptotics of solutions of the nonlinear equation with dissipation, Mathematicheskie zametki **v.45** (1989), 119–120.
[2] T. Taniuti and A. Washimi, Self-trapping and instability of Hydromagnetic waves along the magnetic field in a cold plasma, Phys. Rev. Letters **v.21** (1968), 32.
[3] Y. Kuramoto, Chemical oscillations, waves and turbulence, Berlin: Springer, (1982).
[4] P. L. Kelley, Self-focusing of optical beams, Phys. Rev. Letters **v.15.** (1965), 1005–1008.
[5] A. C. Newel and J. A. Whitehead, Finite bandwidth, finite amplitude convection, J. Fluid Mech. **v.38, pt. 2** (1969), 279–303.
[6] K. Nozaki and N. Bekki, Exact solutions of the generalized Landau-Ginzburg equation, J. Phys. Soc. Jap. **53, No 5** (1984), 1581.
[7] E. I. Kaikina, The Cauchy problem for the Nonlocal Schrödinger equation. I. Mat. modelirovanie, **v.3, No 11** (1991), 83–95. (in Russian)
[8] E. I. Kaikina, The Cauchy problem for Nonlocal Schrödinger equation. II. Mat. modelirovanie, **v.3, No 11** (1991), 96–108. (in Russian)
[9] K. Hayashi, K. Nakamitsu and M. Tsutsumi, On solutions of the initial value problem for the nonlinear Schrödinger equations, J. Funct. Anal. **71** (1987), 218–245.
[10] K. Hayashi, K. Nakamitsu and M. Tsutsumi, Nonlinear Schrödinger equation in weighted Sobolev spaces, Funkcialaj Ekvacioj, **31** (1988), 363–381.

[11] N. Hayashi and M. Tsutsumi, $L^\infty(R^m)$-decay of classical solutions of nonlinear Schrödinger equations, Proc. Roy. Soc. Edinburgh **108A** (1986), 309–327.
[12] P. I. Naumkin and I. A. Shishmarev, Nonlinear Nonlocal Equations in the Theory of Waves., Translations of Mathematical Monographs, AMS, Providence, Rhode Island, vol. 133, 1996.
[13] P. I. Naumkin and I. A. Shishmarev, Large time asymptotics of the solution of nonlinear equation with weak dissipation and dispersion, Izvestiya RAI, ser. math. 1993, 52–63. (in Russian)
[14] P. I. Naumkin and I. A. Shishmarev, Large time asymptotics of solutions of nonlinear equations with large initial data, Math. Zametki **59** No. 6 1996, 855–864. (in Russian)

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