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An inverse problem in quantum field theory and canonical correlation functions

An application of a solvable model called the rotating wave approximation.

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Abstract. In this paper, we treat a quantum harmonic oscillator in thermal equilibrium with any systems in certain classes of bosons with infinitely many degrees of freedom. We describe the following results: (i) when a canonical correlation function is given, we so reconstruct a Hamiltonian by the rotating wave approximation from it that the Hamiltonian restores it. Namely, we solve an inverse problem in the quantum field theory at finite temperature in a finite volume. (ii) Taking an infinite volume limit for the result in (i), we consider long-time behavior of the canonical correlation function in the finite volume limit.

1. Introduction

In this paper, we shall treat long-time behavior of the canonical correlation function in an infinite volume limit. For that purpose, we shall apply Arai's results [**5**] concerning long-time behavior of two-point functions to a class of canonical correlation functions of position and momentum operators in the infinite volume limit. In [**5**], Arai argued long-time behavior of two-point functions of position operators for some models of a quantum harmonic oscillator interacting with bosons.

We consider a quantum harmonic oscillator in thermal equilibrium with any system in certain classes of bosons with infinitely many degrees of freedom in a finite volume $V > 0$. Our models describe photons in a laser interacting with oscillation caused by a heat bath, which can be observed when the laser passes in the heat bath as well as photons in a laser interacting with oscillation caused by phonons on the surface of a material, which can be observed when we irradiate the weak laser on the surface.

When a two-point function (or canonical correlation function) $R^V(t_1, t_2)$ of the position operator or momentum operator of the harmonic oscillator is given by an observation at a temperature, we take the infinite volume limit, $V \to \infty$, for $R^V(t_1, t_2)$, and get $R_\beta^\infty(t_1, t_2) \equiv \lim_{V\to\infty} R^V(t_1, t_2)$ at the inverse temperature, β, under suitable conditions. And we argue long-time behavior of $R_\beta^\infty(t_1, t_2)$.

For a positive parameter $V > 0$, we set $\Gamma_V \equiv 2\pi \boldsymbol{Z}/V$:

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$$\Gamma_V \stackrel{\text{def}}{=} \frac{2\pi \boldsymbol{Z}}{V} \equiv \left\{ k \,\middle|\, k = \frac{2\pi n}{V}, n = 0, \pm 1, \pm 2, \cdots \right\}. \tag{1.1}$$

The Hilbert space of state vectors of our system in the finite volume case is taken to be the symmetric Fock space $\mathscr{F}$ over $\boldsymbol{C} \oplus \ell^2(\Gamma_V)$. We denote by a, a^* the annihilation and creation operators of the quantum harmonic oscillator, respectively, and by b_k, b_k^* ($k \in \Gamma_V$) those of bosons, acting in $\mathscr{F}$. We assume that there exists a self-adjoint operator H on $\mathscr{F}$, called a Hamiltonian, which governs the time development of the system, such that $e^{-\tau H}$ is trace class for all $\tau \in (0, \beta]$ and

$$R^V(t_1, t_2) \stackrel{\text{def}}{=} \frac{1}{\beta \operatorname{tr}(e^{-\beta H})} \int_0^\beta d\lambda \operatorname{tr}(e^{-(\beta-\lambda)H} e^{iHt_1} B e^{-iHt_1} e^{-\lambda H} e^{iHt_2} B e^{-iHt_2}),$$

where B denotes either the position operator $q = (a + a^*)/\sqrt{2}$ or the momentum operator $p = i(a^* - a)/\sqrt{2}$ of the quantum harmonic oscillator and tr denotes trace on $\mathscr{F}$ (we use units where $\hbar$ (the Planck constant divided by 2π) $= 1$). However, *we do not specify the concrete form of H. The Hamiltonian H may be a complicated function of a, a^*, b_k and b_k^* ($k \in \Gamma_V$).*

To apply Arai's result [**5**, Theorem 1.3] to our case, we first solve the following inverse problem: In terms of $R^V(t_1, t_2)$ only, determine positive frequencies x^0, x_k ($k \in \Gamma_V$) of the quantum harmonic oscillator and scalar bosons, respectively, and coupling constants $y_k \in \boldsymbol{C}$ ($k \in \Gamma_V$), appearing in the Hamiltonian of the rotating wave approximation (RWA),

$$H_{\mathrm{RWA}}(x, y) \stackrel{\text{def}}{=} x^0 a^* a + \sum_{k \in \Gamma_V} x_k b_k^* b_k + \sum_{k \in \Gamma_V} (y_k a^* b_k + \overline{y_k} b_k^* a), \tag{1.2}$$

$$x \stackrel{\text{def}}{=} x^0 \oplus (x_k)_{k \in \Gamma_V}, \qquad y \stackrel{\text{def}}{=} 0 \oplus (y_k)_{k \in \Gamma_V},$$

(where $\overline{c}$ means the complex conjugate of $c \in \boldsymbol{C}$) such that the Hamiltonian $H_{\mathrm{RWA}}(x, y)$ recovers $R^V(t_1, t_2)$ in the following sense:

$\{$energy levels of $H_{\mathrm{RWA}}(x, y)\} \backslash \{0\}$

(where 'energy level' is the notion in the quantum theory of

many-particle systems [**32**, §2-1-a])

$= \{$spectra (energy levels) of the one particle Hamiltonian h_{RWA} of $H_{\mathrm{RWA}}(x, y)$

more than its lowest one$\}$

(where this 'energy level' is the notion in the quantum mechanics [**33**, §1-3])

$$= \left\{ \text{positive poles of the meromorphic function } \int_0^\infty dt e^{itz} R^V(0, t) \right\}$$

with h_{RWA} a self-adjoint operator on $\boldsymbol{C} \oplus \ell^2(\Gamma_V)$ such that $H_{\mathrm{RWA}} = d\Gamma(h_{\mathrm{RWA}})$, the second quantization of h_{RWA} (see §2.2), and

$$R^V(t_1, t_2) = \textit{a representation in terms of } W^V(t_1, t_2), \tag{1.3}$$

with

$$W^V(t_1,t_2) \overset{\text{def}}{=} (\Omega_0, e^{iH_{\mathrm{RWA}}(x,y)t_1} B e^{-iH_{\mathrm{RWA}}(x,y)t_1} e^{iH_{\mathrm{RWA}}(x,y)t_2} B e^{-iH_{\mathrm{RWA}}(x,y)t_2} \Omega_0)_{\mathscr{F}},$$

the two-point vacuum expectation value (two-point function) of B, with the Hamiltonian $H_{\mathrm{RWA}}(x,y)$, where Ω_0 is the Fock vacuum in $\mathscr{F}$ (in fact, it is the vacuum (ground state) of $H_{\mathrm{RWA}}(x,y)$), and $(\ ,\ )_{\mathscr{F}}$ is the inner product of $\mathscr{F}$.

There are some negative criticisms in a physical sense against RWA [**13**, §V.D], claiming that a model, called the independent-oscillator model, is more suitable and useful in physics than RWA model [**12**, **13**, **22**]. But, in this paper, we use RWA model, since we find that it is suitable for our purpose of investigating the long-time behavior of $R^V(t_1,t_2)$ in the infinite volume limit. The reason why we employ RWA is nothing but easy to argue the infinite volume limit for the Hamiltonian of RWA in a mathematically rigorous way, and mathematical theories on RWA model are established in [**4**, **5**, **19**]. If the inverse problem stated above is solved, then we can investigate the long-time behavior of the infinite volume limit of $R^V(t_1,t_2)$ through the infinite volume limit $W^\infty(t_1,t_2)$ of $W^V(t_1,t_2)$ with representation (1.3). On the other hand, the long-time behavior of the latter function is investigated in detail by Arai [**5**].

An answer to the inverse problem above is given by Theorem 3.1 in this paper. By representation (1.3), we can consider the infinite volume limit $R_\beta^\infty(t_1,t_2)$ of $R^V(t_1,t_2)$, through the right hand side of (1.3). Then, we have a representation of $R_\beta^\infty(t_1,t_2)$ by using $W^\infty(t_1,t_2)$ at (4.50) in §4.2. And, applying Arai's results in [**5**] to the representation, we consider the long-time behavior of $R_\beta^\infty(t) \equiv R_\beta^\infty(0,t)$ for B in Theorem 3.2 of this paper.

The present paper is organized as follows. In §2 we review some basic facts on the Liouville space and RWA. In §3 we state our main results. In §4 we give proofs for the main results. In the last section, as two appendices, we review Mori's memory kernel equation, and find an example.

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2. Preliminaries

2.1. Liouville space

We give a complex Hilbert space $\ell^2(\Gamma_V)$ by $\ell^2(\Gamma_V) \overset{\text{def}}{=} \{(c_k)_{k\in\Gamma_V} \mid c_k \in \boldsymbol{C}, k \in \Gamma_V, \sum_{k\in\Gamma_V} |c_k|^2 < \infty\}$. For each $f \in \boldsymbol{C} \oplus \ell^2(\Gamma_V)$, we denote f by $f^0 \oplus (f_k)_{k\in\Gamma_V}$, where

$f^0 \in \boldsymbol{C}$ and $(f_k)_{k \in \Gamma_V} \in \ell^2(\Gamma_V)$. An inner product $(\ ,\ )_{\boldsymbol{C}\oplus\ell^2}$ of $\boldsymbol{C} \oplus \ell^2(\Gamma_V)$ is given by $(f, g)_{\boldsymbol{C}\oplus\ell^2} \stackrel{\text{def}}{=} \overline{f^0} g^0 + \sum_{k \in \Gamma_V} \bar{f}_k g_k$ $(f, g \in \boldsymbol{C} \oplus \ell^2(\Gamma_V))$. We denote the symmetric Fock space over $\boldsymbol{C} \oplus \ell^2(\Gamma_V)$ by

$$\mathscr{F} \stackrel{\text{def}}{=} \bigoplus_{n=0}^{\infty} S_n(\boldsymbol{C} \oplus \ell^2(\Gamma_V))^n, \tag{2.1}$$

where $S_n(\boldsymbol{C} \oplus \ell^2(\Gamma_V))^n$ is the n-fold symmetric tensor product of $\boldsymbol{C} \oplus \ell^2(\Gamma_V)$ for each $n \in \boldsymbol{N}$ and $S_0(\boldsymbol{C} \oplus \ell^2(\Gamma_V))^0 \stackrel{\text{def}}{=} \boldsymbol{C}$ (see [**29**, p. 53, Example 2]).

We denote by $a(f)$ $(f \in \boldsymbol{C} \oplus \ell^2(\Gamma_V))$ the (smeared) annihilation operator on $\mathscr{F}$ and set $a = a(1 \oplus 0)$, $b_k = a(0 \oplus e_k)$ $(k \in \Gamma_V)$ with $e_k \in \ell^2(\Gamma_V)$ such that the k'-th component of e_k is given by $(e_k)_{k'} = \delta_{kk'}$ (the Kronecker delta) for $k, k' \in \Gamma_V$. The operators a and a^* (the adjoint of a) physically denote the annihilation and creation operators of the quantum harmonic oscillator, respectively, and b_k, b_k^* $(k \in \Gamma_V)$ those of free bosons.

We consider a quantum harmonic oscillator in thermal equilibrium at an inverse temperature $\beta > 0$ with a system of bosons with infinitely many degrees of freedom in a finite volume. We take the Hilbert space of state vectors of the system to be $\mathscr{F}$.

Let H be a strictly positive self-adjoint operator on $\mathscr{F}$ satisfying the condition

(**H**) $e^{-\tau H}$ is a trace class operator on $\mathscr{F}$ for every $\tau \in (0, \beta]$.

The operator H physically denotes the Hamiltonian of the quantum system under consideration. Condition (**H**) implies that the spectrum of H consists of only eigenvalues, each having a finite multiplicity. We set $\boldsymbol{N}^* \stackrel{\text{def}}{=} \{0, 1, 2, \ldots\}$. We denote the eigenvalues of H by λ_n $(n \in \boldsymbol{N}^*)$ with order $0 < \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n \leq \lambda_{n+1} \leq \ldots$, $\lambda_n \nearrow \infty$ as $n \nearrow \infty$, counting multiplicities. We can take a complete orthonormal system $\{\varphi_n \,|\, n \in \boldsymbol{N}^*\}$ of $\mathscr{F}$ in such a way that each φ_n is a normalized eigenvector of H with eigenvalue $\lambda_n : H\varphi_n = \lambda_n \varphi_n$.

For the Hamiltonian H, we can construct a space $\mathbf{X}_c(H)$ consisting of suitable operators on $\mathscr{F}$ [**18**, **19**, **20**], called a Liouville space. We denote the linear hull of $\{\varphi_n \,|\, n \in \boldsymbol{N}^*\}$ by $\mathbf{D}$, i.e.,

$$\mathbf{D} \stackrel{\text{def}}{=} \mathbf{L.h.}\ [\{\varphi_n \,|\, n \in \boldsymbol{N}^*\}]. \tag{2.2}$$

From here on, we denote the linear hull of a set S by $\mathbf{L.h.}\ [S]$. Obviously $\mathbf{D}$ is dense in $\mathscr{F}$. Further, we denote by $\mathbf{B}(\mathbf{D}, \mathscr{F})$ the space of bounded linear operators from $\mathbf{D}$ to $\mathscr{F}$. Every element A in $\mathbf{B}(\mathbf{D}, \mathscr{F})$ has a unique extension to an element in $\mathbf{B}(\mathscr{F})$, the space of bounded linear operators on $\mathscr{F}$. We denote the extension of A by A^-, and $A^* \lceil \mathbf{D}$ (the restriction of A^* to $\mathbf{D}$) by A^+.

We first define a class $\mathbf{T}(H)$ of linear operators on $\mathscr{F}$. For a linear operator A on $\mathscr{F}$, we denote its domain by $\mathrm{D}(A)$.

We say that a linear operator A on $\mathscr{F}$ is in the class $\mathbf{T}(H)$ if the following conditions (**T.1**) and (**T.2**) are satisfied:

(**T.1**) $D(A) = \mathbf{D}$ and $D(A^*) \supset \mathbf{D}$.

When $\mathrm{D}(A) \supset \mathbf{D}$, we regard A as $A \lceil \mathbf{D}$ and denote $A \lceil \mathbf{D}$ by A only.

(**T.2**) For all τ in $(0,\beta]$, $e^{-\tau H}A$ and $Ae^{-\tau H}$ are in $\mathbf{B}(\mathbf{D},\mathscr{F})$ with $(e^{-\tau H}A)^{-}$ and $(Ae^{-\tau H})^{-}$ being Hilbert-Schmidt operators on $\mathscr{F}$.

It is easy to see that $\mathbf{T}(H)$ is a complex vector space with the natural operations of addition and scalar multiplication. We also remark that $\mathbf{T}(H)$ may contain not only bounded operators but also unbounded ones.

We can introduce an inner product in $\mathbf{T}(H)$, called the Bogoliubov (Kubo-Mori) scalar product, by

$$\langle A;B\rangle_H \stackrel{\text{def}}{=} \frac{1}{\beta Z(\beta)}\int_0^\beta d\lambda\,\mathrm{tr}((e^{-(\beta-\lambda)H}A^*)^{-}(e^{-\lambda H}B)^{-}) \tag{2.3}$$

for $A,B\in\mathbf{T}(H)$ with

$$Z(\beta) \stackrel{\text{def}}{=} \mathrm{tr}(e^{-\beta H}) \tag{2.4}$$

(see [**19**, Lemma 3.2]).

We denote by $\mathbf{X}_c(H)$ the completion of $\mathbf{T}(H)$ in the norm $\|\cdot\|_H$ of $\mathbf{T}(H)$ induced by the inner product $\langle\cdot;\cdot\rangle_H$. The Hilbert space $\mathbf{X}_c(H)$ has a structure of partial $*$-algebra ([**19**, Proposition 3.14]). We also note here that an element in $\mathbf{X}_c(H)$ is not always an operator acting in $\mathscr{F}$. It is noteworthy that Naudts et al. attempted to argue in general about linear response theory on the Hilbert space which is constructed by a completion of a von Neumann algebra with KMS-state [**27**]. Similarly we can formulate Mori's theory in statistical physics in terms of $\mathbf{X}_c(H)$.

We define an operator $\mathscr{L}$ acting in $\mathbf{X}_c(H)$, called the Liouville operator in physics, by

$$\mathrm{D}(\mathscr{L}) \stackrel{\text{def}}{=} \{A\in\mathbf{T}(H)\,|\,HA, AH\lceil\mathbf{D}\in\mathbf{T}(H)\}, \tag{2.5}$$

$$\mathscr{L}A \stackrel{\text{def}}{=} [H,A] = HA - AH, \quad A\in\mathrm{D}(\mathscr{L}). \tag{2.6}$$

It is shown that $\mathscr{L}$ is essentially self-adjoint [**19**, Lemma 3.8]. We denote the closure of $\mathscr{L}$ by the same symbol.

The spectral properties of $\mathscr{L}$ can exactly be known as is shown below. Let $\Phi_{m,n}$: $\mathbf{D}\to\mathbf{D}$ be an operator defined by

$$\mathrm{D}(\Phi_{m,n}) \stackrel{\text{def}}{=} \mathbf{D}, \tag{2.7}$$

$$\Phi_{m,n}x \stackrel{\text{def}}{=} \beta^{1/2}Z(\beta)^{1/2}W_{m,n}^{1/2}(\varphi_n,x)_{\mathscr{F}}\varphi_m, \qquad x\in\mathbf{D};\ m,n\in\boldsymbol{N}^*, \tag{2.8}$$

where

$$W_{m,n} \stackrel{\text{def}}{=} \begin{cases} \dfrac{\lambda_n-\lambda_m}{e^{-\beta\lambda_m}-e^{-\beta\lambda_n}} & \text{if } \lambda_m\neq\lambda_n, \\ \beta^{-1}e^{\beta\lambda_m} & \text{if } \lambda_m=\lambda_n. \end{cases} \tag{2.9}$$

Note that

$$W_{m,n} > 0, \quad m,n\in\boldsymbol{N}^*, \tag{2.10}$$

$$W_{m,n} = W_{n,m}, \quad m,n\in\boldsymbol{N}^*. \tag{2.11}$$

It can be shown that $\{\Phi_{m,n}\}_{m,n\in \boldsymbol{N}^*}$ is a complete orthonormal system of $\mathbf{X}_c(H)$ with

$$\mathscr{L}\Phi_{m,n} = (\lambda_m - \lambda_n)\Phi_{m,n}, \qquad m,n \in \boldsymbol{N}^*, \tag{2.12}$$

$$\Phi^+_{m,n} = \Phi_{n,m}, \qquad m,n \in \boldsymbol{N}^*, \tag{2.13}$$

(see [**20**, Proposition 3.9]). Moreover, we can prove

$$\sigma(\mathscr{L}) = \{\lambda_m - \lambda_n \,|\, m,n \in \boldsymbol{N}^*\}, \tag{2.14}$$

$$\sigma_{ess}(\mathscr{L}) = \{\varepsilon \in \sigma(\mathscr{L}) \,|\, \varepsilon \text{ is an eigenvalue of } \mathscr{L} \text{ of infinite multiplicity}\}, \tag{2.15}$$

where $\sigma(S)$ (resp. $\sigma_{ess}(S)$) denotes the (resp. essential) spectrum of an operator S.

REMARK. One can easily show that the set of possible limit points of $\sigma(\mathscr{L})$ is $\{0\}$. The referee remarked this fact to the author. The author is thankful to the referee for it. We note here that 0 is an eigenvalue of $\mathscr{L}$ of infinite multiplicity.

For every $A \in \mathbf{X}_c(H)$ and $t \in \boldsymbol{R}$, we define

$$A(t) \stackrel{\text{def}}{=} e^{i\mathscr{L}t}A, \tag{2.16}$$

$$R_A(t_1,t_2) \stackrel{\text{def}}{=} \langle A(t_1); A(t_2)\rangle_H, \qquad t_1,t_2 \in \boldsymbol{R}. \tag{2.17}$$

REMARK. The vector $A(t)$ can be regarded as a generalization of the Heisenberg operator for A with the Hamiltonian H [**19**, Proposition 3.13]. With an interpretation that A is an observable in the quantum system under consideration, the function $R_A(t_1,t_2)$ physically means a two-point correlation function of the Heisenberg operator $A(t)$ of A at inverse temperature β.

Putting

$$R_A(t) = R_A(0,t), \tag{2.18}$$

we have

$$R_A(t_1,t_2) = R_A(t_2 - t_1). \tag{2.19}$$

Thus, as for the correlation function $R_A(t_1,t_2)$, we need only to consider $R_A(t)$.

By (2.12) and (2.14), we have

$$R_A(t) = \sum_{m,n=0}^{\infty} A_{m,n}e^{it(\lambda_m-\lambda_n)} \tag{2.20}$$

with

$$A_{m,n} = |\langle\Phi_{m,n}; A\rangle_H|^2 \geq 0. \tag{2.21}$$

We can enumerate elements of the set

$$\{\varepsilon_p\}_{p=-\infty}^{\infty} \stackrel{\text{def}}{=} \sigma(\mathscr{L}) \cap (0,\infty) \tag{2.22}$$

with order $0 < \cdots < \varepsilon_{-(p+1)} < \varepsilon_{-p} < \cdots < \varepsilon_{-1} < \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_p < \varepsilon_{p+1} < \cdots$,

$\varepsilon_p \to \infty$ as $p \to \infty$. It is easy to see that $\sigma(\mathscr{L})$ is reflection symmetric with respect to the origin, so that we have

$$\{-\varepsilon_p\}_{p=-\infty}^{\infty} \stackrel{\text{def}}{=} \sigma(\mathscr{L}) \cap (-\infty, 0). \tag{2.23}$$

Introducing

$$A_p^{(\pm)} = \sum_{m,n;\, \lambda_m - \lambda_n = \pm\varepsilon_p} A_{m,n}, \quad p \in \boldsymbol{Z}, \tag{2.24}$$

$$R_A^0 = \sum_{m,n;\, \lambda_m - \lambda_n = 0} A_{m,n}, \tag{2.25}$$

we can write

$$R_A(t) = R_A^-(t) + R_A^+(t) + R_A^0, \tag{2.26}$$

with

$$R_A^{\pm}(t) = \sum_{p=-\infty}^{\infty} A_p^{(\pm)} e^{\pm it\varepsilon_p}.$$

Here we set the following condition:

(**A.0**) There exists $p_0 \in \boldsymbol{N}$ such that $A_{-p}^{(\pm)} = 0$ for each p with $-p < -p_0$.

Let

$$\Gamma_V^+ \equiv \{k \in \Gamma_V \,|\, k \geq 0\}, \quad \Gamma_V^- \equiv \{k \in \Gamma_V \,|\, k \leq 0\}. \tag{2.27}$$

Under assumption (**A.0**), we reorder $\{\varepsilon_p\}_{p=-p_0}^{\infty}$ as $\varepsilon_k \equiv \varepsilon_p$ for $k \in \Gamma_V^+$ and $p \in \{-p_0, -p_0+1, -p_0+2, \cdots\}$ with $k = 2\pi(p+p_0)/V$. We note here that for $k \in \Gamma_V^-$, $\varepsilon_k = \varepsilon_{-|k|} = -\varepsilon_{|k|} (-k = |k| \in \Gamma_V^+)$ by the reflective symmetry above. Thus, under (**A.0**), we have

$$R_A^{\pm}(t) = \sum_{k \in \Gamma_V^+} A_k^{(\pm)} e^{\pm it\varepsilon_k}. \tag{2.28}$$

We define a self-adjoint operator $\tilde{\mathscr{L}}$ by

$$\tilde{\mathscr{L}} \stackrel{\text{def}}{=} \mathscr{P}\mathscr{L}\mathscr{P}, \tag{2.29}$$

where $\mathscr{P}$ is the orthogonal projection onto

$$\overline{\mathbf{L.h.}\ [\{\Phi_{m,m} | m \in \boldsymbol{N}^*\} \cup \{\Phi_{m,n} | m, n \in \boldsymbol{N}^* \text{ with } \lambda_m - \lambda_n = 0, \pm\varepsilon_k, k \in \Gamma_V^+\}]}.$$

Then, under (**A.0**), we have

$$A(t) = e^{i\tilde{\mathscr{L}}t} A, \qquad t \in \boldsymbol{R}. \tag{2.30}$$

Thus, *as far as the operator A satisfying* (**A.0**) *is concerned*, it is sufficient to consider $\tilde{\mathscr{L}}$ instead of $\mathscr{L}$. Since we are concerned with only A satisfying (**A.0**) in what follows, henceforth

$$\text{we denote } \tilde{\mathscr{L}} \text{ by } \mathscr{L} \text{ under } (\mathbf{A.0}). \tag{2.31}$$

For a bounded measurable function f on $[0, \infty)$, we can define the Fourier-Laplace transform $[f](z)$ $(\Im z > 0)$ by

$$[f](z) \overset{\text{def}}{=} \int_0^\infty dt\, e^{itz} f(t), \qquad \Im z > 0. \tag{2.32}$$

Note that

$$R_A(0) = \sum_{m,n=0}^{\infty} A_{m,n} = \|A\|_H^2 < \infty, \tag{2.33}$$

and hence

$$|R_A(t)|,\ |R_A^{\pm}(t)| \le \|A\|_H^2, \quad t \in \boldsymbol{R}. \tag{2.34}$$

By this fact, $[R_A]$ and $[R_A^{\pm}]$ exist. For all $M, N \in \boldsymbol{N}$ and $t > 0$, we have

$$\left| \sum_{m=0}^{M} \sum_{n=0}^{N} A_{m,n} e^{it(\lambda_m - \lambda_n)} e^{itz} \right| \le \|A\|_H^2 e^{-t\Im z}.$$

Hence, by the Lebesgue dominated convergence theorem, under (**A.0**) we obtain

$$[R_A](z) = [R_A^-](z) + [R_A^+](z) + \frac{iR_A^0}{z} \tag{2.35}$$

with

$$[R_A^{\pm}](z) = i \sum_{k \in \Gamma_V^+} \frac{A_k^{(\pm)}}{z \pm \varepsilon_k}, \qquad \Im z > 0. \tag{2.36}$$

The function on the right hand side of (2.35) is obviously meromorphic on $\boldsymbol{C}$ with possible poles $\{0, \pm\varepsilon_k \,|\, k \in \Gamma_V^+\}$. Thus, $[R_A](z)$ can be extended uniquely to a meromorphic function on $\boldsymbol{C}$. We denote the extension of $[R_A](z)$ by the same symbol.

It follows from (2.35) and (2.36) that

$$A_k^{(\pm)} = \lim_{z \to \mp\varepsilon_k} \frac{1}{i}(z \pm \varepsilon_k)[R_A](z), \qquad k \in \Gamma_V^+, \tag{2.37}$$

$$R_A^0 = \lim_{z \to 0} \frac{1}{i} z [R_A](z). \tag{2.38}$$

LEMMA 2.1. *Assume* (**A.0**). *Then the function $R_A(t)$ is twice continuously differentiable if and only if*

$$\sum_{k \in \Gamma_V^+} \left(\lim_{z \to \varepsilon_k} \frac{1}{i}(z - \varepsilon_k)[R_A](z) \right) \varepsilon_k^2 < \infty$$

and

$$\sum_{k \in \Gamma_V^+} \left(\lim_{z \to -\varepsilon_k} \frac{1}{i}(z + \varepsilon_k)[R_A](z) \right) \varepsilon_k^2 < \infty.$$

Proof. This lemma follows from the following well-known lemma and (2.37). $\square$

Lemma 2.2. *Let μ be a finite measure on a measure space X. Let $C(t) = \int_X e^{itx} d\mu(x)$, $t \in \boldsymbol{R}$. Then $C(t)$ is twice continuously differentiable if and only if*

$$\int_X x^2 \, d\mu(x) < \infty.$$

Here we note the following lemma:

Lemma 2.3. *Assume* (**A.0**). *If $A \in \mathrm{D}(\mathscr{L})$, then $R_A(t)$ is twice continuously differentiable.*

Proof. Let $A \in \mathrm{D}(\mathscr{L})$. Then we have

$$0 < \sum_{m,n=0}^{\infty} (\lambda_m - \lambda_n)^2 A_{m,n} = \|\mathscr{L}A\|_H^2 < \infty$$

by (2.21). Thus, our lemma follows from (2.14), (2.22), (2.24), (2.28), (2.37), (**A.0**) and Lemma 2.1. $\square$

Let, for $k \in \Gamma_V^+, \mathbf{X}_-^{(k)}(H)$ be the closed subspace generated by $\{\Phi_{m,n} \mid m, n \in \boldsymbol{N}^* \text{ with } \lambda_m - \lambda_n = -\varepsilon_k\}$ and $P_-^{(k)}$ be the orthogonal projection from $\mathbf{X}_c(H)$ onto $\mathbf{X}_-^{(k)}(H)$.

Lemma 2.4. *Assume* (**A.0**). *Suppose that $P_-^{(k)} A \neq 0$ for all $k \in \Gamma_V^+$. Then, for each $k \in \Gamma_V^+$, there exists a simple zero ω_k of $[R_A^-](z)$ such that $\varepsilon_k < \omega_k < \varepsilon_{k+2\pi/V}$. Moreover, the set of zeros of $[R_A^-](z)$ is $\{\omega_k\}_{k \in \Gamma_V^+}$.*

Proof. Let $[R_A^-](c) = 0$. Then, by (2.36), we have $\sum_{k \in \Gamma_V^+} A_k^{(-)} (c - \varepsilon_k)^{-1} = 0$. It is easy to see that c must be positive. The function $f(z) = \sum_{k \in \Gamma_V^+} A_k^{(-)} (z - \varepsilon_k)^{-1}$ is holomorphic on $\boldsymbol{C} \backslash \{\varepsilon_k\}_{k \in \Gamma_V^+}$ and $f'(z) = -\sum_{k \in \Gamma_V^+} A_k^{(-)} (z - \varepsilon_k)^{-2}$. It follows from this and the assumption $P_-^{(k)} A \neq 0$ $(k \in \Gamma_V^+)$ that, in each interval $I_k \equiv (\varepsilon_k, \varepsilon_{k+2\pi/V}), f(z)$ is monotone decreasing with $f(z) \to \infty$ as $z \downarrow \varepsilon_k$ and $f(z) \to -\infty$ as $z \uparrow \varepsilon_{k+2\pi/V}$. Hence there exists a unique ω_k in I_k such that $f(\omega_k) = 0$. It is easy to check that ω_k is simple and that there are no zeros of $f(z)$ other than $\{\omega_k\}_{k \in \Gamma_V^+}$. $\square$

Under assumption (**A.0**) and that $R_A(t)$ is twice continuously differentiable and $A_k^{(-)} \neq 0$ for some k, we can define

$$\omega_V^0 \stackrel{\text{def}}{=} \frac{\sum_{k \in \Gamma_V^+} \varepsilon_k \left(\lim_{z \to \varepsilon_k} \frac{1}{i} (z - \varepsilon_k)[R_A](z) \right)}{\sum_{k \in \Gamma_V^+} \left(\lim_{z \to \varepsilon_k} \frac{1}{i} (z - \varepsilon_k)[R_A](z) \right)} = \frac{\sum_{k \in \Gamma_V^+} \varepsilon_k A_k^{(-)}}{\sum_{k \in \Gamma_V^+} A_k^{(-)}}. \tag{2.39}$$

Remark. We here note that the constant ω_V^0 is determined by $[R_A](z)$ only.

2.2. The Hamiltonian of RWA.

In this subsection, we introduce the Hamiltonian of RWA. Let $x = x^0 \oplus (x_k)_{k \in \Gamma_V} \in \boldsymbol{C} \oplus \ell^2(\Gamma_V)$ be a sequence of real numbers with $0 < x_k < x_{k'}$ for $k, k' \in$

Γ_V^+ with $k < k'$; $x_k = x_{-k}$ for $k \in \Gamma_V^-$ $(-k \in \Gamma_V^+)$, and $\lim_{|k|\to\infty} x_k = \infty$. Then, for this $x = x^0 \oplus (x_k)_{k\in\Gamma_V}$ and a sequence of complex numbers, $y = 0 \oplus (y_k)_{k\in\Gamma_V} \in \boldsymbol{C} \oplus \ell^2(\Gamma_V)$, we define operators h_0 and h_{RWA} by

$$h_0(1\oplus 0) \stackrel{\mathrm{def}}{=} x^0\, 1 \oplus 0, \tag{2.40}$$

$$h_0(0\oplus e_k) \stackrel{\mathrm{def}}{=} x_k\, 0 \oplus e_k, \qquad k \in \Gamma_V, \tag{2.41}$$

$$h_{\mathrm{RWA}} \stackrel{\mathrm{def}}{=} h_0 + (y,\cdot)_{\boldsymbol{C}\oplus\ell^2} 1\oplus 0 + (1\oplus 0,\cdot)_{\boldsymbol{C}\oplus\ell^2}\, y. \tag{2.42}$$

We here assume the following condition:

$$|y_k| = |y_{-k}| \neq 0, \qquad k \in \Gamma_V^+. \tag{2.43}$$

We now define a function $D(z)$ for every $z \in \boldsymbol{C}\backslash\{x_k\}_{k\in\Gamma_V}$ by

$$D(z) \stackrel{\mathrm{def}}{=} z - x^0 + \sum_{k\in\Gamma_V} \frac{|y_k|^2}{x_k - z} = z - x^0 + \sum_{k\in\Gamma_V^+} \frac{\widetilde{y_k}^2}{x_k - z}, \tag{2.44}$$

where $\tilde{y}_0 = |y_0|$ and $\tilde{y}_k = \sqrt{2}|y_k|$ ($k \in \Gamma_V^+$ with $k > 0$).

Then the following lemmas are well-known ([**4**, §II], [**19**, Lemma 4.1]):

Lemma 2.5. *For each $k \in \Gamma_V^+$, there exists a simple zero E_k of $D(z)$ such that $E_0 < x_0 \equiv \inf_{k\in\Gamma_V} x_k$, and $x_{k-2\pi/V} < E_k < x_k$ for $k \in \Gamma_V^+$ with $k \geq 2\pi/V$. Moreover the set of zeros of $D(z)$ is $\{E_k\}_{k\in\Gamma_V^+}$.*

Lemma 2.6. *h_0 and h_{RWA} are self-adjoint, and $\sigma(h_{\mathrm{RWA}}) = \{E_k\}_{k\in\Gamma_V^+}$ with each E_k simple.*

As [**4**, **19**], we define the Hamiltonian of RWA by

$$H_{\mathrm{RWA}}(x,y) \stackrel{\mathrm{def}}{=} d\Gamma(h_{\mathrm{RWA}}), \tag{2.45}$$

where $d\Gamma(h_{\mathrm{RWA}})$ is the second quantization of h_{RWA}.

We define two operators acting in $\mathscr{F}$ by

$$H_0 = d\Gamma(h_0) = x^0 a^* a + \sum_{k\in\Gamma_V} x_k b_k^* b_k, \tag{2.46}$$

$$H_I^{\mathrm{RWA}} = \sum_{k\in\Gamma_V} (\overline{y_k} a^* b_k + y_k b_k^* a), \tag{2.47}$$

where a, b_k ($k \in \Gamma_V$) are the annihilation operators, and a^*, b_k^* ($k \in \Gamma_V$) the creation operators acting in $\mathscr{F}$ ([**11**, §IV]). Then, it follows that $D(H_I^{\mathrm{RWA}}) \supset D(H_0)$, and

$$H_{\mathrm{RWA}}(x,y) = H_0 + H_I^{\mathrm{RWA}}$$

on $D(H_0)$ ([**19**, §IV]).

Remark. The Hamiltonian $H_{\mathrm{RWA}}(x,y)$ of RWA is often used in the quantum optics. $H_{\mathrm{RWA}}(x,y)$ is derived from the Hamiltonian $H_{\mathrm{LC}}(x,y)$ of the linear coupling model,

$$H_{\mathrm{LC}}(x,y) \stackrel{\mathrm{def}}{=} H_0 + \sum_{k\in\Gamma_V}(a^*+a)(y_k b_k^* + \overline{y_k} b_k), \tag{2.48}$$

as follows: now we regard H_0 and $\sum_{k\in\Gamma_V}(a^*+a)(y_k b_k^* + \overline{y_k} b_k)$ as the free part and interaction one, respectively. We have

$$\begin{aligned}
e^{iH_0t}a^*b_k^*e^{-iH_0t} &= e^{i(x^0+x_k)t}a^*b_k^*,\\
e^{iH_0t}ab_ke^{-iH_0t} &= e^{-i(x^0+x_k)t}ab_k,\\
e^{iH_0t}a^*b_ke^{-iH_0t} &= e^{i(x^0-x_k)t}a^*b_k,\\
e^{iH_0t}ab_k^*e^{-iH_0t} &= e^{-i(x^0-x_k)t}ab_k^*
\end{aligned}$$

for each $k\in\Gamma_V$. We here assume that the time t_0 spent in observing particles in an experimental equipment satisfies the inequality,

$$\frac{1}{|x_0+x_k|} \ll t_0 \ll \frac{1}{|x_0-x_k|}, \quad k\in\Gamma_V.$$

Then the equipment can find the effect of the terms for a^*b_k, ab_k^* $(k\in\Gamma_V)$, but it cannot find that of the terms for $a^*b_k^*, ab_k$ $(k\in\Gamma_V)$. So, in the observation using the equipment, we can neglect the terms for $a^*b_k^*, ab_k$ $(k\in\Gamma_V)$ from (2.48), and we obtain $H_{\mathrm{RWA}}(x,y)$. Thus $H_{\mathrm{RWA}}(x,y)$ is very unnatural Hamiltonian, which brings some troubles into physics (see [**13**]).

We now introduce the position and momentum operators exactly. The position operator q and the momentum operator p are given by

$$q \stackrel{\mathrm{def}}{=} \frac{1}{\sqrt{2}}(a+a^*), \qquad p \stackrel{\mathrm{def}}{=} \frac{i}{\sqrt{2}}(a^*-a). \tag{2.49}$$

For $B=q,p$, we define the two-point function $W_B(t_1,t_2)$ by

$$W_B(t_1,t_2) \stackrel{\mathrm{def}}{=} (\Omega_0, e^{iH_{\mathrm{RWA}}(x,y)t_1}Be^{-iH_{\mathrm{RWA}}(x,y)t_1}e^{iH_{\mathrm{RWA}}(x,y)t_2}Be^{-iH_{\mathrm{RWA}}(x,y)t_2}\Omega_0)_{\mathscr{F}}. \tag{2.50}$$

So putting

$$W(t) \stackrel{\mathrm{def}}{=} \tfrac{1}{2}(a^*\Omega_0, e^{iH_{\mathrm{RWA}}(x,y)t}a^*e^{-iH_{\mathrm{RWA}}(x,y)t}\Omega_0)_{\mathscr{F}}, \tag{2.51}$$

we have

$$W_q(t_1,t_2) = W_p(t_1,t_2) = W(t_2-t_1). \tag{2.52}$$

Then, for instance, by [**19**, Lemma 4.5(b), (60)], we obtain

$$W(t) = \frac{1}{2}\sum_{k\in\Gamma_V^+}\frac{e^{iE_kt}}{D'(E_k)}. \tag{2.53}$$

3. Statement of the main results.

Now, we can state our main results:

Theorem 3.1. *Assume that* (**H**), (**A.0**) *and* $A \in \mathbf{T}(H) \cap \mathrm{D}(\mathscr{L})$ *is symmetric with* $P_-^{(k)} A \neq 0$ *for all* $k \in \Gamma_V^+$. *Let* ω_V^0 *be given* (2.39) *and* $\{\omega_k\}_{k \in \Gamma_V^+}$ *be the set of zeros of* $[R_A^-](z)$ (*Lemma* 2.4). *Set* $x = x^0 \oplus (x_k)_{k \in \Gamma_V}$ *and* $y = 0 \oplus (y_k)_{k \in \Gamma_V}$ *with*

$$x^0 = \omega_V^0, \ x_k = x_{-k} = \omega_k, \quad k \in \Gamma_V^+ \ (so \ -k \in \Gamma_V^-), \tag{3.1}$$

$$y_k = \sqrt{2\pi}\rho_k/\sqrt{V}, \quad k \in \Gamma_V, \tag{3.2}$$

where

$$\rho_0 \equiv \tilde{\rho}_0, \qquad \rho_k = \rho_{-k} \equiv \tilde{\rho}_k/\sqrt{2}, \quad k \in \Gamma_V^+ \ (so \ -k \in \Gamma_V^-) \tag{3.3}$$

$$(i.e., \ \tilde{y}_k = \sqrt{2\pi}\tilde{\rho}_k/\sqrt{V})$$

and

$$\tilde{\rho}_k \equiv \sqrt{\frac{VR_A^+(0)}{2\pi i [R_A^+]'(-\omega_k)}}, \quad k \in \Gamma_V^+. \tag{3.4}$$

Then;

(i) $\sum_{k \in \Gamma_V} |y_k|^2 = \frac{2\pi}{V} \sum_{k \in \Gamma_V^+} \tilde{\rho}_k^2 < \infty$ *and*

$$\sigma(h_{\mathrm{RWA}}) \backslash \{\inf \sigma(h_{\mathrm{RWA}})\} = \{\text{positive poles of } [R_A](z)\},$$

where h_{RWA} *is defined by* (2.42) *with* x *and* y *given above.*

(ii) *For all* $t_1, t_2 \in \boldsymbol{R}$ *and* $B = q, p$,

$$R_A(t_1, t_2) = 2(R_A(0) - R_A^0)\Re W_B(t_1, t_2) + R_A^0,$$

where $W_B(t_1, t_2)$ *is defined by* (2.50) *with* x *and* y *given above.*

Remark. In Theorem 3.1, the assumption with respect to Hamiltonian H is (**H**) only. The other assumptions are concerned with observable A.

From now on, we consider the case that the operator A is given by $B = q, p$. So we omit the index "$B(= q, p)$" in $R_B(t)$ and $R_B^{\pm}(t)$. Moreover, we write clearly $V > 0$ in $R_B(t)$ and $R_B^{\pm}(t)$. That is $R^V(t) \equiv R_B(t)$ and $R^{\pm,V}(t) \equiv R_B^{\pm}(t)$. And besides, we set $R^{0,V} \equiv R_B^0$.

We find functions $\omega_\beta(k)$ and $\rho_\beta(k)$ $(k \in \boldsymbol{R})$ such that data ω_k and ρ_k $(k \in \Gamma_V)$ derived from $R^V(t)$ are distributed around $\omega_\beta(k)$ and $\rho_\beta(k)$ respectively. We set

$$\omega_{-k} \equiv \omega_k, \quad k \in \Gamma_V^+. \tag{3.5}$$

So, we extend $\{\omega_k\}_{k \in \Gamma_V^+}$ to the sequence $\{\omega_k\}_{k \in \Gamma_V}$. In order to find such functions, we assume the following technical conditions for existence of the infinite volume limit:

(**A.1**) $\omega_V^0 \to \omega_\beta^0 > 0$ as $V \to \infty$.
(**A.2**) There exist a non-negative, continuously differentiable function ω_β, and a real-valued continuous function $\rho_\beta \in L^2(\boldsymbol{R})$, which satisfy the following conditions;

$$\omega_\beta(k') < \omega_\beta(k), \quad 0 \le k' < k, \tag{3.6}$$

$$\omega_\beta(-k) = \omega_\beta(k), \quad k \in \boldsymbol{R}, \tag{3.7}$$

$$\lim_{k \to \pm\infty} \omega_\beta(k) = \infty, \tag{3.8}$$

$$m \equiv \omega_\beta(0) = \inf_{-\infty < k < \infty} \omega_\beta(k) > 0, \tag{3.9}$$

and there exist constants $\alpha_0 > 1/2, K_0 > 0$, and $\alpha_i > 0, C_i > 0$ $(i = 1, 2)$ such that

$$|\rho_\beta(k)| \le \frac{\Lambda}{1 + |k|^{\alpha_0}} \tag{3.10}$$

for all $|k| \ge K_0$ with Λ a constant (which may depend on α_0 and K_0).

$$|\omega_\beta(k) - \omega_\beta(k')| \le C_1 |k - k'|^{\alpha_1} (1 + \omega_\beta(k) + \omega_\beta(k')), \qquad k, k' \in \boldsymbol{R}, \tag{3.11}$$

$$|\omega_\beta(k) - \omega_k| \le C_2 \left(\frac{\pi}{V}\right)^{\alpha_2}, \qquad k \in \Gamma_V, \tag{3.12}$$

$$\sqrt{\frac{2}{\omega_\beta^0}} \left\| \frac{\rho_\beta}{\sqrt{\omega_\beta}} \right\|_{L^2} < 1, \tag{3.13}$$

$$\sum_{k \in \Gamma_V} \omega_\beta(k)^2 \rho_\beta(k)^2 < \infty, \tag{3.14}$$

$$\int_{-\infty}^{\infty} \omega_\beta(k)^2 \rho_\beta(k)^2 dk < \infty, \tag{3.15}$$

which implies

$$\omega_\beta^j \rho_\beta \in L^2(\boldsymbol{R}), \qquad j = -1, -1/2, 0, 1/2, 1. \tag{3.16}$$

Furthermore, for every $V > 0$, there exists a sequence $\Delta\rho_k \in \boldsymbol{R}$ $(k \in \Gamma_V)$ such that

$$|\rho_\beta(k) - \rho_k| \le \Delta\rho_k, \qquad k \in \Gamma_V, \tag{3.17}$$

$$\sum_{k \in \Gamma_V} \omega_k^2 (\Delta\rho_k)^2 < \infty, \tag{3.18}$$

$$\lim_{V \to \infty} \frac{1}{V} \sum_{k \in \Gamma_V} (\Delta\rho_k)^2 = 0. \tag{3.19}$$

Then, we have

Lemma 3.1. *If $R_\beta^\infty(0) \equiv \lim_{V \to \infty} R^V(0)$ and $R_{\beta,0}^\infty \equiv \lim_{V \to \infty} R^{0,V}$ exist, then $[R_\beta^{+,\infty}](z) \equiv \lim_{V \to \infty} [R^{+,V}](z)$ also exists.*

PROOF. We note $R^{\pm,V}(0) \in \boldsymbol{R}$, so by (4.2), (4.10), (4.11) and (4.33) below, we have

$$[R^{+,V}](z) = -i\left(\frac{R^V(0) - R^{0,V}}{2}\right)\left(-z - \omega_V^0 + \frac{2\pi}{V}\sum_{k \in \Gamma_V^+} \frac{\widetilde{\rho_k}^2}{\omega_k + z}\right)^{-1}. \tag{3.20}$$

So, we have our lemma. □

Here, to use Theorem 3.1 and Lemma 3.1, we assume that

(**A.3**) For every $V > 0, R^V(t)$ is twice continuously differentiable. And $R_\beta^\infty(0) \equiv \lim_{V\to\infty} R^V(0)$ and $R_{\beta,0}^\infty \equiv \lim_{V\to\infty} R_0^V$ exist.

We define a function $D_{\mathrm{RWA}}^\beta(z)$ by

$$D_{\mathrm{RWA}}^\beta(z) \stackrel{\mathrm{def}}{=} \left(\frac{R_\beta^\infty(0) - R_{\beta,0}^\infty}{2}\right)\left(\overline{i[R_\beta^{+,\infty}](-\bar{z})}\right)^{-1}. \tag{3.21}$$

REMARK. $D_{\mathrm{RWA}}^\beta(z)$ is Arai's original function [5, (1.14)] multiplied by -1 (see (4.51) below).

It is clear that there exists the inverse function $\varphi_\beta(x)$ of ω_β such that $\varphi_\beta(x)$ is differentiable and monotone increasing in (m, ∞) with

$$\lim_{x\downarrow m} \varphi_\beta(x) = 0, \quad \varphi_\beta'(x) = (\omega_\beta'(\varphi_\beta(x)))^{-1}, \quad x > m.$$

To use Arai's results in [**5**], we assume a little more assumptions:

(**A.4**) (see [**5**, (AI)])

$$\sup_{0<\varepsilon, m \le x}\left|\int_{-\infty}^\infty \frac{\rho_\beta(k)^2}{(x - i\varepsilon) - \omega_\beta(k)}dk\right| < \infty, \inf_{0<\varepsilon, m \le x}\left|D_{\mathrm{RWA}}^\beta(x - i\varepsilon)\right| < \infty.$$

(**A.5**) (see [**5**, (AII) and §IV]) There exists a constant $\theta(\beta) \in (0, \pi/2)$ such that the function $\varphi_\beta'(x)\rho_\beta(\varphi_\beta(x))^2$ has an analytic continuation $I_\beta^{(0)}(z)$ onto the domain

$$\mathbf{D}_{m,\theta}^\beta \stackrel{\mathrm{def}}{=} \{z \in \boldsymbol{C} \mid \Re z > m, -\theta(\beta) < \arg z < 0\} \tag{3.22}$$

with the following properties:

$$\lim_{\varepsilon\downarrow 0} I_\beta^{(0)}(x - i\varepsilon) = I_\beta^{(0)}(x), \quad x > m, \tag{3.23}$$

$$|I_\beta^{(0)}(z)| \le \mathrm{const}|z|^{-\mathrm{q}_0(\beta)}, \tag{3.24}$$

for all sufficiently large $|z|$ $(z \in \mathbf{D}_{m,\theta}^\beta)$ with a constant $\mathrm{q}_0(\beta) \ge 0$. Moreover,

$$\lim_{z\to 0;\, z \in \mathbf{D}_{m,\theta}^\beta} \frac{I_\beta^{(0)}(m + z)}{z^{\mathrm{p}_0(\beta;m)}} = A_m^{(0)}(\beta) \tag{3.25}$$

with constant $A_m^{(0)}(\beta) \neq 0$ and $\mathrm{p}_0(\beta; m) \geq 0$, and

$$\inf_{0<\varepsilon<\varepsilon_0;\, m \leq x} |D_{\mathrm{RWA}}^{\beta}(x - i\varepsilon) - 2i\pi I_{\beta}^{(0)}(x - i\varepsilon)| > 0 \tag{3.26}$$

for all sufficiently small $\varepsilon_0 > 0$.

So by using Arai's result [**5**, Theorem 1.3], we obtain the following theorem:

THEOREM 3.2. *For $B = q, p$, there exists $R_\beta^\infty(t_1, t_2) \equiv \lim_{V\to\infty} R^V(t_1, t_2)$. Let $B_m^{(0)}(\beta) \stackrel{\mathrm{def}}{=} -(D_{\mathrm{RWA}}^{\beta}(m) + 2i\pi\delta_{0,\mathrm{p}_0(m)} A_m^{(0)}) D_{\mathrm{RWA}}^{\beta}(m)$, and $R_\beta^\infty(t) \equiv R_\beta^\infty(0, t)$.*

(i) *If $R_{\beta,0}^\infty \neq 0$, then $\lim_{t\to\infty} R_\beta^\infty(t) = R_{\beta,0}^\infty$.*

(ii) *If $R_{\beta,0}^\infty = 0$, then*

$$R_\beta^\infty(t) = R_\beta^{+,\infty}(t) + R_\beta^{+,\infty}(-t),$$

$$R_\beta^{+,\infty}(t) \underset{t\to\infty}{\sim} (R_\beta^\infty(0) - R_{\beta,0}^\infty) \frac{A_m^{(0)}(\beta) e^{-i\pi(\mathrm{p}_0(\beta;m)+1)/2} \Gamma(\mathrm{p}_0(\beta; m) + 1)}{B_m^{(0)}(\beta)}$$
$$\times e^{-itm} t^{-(\mathrm{p}_0(\beta;m)+1)},$$

where $\Gamma(z)$ is the gamma function.

REMARK. Concerning part (i), if the condition that $R_{\beta,0}^\infty \neq 0$ occurs, it may be the case where there are infinitely many elements in the thermal states for every $V > 0$ such that the elements are not orthogonal to B just like the superfluidity at $T = 0$. Here the thermal states is a physical notion given by $\overline{\mathbf{L.h.}\ [\{\Phi_{n,n}\}_{n=0,1,\dots}]}$ (i.e., $\mathscr{L}$ (*the thermal state*) $= 0$) in thermo field dynamics (e.g. [**18**]).

4. Proofs of main results.

4.1. Proof of Theorem 3.1.

In this subsection, we prove our main theorem, Theorem 3.1, by using the mathematical structure of Mori's memory kernel equation for quantum statistical physics [**19**, **20**, **21**].

LEMMA 4.1. *For all $C \in \mathbf{T}(H)$ and $m, n \in \boldsymbol{N}^*$,*

$$\langle \Phi_{m,n}; C \rangle_H = \overline{\langle \Phi_{n,m}; C^+ \rangle_H}$$

PROOF. By (2.8), (2.11) and (2.13), we have

$$\beta^{-1/2} Z(\beta)^{-1/2} W_{m,n}^{-1/2} \langle \Phi_{m,n}; C \rangle_H = (\varphi_m, C\varphi_n)_{\mathscr{F}}$$
$$= (C^+\varphi_m, \varphi_n)_{\mathscr{F}} = \overline{(\varphi_n, C^+\varphi_m)_{\mathscr{F}}} = \beta^{-1/2} Z(\beta)^{-1/2} W_{m,n}^{-1/2} \overline{\langle \Phi_{n,m}; C^+ \rangle_H}. \qquad \square$$

It follows from Lemma 4.1 and $A = A^+$ (by the present assumption in Theorem 3.1) that $A_{m,n} = A_{n,m}$ for each $m, n \in \boldsymbol{N}^*$, which implies that $A_k^{(+)} = A_k^{(-)}$ for all $k \in \Gamma_V^+$. Hence we have

$$R_A^-(t) = \overline{R_A^+(t)} \tag{4.1}$$

so that

$$R_A(t) = 2\Re R_A^+(t) + R_A^0. \tag{4.2}$$

Let $\mathbf{X}_+(H)$ be the closed subspace generated by the vectors $\Phi_{m,n}$ with $\lambda_m - \lambda_n > 0$ and P_+ be the orthogonal projection from $\mathbf{X}_c(H)$ onto $\mathbf{X}_+(H)$. It follows that $P_+\mathscr{L} \subset \mathscr{L}P_+$. Hence the vector

$$A_+ \stackrel{\text{def}}{=} P_+A \tag{4.3}$$

is in $\mathrm{D}(\mathscr{L})$, since A is in $\mathrm{D}(\mathscr{L})$. Using Parseval's formula with respect to the complete orthonormal system $\{\Phi_{m,n} \mid m, n \in \boldsymbol{N}^*\}$, one can easily show that

$$\|A_+\|_H^2 = \sum_{k \in \Gamma_V^+} A_k^{(+)} = R_A^{(+)}(0),$$

$$\langle A_+; \mathscr{L}A_+\rangle_H = \sum_{k \in \Gamma_V^+} \varepsilon_k A_k^{(+)}.$$

Hence

$$\omega_V^0 = \frac{\langle A_+; \mathscr{L}A_+\rangle_H}{\|A_+\|_H^2} \tag{4.4}$$

since $A_k^{(+)} = A_k^{(-)}$ $(k \in \Gamma_V^+)$, where ω_V^0 is defined by (2.39).

Let Π_0 be the orthogonal projection from $\mathbf{X}_c(H)$ onto the one-dimensional subspace $\{\alpha A_+ \mid \alpha \in \boldsymbol{C}\}$. We can define a self-adjoint operator $\mathscr{L}_1$ acting in $(I - \Pi_0)\mathbf{X}_c(H)$, called the projected Liouville operator, by

$$\mathrm{D}(\mathscr{L}_1) \stackrel{\text{def}}{=} \mathrm{D}(\mathscr{L}) \cap (I - \Pi_0)\mathbf{X}_c(H), \tag{4.5}$$

$$\mathscr{L}_1 \stackrel{\text{def}}{=} (I - \Pi_0)\mathscr{L} \tag{4.6}$$

(see [**19**, **20**, **28**]).

LEMMA 4.2. *Assume* (**A.0**). *Then, under identification* (2.31), *the spectrum of the projected Liouville operator* $\mathscr{L}_1$ *consists of isolated points only, and* $\sigma_{ess}(\mathscr{L}_1)$ *is a set of all eigenvalues* $\lambda_m - \lambda_n \neq 0$ $(m, n \in \boldsymbol{N}^*)$ *with infinitely dimensional eigenspace of* $\mathscr{L}$.

PROOF. We now define symmetric operator $\mathscr{V}$ acting in $\mathbf{X}_c(H)$ by

$$\mathrm{D}(\mathscr{V}) \stackrel{\text{def}}{=} \mathrm{D}(\mathscr{L}),$$

$$\mathscr{V} \stackrel{\text{def}}{=} -(\Pi_0\mathscr{L} + \mathscr{L}\Pi_0) + \Pi_0\mathscr{L}\Pi_0.$$

We note here that $\Pi_0 C \in \mathrm{D}(\mathscr{L})$ for every $C \in \mathbf{X}_c(H)$ since $A_+ \in \mathrm{D}(\mathscr{L})$. It is evident that $(1 - \Pi_0)\mathscr{L}(1 - \Pi_0) = \mathscr{L} + \mathscr{V}$ on $\mathrm{D}(\mathscr{L})$. Let $C_n \in \mathrm{D}(\mathscr{L})$ $(n \in \boldsymbol{N})$ with conditions $\sup_{n=1,2,\ldots}\|C_n\|_H < \infty$ and $\sup_{n=1,2,\ldots}\|\mathscr{L}C_n\|_H < \infty$. Then, there exist subsequences $\{C_\kappa\}_\kappa \subset \{C_n\}_n$ and vectors $B_1, B_2 \in \mathbf{X}_c(H)$ such that $\text{w-lim}_{\kappa\to\infty} C_\kappa = B_1$ and $\text{w-lim}_{\kappa\to\infty} \mathscr{L}C_\kappa = B_2$. It follows that $\text{s-lim}_{\kappa\to\infty} \Pi_0 C_\kappa = \Pi_0 B_1$ and $\text{s-lim}_{\kappa\to\infty} \Pi_0\mathscr{L}C_\kappa =$

$\Pi_0 B_2$ since Π_0 is a finite rank operator, so a compact operator. And besides, we have

$$\mathscr{L}\Pi_0 C_\kappa = \frac{\langle A_+; C_\kappa\rangle_H}{\langle A_+; A_+\rangle_H}\mathscr{L}A_+ \to \frac{\langle A_+; B_1\rangle_H}{\langle A_+; A_+\rangle_H}\mathscr{L}A_+$$

as $\kappa \to \infty$. Therefore, we have

$$\mathscr{V}C_\kappa \to -\Pi_0 B_2 + \frac{\langle A_+; B_1\rangle_H}{\langle A_+; A_+\rangle_H}(-\mathscr{L}A_+ + \Pi_0\mathscr{L}A_+)$$

as $\kappa \to \infty$. Thus, $\mathscr{V}$ is relatively compact with respect to $\mathscr{L}$, so that $\mathscr{L}+\mathscr{V}$ is self-adjoint, and $\sigma_{ess}(\mathscr{L}) = \sigma_{ess}(\mathscr{L}+\mathscr{V})$. Hence it follows our lemma from (2.14), (2.15) and (2.31). □

Let

$$A_+(t) \stackrel{\text{def}}{=} e^{i\mathscr{L}t}A_+, \qquad t \in \boldsymbol{R}. \tag{4.7}$$

Then we have

$$R_A^+(t) = \langle A_+; A_+(t)\rangle_H.$$

Let

$$I_+(t) \stackrel{\text{def}}{=} ie^{i\mathscr{L}_1 t}(I-\Pi_0)\mathscr{L}A_+, \qquad t \in \boldsymbol{R}, \tag{4.8}$$

$$\phi_+(t) \stackrel{\text{def}}{=} \frac{\langle I_+(0); I_+(t)\rangle_H}{\|A_+\|_H^2}, \qquad t \in \boldsymbol{R}. \tag{4.9}$$

Then $A_+(t)$ satisfies Mori's memory kernel equation

$$\frac{d}{dt}A_+(t) = i\omega_V^0 A_+(t) - \int_0^t \phi_+(t-s)A_+(s)\,ds + I_+(t), \qquad t \in \boldsymbol{R},$$

(see Theorem A.1(iii) in §5. Appendix). Since $\langle A_+; I_+(t)\rangle_H = 0$, it follows that, by Theorem A.1(i) in §5. Appendix,

$$\frac{d}{dt}R_A^+(t) = i\omega_V^0 R_A^+(t) - \int_0^t \phi_+(t-s)R_A^+(s)\,ds. \tag{4.10}$$

Lemma 4.3. *Assume* (**A.0**). *Let ω_k $(k \in \Gamma_V^+)$ be the zeros of $[R_A^-](z)$ (Lemma 2.4). Then $\{\omega_k\}_{k\in\Gamma_V^+} \subset \sigma(\mathscr{L}_1)$ and*

$$\phi_+(t) = \sum_{k\in\Gamma_V^+}\frac{R_A^+(0)}{i[R_A^+]'(-\omega_k)}e^{it\omega_k} = \sum_{k\in\Gamma_V^+}\tilde{y}_k^2 e^{it\omega_k} = \frac{2\pi}{V}\sum_{k\in\Gamma_V^+}\tilde{\rho}_k^2 e^{it\omega_k} \tag{4.11}$$

with

$$\sum_{k\in\Gamma_V^+}\left|\frac{1}{[R_A^+]'(-\omega_k)}\right| < \infty. \tag{4.12}$$

PROOF. By Lemma 4.2 and the fact $\dim(I - \Pi_0)\mathbf{X}_c(H) = \infty$, there exist real constants γ_k $(k \in \Gamma_V^+)$ and vectors $\Psi_k \in \mathrm{D}(\mathscr{L}_1)$ such that $\mathscr{L}_1 \Psi_k = \gamma_k \Psi_k$ $(k \in \Gamma_V^+)$; $\gamma_k \neq \gamma_j$ if $k \neq j$; and $\sigma(\mathscr{L}_1) = \{\gamma_k\}_{k \in \Gamma_V^+}$. Then we have

$$\phi_+(t) = \frac{1}{R_A^+(0)} \sum_{k \in \Gamma_V^+} a_k e^{it\gamma_k} \tag{4.13}$$

with $a_k = |\langle \Psi_k;\ (I - \Pi_0)\mathscr{L}A_+\rangle_H|^2$. It is easy to see that $[\phi_+](z)$ $(\Im z > 0)$ can be analytically continued as a meromorphic function on $\boldsymbol{C}$ with

$$[\phi_+](z) = \frac{i}{R_A^+(0)} \sum_{k \in \Gamma_V^+} \frac{a_k}{z + \gamma_k}. \tag{4.14}$$

On the other hand, by Theorem A.1(ii) in §5. Appendix, (4.10) gives

$$[\phi_+](z) = i(z + \omega_V^0) + \frac{R_A^+(0)}{[R_A^+](z)}. \tag{4.15}$$

From this and (4.14), it follows that $-\gamma_k$ with $a_k \neq 0$ is a zero of $[R_A^+](z)$ and we have

$$a_k = \frac{R_A^+(0)^2}{i[R_A^+]'(-\gamma_k)} > 0. \tag{4.16}$$

It is obvious that any zero of $[R_A^+](z)$ is a pole of $[\phi_+](z)$ and hence it is equal to one of $-\gamma_k$'s with $a_k \neq 0$. Since $[R_A^-](z) = \overline{[R_A^+](-\bar{z})}$ by (4.1), we have $\gamma_k = \omega_k$, namely the set of zeros of $[R_A^+](z)$ is equal to $\{-\omega_k\}_{k \in \Gamma_V^+}$. Thus the first half of lemma follows. Putting (4.16) and $\gamma_k = \omega_k$ into (4.13), we obtain (4.11). Since $\sum_{k \in \Gamma_V^+} a_k = \|(I - \Pi_0)\mathscr{L}A_+\|_H^2 < \infty$, we have (4.12). $\square$

PROOF OF PART (i) OF THEOREM 3.1. The fact $\sum_{k \in \Gamma_V^+} |y_k|^2 < \infty$ follows from (4.12). It is clear that the set of positive poles of $[R_A](z)$ is equal to the set of poles of $[R_A^-](z)$, i.e., $\{\varepsilon_k\}_{k \in \Gamma_V^+}$. By (3.20), (4.14), (4.15), Lemma 4.3, and the fact $\lim_{z \to -\varepsilon_k} 1/[R_A^+](z) = 0$, we have

$$\varepsilon_k - \omega_V^0 + \sum_{\ell \in \Gamma_V^+} \frac{R_A^+(0)}{i[R_A^+]'(-\omega_\ell)} \frac{1}{\omega_\ell - \varepsilon_\ell} = 0,$$

i.e., ε_k is a solution to $D(z) = 0$ with $x^0 = \omega_V^0$, $x_k = x_{-k} = \omega_k$ $(k \in \Gamma_V^+)$, and $y_0 = \tilde{y}_0$, $y_\ell = y_{-\ell} = \sqrt{2}\tilde{y}_\ell$, where $\tilde{y}_\ell{}^2 = R_A^+(0)/i[R_A^+]'(-\omega_\ell)$ $(\ell \in \Gamma_V^+)$. Hence $\varepsilon_k = E_k$ $(k \in \Gamma_V^+$: see Lemma 2.5 in §2.2). Thus, by Lemma 2.6 in §2.2, we obtain the desired result.

PROOF OF PART (ii) OF THEOREM 3.1. Let $W(t)$ be given by (2.51) in §2.2. Let

$$a^*(t) = e^{iH_{\mathrm{RWA}}(x,y)t} a^* e^{-iH_{\mathrm{RWA}}(x,y)t}, \quad t \in \boldsymbol{R}.$$

Then we can show that

$$\frac{d}{dt} a^*(t) = ix^0 a^*(t) - \int_0^t \phi_{\mathrm{RWA}}(t-s) a^*(s)\, ds + I_{\mathrm{RWA}}(t),$$

where

$$I_{\mathrm{RWA}}(t) = i \sum_{k \in \Gamma_V} e^{ix_k t} y_k b_k^*, \quad \phi_{\mathrm{RWA}}(t) = \sum_{k \in \Gamma_V} y_k^2 e^{ix_k t}$$

(see [**17**], [**19**, (120)]). Since $(a^* \Omega_0, I_{\mathrm{RWA}}(t)\Omega_0)_{\mathscr{F}} = 0$, it follows that

$$\frac{d}{dt} W(t) = ix^0 W(t) - \int_0^t \phi_{\mathrm{RWA}}(t-s) W(s)\, ds.$$

Note that $W(0) = 1/2$. Take x^0, x_k and y_k as in the assumption of Theorem 3.1. Then $\phi_+(t) = \phi_{\mathrm{RWA}}(t)$ $(t \in \boldsymbol{R})$. Hence $W(t)$ obeys the same equation as that $R_A^+(t)$ satisfies (see (4.10)). Thus, by the uniqueness of solution to Eq. (4.10), we have

$$R_A^+(t) = 2R_A(0) W(t), \quad t \in \boldsymbol{R}.$$

Putting this expression into (4.2), we obtain the desired result. Note that, by (4.1), $2R_A^+(0) = R_A(0) - R_A^0$ and $R_A^{\pm}(0) \in \boldsymbol{R}$.

4.2. Proof of Theorem 3.2.

We prepare basic tools from [**7**, **8**, **16**]. Let $\mathscr{F}_b^\infty$ be the symmetric Fock space given by standing $\ell^2(\Gamma_V)$ for $L^2(\boldsymbol{R})$ in (2.1), i.e.,

$$\mathscr{F}_b^\infty \stackrel{\text{def}}{=} \bigoplus_{n=0}^{\infty} S_n(L^2(\boldsymbol{R}))^n. \tag{4.17}$$

We use the operator-valued distribution kernels $b(k)$ and $b(k)^*$ of the standard smeared annihilation and creation operators, respectively [**11**, (4.3.13)–(4.3.15)].

We define a Hamiltonian H_b of boson free particles in the infinite volume by

$$H_b \stackrel{\text{def}}{=} d\Gamma(\omega_\beta) = \int_{-\infty}^{\infty} \omega_\beta(k) b(k)^* b(k)\, dk. \tag{4.18}$$

We can define the Fock space $\mathscr{F}_b^V$ for the volume V by standing $\ell^2(\Gamma_V)$ for $\boldsymbol{C} \oplus \ell^2(\Gamma_V)$ in (2.1), i.e.,

$$\mathscr{F}_b^V \stackrel{\text{def}}{=} \bigoplus_{n=0}^{\infty} S_n(\ell^2(\Gamma_V))^n, \tag{4.19}$$

which describes state vectors of bosons in the finite box $[-V/2, V/2]$. Then we can identify $\mathscr{F}_b^V$ with the subspace of $\mathscr{F}_b^\infty$ since each element in $S_n(\ell^2(\Gamma_V))^n$ can be identified with a piecewise constant function in $S_n(L^2(\boldsymbol{R}))^n$ which is a constant on each cube of volume $(2\pi/V)^n$ centered about a lattice point $(k_1, \dots, k_n) \in \Gamma_V^n \equiv \Gamma_V \times \Gamma_V \times \cdots \times \Gamma_V$.

The periodic annihilation and creation operator-valued distribution kernels $b_V(k)$ and $b_V(k)^*$ are defined by

$$b_V(k) \stackrel{\text{def}}{=} \left(\frac{V}{2\pi}\right)^{1/2} \int_{-\pi/V}^{\pi/V} b(k+l) dl, \tag{4.20}$$

$$b_V(k)^* \stackrel{\text{def}}{=} \left(\frac{V}{2\pi}\right)^{1/2} \int_{-\pi/V}^{\pi/V} b(k+l)^* dl \tag{4.21}$$

acting in $\mathscr{F}_b^V$.

We define functions ω_β^V and $\omega_\beta^{(V)}$ by

$$\omega_\beta^V(k) \stackrel{\text{def}}{=} \omega_\beta(k_V), \tag{4.22}$$

$$\omega_\beta^{(V)}(k) \stackrel{\text{def}}{=} \omega_{k_V} \tag{4.23}$$

for $k \in \boldsymbol{R}$ with k_V a discrete point closed to k:

$$k_V \in \Gamma_V, \quad \text{with } |k - k_V| \le \frac{\pi}{V}. \tag{4.24}$$

We define Hamiltonians H_b^V and $H_b^{(V)}$ of boson free particles in the finite volume $V > 0$ by

$$H_b^V \stackrel{\text{def}}{=} d\Gamma(\omega_\beta^V) = \int_{-\infty}^{\infty} \omega_\beta^V(k) b(k)^* b(k)\, dk, \tag{4.25}$$

$$H_b^{(V)} \stackrel{\text{def}}{=} d\Gamma(\omega_\beta^{(V)}) = \int_{-\infty}^{\infty} \omega_\beta^{(V)}(k) b(k)^* b(k)\, dk. \tag{4.26}$$

We set

$$C_1^V \equiv C_1 \left(\frac{\pi}{V}\right)^{\alpha_1} \left(\frac{1}{2m} + 1\right), \quad C_2^V \equiv C_2 \max\left\{\frac{1}{m}, \frac{1}{\varepsilon_0}\right\} \left(\frac{\pi}{V}\right)^{\alpha_2}. \tag{4.27}$$

In what follows we assume that

$$C_i^V < 1, \quad i = 1, 2 \tag{4.28}$$

since this is satisfied for all sufficiently large V. Here we note that (3.11) implies

$$|\omega_\beta(k) - \omega_\beta^V(k)| \le \frac{2C_1^V}{1 - C_1^V} \omega_\beta(k), \quad k \in \boldsymbol{R} \tag{4.29}$$

$$|\omega_\beta^V(k) - \omega_\beta^{(V)}(k)| \le C_2^V \omega_\beta^V(k), \quad k \in \boldsymbol{R} \tag{4.30}$$

(see [**8**, proof of Lemma 3.1]).

In the same way as [**7**, Lemma 3.1] and [**8**, Lemma 3.1], by (3.11) with (4.29), (3.12) with (4.30), (3.9) and (4.24) we have the following lemma:

Lemma 4.4. $D(H_b^V) = D(H_b^{(V)}) = D(H_b)$ *and*

$$\|(H_b - H_b^V)\Psi\|_{\mathscr{F}_b^\infty} \le \frac{2C_1^V}{1 - C_1^V} \|H_b \Psi\|_{\mathscr{F}_b^\infty},$$

$$\|(H_b^V - H_b^{(V)})\Psi\|_{\mathscr{F}_b^\infty} \le C_2^V \|H_b^V \Psi\|_{\mathscr{F}_b^\infty}$$

for $\Psi \in D(H_b)$.

In the same way as [**7**, Lemma 3.3 and (3.12)], by (3.10), we can define a function $\rho_\beta^V \in L^2(\boldsymbol{R})$ as

$$\rho_\beta^V \overset{\text{def}}{=} L^2\!-\lim_{K\to\infty} \sum_{\ell \in \Gamma_V, |\ell| \le K} \rho_\beta(\ell)\chi_{[\ell-\pi/V, \ell+\pi/V]} \tag{4.31}$$
$$= \sum_{\ell \in \Gamma_V} \rho_\beta(\ell)\chi_{[\ell-\pi/V, \ell+\pi/V]},$$

where χ_I denotes the characteristic function of an interval I. Moreover, we define $\rho_\beta^{(V)}$ on $\boldsymbol{R}$ by

$$\rho_\beta^{(V)} \overset{\text{def}}{=} \sum_{\ell \in \Gamma_V} \rho_\ell \chi_{[\ell-\pi/V, \ell+\pi/V]}, \tag{4.32}$$

where ρ_ℓ's were given in (3.3). We note that $\rho_\beta^{(V)} \in L^2(\boldsymbol{R})$ by Theorem 3.1(i) and (3.2).

Lemma 4.5.
(i) $\lim_{V\to\infty} \|\rho_\beta^V - \rho_\beta\|_{L^2} = 0$.
(ii) $\lim_{V\to\infty} \|\rho_\beta^V - \rho_\beta^{(V)}\|_{L^2} = 0$.
(iii) $\lim_{V\to\infty} \left\| \frac{\rho_\beta^V}{\sqrt{\omega_\beta^V}} - \frac{\rho_\beta}{\sqrt{\omega_\beta}} \right\|_{L^2} = 0$.
(iv) $\lim_{V\to\infty} \left\| \frac{\rho_\beta^V}{\sqrt{\omega_\beta^V}} - \frac{\rho_\beta^{(V)}}{\sqrt{\omega_\beta^{(V)}}} \right\|_{L^2} = 0$.
(v) *For every* $z \in \boldsymbol{C}$ *with* $\Im z \neq 0$,

$$\lim_{V\to\infty} \frac{2\pi}{V} \sum_{\ell \in \Gamma_V^+} \frac{\tilde{\rho}_\ell{}^2}{\omega_\ell + z} = \lim_{V\to\infty} \int_{-\infty}^{\infty} \frac{\rho_\beta^{(V)}(k)^2}{\omega_\beta^{(V)}(k) + z}\, dk = \int_{-\infty}^{\infty} \frac{\rho_\beta(k)^2}{\omega_\beta(k) + z}\, dk \tag{4.33}$$

Proof. Since we can use (3.10) as a growth condition for ρ_β, in the same way as [**7**, Lemma 4.2], we have part (i). By (3.9) and (3.10), we have

$$\left| \frac{\rho_\beta(k)}{\sqrt{\omega_\beta(k)}} \right| \le \frac{\Lambda}{\sqrt{m}} \frac{1}{1 + |k|^{\alpha_0}},$$

which is a growth condition for $\rho_\beta/\sqrt{\omega_\beta}$. So we have part (iii) in the same way as part (i). Part (ii) follows from (3.19). We have the following inequality by Lemma 2.4, (3.12), (3.19) and part (i):

$$\begin{aligned}
&\left\| \frac{\rho_\beta^V}{\sqrt{\omega_\beta^V}} - \frac{\rho_\beta^{(V)}}{\sqrt{\omega_\beta^{(V)}}} \right\|_{L^2}^2 \\
&\le \frac{2}{\sqrt{\varepsilon_0}} \frac{2\pi}{V} \sum_{\ell \in \Gamma_V} (\triangle\rho_\ell)^2 + \frac{2}{\sqrt{m\varepsilon_0}} \frac{2\pi}{V} \sum_{\ell \in \Gamma_V} \left| \sqrt{\omega_\ell} - \sqrt{\omega_\beta(\ell)} \right|^2 \rho_\beta(\ell)^2 \\
&\le \frac{2}{\sqrt{\varepsilon_0}} \frac{2\pi}{V} \sum_{\ell \in \Gamma_V} (\triangle\rho_\ell)^2 + \frac{2}{\sqrt{m\varepsilon_0}} \frac{C_2^2}{(\sqrt{\varepsilon_0} + \sqrt{m})^2} \left(\frac{\pi}{V}\right)^{2\alpha_2} \|\rho_\beta^V\|_{L^2}^2 \to 0
\end{aligned}$$

as $V \to \infty$, which is a proof of part (iv). For every $z \in \boldsymbol{C}$ with $\Im z \neq 0$, we have

(4.34)

$$\begin{aligned}
J(z) &\equiv \left| \int_{-\infty}^{\infty} \left(\frac{\rho_\beta^{(V)}(k)^2}{\omega_\beta^{(V)}(k) + z} - \frac{\rho_\beta(k)^2}{\omega_\beta(k) + z} \right) dk \right| \\
&\leq \left| \int_{-\infty}^{\infty} \left\{ \frac{1}{\omega_\beta^{(V)}(k) + z} (\rho_\beta^{(V)}(k)^2 - \rho_\beta^V(k)^2) \right. \right. \\
&\quad \left. \left. + \left(\frac{1}{\omega_\beta^{(V)}(k) + z} - \frac{1}{\omega_\beta^V(k) + z} \right) \rho_\beta^V(k)^2 \right\} dk \right| \\
&\quad + \left| \int_{-\infty}^{\infty} \left\{ \frac{1}{\omega_\beta^V(k) + z} (\rho_\beta^V(k)^2 - \rho_\beta(k)^2) \right. \right. \\
&\quad \left. \left. + \left(\frac{1}{\omega_\beta^V(k) + z} - \frac{1}{\omega_\beta(k) + z} \right) \rho_\beta(k)^2 \right\} dk \right| \\
&\leq \frac{1}{|\Im z|} \left(\int_{-\infty}^{\infty} |\rho_\beta^{(V)}(k)^2 - \rho_\beta^V(k)^2| \, dk + \int_{-\infty}^{\infty} |\rho_\beta^V(k)^2 - \rho_\beta(k)^2| \, dk \right) \\
&\quad + \frac{1}{|\Im z|^2} \left(\int_{-\infty}^{\infty} |\omega_\beta^V(k) - \omega_\beta^{(V)}(k)| \rho_\beta^V(k)^2 dk + \int_{-\infty}^{\infty} |\omega_\beta(k) - \omega_\beta^V(k)| \rho_\beta(k)^2 dk \right).
\end{aligned}$$

By Schwarz's inequality,

$$\int_{-\infty}^{\infty} |\rho_\beta^{(V)}(k)^2 - \rho_\beta^V(k)^2| \, dk \leq (\|\rho_\beta^{(V)}\|_{L^2} + \|\rho_\beta^V\|_{L^2}) \|\rho_\beta^{(V)} - \rho_\beta^V\|_{L^2}, \tag{4.35}$$

$$\int_{-\infty}^{\infty} |\rho_\beta^V(k)^2 - \rho_\beta(k)^2| \, dk \leq (\|\rho_\beta^V\|_{L^2} + \|\rho_\beta\|_{L^2}) \|\rho_\beta^V - \rho_\beta\|_{L^2}. \tag{4.36}$$

By (3.12), (3.16) and (4.29),

$$\int_{-\infty}^{\infty} |\omega_\beta^V(k) - \omega_\beta^{(V)}(k)| \rho_\beta^V(k)^2 \leq C_2 \left(\frac{\pi}{V} \right)^{\alpha_2} \|\rho_\beta^V\|_{L^2}^2, \tag{4.37}$$

$$\int_{-\infty}^{\infty} |\omega_\beta(k) - \omega_\beta^V(k)| \rho_\beta(k)^2 \leq \frac{2C_1^V}{1 - C_1^V} \|\sqrt{\omega_\beta} \rho_\beta\|_{L^2}^2. \tag{4.38}$$

By (4.34)–(4.38),

$$\begin{aligned}
J(z) &\leq \frac{1}{|\Im z|} \{ (\|\rho_\beta^{(V)}\|_{L^2} + \|\rho_\beta^V\|_{L^2}) \|\rho_\beta^{(V)} - \rho_\beta^V\|_{L^2} + (\|\rho_\beta^V\|_{L^2} + \|\rho_\beta\|_{L^2}) \|\rho_\beta^V - \rho_\beta\|_{L^2} \} \\
&\quad + \frac{1}{|\Im z|^2} \left\{ C_2 \left(\frac{\pi}{V} \right)^{\alpha_2} \|\rho_\beta^V\|_{L^2}^2 + \frac{2C_1^V}{1 - C_1^V} \|\sqrt{\omega_\beta} \rho_\beta\|_{L^2}^2 \right\} \to 0
\end{aligned}$$

as $V \to \infty$ since $\lim_{V\to\infty}\|\rho_\beta^{(V)}\|_{L^2} = \|\rho_\beta\|_{L^2} = \lim_{V\to\infty}\|\rho_\beta^V\|_{L^2}$ and $\lim_{V\to\infty}\|\rho_\beta^{(V)} - \rho_\beta^V\|_{L^2} = 0 = \lim_{V\to\infty}\|\rho_\beta^V - \rho_\beta\|_{L^2}$ by parts (i) and (ii). So we obtain part (v). □

Let H_a^V and H_a be self-adjoint operators defined as the closure of $\omega_V^0 a^* a$ and $\omega_\beta^0 a^* a$, respectively, where a and a^* above are annihilation and creation operators acting in $L^2(\boldsymbol{R})$ defined by $af_n \stackrel{\text{def}}{=} \sqrt{n} f_{n-1}$ and $a^* f_n \stackrel{\text{def}}{=} \sqrt{n+1} f_{n+1}$, respectively, for a complete orthonormal basis $\{f_n\}_{n\in \boldsymbol{N}^*}$ of $L^2(\boldsymbol{R})$.

Let

$$\mathscr{F}^\infty \stackrel{\text{def}}{=} L^2(\boldsymbol{R}) \otimes \mathscr{F}_b^\infty. \tag{4.39}$$

We define Hamiltonians H_{RWA}, H_{RWA}^V, and $H_{\text{RWA}}^{(V)}$ by

$$\begin{aligned} H_{\text{RWA}} &\stackrel{\text{def}}{=} H_0 + H_I, \\ H_0 &\equiv H_a \otimes I + I \otimes H_b, \\ H_I &\equiv \int_{-\infty}^{\infty} \rho_\beta(k)(a^* \otimes b(k) + a \otimes b(k)^*)\, dk, \end{aligned} \tag{4.40}$$

$$\begin{aligned} H_{\text{RWA}}^V &\stackrel{\text{def}}{=} H_0^V + H_I^V, \\ H_0^V &\equiv H_a^V \otimes I + I \otimes H_b^V, \\ H_I^V &\equiv \int_{-\infty}^{\infty} \rho_\beta^V(k)(a^* \otimes b(k) + a \otimes b(k)^*)\, dk, \end{aligned} \tag{4.41}$$

$$\begin{aligned} H_{\text{RWA}}^{(V)} &\stackrel{\text{def}}{=} H_0^{(V)} + H_I^{(V)}, \\ H_0^{(V)} &\equiv H_a^V \otimes I + I \otimes H_b^{(V)}, \\ H_I^{(V)} &\equiv \int_{-\infty}^{\infty} \rho_\beta^{(V)}(k)(a^* \otimes b(k) + a \otimes b(k)^*)\, dk. \end{aligned} \tag{4.42}$$

We note that, for $v = -1/2, 0, 1$, $(\omega_\beta)^v \rho_\beta \in L^2(\boldsymbol{R})$ by (3.9) and (3.16). Similarly, (3.14) implies $(\omega_\beta^V)^v \rho_\beta^V \in L^2(\boldsymbol{R})$ for $v = -1/2, 0, 1$. By (3.12), (3.17) and (3.18), we have

$$\|\omega_\beta^{(V)} \rho_\beta^{(V)}\|_{L^2}^2 \le \frac{4\pi}{V} \sum_{\ell\in\Gamma_V} \omega_\ell^2 (\Delta\rho_\ell)^2 + 4\|\omega_\beta^V \rho_\beta^V\|_{L^2}^2 + 4C_2^2 \left(\frac{\pi}{V}\right)^{2\alpha_2} \|\rho_\beta^V\|_{L^2}^2,$$

which implies $(\omega_\beta^{(V)})^v \rho_\beta^{(V)} \in L^2(\boldsymbol{R})$ $(v = -1/2, 0, 1)$. So, by Lemma 4.4, and applying [**4**, Proposition2.1] to (3.15) and (3.16),

Lemma 4.6. (i) *$D(H_0^V) = D(H_0^{(V)}) = D(H_0)$, and*

$$\|(H_0^V - H_0)\Psi\| \le \sqrt{2}\max\left\{\frac{2C_1^V}{1-C_1^V}, \frac{|\omega_V^0 - \omega_\beta^0|}{\omega_\beta^0}\right\}\|H_0\Psi\|,$$

$$\|(H_0^{(V)} - H_0^V)\Psi\| \le \sqrt{2}C_2^V\|H_0^V\Psi\|$$

for $\Psi \in D(H_0)$.

(ii) $D(H_0) \subset D(H_{\mathrm{RWA}}^\#)$ *and the closure of* $H_{\mathrm{RWA}}^\# \lceil D(H_0)$ *is essentially self-adjoint on any core for* $H_0^\#$, *where* $H_0^\# = H_0$ (*if* $H_{\mathrm{RWA}}^\# = H_{\mathrm{RWA}}$); $H_0^\# = H_0^V$ (*if* $H_{\mathrm{RWA}}^\# = H_{\mathrm{RWA}}^V$); $H_0^\# = H_0^{(V)}$ (*if* $H_{\mathrm{RWA}}^\# = H_{\mathrm{RWA}}^{(V)}$).

By using well-known inequalities [**11**, (4.3.33) and (4.3.34)] with respect to creation and annihilation operators, and noticing $f/\sqrt{\omega_\beta} \in L^2(\boldsymbol{R})$ for every $f \in L^2(\boldsymbol{R})$ since $m, \varepsilon_0 > 0$, we have

Lemma 4.7. *For* $\Psi \in D(H_0^{\#1/2})$, $f \in L^2(\boldsymbol{R})$,

$$\|a \otimes b(f)^*\Psi\| \le F_\varepsilon^1(\omega_\#^0, \omega_\beta^\#, f)\|H_0^\#\Psi\| + G_\varepsilon^1(\omega_\#^0, f)\|\Psi\|,$$

$$\|a^* \otimes b(f)\Psi\| \le F_{\varepsilon'}^2(\omega_\#^0, \omega_\beta^\#, f)\|H_0^\#\Psi\| + G_{\varepsilon'}^2(\omega_\beta^\#, f)\|\Psi\|,$$

where

$$F_\varepsilon^1(\omega_\#^0, \omega_\beta^\#, f) \equiv \frac{1}{\sqrt{\omega_\#^0}}\left(\frac{1}{2}\|f/\sqrt{\omega_\beta^\#}\|_{L^2} + \varepsilon\|f\|_{L^2}\right),$$

$$F_{\varepsilon'}^2(\omega_\#^0, \omega_\beta^\#, f) \equiv \left(\sqrt{\frac{1}{2\omega_\#^0}} + \varepsilon'\right)\|f/\sqrt{\omega_\beta^\#}\|_{L^2},$$

$$G_\varepsilon^1(\omega_\#^0, f) \equiv \frac{\|f\|_{L^2}}{4\varepsilon\sqrt{\omega_\#^0}},$$

$$G_{\varepsilon'}^2(\omega_\beta^\#, f) \equiv \frac{\|f/\sqrt{\omega_\beta^\#}\|_{L^2}}{4\varepsilon'}.$$

for every $\varepsilon, \varepsilon' > 0$, *where* $\omega_\#^0 = \omega_\beta^0$ (*if* $H_0^\# = H_0$); $\omega_\#^0 = \omega_V^0$ (*if* $H_0^\# = H_0^V, H_0^{(V)}$), *and* $\omega_\beta^\# = \omega_\beta$ (*if* $H_0^\# = H_0$); $\omega_\beta^\# = \omega_\beta^V$ (*if* $H_0^\# = H_0^V$); $\omega_\beta^\# = \omega_\beta^{(V)}$ (*if* $H_0^\# = H_0^{(V)}$).

We set

$$\begin{aligned}
(4.43)\qquad F_{\varepsilon,\varepsilon'}(\omega_\#^0, \omega_\beta^\#, f) &\equiv F_\varepsilon^1(\omega_\#^0, \omega_\beta^\#, f) + F_{\varepsilon'}^2(\omega_\#^0, \omega_\beta^\#, f)\\
&= \sqrt{\frac{2}{\omega_\#^0}}\|f/\sqrt{\omega_\beta^\#}\|_{L^2} + \frac{\varepsilon}{\sqrt{\omega_\#^0}}\|f\|_{L^2} + \varepsilon'\|f/\sqrt{\omega_\beta^\#}\|_{L^2}\\
&\le \left\{\left(\sqrt{\frac{2}{\omega_\#^0}} + \varepsilon'\right)\max\left\{\frac{1}{\sqrt{m}}, \frac{1}{\sqrt{\varepsilon_0}}\right\} + \frac{\varepsilon}{\sqrt{\omega_\#^0}}\right\}\|f\|_{L^2},
\end{aligned}$$

$$
\begin{aligned}
(4.44)\qquad G_{\varepsilon,\varepsilon'}(\omega^0_{\#}, \omega^{\#}_{\beta}, f) &\equiv G^1_{\varepsilon}(\omega^0_{\#}, f) + G^2_{\varepsilon'}(\omega^{\#}_{\beta}, f) \\
&= \frac{1}{4}\left(\frac{\|f\|_{L^2}}{\varepsilon\sqrt{\omega^0_{\#}}} + \frac{\|f/\sqrt{\omega^{\#}_{\beta}}\|_{L^2}}{\varepsilon'}\right) \\
&\le \frac{1}{4}\left\{\frac{1}{\varepsilon\sqrt{\omega^0_{\#}}} + \frac{1}{\varepsilon'}\max\left\{\frac{1}{\sqrt{m}}, \frac{1}{\sqrt{\varepsilon_0}}\right\}\right\}\|f\|_{L^2}.
\end{aligned}
$$

By Lemmas 4.5, 4.6, 4.7 with (**A.1**) and (3.13), similarly to [**8**, Lemma 3.4],

Lemma 4.8. (i) *There exist constant* $c_1 > 0$ *and* $d_1 > 0$ (c_1 *and* d_1 *may depend on* ε, $\varepsilon' > 0$ *in Lemma* 4.7) *such that*

$$\|H_0\Psi\| \le c_1\|H_{\mathrm{RWA}}\Psi\| + d_1\|\Psi\|, \quad \Psi \in D(H_0).$$

In particular, for all $z \in \boldsymbol{C}\backslash\boldsymbol{R}$, $H_0(H_{\mathrm{RWA}} - z)^{-1}$ *is bounded.*

(ii) *There exist constant* $c_2 > 0$ *and* $d_2 > 0$ *independent of* V (c_2 *and* d_2 *may depend on* ε, $\varepsilon' > 0$ *in Lemma* 4.7) *such that*

$$\|H^V_0\Psi\| \le c_2\|H^V_{\mathrm{RWA}}\Psi\| + d_2\|\Psi\|, \quad \Psi \in D(H_0)$$

for sufficiently large $V > 0$. *In particular, for all* $z \in \boldsymbol{C}\backslash\boldsymbol{R}$, $H^V_0(H^V_{\mathrm{RWA}} - z)^{-1}$ *is bounded.*

Lemma 4.9. *For all* $z \in \boldsymbol{C}\backslash\boldsymbol{R}$,

$$\lim_{V\to\infty}\|(H^{(V)}_{\mathrm{RWA}} - z)^{-1} - (H^V_{\mathrm{RWA}} - z)^{-1}\| = 0,$$

$$\lim_{V\to\infty}\|(H^V_{\mathrm{RWA}} - z)^{-1} - (H_{\mathrm{RWA}} - z)^{-1}\| = 0,$$

so

$$\lim_{V\to\infty}\|(H^{(V)}_{\mathrm{RWA}} - z)^{-1} - (H_{\mathrm{RWA}} - z)^{-1}\| = 0.$$

Proof. We can prove our lemma in the same way as [**8**, Lemma 3.5]. We have

$$(H^{(V)}_{\mathrm{RWA}} - z)^{-1} - (H^V_{\mathrm{RWA}} - z)^{-1} = L_1(V) + L_2(V),$$

where

$$L_1(V) \equiv (H^{(V)}_{\mathrm{RWA}} - z)^{-1}(H^V_0 - H^{(V)}_0)(H^V_{\mathrm{RWA}} - z)^{-1},$$

$$L_2(V) \equiv (H^{(V)}_{\mathrm{RWA}} - z)^{-1}(H^V_I - H^{(V)}_I)(H^V_{\mathrm{RWA}} - z)^{-1}.$$

By Lemma 4.8(ii) and using $\|(H^V_{\mathrm{RWA}} - z)^{-1}\| \le |\Im z|^{-1}$, we have

$$(4.45)\qquad \|H^V_0(H^V_{\mathrm{RWA}} - z)^{-1}\| \le c_2\left(1 + \frac{|z|}{|\Im z|}\right) + \frac{d_2}{|\Im z|},$$

where c_2 and d_2 are in Lemma 4.8(ii), so they are independent of $V > 0$. By Lemma 4.6(i), (4.45) and using $\|(H_{\mathrm{RWA}}^{(V)} - z)^{-1}\| \le |\Im z|^{-1}$, we have

$$\|L_1(V)\| \le \frac{\sqrt{2}}{|\Im z|} C_2^V \left\{ c_2 \left(1 + \frac{|z|}{|\Im z|} \right) + \frac{d_2}{|\Im z|} \right\} \to 0$$

as $V \to \infty$. By (4.43), (4.44), (4.45) and applying Lemma 4.7 to $f = \rho_\beta^V - \rho_\beta^{(V)} \in L^2(\boldsymbol{R})$, we have

$$\begin{aligned}
&\|(H_I^V - H_I^{(V)})(H_{\mathrm{RWA}}^V - z)^{-1}\| \\
&\quad \le F_{\varepsilon,\varepsilon'}(\omega_V^0, \omega_\beta^V, \rho_\beta^V - \rho_\beta^{(V)}) \|H_0^V (H_{\mathrm{RWA}}^V - z)^{-1}\| \\
&\quad\quad + G_{\varepsilon,\varepsilon'}(\omega_V^0, \omega_\beta^V, \rho_\beta^V - \rho_\beta^{(V)}) \|(H_{\mathrm{RWA}}^V - z)^{-1}\| \\
&\quad \le \left\{ \left(\sqrt{\frac{2}{\omega_V^0}} + \varepsilon' \right) \max\left\{ \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{\varepsilon_0}} \right\} + \frac{\varepsilon}{\sqrt{\omega_V^0}} \right\} \left\{ c_2 \left(1 + \frac{|z|}{|\Im z|} \right) + \frac{d_2}{|\Im z|} \right\} \|\rho_\beta^V - \rho_\beta^{(V)}\|_{L^2} \\
&\quad\quad + \frac{1}{4|\Im z|} \left\{ \frac{1}{\varepsilon \sqrt{\omega_V^0}} + \frac{1}{\varepsilon'} \max\left\{ \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{\varepsilon_0}} \right\} \right\} \|\rho_\beta^V - \rho_\beta^{(V)}\|_{L^2}.
\end{aligned}$$

By the inequality above, Lemma 4.5(ii), (**A.1**) and using $\|(H_{\mathrm{RWA}}^{(V)} - z)^{-1}\| \le |\Im z|^{-1}$, we have $\lim_{V\to\infty} \|L_2(V)\| = 0$. Thus, we obtain the first statement of our lemma. The second statement of our lemma can be proven similarly to the first one, and the last statement follows from the first and second ones. □

Let

$$\mathscr{F}^V \equiv L^2(\boldsymbol{R}) \otimes \mathscr{F}_b^V.$$

By using the same way as the proof in [**8**, Lemmas 3.6, 3.7] with Lemma 4.6(ii), we can show the following lemma:

LEMMA 4.10. (i) *The operator* $H_b^{(V)}$ *is reduced by* $\mathscr{F}_b^V$ *and*

$$H_b^{(V)} \lceil \mathscr{F}_b^V = \sum_{k \in \Gamma_V} \omega_k b_V(k)^* b_V(k),$$

the second quantization of $\omega_\beta^{(V)} \lceil \ell^2(\Gamma_V) = \{\omega_k\}_{k \in \Gamma_V}$ *in* $\mathscr{F}_b^V$.

(ii) *The operator* $H_{\mathrm{RWA}}^{(V)}$ *is reduced by* $\mathscr{F}^V$.

It is well known that

$$\mathscr{F} = \bigoplus_{n=0}^{\infty} S_n(\boldsymbol{C} \oplus \ell^2(\Gamma_V))^n \cong L^2(\boldsymbol{R}) \otimes \bigoplus_{n=0}^{\infty} S_n(\ell^2(\Gamma_V))^n = L^2(\boldsymbol{R}) \otimes \mathscr{F}_b^V = \mathscr{F}^V$$

since $\boldsymbol{C}$ and $\ell^2(\Gamma_V)$ intersect orthogonally in $\boldsymbol{C} \oplus \ell^2(\Gamma_V)$. So, by Lemma 4.10, we can identify $H_{\mathrm{RWA}}(x, y)$ acting in $\mathscr{F}$, given by (2.1) in Section 2, as $H_{\mathrm{RWA}}^{(V)}$ acting in $\mathscr{F}^V$.

For simplicity, we set

$$W^V(t_1, t_2) \equiv W_B(t_1, t_2) = (\Omega, e^{iH^{(V)}_{\mathrm{RWA}} t_1} B e^{-iH^{(V)}_{\mathrm{RWA}} t_1} e^{iH^{(V)}_{\mathrm{RWA}} t_2} B e^{-iH^{(V)}_{\mathrm{RWA}} t_2} \Omega)_{\mathscr{F}^V}, \tag{4.46}$$

$$W^V(t) \equiv W(t) = \frac{1}{2}(a^* \otimes I\Omega, e^{iH^{(V)}_{\mathrm{RWA}} t}(a^* \otimes I) e^{-iH^{(V)}_{\mathrm{RWA}} t}\Omega)_{\mathscr{F}^V}, \tag{4.47}$$

$$W^\infty(t_1, t_2) \equiv (\Omega, e^{iH_{\mathrm{RWA}} t_1} B e^{-iH_{\mathrm{RWA}} t_1} e^{iH_{\mathrm{RWA}} t_2} B e^{-iH_{\mathrm{RWA}} t_2} \Omega)_{\mathscr{F}^\infty} \tag{4.48}$$

$$W^\infty(t) \equiv \frac{1}{2}(a^* \otimes I\Omega, e^{iH_{\mathrm{RWA}} t}(a^* \otimes I) e^{-iH_{\mathrm{RWA}} t}\Omega)_{\mathscr{F}^\infty}, \tag{4.49}$$

where $\Omega \equiv f_0 \otimes \Omega_0$, Ω_0 is the Fock vacuum of $\mathscr{F}_b^V$ and $\mathscr{F}_b^\infty$. We remember that $W^V(t_1, t_2) = W^V(t_2 - t_1)$ (see (2.52)), and similarly we have $W^\infty(t_1, t_2) = W^\infty(t_2 - t_1)$.

Proof of Theorem 3.2.

By Lemma 4.9, for (x, y) given in Theorem 3.1, $H^{(V)}_{\mathrm{RWA}} \to H_{\mathrm{RWA}}$ in the norm resolvent sense as $V \to \infty$ through the identification in the remark after Lemma 4.10. So, by [**29**, Theorem VIII.21], we have

$$W^\infty(t_1, t_2) = \lim_{V \to \infty} W^V(t_1, t_2),$$

so by Theorem 3.1, there exists

$$R^\infty_\beta(t_1, t_2) \equiv \lim_{V \to \infty} R^V(t_1, t_2)$$

such that

$$R^\infty_\beta(t_1, t_2) = (R^\infty_\beta(0) - R^\infty_{\beta,0})(W^\infty(t_1, t_2) + W^\infty(t_2, t_1)) + R^\infty_{\beta,0}, \tag{4.50}$$

where we note $2\Re W^\infty(t_1, t_2) = W^\infty(t_1, t_2) + W^\infty(t_2, t_1)$.

On the other hand, like as (3.20), by (4.1), (4.10), (4.11) and (4.33), we have

$$D^\beta_{\mathrm{RWA}}(z) = z - \omega^0_\beta + \int_{-\infty}^{\infty} \frac{\rho_\beta(k)^2}{\omega_\beta(k) - z}\, dk. \tag{4.51}$$

Therefore, Theorem 3.2 follows from Arai's result [**5**, (1.5) and Theorem 1.3(a)].

5. Appendix

5.1. Mori's memory kernel equation.

In this subsection, we recall briefly Mori's memory kernel equation [**25**, **26**]. Let $\mathbf{X}$ be a Hilbert space with an inner product $(\ ,\ )_{\mathbf{X}}$, $\mathscr{L}$ a self-adjoint operator in $\mathbf{X}$ with domain $\mathrm{D}(\mathscr{L})$, and A a non-zero element in $\mathrm{D}(\mathscr{L})$, where the inner product $(\ ,\ )_{\mathbf{X}}$ is linear in the right vector.

We consider a stationary curve $\{A(t) \mid t \in \boldsymbol{R}\}$ defined by $A(t) \stackrel{\text{def}}{=} e^{i\mathscr{L}t}A$ $(t \in \boldsymbol{R})$ and the autocorrelation function R_A of A given by $R_A(t) \stackrel{\text{def}}{=} (A(0), A(t))_{\mathbf{X}}$.

Let $\mathbf{X}_0$ be the closed subspace generated by A, and Π_0 and $\mathbf{X}_1$ the orthogonal projection operator on $\mathbf{X}_0$ and the complementary subspace of $\mathbf{X}_0$ in $\mathbf{X}$, respectively. Then we define a linear operator $\mathscr{L}_1$ on the Hilbert space $\mathbf{X}_1$ by

$$\mathrm{D}(\mathscr{L}_1) \stackrel{\text{def}}{=} (1 - \Pi_0)\mathbf{X} \cap \mathrm{D}(\mathscr{L})$$

$$\mathscr{L}_1 x \stackrel{\text{def}}{=} (1 - \Pi_0)\mathscr{L}x, \quad x \in \mathrm{D}(\mathscr{L}_1).$$

From this, we note that $\mathscr{L}_1$ is a self-adjoint operator acting in the Hilbert space $\mathbf{X}_1$ [**25**, **28**]. And we define Mori's frequency $\omega_{_A}$, fluctuation $I_A(t)$ $(t \in \boldsymbol{R})$ and memory function $\phi_{_A}$ by

$$\omega_{_A} \stackrel{\text{def}}{=} -(A(0), \mathscr{L}A(0))_{\mathbf{X}}(A(0), A(0))_{\mathbf{X}}^{-1}, \tag{5.1}$$

$$I_A(t) \stackrel{\text{def}}{=} ie^{i\mathscr{L}_1 t}(1 - \Pi_0)\mathscr{L}A, \quad t \in \boldsymbol{R}, \tag{5.2}$$

$$\phi_{_A}(t) \stackrel{\text{def}}{=} (I_A(0), I_A(t))_{\mathbf{X}}(A(0), A(0))_{\mathbf{X}}^{-1}, \quad t \in \boldsymbol{R}. \tag{5.3}$$

We note here that we change the original definition of Mori's frequency into (5.1) to discuss our argument. Then we have the following theorem:

THEOREM A.1. ([**25**, **26**, **28**]). (i) *For all* $t \in \boldsymbol{R}$,

$$\frac{d}{dt}R_A(t) = -i\omega_{_A}R_A(t) - \int_0^t ds\, \phi_{_A}(t-s)R_A(s).$$

(ii) *For all* $z \in \boldsymbol{C}^+ \stackrel{\text{def}}{=} \{z \in \boldsymbol{C} | \Im z > 0\}$,

$$\int_0^\infty dt\, e^{itz}R_A(t) = R_A(0)\frac{1}{i\omega_{_A} - iz + \displaystyle\int_0^\infty dt\, e^{itz}\phi_{_A}(t)}.$$

(iii) *For all* $t \in \boldsymbol{R}$,

$$\frac{d}{dt}A(t) = -i\omega_{_A}A(t) - \int_0^t ds\, \phi_{_A}(t-s)A(s) + I_A(t). \tag{5.4}$$

Equation (5.4) is *Mori's memory kernel equation*, or *Mori's Langevin equation*.

5.2. An example of the Hamiltonian H and the observable A in Theorem 3.1.

In this subsection, we give an example of the Hamiltonian H and the observable A in Theorem 3.1. We find the example in [**6**].

There are many examples in models with free or quasi-free Hamiltonians which are solvable. However, it is worth applying our theory to non-quasi-free (unsolvable) models rather than quasi-free (solvable) ones.

Let $\mathscr{H}$ be a separable real Hilbert space with inner product $(\ ,\ )_{\mathscr{H}}$, and h_B a strictly positive self-adjoint operator acting in $\mathscr{H}$ such that

$(\mathbf{h})_B$ for some constant $\alpha > 0$, $h_B^{-\alpha/2}$ is Hilbert-Schmidt on $\mathscr{H}$.

Then we obtain a real Hilbert space $\mathscr{H}_{-1}$ by the completion of $\mathscr{H}$ with respect to the inner product

$$(f,g)_{-1} \overset{\text{def}}{=} (h_B^{-1/2}f, h_B^{-1/2}g)_{\mathscr{H}}, \quad f,g \in \mathscr{H}. \tag{5.5}$$

We consider the Gaussian mean zero random process $\{\phi(f)\,|\,f \in \mathscr{H}_{-1}\}$ indexed by $\mathscr{H}_{-1}$. We denote by Q (resp. $d\mu_0$) the underlying measure space (resp. the Gaussian probability measure) and by $\langle\ \rangle$ the expectation with respect to $d\mu_0$. Then we have

$$\langle\phi(f)\phi(g)\rangle = (f,g)_{-1}, \quad f,g \in \mathscr{H}_{-1}. \tag{5.6}$$

The symmetric Fock space $\mathscr{F}_B$ over $\mathscr{H}_{-1}$ is given by

$$\mathscr{F}_B \overset{\text{def}}{=} L^2(Q, d\mu_0). \tag{5.7}$$

It is well-known [**11**, Theorem 3.2.10] that $\mathscr{F}_B$ is written as follows:

$$\mathscr{F}_B = \bigoplus_{n=0}^{\infty} \Gamma_n(\mathscr{H}_{-1}), \tag{5.8}$$

$$\Gamma_0(\mathscr{H}_{-1}) = \boldsymbol{C},$$

$$\Gamma_n(\mathscr{H}_{-1}) = \overline{\mathbf{L.h.}\,\{:\phi(f_1)\cdots\phi(f_n):\,|\,f_j \in \mathscr{H}_{-1}, j=1,\dots,n\}}^{\text{closure}},$$

where $:\phi(f_1)\cdots\phi(f_n):$ is the Wick product of random variables $\phi(f_1)\cdots\phi(f_n)$, and $\overline{\mathbf{L.h.}\,\{\ \}}^{\text{closure}}$ denotes the closure of $\mathbf{L.h.}\,\{\ \}$ in $L^2(Q, d\mu_0)$. We define a subspace $\Gamma_{alg}(\mathscr{H}_{-1})$ by

$$\begin{aligned} \Gamma_{alg}(\mathscr{H}_{-1}) &= \bigoplus_{n=0}^{\infty} \Gamma_n^{(0)}(\mathscr{H}_{-1}), \\ \Gamma_0^{(0)}(\mathscr{H}_{-1}) &= \boldsymbol{C}, \\ \Gamma_n^{(0)}(\mathscr{H}_{-1}) &= \mathbf{L.h.}\,\{:\phi(f_1)\cdots\phi(f_n):\,|\,f_j \in \mathscr{H}_{-1}, j=1,\dots,n\}, \end{aligned} \tag{5.9}$$

then $\Gamma_{alg}(\mathscr{H}_{-1})$ is dense in $\mathscr{F}_B$.

It is evident that h_B has a unique extension $\hat{h}_B$ to $\mathscr{H}_{-1}$ such that

$$(g, \hat{h}_B f)_{-1} = (g,f)_{\mathscr{H}}, \quad f,g \in \mathrm{D}(h_B). \tag{5.10}$$

For each $f \in \mathrm{D}(\hat{h}_B^{1/2})$, we define the annihilation operator $b(f)$ on $\Gamma_{alg}(\mathscr{H}_{-1})$ by [**6**, (2.7)]

$$\begin{aligned} b(f) &:\phi(f_1)\cdots\phi(f_n): \\ &= \sum_{j=1}^{n} (\hat{h}_B^{1/2} f, f_j)_{-1} :\phi(f_1)\cdots\phi(f_{j-1})\phi(f_{j+1})\cdots\phi(f_n): \end{aligned} \tag{5.11}$$

and extending it by linearity to $\Gamma_{alg}(\mathscr{H}_{-1})$.

The creation operator $b^+(f)$ is defined as the adjoint of $b(f) \lceil \Gamma_{alg}(\mathscr{H}_{-1})$:

$$b^+(f) \overset{\text{def}}{=} (b(f) \lceil \Gamma_{alg}(\mathscr{H}_{-1}))^*. \tag{5.12}$$

Then, $\mathrm{D}(b^+(f)) \supset \Gamma_{alg}(\mathscr{H}_{-1})$ and

$$b^+(f) : \phi(f_1)\cdots\phi(f_n) :=: \phi(\hat{h}_B^{1/2}f)\phi(f_1)\cdots\phi(f_n) : \tag{5.13}$$

hold [**6**, (2.8)]. Therefore, $b(f)$ is closable and we denote the closure by the same symbol. Both $b(f)$ and $b^+(f)$ leave $\Gamma_{alg}(\mathscr{H}_{-1})$ invariant and satisfy the canonical commutation relation on $\Gamma_{alg}(\mathscr{H}_{-1})$ (see [**6**, (2.9)]).

We denote by $b^\#$ either b or b^+. In terms of $b^\#(f)$, $\phi(f)$ is written as

$$\phi(f) = b(\hat{h}_B^{-1/2}f) + b^+(\hat{h}_B^{-1/2}f), \quad f \in \mathscr{H}_{-1} \tag{5.14}$$

on $\Gamma_{alg}(\mathscr{H}_{-1})$ [**6**, (2.12)]. For $f \in \mathrm{D}(\hat{h}_B)$ we define the canonical conjugate momentum operator $\pi(f)$ by

$$\pi(f) \stackrel{\text{def}}{=} \frac{1}{2i}(b(\hat{h}_B^{-1/2}f) - b^+(\hat{h}_B^{-1/2}f)). \tag{5.15}$$

As [**6**, p. 334], we can show that $\phi(f)$ and $\pi(f)$ are essentially self-adjoint on $\Gamma_{alg}(\mathscr{H}_{-1})$ with the canonical commutation relation on it, and we denote self-adjoint operators as their closure by the same symbols, respectively.

Let

$$H_{0B} \stackrel{\text{def}}{=} d\Gamma(\hat{h}_B), \tag{5.16}$$

where of course $d\Gamma(\hat{h}_B)$ is the second quantization of $\hat{h}_B$.

By $(\mathbf{h})_B$, for all $\tau > 0$, $\exp(-\tau\hat{h}_B)$ is trace class on $\mathscr{H}_{-1}$, and hence compact. So that $\hat{h}_B$ has a purely discrete spectrum $\{w_n\}_{n=0}^\infty$ with $0 < w_0 \leq w_2 \leq \cdots \leq w_n < w_{n+1} \leq \cdots$, $w_n \nearrow \infty$ as $n \to \infty$. Let $\{e_n\}_{n=0}^\infty$ be the complete orthonormal system of the corresponding eigenvectors in $\mathscr{H}_{-1}$:

$$\hat{h}_B e_n = w_n e_n, \quad n \in \boldsymbol{N}^*. \tag{5.17}$$

Let

$$q \equiv \phi_0 \stackrel{\text{def}}{=} \phi(e_0),\ p \equiv \pi_0 \stackrel{\text{def}}{=} \pi(e_0), \tag{5.18}$$

$$\phi_n \stackrel{\text{def}}{=} \phi(e_n),\ \pi_n \stackrel{\text{def}}{=} \pi(e_n), \quad n \in \boldsymbol{N}. \tag{5.19}$$

Then we have, for $m, n \in \boldsymbol{N}^*$,

$$[\phi_m, \pi_n] = i\delta_{mn}, \tag{5.20}$$

$$[\phi_m, \phi_n] = 0 = [\pi_m, \pi_n]. \tag{5.21}$$

Let v be a polynomially bounded continuous real function on $\boldsymbol{R}$ and bounded below. Let $(Y, d\sigma(y))$ be a finite measure space with Y being compact Hausdorff (the σ-field is omitted) and γ be an $\mathscr{H}_{-1}$-valued strongly continuous function on Y. Then $v(\phi(\gamma)) \in L^2(Q, d\mu_0)$ for all $y \in Y$, and that the Bochner integral

$$V(\phi) \stackrel{\text{def}}{=} \int_Y v(\phi(\gamma(y)))d\sigma(y) \in L^2(Q, d\mu_0)$$

is defined. Since $\sigma(Y)$ is finite, $V(\phi)$ is bounded below. Thus, by a general theorem [**30**, p. 265,Theorem X.59], the operator $H_{0B} + V(\phi)$ is essentially self-adjoint on $C^\infty(H_{0B}) \cap \mathrm{D}(V(\phi))$ and bounded below. We set for fixed constant $c_I > 0$

$$H_I \stackrel{\text{def}}{=} V(\phi) - \inf \sigma(V(\phi)) + c_I > 0, \tag{5.22}$$

where $\sigma(V(\phi))$ is the spectrum of $V(\phi)$. And let

$$H \stackrel{\text{def}}{=} H_{0B} + H_I. \tag{5.23}$$

Then H is essentially self-adjoint on $C^\infty(H_{0B}) \cap \mathrm{D}(H_I)$. We denote the closure of H by the same symbol.

Proposition B.1. (i) e^{-tH} *is trace class for every* $t > 0$.
(ii) *For* $B = q, p$, $B \in \mathbf{T}(H)$.

Proof. First part (i) follows from Golden-Thompson inequality [**31**, Corollary, p320]. So we can take $\mathbf{D}$ as (2.2) for the present H. It is well known that $\mathrm{D}(H_{0B}^{1/2}) \subset \mathrm{D}(p) \cap \mathrm{D}(q)$ and

$$\|qx\| \le C(\|H_{0B}^{1/2}x\| + \|x\|), \quad \|px\| \le C(\|H_{0B}^{1/2}x\| + \|x\|)$$

for $x \in \mathrm{D}(H_{0B}^{1/2})$, where $C > 0$ is a constant. By the fact that $H_I > 0$, we have for all $x \in \mathrm{D}(H_{0B}) \cap \mathrm{D}(H_I)$, $(x, H_{0B}x) \le (x, Hx)$. Since $\mathrm{D}(H_{0B}) \cap \mathrm{D}(H_I)$ is a core for H, it is a core for $H^{1/2}$. Hence we can extend, by a limiting argument, this inequality to all $x \in \mathrm{D}(H^{1/2})$, showing that $\mathrm{D}(H^{1/2}) \subset \mathrm{D}(H_{0B}^{1/2})$ and $\|H_{0B}^{1/2}x\| \le \|H^{1/2}x\|$ $(x \in \mathrm{D}(H^{1/2}))$. Hence $\mathrm{D}(H^{1/2}) \subset \mathrm{D}(p) \cap \mathrm{D}(q)$ and

$$\|qx\| \le C(\|H^{1/2}x\| + \|x\|), \quad \|px\| \le C(\|H^{1/2}x\| + \|x\|) \tag{5.24}$$

for $x \in \mathrm{D}(H^{1/2})$. In particular $\mathbf{D} \subset \mathrm{D}(q) \cap \mathrm{D}(p)$. Since q and p are self-adjoint, $B = q$ (or $B = p$) satisfies (**T.1**). Since $e^{-\tau H}x \in \mathrm{D}(H^{1/2})$ $(\tau > 0)$, by (5.24), we have for all $\tau > 0$

$$\|qe^{-\tau H}x\| \le C(\|H^{1/2}e^{-\tau H}x\| + \|e^{-\tau H}x\|), \quad x \in \mathscr{F}_B.$$

Since $H^{1/2}e^{-\tau H}$ and $e^{-\tau H}$ are Hilbert-Schmidt, it follows that $qe^{-\tau H}$ is also Hilbert-Schmidt. Hence $(qe^{-\tau H})^*$ is Hilbert-Schmidt. We have $(e^{-\tau H}q)^- = (qe^{-\tau H})^*$. Hence $(e^{-\tau H}q)^-$ is Hilbert-Schmidt. Thus $B = q$ satisfies (**T.2**). Similarly we can prove that $p \in \mathbf{T}(H)$. □

Example. Fix arbitrary natural number $M \in N$ and positive number $\varepsilon > 0$. For $B = q, p$, by using (2.8) and (2.12) we set

$$A(M, \varepsilon) \stackrel{\text{def}}{=} \sum_{0 \le m, n \le M;\, \varepsilon \le |\lambda_m - \lambda_n|} \langle \Phi_{m,n}; B \rangle_H \Phi_{m,n}. \tag{5.25}$$

Then $A(M, \varepsilon)$ satisfies (**A.0**), and $A(M, \varepsilon) \in \mathrm{D}(\mathscr{L})$. Therefore H and $A(M, \varepsilon)$ give an example for Theorem 3.1 such that

$$\lim_{\varepsilon \downarrow 0} \lim_{M \to \infty} A(M, \varepsilon) = B \tag{5.26}$$

in $\mathbf{X}_c(H)$.

REMARK. Actually, we can have finite numbers of data from an experiment. So we have to build a model $A(M, \varepsilon)$ for the observable B using the data. Thus, we can say that $M \in \boldsymbol{N}$ and $\varepsilon > 0$ mean parameters representing precision of an experimental equipment. So, $M \to \infty$ and $\varepsilon \downarrow 0$ in (5.26) correspond to making the precision better.

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