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Complex varieties of general type whose canonical systems are composed with pencils

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Abstract. This paper aims to study a variety of general type whose canonical system is composed with a pencil. This kind of variety admits a natural fibration onto a nonsingular curve. A natural problem is whether the geometric genus of the general fiber of this fibration is bounded. A simple classification is given in this paper. When the object is a nonsingular minimal 3-fold of general type, if the canonical system is composed of an irrational pencil, then the geometric genus of the general fibre is bounded. If the canonical system is composed of a rational pencil, it seems that the geometric genus of the general fibre is not bounded though no counter examples have been found.

Throughout this paper, most our notations and terminologies are standard within algebraic geometry except for the following which we are in favour of:

$:=$ —definition;

$\sim_{\text{lin}}$—linear equivalence;

$\sim_{\text{num}}$—numerical equivalence.

Let X be a complex nonsingular projective variety of general type with dimension d ($d \geq 2$). Suppose $\dim \Phi_{|K_X|}(X) = 1$, we usually say that the canonical system $|K_X|$ is composed with a pencil. Taking possible blow-ups $\pi : X' \to X$ according to Hironaka such that $g := \Phi_{|K_X|} \circ \pi$ is a morphism. We have the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{f} & C \\ \| & & \downarrow{\scriptstyle \psi} \\ X' & \xrightarrow[g]{} & W_1 \subset \boldsymbol{P}^{p_g-1} \\ {\scriptstyle \pi}\downarrow & & \\ X & & \end{array}$$

where we set $W_1 := \overline{\Phi_{|K_X|}(X)}$ and let $g := \psi \circ f$ be the Stein factorization of g. Note that f is a fibration onto a nonsingular curve C. Let F be a general fiber of f, then F is a nonsingular projective variety of dimension $d-1$. We also say that f is a derived fibration of the canonical map. Denote $b := g(C)$, the genus of C.

The aim of this note is to build the following theorems.

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Theorem 1. *Let X be a complex nonsingular projective variety of general type with dimension d $(d \geq 3)$. Suppose the canonical system $|K_X|$ be composed with a pencil, using the above diagram and notations, and assume that $b \geq 2$, then either*

$$p_g(F) = 1, \ p_g(X) \geq b - 1,$$

or

$$b = p_g(F) = p_g(X) = 2.$$

Remark 1. In the case of dimension 2, one can refer to [**10**].

Theorem 2. *Under the same assumption as in Theorem* 1, *assume in addition that* $\dim X = 3$ *and K_X is nef and big, set $F_1 := \pi_* F$, then*

(1) *If $b = 1$, then $p_g(F) \leq 38$;*

(2) *If $b = 0$, $K_X \cdot F_1^2 = 0$ and $p_g(X) \geq 20$, then*

$$p_g(F) \leq 38 + \frac{756}{p_g(X) - 19};$$

(3) *If $b = 0$ and $K_X \cdot F_1^2 > 0$, then*

$$p_g(F) \geq q(F) \geq \frac{1}{36}(p_g(X) - 1)(p_g(X) - 37).$$

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§1. Proof of Theorem 1.

Proof of Theorem 1. Using the first commutative diagram of this paper, we have the fibration $f : X' \to C$. Denote by $\mathscr{L}$ the saturated subbundle of $f_*\omega_{X'}$ which is generated by $H^0(C, f_*\omega_{X'})$, $\mathscr{L}$ is of rank one under the assumption of the theorem. Thus we obtain the following exact sequence

$$0 \to \mathscr{L} \to f_*\omega_{X'} \to \mathscr{Q} \to 0.$$

We know that $R^i f_*\omega_{X'/C}$ is a semi-positive vector bundle (Griffith-Fujita-Kawamata-Ohno semipositivity). $\mathscr{Q} \otimes \omega_C^{-1}$ is automatically semi-positive. Since $H^0(C, \mathscr{L}) = H^0(C, f_*\omega_{X'})$, $H^0(C, \mathscr{Q})$ injects into $H^1(C, \mathscr{L})$. By Riemann-Roch, $h^0(C, \mathscr{Q}) \geq (b-1) \cdot (p_g(F) - 1)$, $h^1(C, \mathscr{L}) \leq b - 1$, and hence we get $p_g(F) \leq 2$.

If $p_g(F) = 1$, then $p_g(X) = h^0(C, \mathscr{L}) \geq b - 1$ by the semipositivity of $\mathscr{L} \otimes \omega_C^{-1} = f_*\omega_{X'/C}$.

If $p_g(F) = 2$, then we must have $h^0(\mathscr{Q}) = h^1(\mathscr{L}) = b - 1$, so that $\mathscr{Q} \otimes \omega_C^{-1}$ is of degree ≤ 0. Since $f_*\omega_{X'} \otimes \omega_C^{-1}$ is semi-positive, this means that $\mathscr{L}$ is of degree $\geq 2b - 2$, with non-trivial H^1. Hence $\mathscr{L} \cong \omega_C$, and $1 = h^1(C, \mathscr{L}) \geq h^0(\mathscr{Q}) = b - 1$. This shows that $b = 2$, $p_g(X) = h^0(C, \mathscr{L}) = 2$, completing the proof of Theorem 1.

§2. Proof of Theorem 2.

At first, let us recall Miyaoka's inequality as follows.

Fact. ([**7**]) Let X be a nonsingular projective threefold with nef and big canonical divisor K_X. Then $3c_2 - c_1^2$ is pseudo-effective, where c_i's are Chern numbers of X.

Proposition 2.1. *Let X be a nonsingular projective 3-fold with nef and big canonical divisor K_X. Assume that $|K_X|$ be composed with a pencil and that $K_X \cdot F_1^2 = 0$, then*

$$\mathcal{O}_F(\pi^*(K_X)|_F) \cong \mathcal{O}_F(\sigma^*(K_{F_0})),$$

where $\sigma : F \to F_0$ is the contraction onto the minimal model.

Proof. This can be obtained by a similar argument as that of Theorem 7 in [**6**].

Proof of Theorem 2. We use the same exact sequence as in the proof of Theorem 1. It is easy to see that $K_{X'} \sim_{\mathrm{num}} (\deg \mathscr{L})F + Z$, where Z is the fixed part.

Case (1). $b = 1$. In this case, we can suppose $X' = X$. $\mathscr{Q}$ is semi-positive. Note that $\deg \mathscr{Q} = 0$; otherwise, by Riemann-Roch, we should have

$$h^0(C, f_*\omega_X) \geq \deg \mathscr{L} + \deg \mathscr{Q} > \deg \mathscr{L} = h^0(C, \mathscr{L}),$$

a contradiction. $R^1 f_*\omega_X$ is semi-positive, while $R^2 f_*\omega_X \cong \mathcal{O}_C$ by duality. Thus

$$\begin{aligned}\chi(X, \omega_X) &= \chi(C, f_*\omega_X) - \chi(C, R^1 f_*\omega_X) + \chi(C, R^2 f_*\omega_X)\\ &\leq \chi(C, f_*\omega_X) = \deg \mathscr{L} = p_g(X).\end{aligned}$$

It follows from Miyaoka's inequality that $K_X^3 \leq 72 \deg \mathscr{L}$. On the other hand,

$$\begin{aligned}K_X^3 &\geq (\deg \mathscr{L}) K_X^2 \cdot F\\ &= (\deg \mathscr{L}) K_F^2 \geq 2(\deg \mathscr{L})(p_g(F) - 2).\end{aligned}$$

This implies that $(\deg \mathscr{L})(p_g(F) - 2) \leq 36 \deg \mathscr{L}$, i.e. $p_g(F) \leq 38$.

Case (2). $b = 0$ and $K_X \cdot F_1^2 = 0$. In this case, any vector bundle on C is a direct sum of line bundles. $f_*\omega_{X'}$ is a direct sum of $\mathscr{L}$ and $p_g(F) - 1$ line bundles of degree -1 or -2, while $R^1 f_*\omega_{X'}$ is a direct sum of $q(F)$ line bundles of degree ≥ -2. Hence

$$\begin{aligned}\chi(X', \omega_{X'}) &= \chi(C, f_*\omega_{X'}) - \chi(C, R^2 f_*\omega_{X'}) + \chi(C, \omega_C)\\ &\leq \deg \mathscr{L} + 1 + q(F) - 1.\end{aligned}$$

Then, by Proposition 2.1, we get

$$(\deg \mathscr{L})(p_g(F) - 2) \leq \frac{1}{2}(\deg \mathscr{L}) K_{F_0}^2 \leq 36(\deg \mathscr{L} + q(F)).$$

Apply the inequality $p_g(F) \geq 2q(F) - 4$ ([**2**]), we get

$$(\deg \mathscr{L} - 18) p_g(F) \leq 38 \deg \mathscr{L} - 72.$$

We obtain the desired inequality by substituting $\deg \mathscr{L}$ by $p_g(X) - 1$.

Case (3). $b = 0$ and $K_X \cdot F_1^2 > 0$. It is easy to check that $K_X \cdot F_1^2$ is an even number. We get the following inequality

$$K_X^3 \geq (\deg \mathscr{L})^2 K_X \cdot F_1^2 \geq 2(\deg \mathscr{L})^2.$$

Therefore we have

$$2(\deg \mathscr{L})^2 \leq 72(\deg \mathscr{L} + q(F)),$$

which directly induces what we want. The proof is completed.

§3. Examples.

3.1 Examples with $p_g(F) = 1$ and $p_g(X) \geq b$.

Let S be a minimal surface with $K_S^2 = 1$ and $p_g(S) = q(S) = 0$. We chose S in such a way that S admits a torsion element η of order 2, i.e., $\eta \in \mathrm{Pic}^0 S$ and $2\eta \sim_{\mathrm{lin}} 0$. For the existence of this surface, we may refer to [**1**] or [**9**]. Let C be a nonsingular curve of genus $b \geq 1$. Set $Y := C \times S$. Let $p_1 : Y \to C$ and $p_2 : Y \to S$ be the two projection maps. Let D be an effective divisor on C with $\deg D = a > 0$. Let $\delta = p_1^*(D) + p_2^*(\eta)$, $B \sim_{\mathrm{lin}} 2\delta \sim_{\mathrm{lin}} p_1^*(2D)$, we can take B composed of $2a$ distinct points. The pair (δ, B) determines a smooth double cover X over Y. We have the following commutative diagram, where π is the double covering.

$$\begin{array}{ccccc} X & \xrightarrow{\pi} & Y = C \times S & \xrightarrow{p_2} & S \\ \Big\| & & \Big\downarrow{\scriptstyle p_1} & & \\ X & \xrightarrow[\Psi]{} & C & & \end{array}$$

We can see that $\Phi_{|K_X|}$ factors through Ψ. These examples satisfy $p_g(X) = a + b - 1$, $K_X^3 = 6(a + 2b - 2)$, $h^2(\mathcal{O}_X) = 0$, $q(X) = b$, $K_F^2 = 2$, $p_g(F) = 1$ and $q(F) = 0$.

3.2 Examples with $p_g(F) = 1$ and $p_g(X) = b - 1$.

In the construction of 3.1, take C be a hyperelliptic curve of genus $b \geq 3$. Let $\tau = p - q$ be a divisor on C such that $2\tau \sim_{\mathrm{lin}} 0$. Take $\delta = p_1^*(\tau) + p_2^*(\eta)$, then $2\delta \sim_{\mathrm{lin}} 0$. Therefore δ determines an unramified double cover $\pi : X \to Y$. X is an example with $p_g(X) = b - 1$, $h^2(\mathcal{O}_X) = 0$, $q(X) = b$ and $K_X^3 = 6(2b - 2)$.

3.3 Examples with $b = 0$ and $p_g(F) = 2$.

In the construction of 3.1, take S be a minimal surface S with $K_S^2 = 2$ and $p_g(S) = q(S) = 1$ and take $C = \boldsymbol{P}^1$. Take an effective divisor D on C with $\deg D = a \geq 3$, $\eta \in \mathrm{Pic}^0 S$ with $2\eta \sim_{\mathrm{lin}} 0$. Denote by $\delta := p_1^*(D) + p_2^*(\eta)$ and $R \sim_{\mathrm{lin}} 2\delta \sim_{\mathrm{lin}} p_1^*(2D)$. Thus the pair (δ, R) determines a double covering $\pi : X \to Y$. We can check that X is an example with $p_g(X) = a - 1$, $K_X^3 = 6(a - 2)$, $b = 0$, $K_F^2 = 4$, $q(F) = 1$ and $p_g(F) = 2$.

3.3 Examples with $b = 1$ and $p_g(F) = 2$.

Let S be a minimal surface with $p_g(S) = q(S) = 1$ and $K_S^2 = 2$. We know that the albanese map of S is just a genus two fibration onto an elliptic curve. It can be constructed from a double cover onto a ruled surface P which is over the elliptic curve with invariant $e = -1$. Furthermore, all the singularities on the branch locus corresponding to this double cover are negligible. Let $\pi_0 : S \to \tilde{P}$ be this double cover with covering data (δ_0, R_0). Note that $q(P) = 1$.

Take an elliptic curve E and denote $T := E \times \tilde{P}$. Let p_1 and p_2 be two projection maps. Take a 2-torsion element $\eta \in \mathrm{Pic}^0 E$. Since $\tilde{P}$ is a fibration over an elliptic curve, we can take a 2-torsion element $\tau \in \mathrm{Pic}^0 \tilde{P}$ such that $\pi_0^*(\tau) \not\sim_{\mathrm{lin}} 0$ through the double cover π_0.

Let $\delta_1 = p_1^*(\eta) + p_2^*(\delta_0)$ and $R_1 = 2\delta_1$, then the pair (δ_1, R_1) determines a double cover $\Pi_1 : Y \to T$. Let $\phi = p_1 \circ \Pi_1$. Take a divisor A on E with $\deg A = a > 0$. Let $\delta_2 = \phi^*(A) + \Pi_1^* p_2^*(\tau)$ and $R_2 = 2\delta_2$, then the pair (δ_2, R_2) determines a smooth double cover $\Pi_2 : X \to Y$. We can see that X is a minimal threefold of general type and the canonical system $|K_X|$ is composed with a pencil. $\Phi_{|K_X|}$ factors through Π_2 and ϕ. This example satisfies $b = 1$, $p_g(F) = 2$, $q(F) = 1$, $K_F^2 = 16$, $p_g(X) = a$, $q(X) = 2$ and $K_X^3 = 12a$.

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