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# On generalized Igusa local zeta functions associated to simple Chevalley $K$-groups under the adjoint representation

By Roland MARTIN

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**Abstract.** The purpose of this paper is to provide the reader with an introduction to the theory of generalized Igusa local zeta functions associated to irreducible matrix groups. As an application, the generalized Igusa local zeta function $Z_K(s)$ associated to the simple Chevalley $K$-groups of type $B_\ell, C_\ell, D_\ell, E_6, E_7, E_8, F_4$ and $G_2$ under the adjoint representation is explicitly exhibited as a rational function satisfying a certain functional equation.

## 1. Introduction.

The study of Igusa local zeta functions began in 1974 in [**4**] and continues today in the work of a number of authors including D. Meuser, J. Denef and F. Loeser (see [**3**] and [**7**]). One defines these functions in the following way: Take a finite algebraic extension $K$ of $\boldsymbol{Q}_p$. Let $O_K$, $\pi O_K$ denote the ring of integers of $K$, the ideal of nonunits of $O_K$, respectively, and let $\mathrm{card}(O_K/\pi O_K) = q$. Denote by $|\ |_K$ the absolute value on $K$ normalized as $|\pi|_K = q^{-1}$. Take an element $f \in K[x_1, \dots, x_n]$, $f(x) \neq 0$, and let $dx_1 \cdots dx_n$ denote Haar measure on $K^n$ normalized so that $dx_1 \cdots dx_n(O_K^n) = 1$. The Igusa local zeta function is, then, given by

$$Z(s) = \int_{O_K^n} |f(x)|_K^s \, dx_1 \cdots dx_n.$$

A very general result of Denef and Meuser ([**3**]) implies that $Z(s)$ has a finite form that expresses $Z(s)$ as a rational function $Z(q^{-1}, q^{-s})$ satisfying

$$Z(q^{-1}, q^{-s})|_{q \mapsto q^{-1}} = q^{-\deg(f)s} Z(q^{-1}, q^{-s}).$$

There is a natural generalization of the above $Z(s)$. Let $X^*$ denote a closed $K$-analytic subvariety of $K^N$ of dimension $l$. Let $X$ be the set of smooth points of $X^*$ of dimension $l$. Then $X$ is an everywhere $l$-dimensional $K$-analytic submanifold of $K^N$ ([**12**]). Hence, $X$ has a canonical measure $\mu_c$ and $\mu_c(X^0) < \infty$ where $X^0 = X \cap O_K^N$. Take a $K$-analytic function $f$ on $X^*$, nowhere the constant 0, and, for $s \in \boldsymbol{C}$, set

$$Z_K(s) = \int_{X^0} |f(x)|_K^s \mu_c(x).$$

For $\mathrm{Re}(s) > 0$, the above $Z_K(s)$ defines the generalized Igusa local zeta function ([**5**], [**8**]).

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In particular, let $X$ be the $K$-rational points $G_K$ of $G$ where $G$ is a $K$-subgroup of $GL_n$ such that

(1) $G$ is a connected irreducible $K$-subgroup of $GL_n$ not contained in $SL_n$,

(2) $G$ is $K$-split with the subgroup $T$ of all diagonal matrices in $G$ as a maximal $K$-split torus and

(3) $G$ has very good reduction mod $\pi$.

Therefore, by assumption (1), $G$ is the semi-direct product $[G, G]\boldsymbol{G_m}$ where $[G, G]$ denotes the derived group of $G$ and, by assumption (3), $G$ and the $K$-splitting data for $G$ have good reductions mod $\pi$. For the sake of concreteness, we recall the definition of the $K$-splitting data for $G$. Let $R$ be the root system for $G$. The $K$-splitting data for $G$ is the subgroup $T$ together with the $K$-homomorphisms $\theta_\alpha$, $\alpha \in R$, such that each $\theta_\alpha : \boldsymbol{G_a} \to G$ is a $K$-isomorphism onto its image group $\theta_\alpha(\boldsymbol{G_a})$ and $t\theta_\alpha(u)t^{-1} = \theta_\alpha(\alpha(t)u)$ $\forall t \in T$ and $\forall u \in \boldsymbol{G_a}$. Assumption (3) then means that $G$, $T$, a $K$-isomorphism $T = (\boldsymbol{G_m})^{\dim(T)}$ and $\theta_\alpha$, $\forall \alpha \in R$, have good reductions mod $\pi$. Let $f$ be the generator of $\mathrm{Hom}(G, \boldsymbol{G_m})$ of degree $m > 0$. Taking $G^0 = G_K \cap \mathrm{Mat}_n(O_K)$ as $X^0$, then, yields, for $\mathrm{Re}(s) > 0$, the generalized Igusa local zeta function associated to $G$:

$$Z_K(s) = \int_{G^0} |f(g)|_K^s \mu_c(g).$$

It is conjectured by Igusa in [**5**] that $Z_K(s)$ has a finite form that expresses $Z_K(s)$ as a rational function $Z(q^{-1}, q^{-s})$ satisfying

$$Z(q^{-1}, q^{-s})|_{q \mapsto q^{-1}} = q^{-ms} Z(q^{-1}, q^{-s}).$$

In the first two sections of this paper, we will recall some results on the $Z_K(s)$ associated to $l$-dimensional $K$-subgroups $G$ of $GL_n$ satisfying assumptions (1), (2) and (3), which are given by J. I. Igusa in [**5**]. The next two sections will apply this knowledge in the case $G = Ad(G')(\boldsymbol{G_m}\mathbf{1}_{\dim(G')})$ where $G'$ is a simple Chevalley $K$-subgroup of $SL_n$ of type $B_\ell, C_\ell, D_\ell, E_6, E_7, E_8, F_4$ or $G_2$ and $\mathbf{1}_{\dim(G')}$ denotes the identity matrix of size $\dim(G')$, yielding an explicit finite form for each $Z_K(s)$ as a rational function $Z(q^{-1}, q^{-s})$ satisfying

$$Z(q^{-1}, q^{-s})|_{q \mapsto q^{-1}} = q^{-s} Z(q^{-1}, q^{-s}),$$

thereby verifying the above mentioned conjecture of Igusa. Finally, we point out an interesting universal property shared by the $Z_K(s)$ associated to $Ad(G')(\boldsymbol{G_m}\mathbf{1}_{\dim(G')})$, $G'$ as above, and put into perspective the general theory of the $Z_K(s)$ associated to $G$, especially as regards the recent result of J. Denef and D. Meuser in [**3**].

## 2. Canonical Measure.

Given an $l$-dimensional $K$-subgroup $G$ of $GL_n$ satisfying (1), (2) and (3), our goal is to write down a finite form expression for $Z_K(s)$. We will accomplish this in two steps. The first step is to simplify the integral expression for $Z_K(s)$. We recall the definition of canonical measure $\mu_c$ ([**13**]). Let $X$ be an everywhere $l$-dimensional $K$-analytic manifold.

Definition 2.1. $X$ has Riemannian structure if and only if for all $a \in X$ there exists a choice of a lattice $L_a = O_K e_{1,a} + \cdots + O_K e_{l,a}$, the $\{e_{*,a}\}$ a $K$-basis for $T_a(X)$, in $T_a(X)$ such that, given a coordinate map $\phi : U \to V \subset K^l$, $(D\phi)_x(L_x)$ is a lattice in $K^l$ independent of $x \in U$.

If $X$ has Riemannian structure $\{L_a\}_{a \in X}$, then a well-defined measure $\mu_c$ exists on $X$ called the canonical measure. The measure $\mu_c$ is defined locally in a neighborhood $U \ni a$ with coordinate patch $\phi : U \to V \subset K^l$ as follows: Let $\mu_a$ be the unique Haar measure on $T_a(X)$ such that $\mu_a(L_a) = 1$. Then $\mu_c|_U(U') = \nu(\phi(U'))$ where $U' \subset U$ and $\nu$ is the Haar measure on $K^l$ corresponding to $\mu_a$ on $T_a(X)$. The global measure $\mu_c$ on $X$ is the unique extension of $\mu_c|_U$ determined by a partition of unity subordinate to neighborhoods $U$ as above. Note that if $X$ is a $K$-analytic submanifold of $K^N$, then $L_a = T_a(X) \cap O_K^N$ defines a 'locally constant' choice of a lattice on a neighborhood $U$ containing $a$. Such an $X$ has its Riemannian structure induced by the ambient Riemannian structure $O_K^N$ of $K^N$. For example, $G_K$ is an $l$-dimensional $K$-submanifold of $\mathrm{Mat}_n(K)$ and, thus, has its Riemannian structure induced from that on $\mathrm{Mat}_n(K)$.

Since the measure $\mu_c$ on $G_K$ is difficult to work with, we define a function $\Phi^*$ on $G_K$ by

$$\Phi^*(g) = \frac{\mu_c(gG(O_K))}{\mu(G(O_K))}$$

where $\mu$ denotes the Haar measure on $G_K$ normalized to equal canonical measure on $G(O_K) = G_K \cap GL_n(O_K)$.

Proposition 2.1. *$\Phi^*$ is a $G(O_K)$ bi-invariant function on $G_K$ with the property that $\Phi^*(\mathbf{1}_n) = 1$.*

Proof. The hard part of the proposition is to show that $\mu_c(\gamma g G(O_K)) = \mu_c(gG(O_K))$ where $\gamma \in G(O_K)$. Without loss of generality, let $g \in G^0$. By $G^0$ open in $G_K$ and the group property of $G(O_K)$, there is a decomposition

$$G^0 = gG(O_K) + \coprod g\gamma_i G(O_K)$$

with $\gamma_i \notin G(O_K)$. On the other hand, $\mathrm{diam}(\gamma G^0 \gamma^{-1}) = 1 > 0$, $\mathrm{dist}(G^0, G_K \backslash G^0) > 1$ and $\gamma G^0 \gamma^{-1} \cap G^0 \neq \varnothing$ imply $\gamma G^0 \gamma^{-1} \subset G^0$ with set diameter and distance between sets defined with respect to the induced topology from $\mathrm{Mat}_n(K)$. Therefore,

$$\gamma(gG(O_K) + \coprod g\gamma_i G(O_K)) = gG(O_K) + \coprod g\gamma_i G(O_K).$$

Now $\gamma g G(O_K) \not\subset g\gamma_i G(O_K)$ $\forall i$ since $|\det(\gamma g)|_K = |\det(g)|_K$ while $|\det(g\gamma_i)|_K = |\det(g)|_K \cdot |\det(\gamma_i)|_K$ with $\gamma_i \notin G(O_K)$. It follows that $\gamma g G(O_K) \subset gG(O_K)$. The reverse inclusion can be shown similarly so that $\gamma g G(O_K) = gG(O_K)$. It is clear that $\mu_c(\gamma g G(O_K)) = \mu_c(gG(O_K))$. Therefore, given $\gamma, \gamma' \in G(O_K)$,

$$\Phi^*(\gamma g \gamma') = \frac{\mu_c(\gamma g \gamma' G(O_K))}{\mu(G(O_K))} = \frac{\mu_c(gG(O_K))}{\mu(G(O_K))} = \Phi^*(g)$$

$\forall g \in G^0$. The extension to arbitrary $g \in G_K$ is trivial. This is the desired result. $\square$

LEMMA 2.1. *$\mu_c(gH) = \Phi^*(g)\mu(H)$ $\forall g \in G_K$ and any compact open subgroup $H \subset G(O_K)$.*

PROOF. The compactness of $G(O_K)$ implies $G(O_K) = H + \coprod H\gamma_i$ so that $\mu_c(gH) = \Phi^*(g)\mu(H)$ is equivalent to $\mu_c(gH)(\mu(H) + \sum \mu(H\gamma_i)) = (\mu_c(gH) + \sum \mu_c(gH\gamma_i))\mu(H)$. The lemma will follow if $\mu_c(gH) = \mu_c(gH\gamma)$ where $\gamma \in G(O_K)$. Without loss of generality, we choose $H$ small such that canonical measure has the local meaning on $gH$ previously mentioned. Express the Riemannian structure of $G_K$ at $g$ by $T_g(G_K) \cap \mathrm{Mat}_n(O_K)$. By definition, $(D\gamma)_g(T_g(G_K) \cap \mathrm{Mat}_n(O_K)) = (T_g(G_K) \cap \mathrm{Mat}_n(O_K))\gamma = T_{g\gamma}(G_K) \cap \mathrm{Mat}_n(O_K)$. In diagram form we have

$$\begin{array}{cccccc}
G_K & & K^l & K^l & & T_g(gH) \\
\cup & & \cup & \cup & & \cup \\
\phi : gH & \longrightarrow & W & MO_K^l & \longleftarrow & L_g : (D\phi)_g \\
 & & & & & \big\downarrow{\scriptstyle (D\gamma)_g} \\
\phi' : gH\gamma & \longrightarrow & W' & M'O_K^l & \longleftarrow & L_{g\gamma} : (D\phi')_{g\gamma} \\
\cap & & \cap & \cap & & \cap \\
G_K & & K^l & K^l & & T_{g\gamma}(gH)
\end{array}$$

where horizontal maps denote isomorphisms in the category of analytic manifolds or finite dimensional vector spaces and $M, M' \in GL_l(O_K)$. Note that

$$M'O_K^l = (D\phi')_{g\gamma}\gamma(D\phi)_0^{-1}MO_K^l.$$

The measure $\nu$ on $K^l$ corresponding to the Haar measure $\mu_g$ on $T_g(gH)$ such that $\mu_g(L_g) = 1$ is defined by the relation $1 = \nu((D\phi)_g(L_g)) = \lambda \cdot dx_1 \cdots dx_l(MO_K^l)$. Hence, $\lambda = 1$ and $\nu = dx_1 \cdots dx_l$. Similarly, the measure $\nu'$ on $K^l$ corresponding to the Haar measure $\mu_{g\gamma}$ on $T_{g\gamma}(gH\gamma)$ such that $\mu_{g\gamma}(L_{g\gamma}) = 1$ is $dx_1 \cdots dx_l$. By the $p$-adic change of variables formula,

$$\int_{W'} dx_1 \cdots dx_l = \int_W |\det((D\phi')_{g\gamma}\gamma(D\phi)_0^{-1})|_K \, dx_1 \cdots dx_l = \int_W dx_1 \cdots dx_l$$

so that $\nu'(W') = \nu(W)$. It follows that $\mu_c(gH) = \mu_c(gH\gamma)$. □

The lemma may be extended, without difficulty, to the case where a compact open subset $E \subset G(O_K)$ replaces the compact open subgroup $H \subset G(O_K)$. Define a function $\Phi$ on $G_K$ by

$$\Phi(g) = \frac{\Phi^*(g)}{|f(g)|_K^\delta}$$

where $\delta = l/m$.

PROPOSITION 2.2. *$\Phi$ is a $K^\times G(O_K)$ bi-invariant function on $G_K$.*

PROOF. Since $\Phi^*$ is already a $G(O_K)$ bi-invariant function on $G_K$ by PROPOSITION 2.1, the proposition will follow if $\mu_c(\pi^r u\mathbf{1}_n gG(O_K)) = q^{-rl}\mu_c(gG(O_K))$ and $|f(\pi^r u\mathbf{1}_n\gamma)|_K^\delta$

$= q^{-rl}$ where $\pi^r u$ is a generic element in $K^\times$, $\gamma \in G(O_K)$ and $\mathbf{1}_n$ denotes the $n \times n$ identity matrix. The string of equalities

$$|f(\pi^r u\mathbf{1}_n\gamma)|_K^\delta = |f(\pi^r u\mathbf{1}_n)|_K^\delta \cdot |f(\gamma)|_K^\delta = |\det(\pi^r u\mathbf{1}_n)|_K^{l/n} = q^{-rl}$$

holds by $f \in \mathrm{Hom}(G, \boldsymbol{G_m})$ of degree $m > 0$ so that $f^{n/m} = \det$. Express the Riemannian structure of $G_K$ at $g$ by $O_K e_1 + \cdots + O_K e_l$. It follows that the Riemannian structure of $G_K$ at $\pi^r u\mathbf{1}_n g$ is given by $O_K(\pi^r u\mathbf{1}_n e_1) + \cdots + O_K(\pi^r u\mathbf{1}_n e_l)$ or $\pi^r O_K e_1 + \cdots + \pi^r O_K e_l$ from which it is clear that $\mu_c(\pi^r u\mathbf{1}_n gG(O_K)) = q^{-rl}\mu_c(gG(O_K))$. $\square$

It is now possible to write down a simplified expression for $Z_K(s)$ and, thus, complete the first step.

Reduction Theorem. *$\Phi$ is a $K^\times G(O_K)$ bi-invariant function on $G_K$ such that, for* $\mathrm{Re}(s) > 0$,

$$Z_K(s) = Z(s, \Phi) = \int_{G^0} |f(g)|_K^s \Phi(g)\mu_0(g)$$

*where* $\mu_0(g) = |f(g)|_K^\delta \mu(g)$.

Proof. Clearly,

$$Z(s, \Phi) = \int_{G^0} |f(g)|_K^s \Phi^*(g)\mu(g).$$

Write $G^0 = \coprod \gamma_i G(O_K)$ so that

$$Z(s, \Phi) = \sum_i \int_{\gamma_i G(O_K)} |f(g)|_K^s \Phi^*(\gamma_i)\mu(g) = \sum_i \int_{\gamma_i G(O_K)} |f(g)|_K^s \mu_c(\gamma_i g) = Z_K(s)$$

follows by the remarked upon straightforward extension of Lemma 2.1. $\square$

## 3. Computation of $\Phi$.

The second step is composed of writing down an explicit formula for the function $\Phi^*$ and combining this formula with the Reduction Theorem to produce a finite form expression for $Z_K(s)$. Again, let $G$ be a $K$-subgroup of $GL_n$ satisfying (1), (2) and (3). Let $S = \{\alpha_i \mid 1 \le i \le \ell\}$ be a basis for $R$. Note that, with respect to $S$, $R$ decomposes into a direct sum of positive roots $R^+$ and negative roots $R^-$. Let $\omega$ denote the dominant weight of the irreducible representation $g \mapsto {}^t g^{-1}$ ([**1**]). Set $\alpha_0 = f|_T$. By construction, $\alpha_0 \cup S$ and $\omega$ generate $\mathrm{Hom}(T, \boldsymbol{G_m})$ with relation

$$\omega^{\deg(f)} = \alpha_0^{-1} \prod_{i=1}^{\ell} \alpha_i^{b_i}$$

defining positive integers $b_1, \ldots, b_\ell$ ([**5**]). Put $\Xi = \mathrm{Hom}(\boldsymbol{G_m}, T)$ and let $\xi_0, \ldots, \xi_\ell \in \Xi$ be such that $\langle \alpha_i, \xi_j \rangle = 1$ if $i = j$, $0 \le i, j \le \ell$, and zero otherwise, $\langle \cdot, \cdot \rangle$ denoting the natural pairing $\Xi \times \mathrm{Hom}(T, \boldsymbol{G_m}) \to \boldsymbol{Z}$. It follows that $\xi_0, \ldots, \xi_\ell$ generate $\Xi$.

Now let $C$ be the positive Weyl chamber relative to $S$ and put $\Xi^0 = \{\xi \in \Xi \mid \xi(\pi) \in$

$\mathrm{Mat}_n(O_K)\}$. Let $\xi \in \Xi^0 \cap \overline{C}$ and, following [**5**], express $\xi$ as

$$\xi_0^{mn_0} \prod_{i=1}^{\ell} (\xi_0^{b_i} \xi_i)^{n_i} \tag{3.1}$$

where $n_0 \in \boldsymbol{N}$ and $n_i \in \boldsymbol{N}$, $1 \le i \le \ell$. Set $\omega_r(t)$, $1 \le r \le n$, as the diagonal entries of an arbitrary $t \in T$. It is a well known result of E. Cartan that

$$\omega_r = \omega^{-1} \prod_{i=1}^{\ell} \alpha_i^{c_i(r)} \tag{3.2}$$

where the $c(r) = (c_1(r), \dots, c_\ell(r))$, $1 \le r \le n$, form a subset $\Gamma \subset \boldsymbol{N}^\ell$. By the choice of the expression (3.1) for $\xi$,

$$\langle \omega_r, \xi \rangle = n_0 + \sum_{i=1}^{\ell} c_i(r) n_i.$$

For convenience in stating the following theorem, set $\eta = (n_1, \dots, n_\ell) \in \boldsymbol{N}^\ell$.

$\Phi^*$-THEOREM. *For a fixed $\xi \in \Xi^0 \cap \overline{C}$,*

$$\Phi^*(\xi(\pi)) = q^{-(\dim(G) n_0 + \sum_{c \in \Delta_\xi} c \cdot \eta + \sum_{\alpha \in R} c_{\alpha,\xi} \cdot \eta)}$$

*where $\Delta_\xi \subset \Gamma$ is a basis for $\boldsymbol{Q}^\ell$ such that the sum $c \cdot \eta$ for all $c \in \Delta_\xi$ takes the smallest value and we take $\alpha$ from $R$ and determine $r = r(\alpha, \xi)$ such that the $r^{th}$ row of the tangent vector at $\theta_\alpha(u)$ at $u = 0$ is different from zero and such that $c \cdot \eta$ for the corresponding $c = c(\alpha, \xi)$ takes the smallest value.*

We will give a complete proof of the $\Phi^*$-THEOREM. In the course of the proof it is convenient to define the concept of an admissible choice of local coordinates. Let $X \subset K^N$ be an everywhere $l$-dimensional $K$-analytic manifold. Hence, for all $a \in X$, there exist $K$-analytic functions $f_i$, $1 \le i \le N$, defined in a neighborhood $U$ containing $a$ such that

$$\left. \frac{\partial(f_1, \dots, f_N)}{\partial(y_1, \dots, y_N)} \right|_{y=a} \neq 0$$

and $X$ is defined locally by $f_{l+1} = \cdots = f_N = 0$. Write $\{1, 2, \dots, N\}$ as $I \coprod J$, $\mathrm{card}(I) = l$, and set

$$F = \begin{bmatrix} \dfrac{\partial f_{l+1}}{\partial y_1} & \cdots & \dfrac{\partial f_{l+1}}{\partial y_N} \\ \vdots & & \vdots \\ \dfrac{\partial f_N}{\partial y_1} & \cdots & \dfrac{\partial f_N}{\partial y_N} \end{bmatrix} \quad \text{with } F_j = \begin{bmatrix} \dfrac{\partial f_{l+1}}{\partial y_j} \\ \vdots \\ \dfrac{\partial f_N}{\partial y_j} \end{bmatrix};$$

DEFINITION 3.1. The projection $\mathrm{pr}_I : U \to K^l$ given by $\mathrm{pr}_I(x) = (x_i)_{i \in I}$ is said to define an admissible choice of local coordinates on a neighborhood $U$ containing $a$ if and only if $\det((F_j(a))_{j \in J}) \neq 0$.

Note that, in this set-up, canonical measure has the meaning detailed in [**5**]. In particular,

$$\mu_c(U) = \max\{\mu(\mathrm{pr}_I(U))\}$$

where $\mathrm{pr}_I : U \to K^l$ defines an admissible choice of local coordinates on $U$ and $\mu$ is the usual Haar measure on $K^l$.

Proof of the $\Phi^*$-Theorem. Let $U$ be a compact open neighborhood in $G(O_K)$ such that $U$ contains the identity $\mathbf{1}_n$. Recall that, as $G$ is a $K$-subgroup of $GL_n(K)$, an element $g \in G$ has a presentation in the basis $e_{ij}$, $1 \le i, j \le n$, where $e_{ij}$ denotes the $n \times n$ matrix with a 1 in the $(i, j)$th place and zeros elsewhere. Thus, any admissible choice of local coordinates $\mathrm{pr}_I : U \to K^l$ defined by $\mathrm{pr}_I(x) = ((x_{ij})_{ij \in I})$ has image

$$\sum_{ij \in I} \pi^{r_{ij}} O_K e_{ij},$$

where $r_{ij} \in \boldsymbol{N}$. Choosing local coordinates on $U \subset G(O_K)$ corresponds to choosing local coordinates in $T_K \cap U$ and $\theta_\alpha(\boldsymbol{G_a})_K \cap U$. Fix an $\alpha \in R$. Since the dimension of $\theta_\alpha(\boldsymbol{G_a})$ is 1 and $\theta_\alpha(\boldsymbol{G_a}) - \mathbf{1}_n$ is nilpotent, local coordinates in $\theta_\alpha(\boldsymbol{G_a})_K \cap U$ must have the form $x_{ij}, i \ne j, 1 \le i, j \le n$. If $\alpha \in R^+$,

$$\theta_\alpha(u) = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

where the entries above the main diagonal have the form $a_{ij}u$, $i < j$, for fixed $a_{ij} \in K$. Therefore, possible choices for a local coordinate on $\theta_\alpha(\boldsymbol{G_a})_K \cap U$ correspond to the $x_{ij}$, $i < j$, such that $(d/du)\theta_\alpha(u)|_{u=0} \ne 0$. Since $G$ has very good reduction mod $\pi$, $\theta_\alpha$ has good reduction mod $\pi$ and the coefficients $a_{ij}$ satisfy $|a_{ij}|_K = 1$ implying any $x_{ij}$, $i < j$, such that $(d/du)\theta_\alpha(u)|_{u=0} \ne 0$ will suffice to define the local coordinate. The situation is similar for $\alpha \in R^-$.

As $T$ was assumed $K$-split and diagonal, there is a $K$-isomorphism $\gamma : \boldsymbol{G_m}^{\dim(T)} \to T$ and local coordinates in $T_K \cap U$ have the form $x_{ii}$, $1 \le i \le n$. If $\iota : \boldsymbol{G_m} \to \boldsymbol{G_m}^{\dim(T)}$ denotes the inclusion $v \mapsto (1, \dots, 1, v, 1, \dots, 1)$, then

$$\gamma \circ \iota(v) = \begin{bmatrix} \ddots & & 0 \\ & v^{t_i} & \\ 0 & & \ddots \end{bmatrix}$$

where $t_i \in \boldsymbol{Z}$, $1 \le i \le n$. Since $1 + \pi^t O_K$ is a subgroup of $O_K^\times$ when $t$ is an integer greater than or equal 1, possible choices for a local coordinate on $T_K \cap U$ corresponding to $\gamma \circ \iota(\boldsymbol{G_m}) \cap U$ are any entries $x_{ii}$, $1 \le i \le n$, such that $(d/dv)(\gamma \circ \iota)(v)|_{v=1} \ne 0$. Any such choice will do.

Now apply Krasner's Lemma to choose $U$ such that $\mathrm{pr}_I(U)$, for any choice $I$ at $\mathbf{1}_n$, has the form $\sum \pi^r O_K e_{ij}$ where $r$ is a fixed positive integer and the summation is over all

$ij \in I$. By definition,

$$U \ni x \mapsto \begin{bmatrix} \pi^{\langle w_1, \xi \rangle} & & \\ & \ddots & \\ & & \pi^{\langle w_n, \xi \rangle} \end{bmatrix} \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix} = \xi(\pi) x \in \xi(\pi) U.$$

Furthermore, $I$ is admissible at $\mathbf{1}_n$ if and only if $I$ is admissible at $\xi(\pi)$. Therefore, using the freedom in choosing $I$ at $\mathbf{1}_n$,

$$\mu_c(\xi(\pi)U) = \max\{\mu(\mathrm{pr}_I(\xi(\pi)U))\} = q^{-(\dim(G)n_0 + \sum_{c \in \Delta_\xi} c \cdot \eta + \sum_{\alpha \in R} c_{\alpha,\xi} \cdot \eta)} \mu(U)$$

where $\Delta_\xi \subset \Gamma$ and $c_{\alpha,\xi}$ are as in the statement. This is as desired. $\square$

We complete the second step by citing the following theorem of Igusa. The notation in the theorem is as follows: the $a_i$ are defined by $\prod \alpha = \prod \alpha_i^{a_i}$ where the first product is over all $\alpha$ in $R^+$ and the second product is over all $i$, $1 \le i \le \ell$; the $m_i$, $1 \le i \le \ell$, are the exponents of the derived group $[G, G]$ of $G$ ([**2**]); $W$ is the Weyl group of $G$ and an element $w \in W$ acts on a root $\alpha \in R$ as $w(\alpha)(t) = \alpha(w^{-1}(t))$ where $t \in T$; and $\lambda$ is the Bott function defined on $W$ by $\lambda(w) = \mathrm{card}(w^{-1}(R^+) \cap R^-)$.

Theorem 3.1 ([**5**]). *Let $s_i \in \boldsymbol{C}$, $1 \le i \le \ell$. If $\Phi$ is a $K^\times G(O_K)$ bi-invariant function on $G_K$ such that*

$$\Phi(\xi(\pi)) = \prod_{i=1}^{\ell} |\alpha_i(\xi(\pi))|_K^{s_i} \tag{3.3}$$

*for every $\xi \in \Xi^0 \cap \overline{C}$, then*

$$Z(s, \Phi) = \frac{(1 - q^{-1})}{(1 - q^{-m(s+\delta)})} \cdot \prod_{i=1}^{\ell} (1 - q^{-m_i})/(1 - q^{a_i - b_i(s+\delta) - s_i})$$
$$\times \sum_{w \in W} q^{-\lambda(w)} \cdot \prod_{\alpha_i \in w(R^-)} q^{a_i - b_i(s+\delta) - s_i}$$

*is a rational function of $q^{-1}$, $q^{-s}$ and $q^{-s_i}$, $1 \le i \le \ell$, of degree $-m = -\deg(f)$ in $q^{-s}$ with rational coefficients such that*

$$Z(s, \Phi)|_{q \to q^{-1}} = q^{-ms} Z(s, \Phi).$$

A generalization of the above theorem suitable for applications where $\Phi(\xi(\pi))$ does not have the multiplicative expression (3.3) exists. The interested reader is referred to [**11**] for the statement and complete proof of this generalization.

## 4. Weights and Roots.

As an application of the results in the prior two sections, we consider the case $G = Ad(G')(\boldsymbol{G_m} \mathbf{1}_{\dim(G')})$ where $G'$ is a simple Chevalley $K$-subgroup of $SL_n$ of type $B_\ell, C_\ell$, $D_\ell$, $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$. Since our choice of $G$ will satisfy (1), (2) and (3), we are reduced to computing $\Phi(\xi(\pi))$ and verifying that it has the form required to apply

THEOREM 3.1. For notational convenience, write $\Phi(\xi(\pi)) = [\boldsymbol{s}_1, \dots, \boldsymbol{s}_\ell]$ if

$$\Phi(\xi(\pi)) = \prod_{i=1}^{\ell} |\alpha_i(\xi(\pi))|_K^{s_i}.$$

We remark that our computational task was considerably reduced by the MAPLE software package **crystal** ([**6**]).

In the case of $G'$ of exceptional type, $G'$ is assumed generated by a Chevalley basis. The computation of $\Phi(\xi(\pi))$ is, thus, reduced to listing the weights $P$ of $Ad(G')$, computing $\Gamma$ from $P$, computing $\Delta_\xi$ by inspection of $\Gamma$ and computing $c_{\alpha,\xi}$, for $\alpha \in P$ nonzero, by finding $\beta \in P$ such that $\alpha + \beta \in P$ and $\langle \alpha + \beta, \xi \rangle \in \boldsymbol{N}$ is minimal. For example, let $G'$ be the Chevalley $K$-group $G_2$. Let $\{\alpha_1, \alpha_2\}$ be a basis for the root system $R$ of $G_2$. Set $\lambda_0 = -(\lambda_1 + \lambda_2)$, $\lambda_1 = \alpha_1$ and $\lambda_2 = \alpha_1 + \alpha_2$. The weights of $Ad(G_2)$ are, then,

$$P = \{\pm\lambda_i \ (i = 0, 1, 2), \ \pm(\lambda_i - \lambda_j) \ (0 \le i < j \le 2)\} \cup \{0\}.$$

The dominant weight $\omega$ of the irreducible representation $g \mapsto {}^t g^{-1}$ of $Ad(G_2)$ is $\lambda_0 - \lambda_2$. We compute the following elements contained in $\Gamma$ as in (3.2):

| $P$ | $\Gamma$ |
|---|---|
| $\lambda_0 - \lambda_2$ | $(0,0)$ |
| $\lambda_0 - \lambda_1$ | $(0,1)$ |
| $\lambda_0$ | $(1,1)$ |
| $-\lambda_2$ | $(2,1)$ |
| $\lambda_1 - \lambda_2$ | $(3,1)$ |
| $-\lambda_1$ | $(2,2)$ |
| $0$ | $(3,2)$ |

Hence, $\Delta_\xi = (1,2)$ for all $\xi \in \Xi^0 \cap \overline{C}$. It is also easy to see that $c_{\lambda_0-\lambda_2,\xi} = (0,0)$ for all $\xi$ since $0 \in P$ such that $\lambda_0 - \lambda_2 + 0 = \lambda_0 - \lambda_2 \in P$ and $\langle \lambda_0 - \lambda_2, \xi \rangle$ is minimal in $\boldsymbol{N}$. Now $c_{\lambda_2-\lambda_0,\xi} = (3,2)$ for all $\xi$ as, given $\lambda$, $\lambda'$ and $\omega$ in $P$, if $\lambda + \lambda' = \omega$ such that $\omega$ is unique up to multiplicity and $\langle \omega, \xi \rangle$ is minimal in $\boldsymbol{N}$ for all $\xi$, then $-\lambda + \omega = \lambda'$ such that $\lambda'$ is unique up to multiplicity and $\langle \lambda', \xi \rangle$ is minimal in $\boldsymbol{N}$ for all $\xi$. Similarly, $c_{\lambda_0-\lambda_1,\xi} = (0,0)$, $c_{\lambda_1-\lambda_0,\xi} = (3,1)$, $c_{\lambda_0,\xi} = (0,0)$, $c_{-\lambda_0,\xi} = (2,1)$, $c_{-\lambda_2,\xi} = (0,0)$, $c_{\lambda_2,\xi} = (1,1)$, $c_{\lambda_1-\lambda_2,\xi} = (0,0)$, $c_{\lambda_2-\lambda_1,\xi} = (0,1)$, $c_{-\lambda_1,\xi} = (0,1)$ and $c_{\lambda_1,\xi} = (1,1)$ for all $\xi \in \Xi^0 \cap \overline{C}$. Hence,

$$\Phi(\xi(\pi)) = \frac{q^{-15_{n_0} - 11_{n_1} - 10_{n_2}}}{|f(\xi(\pi))|_K^{\delta}} = |\alpha_1(\xi(\pi))|_K^{11} |\alpha_2(\xi(\pi))|_K^{10} = [\mathbf{11}, \mathbf{10}]$$

and THEOREM 3.1 applies.

In a similar manner one finds $\Phi(\xi(\pi)) = [\mathbf{39}, \mathbf{61}, \mathbf{51}, \mathbf{46}]$ for $G'$ the Chevalley $K$-group $F_4$, $\Phi(\xi(\pi)) = [\mathbf{17}, \mathbf{34}, \mathbf{59}, \mathbf{34}, \mathbf{17}, \mathbf{58}]$ for $G'$ the Chevalley $K$-group $E_6$, $\Phi(\xi(\pi)) = [\mathbf{28}, \mathbf{56}, \mathbf{84}, \mathbf{115}, \mathbf{102}, \mathbf{101}, \mathbf{50}]$ for $G'$ the Chevalley $K$-group $E_7$ and $\Phi(\xi(\pi)) = [\mathbf{192}, \mathbf{193}, \mathbf{214}, \mathbf{245}, \mathbf{282}, \mathbf{186}, \mathbf{93}, \mathbf{137}]$ for $G'$ of type $E_8$.

In the case of $G'$ of type $B_\ell$, $C_\ell$ or $D_\ell$, we use the classical presentation of $G'$ as the connected component of the identity of

$$G_f = \{g \in GL_n \mid f(gx) = f(x)\}$$

where $f$ is chosen suitably in $K[x_{11}, \dots x_{nn}]$, i.e., for type $D_\ell$, $B_\ell$, $f(x) = (1/2)^t xSx$ where $S$ equals $\begin{bmatrix} 0 & \mathbf{1}_\ell \\ \mathbf{1}_\ell & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & & \\ & 0 & \mathbf{1}_\ell \\ & \mathbf{1}_\ell & 0 \end{bmatrix}$, respectively, and for $C_\ell$, $f(x, y) = {}^t xJy$ where $J = \begin{bmatrix} 0 & \mathbf{1}_\ell \\ -\mathbf{1}_\ell & 0 \end{bmatrix}$. The conditions of the $\Phi^*$-THEOREM imply there is no loss of generality in computing $\Phi(\xi(\pi))$ via examination of the Lie algebra $\boldsymbol{L}G'$ of $G'$. For example, consider $\boldsymbol{L}SO_{2\ell}$, $\ell \geq 4$. Note that $\boldsymbol{L}SO_{2\ell}$ is the type $D_\ell$ case. Let $V$ be a vector space with basis $v_1, \dots, v_\ell, v_{-1}, \dots, v_{-\ell}$ determined by the condition

$${}^t v_i S v_j = \begin{cases} 0 & \text{if } |j - i| \neq 0 \\ 1 & \text{if } |j - i| = 0. \end{cases}$$

With respect to the given basis for $V$, a representative element $L \in \boldsymbol{L}SO_{2\ell}$ has the form

$$L = \begin{bmatrix} A & B \\ C & -{}^t A \end{bmatrix}$$

where $A$, $B$ and $C$ are in $\mathrm{Mat}_\ell(K)$ and $B$, $C$ are skew-symmetric. Assign the weight of $v_{\pm i}$ as $\pm\lambda_i$. If we let $\mathrm{Ext}(V)$ denote the exterior algebra over $V$ in the given basis and let $\mathrm{Ext}_m(V)$ be the homogeneous subspace of degree $m$, then $ad(\boldsymbol{L}SO_{2\ell})$ has basis the canonical choice of basis for $\mathrm{Ext}_2(V)$. Hence, the weights $P$ of $ad(\boldsymbol{L}SO_{2\ell})$ are

$$\{\pm(\lambda_i - \lambda_j) | 1 \leq i < j \leq \ell\} \cup \{\pm(\lambda_i + \lambda_j) \mid 1 \leq i < j \leq \ell\} \cup \{0\}.$$

The dominant weight $\omega$ of the irreducible representation $L \mapsto -{}^t L$ of $ad(\boldsymbol{L}SO_{2\ell})$ is $\lambda_1 - \lambda_2$. A basis $S$ for the roots $R$ of $\boldsymbol{L}SO_{2\ell}$ is chosen as $\{\alpha_i\}_{1 \leq i \leq \ell}$ where $\alpha_i = \lambda_i - \lambda_{i+1}$, $1 \leq i \leq \ell - 1$, and $\alpha_\ell = \lambda_{\ell-1} + \lambda_\ell$. It follows that

$$(4.1) \qquad \begin{cases} \pm(\lambda_i - \lambda_j) = -\lambda_1 - \lambda_2 + [\lambda_1 + \lambda_2 + (\alpha_i + \cdots + \alpha_{j-1})] \\ \pm(\lambda_i + \lambda_j) = -\lambda_1 - \lambda_2 + [\lambda_1 + \lambda_2 + ((\alpha_i + \cdots + \alpha_{\ell-2}) + (\alpha_j + \cdots + \alpha_\ell))] \end{cases}$$

where $1 \leq i < j \leq \ell$, with the coefficients of the $\alpha_i$, $1 \leq i \leq \ell$, in the bracketed terms determining the subset $\Gamma \subset \boldsymbol{N}^\ell$. We use the above information to compute the $\Delta_\xi$ and $c_{\alpha,\xi}$, $\alpha \in R$, appearing in the $\Phi^*$-THEOREM for $ad(\boldsymbol{L}SO_{2\ell})$. An examination of (4.1) yields $\Delta_\xi$ determined by $-\lambda_1 - \lambda_3, -\lambda_1 - \lambda_4, \dots, -\lambda_1 - \lambda_\ell, -\lambda_2 - \lambda_3$ and $\lambda_\ell - \lambda_1$ for all $\xi \in \Xi^0 \cap \overline{C}$. The representative element $L$ acts on $v_i \wedge v_j \in \mathrm{Ext}_2(V)$ as $L(v_i \wedge v_j) = (Lv_i) \wedge v_j + v_i \wedge (Lv_j)$, $-\ell \leq i < j \leq \ell$. By construction, $c_{-\lambda_i - \lambda_j, \xi}$, $i < j$, corresponds to the row vector of miminal weight in which $\pm c_{ij} \in C$ appears under the action of $L$ on all basis elements of $\mathrm{Ext}_2(V)$. Under the action of $L$, $\pm c_{ij}$ occurs in rows of weight

$$
\begin{aligned}
&-\lambda_i + \lambda_{(\hat{j})}, \quad -\lambda_i - \lambda_{(\hat{i})}, \\
&-\lambda_j + \lambda_{(\hat{i})}, \quad -\lambda_j - \lambda_{(\hat{j})},
\end{aligned}
\tag{4.2}
$$

where $(\hat{r})$ denotes any integer value from 1 to $\ell$ except $r$. The set of weights of the form $-\lambda_{r'} \pm \lambda_{(\hat{r})}$, $r, r' \in \{1, \dots, \ell\}$, is linearly ordered by $<$ the usual partial ordering on roots with respect to $R^+$. Therefore, let

$$
\begin{aligned}
&-\lambda_i + \lambda_u, \quad -\lambda_i - \lambda_v, \\
&-\lambda_j + \lambda_{v'}, \quad -\lambda_j - \lambda_{u'},
\end{aligned}
$$

be the respective minimal weights from (4.2). Then

$$
\begin{aligned}
-\lambda_i + \lambda_u &> -\lambda_i - \lambda_v, \\
& \qquad\quad \wedge \\
-\lambda_j + \lambda_{v'} &> -\lambda_j - \lambda_{u'}.
\end{aligned}
$$

In particular, $-\lambda_i - \lambda_v < -\lambda_j - \lambda_{u'}$ for either $u' = v = 1$, $i < j$, or $i = u' = 1$, $v = 2$ and $j \geq 2$. Hence, $c_{-\lambda_i - \lambda_j, \xi}$, $i < j$, is determined by $-\lambda_i - \lambda_v$ where $v = 1$ if $i \neq 1$ and $v = 2$ if $i = 1$.

Finding the weight determining $c_{\alpha,\xi}$ for $\alpha$ of the form $\lambda_i + \lambda_j$, $i > j$, $\lambda_i - \lambda_j$, $i < j$, and $-\lambda_i + \lambda_j$, $i > j$, is similar. The possible choices for this determining row weight and the determining row weight for the various $\alpha$ are listed below:

$$
\begin{aligned}
\lambda_i + \lambda_j, \ i > j: \quad & \lambda_i - \lambda_{(\hat{j})}, \ \lambda_i + \lambda_{(\hat{i})} \\
& \lambda_j - \lambda_{(\hat{i})}, \ \lambda_j + \lambda_{(\hat{j})}
\end{aligned}
$$

and $c_{\lambda_i + \lambda_j, \xi}$ is determined by $\lambda_j - \lambda_u$ minimal of the form $\lambda_j - \lambda_{(\hat{i})}$;

$$
\begin{aligned}
\lambda_i - \lambda_j, \ i < j: \quad & -\lambda_j + \lambda_d, \ -\lambda_j - \lambda_{(\hat{i},\hat{j})} \\
& \lambda_i + \lambda_{(\hat{i},\hat{j})}, \ \lambda_i - \lambda_d,
\end{aligned}
$$

$d$ arbitrary, $1 \leq d \leq \ell$, and $c_{\lambda_i - \lambda_j, \xi}$, $i < j$, is determined by $-\lambda_j - \lambda_u$ minimal of the form $-\lambda_j - \lambda_{(\hat{i},\hat{j})}$;

$$
\begin{aligned}
-\lambda_i + \lambda_j, \ i > j: \quad & \lambda_i + \lambda_{(\hat{i},\hat{j})}, \ \lambda_i - \lambda_d \\
& -\lambda_j + \lambda_d, \ -\lambda_j - \lambda_{(\hat{i},\hat{j})},
\end{aligned}
$$

$d$ arbitrary, $1 \leq d \leq \ell$, and $c_{-\lambda_i + \lambda_j, \xi}$, $i > j$, is determined by $-\lambda_j - \lambda_u$ minimal of the form $-\lambda_j - \lambda_{(\hat{i},\hat{j})}$. Hence, in the same notation as was used in the exceptional cases,

$$
\begin{aligned}
\Phi(\xi(\pi)) = \Bigg[ & (\mathbf{1}, \ell, \ell - \mathbf{2}, \ell - \mathbf{3}, \dots, \mathbf{3}, \mathbf{1}, \mathbf{1}) \\
& + \left( \mathbf{0}, \frac{(\ell - \mathbf{2})(\ell - \mathbf{3})}{\mathbf{2}}, \frac{(\ell - \mathbf{3})(\ell - \mathbf{4})}{\mathbf{2}}, \dots, \mathbf{1}, \mathbf{0}, \mathbf{0} \right)
\end{aligned}
$$

$$+\left(\boldsymbol{\ell-1,\ldots,\frac{(\ell-i+1)(\ell-i))}{2}+(i-1)\ell,\ldots,\frac{(\ell-2)(\ell-1)}{2},\frac{(\ell-1)\ell}{2}}\right)$$
$$+\left(\boldsymbol{0,\frac{(\ell-2)(\ell-3)}{2}+1,\frac{(\ell-3)(\ell-4)}{2},\ldots,\frac{(\ell-i)(\ell-(i+1))}{2},\ldots,1,0,0}\right)$$
$$\left.+\left(\boldsymbol{\ell-1,\frac{(\ell-1)\ell}{2},\ldots,(i-1)(\ell-i)+\frac{(\ell-i)(\ell-(i+1))}{2},\ldots,\ell-1,0}\right)\right].$$

The $\boldsymbol{LSO}_{2\ell+1}$ or type $B_\ell$, $l \geq 3$, case is similar and

$$\Phi(\xi(\pi)) = \Big[(\boldsymbol{1,\ell,\ell-2,\ldots,2,1})+(\boldsymbol{0,\ell-2,\ell-3,\ldots,1,0})$$
$$+(\boldsymbol{1,\ell,\ell,\ldots,\ell})+\left(\boldsymbol{0,\frac{(\ell-2)(\ell-3)}{2},\frac{(\ell-3)(\ell-4)}{2},\ldots,1,0,0}\right)$$
$$+\left(\boldsymbol{\ell-1,\ldots,\frac{(\ell-i)(\ell-i+1))}{2}+(i-1)\ell,\ldots,\frac{(\ell-1)\ell}{2}}\right)$$
$$+\left(\boldsymbol{0,\frac{(\ell-2)(\ell-3)}{2}+1,\frac{(\ell-3)(\ell-4)}{2},\ldots,\frac{(\ell-i)(\ell-(i+1))}{2},\ldots,1,0,0}\right)$$
$$\left.+\left(\boldsymbol{\ell-1,\frac{(\ell-1)\ell}{2},\ldots,(i-1)(\ell-i)+\frac{(\ell-i)(\ell-(i+1))}{2},\ldots,\ell-1,0}\right)\right].$$

We compute the $\boldsymbol{LSO}_5$ case directly. A simple calculation gives $\Phi(\xi(\pi))$ equal [**4**, **8**].

The $\boldsymbol{LSp}_{2\ell}$ or type $C_\ell$ case, $\ell \geq 3$, may also be handled in a straightforward manner. In this case we have

$$\Phi(\xi(\pi)) = \Big[(\boldsymbol{\ell,\ell-1,\ldots,2,1})+(\boldsymbol{\ell+1,\ell+2,\ldots,\ell+\ell-1,\ell})$$
$$+(\boldsymbol{\ell-1,\ell-2,\ldots,1,0})$$
$$+\left(\boldsymbol{\frac{(\ell-1)\ell}{2},\ldots,\ell(i-1)+\frac{(\ell-i+1)(\ell-i)}{2},\ldots,\frac{(\ell-1)\ell}{2}}\right)$$
$$+\left(\boldsymbol{\frac{(\ell-2)(\ell-1)}{2},\frac{(\ell-3)(\ell-2)}{2},\ldots,1,0,0}\right)$$
$$+\left(\boldsymbol{\frac{(\ell-1)\ell}{2},\ldots,(i-1)(\ell-i)+\frac{(\ell-i+1)(\ell-i)}{2},\ldots,0}\right)$$
$$\left.+\left(\boldsymbol{\frac{(\ell-2)(\ell-1)}{2},\ldots,\frac{(\ell-i+1)(\ell-1)}{2},\ldots,0}\right)\right].$$

It is a remarkable fact that for $G'$ of type $A_\ell$, $\ell > 3$, there does not exist an expression for $\Phi(\xi(\pi))$ suitable for application of THEOREM 3.1. This is because, in contradistinction with the cases of $G'$ just considered, the choice of $\Delta_\xi$ and $c_{\alpha,\xi}$ in this case depend on $\xi$. The interested reader is referred to [**11**] for a detailed examination of this case.

## 5. Results.

In the case $Ad(G')(\boldsymbol{G_m}\mathbf{1}_{\dim(G')})$, where $G'$ is a simple Chevalley $K$-subgroup of $SL_n$ of type $B_\ell$, $C_\ell$, $D_\ell$, $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$, we have now explicitly computed $\Phi(\xi(\pi))$, $\xi \in \Xi^0 \cap \overline{C}$. In addition, each such $\Phi(\xi(\pi))$ was exhibited in the form required for application of Theorem 3.1. We are, thus, led to consider the generalized Igusa local zeta function

$$Z_K(s) = Z(s, \Phi) = \int_{[Ad(G')(\boldsymbol{G_m}\mathbf{1}_{\dim(G')})]^0} |f(g)|_K^s \Phi(g)\mu_0(g).$$

Theorem 5.1. *Let $G'$ be of type $G_2$, $F_4$, $E_6$, $E_7$, $E_8$, $D_\ell$, $B_\ell$ or $C_\ell$. Then $Z_K(s)$ is a rational function of $q^{-1}$, $q^{-s}$ of degree $-1 = -\deg(f)$ in $q^{-s}$ with rational coefficients and it satisfies the functional equation*

$$Z_K(s)|_{q \mapsto q^{-1}} = q^{-s} Z_K(s).$$

Proof. This is an immediate consequence of the forth section. □

This completes the verification in the case $Ad(G')(\boldsymbol{G_m}\mathbf{1}_{\dim(G')})$, $G'$ as above, of the conjecture of Igusa mentioned in the introduction. The straightforwardness of this result was completely unexpected as $\Phi(\xi(\pi))$ rarely has the multiplicative expression (3.3) required by Theorem 3.1. The interested reader is referred to [**9**], [**10**] and [**11**] for examples of more pathological $\Phi(\xi(\pi))$.

The closure of $Ad(G')(\boldsymbol{G_m}\mathbf{1}_{\dim(G')})$ is an absolutely irreducible $(\dim(G') + 1)$-dimensional affine $K$-variety. Taking the $K$-rational points of $Ad(G')(\boldsymbol{G_m}\mathbf{1}_{\dim(G')})$ as the $X$ of the introduction, observe that not only $Z_K(s)$ but also $Z_L(s)$ for every finite extension $L$ of $K$ is defined.

Definition 5.1. The polynomial $Z(u, v) \in \boldsymbol{Q}(u, v)$ is called a universal $p$-adic zeta function of $(G, f)$ if, after a possible extension $K'$ *of* $K$,

$$Z(u, v) = Z_L(s)$$

for some choice of variables $u$, $v$ and all extensions $L$ of $K'$.

Corollary 5.1. *Let $G'$ be of type $G_2$, $F_4$, $E_6$, $E_7$, $E_8$, $D_\ell$, $B_\ell$ or $C_\ell$. If $u$, $v$ are taken as $q^{-1}$, $q^{-s}$, respectively, and $K = \boldsymbol{Q}_p$, $p \neq 2$, the corresponding universal p-adic zeta function of $(Ad(G')(\boldsymbol{G_m}\mathbf{1}_{\dim(G')}), f)$ is*

$$Z(q^{-1}, q^{-s}) = Z_L(s)$$

*with universal functional equation*

$$Z(u^{-1}, v^{-1}) = vZ(u, v).$$

Proof. This is an immediate consequence of Theorem 5.1. □

Finally, we remark that the function $\Phi$ is not 'residual' as defined by J. Denef and D. Meuser in [**3**]. Therefore, the general result Denef and Meuser prove in [**3**] about the finite form and functional equation of an Igusa local zeta function does not apply to a generalized Igusa local zeta function. It is not known how far the theory Denef and Meuser develop in [**3**] extends to generalized Igusa local zeta functions. As one would expect, the major stumbling block to extending this theory is the nontranslation invariance of Serre's canonical measure. The reader is referred to [**5**], [**10**] and, in particular, [**9**] for a look at some of the difficulties one faces in trying to push through for generalized Igusa local zeta functions the Denef-Meuser theory. The reader will also find in [**8**] or [**10**] a complete answer to this extension question for Chevalley $K$-groups.

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Roland MARTIN
Center for Computing Sciences
17100 Science Drive
Bowie, Md. 20715-4300
E-mail: rem@super.org