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Correlated sums of $r(n)$

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Abstract. We prove an asymptotic formula for $\sum_{n\le N} r(n)r(n+m)$ using the spectral theory of automorphic forms and we specially study the uniformity of the error term in the asymptotic approximation when m varies. The best results are obtained under a natural conjecture about the size of a certain spectral mean of the Maass forms.

We also employ large sieve type inequalities for Fourier coefficients of cusp forms to estimate some averages (over m) of the error term.

§1. Introduction.

In this paper we shall deal with the sums

$$S(N,m) = \sum_{n\le N} r(n)r(n+m)$$

where $r(n)$ denotes the number of representations of n as sum of two squares and $m \in \boldsymbol{Z}^+$.

The corresponding sums when $r(n)$ is substituted by the divisor function, $d(n)$, were considered firstly in 1927 by Ingham (see [**In**]), since then they have acquired growing interest because of their relation with Kloosterman sums, spectral theory of automorphic forms and the power moments of the Riemann zeta function (see the introduction and references of [**Mo1**] and [**Mo2**]). Although the similarity between the two fore mentioned sums ($r(n)$ is the divisor function in $\boldsymbol{Z}[i]$), the spectral analysis is different and, as far as we know, a spectral approach to the asymptotics of $S(N,m)$ for all values of m has not been considered before (the odd case is treated in [**Iw**]).

In this paper we give an asymptotic formula with error term for $S(N,m)$ studying its uniformity in m. We also consider average results when m varies. This kind of averages appear in the study of the mean value of the error term in the circle problem (compare with [**Ts**]) which was our initial motivation and we intend to treat in other occasion.

The structure and contents of the subsequent sections are as follows:

In §2 we give in Proposition 2.3 (see also Lemma 2.1 and Lemma 2.2) the spectral expansion of $S(N,m)$ in terms of non-holomorphic modular forms. The proof follows the lines of the Chapter 12 of [**Iw**], the novelty of our argument (apart from some technical variations to include the range $N \le m$) is to cover the case with even m.

In §3 we prove an asymptotic formula with error term for $S(N,m)$ (see Theorem 3.1) assuming the following bound.

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Conjecture 1.1. *Let* $\{u_j(z)\}$ *be the set of Hecke cusp forms of* $\Gamma = \mathrm{PSL}_2(\boldsymbol{Z})$ *or* $\Gamma = \Gamma_0(2)/\{\pm \mathrm{Id}\}$ *with respective eigenvalues* $\{\lambda_j = 1/4 + t_j^2\}$, *then for every* $\varepsilon > 0$ *and* $z \in \boldsymbol{H}$

$$\sum_{T<|t_j|\le 2T} |u_j(z)|^4 = O(T^{2+\varepsilon})$$

where the O-constant depends on ε *and* z.

Changing $|u_j(z)|^4$ by $|u_j(z)|^p$, the bound follows for any $p > 0$ from a general conjecture of "arithmetic quantum chaos" (see Conjecture 3.10 in [**Sa**] or (0.8) in [**Iw-Sa**]). On the other hand, for $p = 2$ the bound is a consequence of Bessel inequality (see Proposition 7.2 of [**Iw**]), and after expanding $|u_j(z)|^2$ into Fourier series, it seems that there is a primary technical difficulty to treat the case $p = 4$, namely, to find a suitable asymptotic formula for the Bessel function $K_{it}(y)$ separating t from y, over all in the range $|t|^{1-\varepsilon} < y < |t|^{1+\varepsilon}$. After overcoming this technical difficulty, perhaps the conjecture could be settled with a convenient application of Bruggeman-Kuznetsov formula. Recently, N. Pitt has proved a summation formula which allows to deduce the conjecture for some special values of z but not including those appearing in our proof of Theorem 3.1 (I thank H. Iwaniec for some comments about this conjecture and N. Pitt for communicating some unpublished results).

The bound of the error term in the asymptotic formula for $S(N,m)$ reads in some ranges as the one proved unconditionally by Y. Motohashi for $\sum d(n)d(n+m)$ (see [**Mo1**] and use [**Bu-Du-Ho-Iw**]), but in our case the range of uniformity is larger. We also obtain a bound assuming Ramanujan-Petersson conjecture (for non-holomorphic cusp forms) instead of Conjecture 1.1.

In §4 we use estimates on linear forms of Fourier coefficients due to W. Luo [**Lu**] (improving those of [**De-Iw**]) to obtain mean results for $S(N,m)$ when m varies.

Finally, in §5 we give a weaker result for the error term in our asymptotic formula but not depending on any conjecture.

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§2. Spectral analysis of $S(N,m)$.

The purpose of this section is to express $S(N,m)$ in terms of Hecke operators acting on automorphic kernels (see Lemma 2.1 and Lemma 2.2 below) and, via spectral analysis, to expand them into eigenfunctions of the Laplace-Beltrami operator on the upper half plane $\boldsymbol{H}$ (see Proposition 2.3).

First of all we shall give some basic definitions on harmonic analysis in Riemann surfaces to fix the notation and facilitate references. We follow the notation of [**Iw**] which is rather standard.

Γ will denote a Fuchsian group of the first kind, in fact in this paper we shall only consider $\Gamma = \mathrm{PSL}_2(\boldsymbol{Z})$ and $\Gamma = \Gamma_0(2)/\{\pm\mathrm{Id}\}$ where $\Gamma_0(2)$ is the Hecke congruence group

$$\Gamma_0(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\boldsymbol{Z}),\ 2|c \right\}.$$

The set of orbits of $\Gamma\backslash\boldsymbol{H}$ has a Riemann surface and Riemannian manifold structure in which the element of area and the distance function are induced by $d\mu$ and ρ, given by

$$d\mu(x+iy) = \frac{dx\,dy}{y^2} \quad \rho(z,w) = \operatorname{arc\,cosh}(1+2u(z,w)) \quad \text{where } u(z,w) = \frac{|z-w|^2}{4\operatorname{Im} z \operatorname{Im} w}.$$

The functions of $L^2(\Gamma\backslash\boldsymbol{H})$ can be analysed in terms of the eigenfunctions of the Laplace-Beltrami operator on $\boldsymbol{H}$. In particular, under suitable regularity and decaying conditions on a function $k : [0,\infty) \to \boldsymbol{C}$, it can be proved the so called *pretrace formula*

$$\sum_{\gamma\in\Gamma} k(u(\gamma z, w)) = \sum_j h(t_j)u_j(z)\overline{u_j(w)} + \frac{1}{4\pi}\sum_a \int_{-\infty}^{\infty} h(t)E_a(z, 1/2+it)\overline{E_a(w, 1/2+it)}\,dt,$$

where $\{u_j(z)\}_{j=0}^{\infty}$ is an orthonormal system of eigenfunctions with respective ordered eigenvalues $\{\lambda_j = 1/4 + t_j^2\}$. This system is generated by Maass cusp forms and residues of Eisenstein series, $E_a(z,\cdot)$, associated to the cusp a. The function h represents the Selberg-Harish-Chandra transform of k, given by

$$h(t) = \int_{\boldsymbol{H}} k(u(z,i))(\operatorname{Im} z)^{1/2+it}\,d\mu(z).$$

If Γ is a congruence group, Hecke operators T_m, $m \in \boldsymbol{Z}^+$, are defined by (a more proper definition can be given in a general context but this matches our purposes)

$$T_m f(z) = \frac{1}{\sqrt{m}} \sum_{\gamma\in\Gamma_1\backslash\Gamma_m} f(\gamma z) = \frac{1}{\sqrt{m}} \sum_{ad=m} \sum_{(b \bmod d)} f\left(\frac{az+b}{d}\right)$$

where $\Gamma_1 = \mathrm{SL}_2(\boldsymbol{Z})$ and Γ_m is the set of 2×2 integral matrices whose determinant equals m.

Hecke operators constitute one of the most important links between automorphic forms and arithmetic. From the point of view of spectral theory, T_m is self-adjoint in $L^2(\Gamma\backslash\boldsymbol{H})$ if m and the level of Γ are coprime, moreover, T_m commutes with Laplace-Beltrami operator, hence, for a suitable choice of $\{u_j(z)\}$, it holds

$$T_m u_j(z) = \lambda_j(m)u_j(z) \quad \text{and} \quad T_m E_a(z, 1/2+it) = \eta_t(m)E_a(z, 1/2+it).$$

The so chosen cusp forms are called *Hecke cusp forms* and Ramanujan-Petersson conjecture in this context, asserts that $|\lambda_j(m)| \le d(m)$, but this bound is out of reach with current methods. The best known bound, $|\lambda_j(m)| \le m^{5/28}d(m)$, is proved in [**Bu-Du-Ho-Iw**] using quite advanced arguments. On the other hand the formula $\eta_t(m) = \sum_{d|m}(m/d^2)^{it}$ implies at once $|\eta_t(m)| \le d(m)$.

A simple and direct relation between Hecke operators and $S(N,m)$ is the content of the following result.

Lemma 2.1. *Let k be the characteristic function of $[0,N]$ and $\Gamma = \Gamma_0(2)/\{\pm\mathrm{Id}\}$, then for any odd positive integer, m,*

$$S(N,m) = 2\sqrt{m}T_m|_{z=z_0} \sum_{\gamma\in\Gamma} k(mu(\gamma z_0, z))$$

where $z_0 = (i-1)/2$.

PROOF. It is not difficult to check that

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\boldsymbol{Z}) \Rightarrow 4u(\gamma i, i) + 2 = a^2 + b^2 + c^2 + d^2.$$

To specify that $a+d$ and $b+c$ are even one can simply write

$$\gamma \in \tau^{-1}\Gamma_0(2)\tau \quad \text{with } \tau = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

Note that

$$\begin{cases} a^2+b^2+c^2+d^2 = 4n+2 \\ ad-bc = 1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \left(\dfrac{a+d}{2}\right)^2 + \left(\dfrac{c-b}{2}\right)^2 = n+1 \\ \left(\dfrac{a-d}{2}\right)^2 + \left(\dfrac{c+b}{2}\right)^2 = n. \end{cases}$$

Hence, if k is the characteristic function of $[0, N]$ we have

$$\sum_{\gamma \in \Gamma_0(2)} k(u(\tau^{-1}\gamma\tau i, i)) = \sum_{n \le N} r(n)r(n+1).$$

By technical reasons we prefer to consider $\Gamma = \Gamma_0(2)/\{\pm \mathrm{Id}\}$ which is covered twice by $\Gamma_0(2)$. Substituting $\tau i = (i-1)/2$ we can write the previous formula as

$$S(N,1) = 2\sum_{\gamma \in \Gamma} k\left(u\left(\gamma\frac{i-1}{2}, \frac{i-1}{2}\right)\right).$$

The same argument can be repeated replacing a, b, c, d by $\alpha/\sqrt{m}$, $\beta/\sqrt{m}$, $\gamma/\sqrt{m}$, $\delta/\sqrt{m}$ with $\alpha\delta - \beta\gamma = m$ and $\alpha, \beta, \gamma, \delta \in \boldsymbol{Z}$, getting

$$S(N,m) = \sum_{\gamma \in \Gamma'_m} k\left(mu\left(\gamma\frac{i-1}{2}, \frac{i-1}{2}\right)\right)$$

where Γ'_m is the set of matrices $(a_{ij}) \in \Gamma_m$ such that a_{21} is even. Note that the representatives of $\Gamma_1 \backslash \Gamma_m$ chosen in the definition of T_m, say γ_i, are also representatives of $\Gamma_0(2)\backslash\Gamma'_m$, i.e. $\Gamma'_m = \bigcup \Gamma_0(2)\gamma_i$, and the result follows. □

Our objective is to expand $T_m k$ in terms of eigenfunctions but the lack of regularity of k leads to a non-absolutely convergent series. Another more serious difficulty is that, as we noticed before, T_m only behaves as a multiplier when m and the level are coprime, i.e. when m is odd, which limits the interest of generalizing Lemma 2.1. This problem is overcome thanks to the following result.

LEMMA 2.2. *For $m \in \boldsymbol{Z}^+$ even, let 2^k be the greatest power of 2 dividing m, then*

$$S(N,m) = S(N/2, m/2) \quad \text{if } k = 1,$$

and

$$S(N,m) = 2A(N,m) - 2A(N/2, m/2) + 2A(N/4, m/4) + \cdots + 2(-1)^{k-2}A(N/2^{k-2}, m/2^{k-2}) + (-1)^{k-1}S(N/2^{k-1}, m/2^{k-1}) \quad \text{if } k > 1.$$

where

$$A(4N,4m) = 2\sqrt{m}T_m|_{z=i} \sum_{\gamma \in \Gamma} k(mu(\gamma i, z))$$

with $\Gamma = \mathrm{PSL}_2(\boldsymbol{Z})$ *and* k *the characteristic function of* $[0,N]$.

Proof. If m is even, $4 \nmid m$ and $r(n)r(n+m) \neq 0$ then n is even and the equality $r(2l) = r(l)$ implies the first part of the lemma.

For the second part it is enough to prove

$$2A(N,m) = S(N,m) + S(N/2,m/2) \quad \text{for } k > 1. \tag{2.1}$$

Let $a(n,m)$ be the number of integral solutions a,b,c,d of

$$\begin{cases} a^2+b^2+c^2+d^2 = n+m/2 \\ ad-bc = m/4. \end{cases} \tag{2.2}$$

Proceeding as in the proof of Lemma 2.1

$$\sum_{n \le N} a(n,m) = A(N,m).$$

Writing $A = a+d$, $B = c-b$, $C = a-d$, $D = c+b$ in (2.2), we have that $a(n,m)$ is the number of integral solutions A,B,C,D, of

$$\begin{cases} A^2+B^2 = n+m \\ C^2+D^2 = n \\ 2|A-C,\ 2|B-D. \end{cases} \tag{2.3}$$

If $n \equiv 2\ (4)$ or $n \equiv 0\ (4)$ then A,B,C,D are simultaneously odd or even. Hence the third condition in (2.3) is superflous and we get

$$a(n,m) = r(n)r(n+m) = r(n/2)r(n/2+m/2) \quad \text{if } n \text{ is even.} \tag{2.4}$$

If n is odd then $A \not\equiv B\ (2)$ and $C \not\equiv D\ (2)$. Hence, perhaps exchanging C and D, we can omit the third condition and we have

$$a(n,m) = \frac{1}{2} r(n)r(n+m) \quad \text{if } 2 \nmid n. \tag{2.5}$$

Note that

$$\begin{aligned} \sum_{2 \nmid n} r(n)r(n+m) &= \sum_{n} r(n)r(n+m) - \sum_{2|n} r(n)r(n+m) \\ &= \sum_{n} r(n)r(n+m) - \sum_{2|n} r(n/2)r(n/2+m/2). \end{aligned}$$

Hence, from (2.4) and (2.5) it is deduced (2.1). □

The following result performs the needed smoothing and gives the spectral expansion of the involved automorphic kernels.

PROPOSITION 2.3. *Let Γ be $\mathrm{PSL}_2(\boldsymbol{Z})$ or $\Gamma_0(2)/\{\pm\mathrm{Id}\}$. Let k be the characteristic function of $[0,N]$ and $X = \operatorname{arc\,cosh}(1+2N/m)$. Then for every $0 < \Delta \le \min(1, X/2)$ there exists Δ_0, $|\Delta_0| \le \Delta$, such that*

$$T_m|_{w=z} \sum_{\gamma\in\Gamma} k(mu(\gamma z, w)) = \sum_j \lambda_j(m) H(t_j)|u_j(z)|^2 + \frac{1}{4\pi}\sum_a \int \eta_t(m) H(t)|E_a(z, 1/2+it)|^2\, dt$$

where $m \in \boldsymbol{Z}^+$, $2 \nmid m$ if $\Gamma = \Gamma_0(2)/\{\pm\mathrm{Id}\}$, and $H(t)$ is the Selberg-Harish-Chandra transform of the kernel K defined by

$$K(u(z,w)) = \frac{1}{2\pi \sinh^2(\Delta/2)} \int_{\boldsymbol{H}} k\left(\frac{Nu(z,v)}{\sinh^2((X+\Delta_0)/2)}\right) k\left(\frac{Nu(v,w)}{\sinh^2(\Delta/2)}\right) d\mu(v).$$

Although the definition of K seems very complicated, its Selberg-Harish-Chandra transform, H, can be easily described in terms of special functions which can be estimated by standard arguments. All the needed bounds are contained in the following result.

LEMMA 2.4. *Let H be as in Proposition 2.3, then*
a) $H(i/2) = 4\pi N m^{-1} + O(\Delta N m^{-1} + \Delta N^{1/2} m^{-1/2})$,
b) $H(t) \ll (1+|t|)^{-3/2} N^{1/2} m^{-1/2} \min(\log N, (\Delta|t|)^{-3/2})$ *if $N \ge m$ and $t \in \boldsymbol{R}$*
c) $H(t) \ll N m^{-1}(1+Nt^2/m)^{-3/4} \min(1, (\Delta|t|)^{-3/2})$ *if $N \le m$ and $t \in \boldsymbol{R}$.*

PROOF. Note that K is the convolution of two kernels, then its Selberg-Harish-Chandra transform is given by (see (2.13) in [**Ch**])

$$H(t) = (4\pi)^{-1} \sinh^{-2}(\Delta/2) h_1(t) h_2(t)$$

where h_1 and h_2 are respectively the Selberg-Harish-Chandra transforms of the characteristic functions of the intervals $[0, \sinh^2 R_1]$ and $[0, \sinh^2 R_2]$ with $R_1 = (X+\Delta_0)/2$ and $R_2 = \Delta/2$.

In Lemma 2.4 of [**Ch**] asymptotics formulas are given for the Selberg-Harish-Chandra transform of a characteristic function in several ranges which imply the needed bounds. □

We conclude this section proving Proposition 2.3.

PROOF OF PROPOSITION 2.3. We proceed as in Lemma 2.3 of [**Ch**] considering K^+ and K^- defined as K but replacing Δ_0 by Δ and $-\Delta$, respectively.

Note that $f(v) = k(Nu(v,w)/\sinh^2(\Delta/2))$ vanishes if $\rho(v,w) \ge \Delta$. Using the triangle inequality for ρ one deduces that the supports of $K^-(u(z,w))$ and $K^+(u(z,w))$ are $\rho(z,w) \le X$ and $\rho(z,w) \le X + 2\Delta$, respectively, which combined with $\int f(v)\, d\mu(v) = 4\pi \sinh^2(\Delta/2)$ proves

$$\sum_\gamma K^-(u(\gamma z, w)) \le \sum_\gamma k(mu(\gamma z, w)) \le \sum_\gamma K^+(u(\gamma z, w)).$$

Applying T_m (note that it is a monotone operator) by mean value theorem there exists Δ_0, $-\Delta \le \Delta_0 \le \Delta$, such that

$$T_m\big|_{z=w}\sum_{\gamma\in\Gamma} k(mu(\gamma w, z)) = T_m\big|_{z=w}\sum_{\gamma\in\Gamma} K(u(\gamma w, z)),$$

and the result is a consequence of the pretrace formula. □

§3. An asymptotic formula with error term.

The main results of this section are the following theorems.

THEOREM 3.1. *If 2^k is the greatest power of two dividing m, we have under* Conjecture 1.1

$$S(N,m) = 8|2^{k+1}-3|\sigma\left(\frac{m}{2^k}\right)\frac{N}{m} + E(N,m)$$

where σ indicates the sum of positive divisors and

$$E(N,m) \ll_\varepsilon N^{2/3+\varepsilon} + N^{1/2+\varepsilon}m^{3/14} + \min(N^{1/2}m^{1/4+\varepsilon}, N^{1/4}m^{13/28+\varepsilon})$$

for every $\varepsilon > 0$.

REMARK. The bound for the error term is quite similar to the one for $\sum d(n)d(n+m)$ obtained from the work of Y. Motohashi [**Mo1**] after substituting the bound of [**Bu-Du-Ho-Iw**] in his Theorem 5. In our case, not only the statement of the result, but the proof is technically much simpler (although the underlying ideas are similar). We find it surprising taking into account the analogies between $r(n)$ and $d(n)$ and perhaps it reveals an unsuspected straight relation in this context between Bruggeman-Kuznetsov formula and pretrace formula.

Compare the previous result with Theorem 12.5 of [**Iw**] and note the absence of a 2 factor because in Chapter 12 of [**Iw**] the spectral calculations are implicitly done in subgroups of $\mathrm{PSL}_2(\boldsymbol{Z})$ and each element corresponds to two integral matrices in $\mathrm{SL}_2(\boldsymbol{Z})$.

THEOREM 3.2. *If 2^k is the greatest power of two dividing m, we have under Ramanujan-Petersson conjecture*

$$S(N,m) = 8|2^{k+1}-3|\sigma\left(\frac{m}{2^k}\right)\frac{N}{m} + O_\varepsilon(N^{2/3+\varepsilon} + N^{1/3}m^{1/3+\varepsilon})$$

for every $\varepsilon > 0$.

The auxiliary results that we shall use in the proofs of these theorems are summarized in the following lemma.

LEMMA 3.3. *If $z = i$ and $\Gamma = \mathrm{PSL}_2(\boldsymbol{Z})$ (or $z = (i-1)/2$ and $\Gamma = \Gamma_0(2)/\{\pm\mathrm{Id}\}$), let us define for $T \ge 1$*

$$\mathscr{S} = \sum_{T<|t_j|\le 2T} |\lambda_j(m)|\,|u_j(z)|^2 \qquad t_j \in \boldsymbol{R}$$

then for every $\varepsilon > 0$

$$\mathscr{S} \ll m^{5/28+\varepsilon} T^{2+\varepsilon}$$

and under Conjecture 1.1

$$\mathscr{S} \ll m^{\varepsilon} T^{2+\varepsilon} + m^{1/4+\varepsilon} T^{1+\varepsilon}.$$

REMARK. The second bound of this lemma is somehow weaker than Conjecture 1.1 but, actually, it is the only conjectural result needed in the proof of Theorem 3.1.

PROOF. By the result of [**Bu-Du-Ho-Iw**] (in the general form stated in p. 128 of [**Iw**]) we have

$$\mathscr{S} \ll m^{5/28+\varepsilon} \sum_{T<|t_j|\le 2T} |u_j(z)|^2$$

and the first bound follows from Proposition 7.2 of [**Iw**].

As a consequence of Bruggeman-Kuznetsov formula (apply Theorem 9.3 of [**Iw**], (8.43) and Weil's bound for Kloosterman sums) one gets the following estimate due to N. V. Kuznetsov

$$\sum_{T<|t_j|\le 2T} |\lambda_j(m)|^2 \ll T^{2+\varepsilon} + m^{1/2+\varepsilon},$$

and the result follows from Conjecture 1.1 after Cauchy's inequality. □

PROOF OF THEOREM 3.1. After Lemma 2.1, Lemma 2.2 and Proposition 2.3, we can write $S(N,m)$ as a sum involving Maass wave forms and integrals of Eisenstein series, coming respectively from the discrete and continuous part in spectral resolution of Laplace-Beltrami operator. For a sake of notational simplicity we shall separate these contributions writing

$$S(N,m) = \mathscr{D} + \mathscr{C}.$$

Obviously, the result follows from

$$\mathscr{D} = 8|2^{k+1} - 3|\sigma\left(\frac{m}{2^k}\right)\frac{N}{m} + O(\mathscr{R}) \tag{3.1}$$

and

$$\mathscr{C} = O(\mathscr{R}) \tag{3.2}$$

where

$$\mathscr{R} = N^{2/3+\varepsilon} + N^{1/2+\varepsilon} m^{3/14} + \min(N^{1/2} m^{1/4+\varepsilon}, N^{1/4} m^{13/28+\varepsilon}).$$

Note that the contribution in Proposition 2.3 of zero eigenvalue of Laplace-Beltrami operator (corresponding to $t_0 = i/2$, $u_0 = (|\Gamma\backslash \boldsymbol{H}|)^{-1/2}$) is by Lemma 2.4 a))

$$\lambda_0(m) H(t_0) |u_0(z)|^2 = \frac{\sigma(m)}{|\Gamma\backslash \boldsymbol{H}|\sqrt{m}} H(t_0) = \frac{4\pi\sigma(m)N}{|\Gamma\backslash \boldsymbol{H}| m^{3/2}} + O(\Delta N m^{-1/2+\varepsilon} + \Delta N^{1/2} m^{\varepsilon}).$$

Substituting in Lemma 2.1 and Lemma 2.2, after some calculations (use $|\mathrm{PSL}_2(\boldsymbol{Z})\backslash\boldsymbol{H}| = \pi/3$ and $|(\Gamma_0(2)/\{\pm\mathrm{Id}\})\backslash\boldsymbol{H}| = \pi$), the sums of the main terms is for $k > 1$

$$\frac{N}{m}(48\sigma(m/4) - 48\sigma(m/8) + \cdots + (-1)^{k-2}48\sigma(m/2^k) + (-1)^{k-1}8\sigma(m/2^k))$$

which equals $8(2^{k+1} - 3)\sigma(m/2^k)N/m$. The cases $k = 0$ and $k = 1$ are easily treated separately giving $8\sigma(m)N/m$ and $8\sigma(m/2)N/m$. After these considerations it is enough to prove that each error term is $O(\mathscr{R})$ which, after dividing into dyadic intervals in Proposition 2.3, reduces (3.1) to prove

$$N\Delta + N^{1/2}m^{1/2}\Delta + \sqrt{m}\sup_T|\mathscr{E}(N,m,T)| = O(\mathscr{R}) \tag{3.3}$$

where

$$\mathscr{E}(N,m,T) = \sum_{\substack{T<|t_j|\le 2T\\ j\ne 0}} \lambda_j(m)H(t_j)|u_j(w)|^2$$

with Δ and H as in Proposition 2.3 and $w = i$ or $w = (i-1)/2$.

It is known (see for instance Theorem 11.4 of [**Iw**]) that for $j \ne 0$, $1/4 + t_j^2 > 1/4 + c^2$ with $c > 0$ in $\Gamma = \mathrm{PSL}_2(\boldsymbol{Z})$ and $\Gamma = \Gamma_0(2)/\{\pm\mathrm{Id}\}$, then we can assume $t_j \in \boldsymbol{R} - \{0\}$.

If $N \ge m$, Lemma 2.4 implies

$$\mathscr{E}(N,m,T) \ll N^{1/2+\varepsilon}m^{-1/2}T^{-3/2}\min(1,(\Delta T)^{-3/2})\sum_{T<|t_j|\le 2T}|\lambda_j(m)|\,|u_j(w)|^2. \tag{3.4}$$

By Lemma 3.3 and choosing $\Delta = N^{-1/3}$, we get

$$\mathscr{E}(N,m,T) \ll N^{1/2+\varepsilon}m^{-1/2}T^{-3/2+\varepsilon}\min(1,N^{1/2}T^{-3/2})\min(m^{5/28}T^2, T^2 + m^{1/4}T).$$

The size of the maximum of the right hand side is reached at $T = N^{1/3}$ or $T = m^{1/14}$, getting (3.3) and hence (3.1) (in fact, only the two first terms of $\mathscr{R}$ are needed).

The case $N < m$ is formally similar but replacing (3.4) by (see Lemma 2.4)

$$\mathscr{E}(N,m,T) \ll Nm^{-1}(1 + NT^2/m)^{-3/4}\min(1,(\Delta T)^{-3/2})\sum_{T\le|t_j|<2T}|\lambda_j(m)|\,|u_j(w)|^2. \tag{3.5}$$

We can assume $m \le N^2$ (otherwise the result is trivial) and then $\Delta = N^{-1/6}m^{-1/6}$ is under the hypothesis of Proposition 2.3. With this choice of Δ, by Lemma 3.3 we have

$$\mathscr{E}(N,m,T) \ll Nm^{-1}(1+NT^2/m)^{-3/4}\min(1,N^{1/4}m^{1/4}T^{-3/2})m^{5/28+\varepsilon}T^2 \text{ if } T \le m^{1/14}$$

$$\mathscr{E}(N,m,T) \ll Nm^{-1}(1+NT^2/m)^{-3/4}\min(1,N^{1/4}m^{1/4}T^{-3/2})(T^{2+\varepsilon}+m^{1/4}T^{1+\varepsilon}) \text{ if } T \ge m^{1/14}.$$

The right hand side of the first inequality reaches its maximum at $T = m^{1/14}$, then it is enough to consider the second inequality (because both bounds coincide when $T = m^{1/14}$), in which the maximum order is reached at $T = m^{1/2}N^{-1/2}$ if $N < m^{6/7}$ and at $T = m^{1/14}$ if $N > m^{6/7}$. These maximal orders are $N^{1/2}m^{-1/4+\varepsilon}$ and $N^{1/4}m^{-1/28+\varepsilon}$

respectively, then we can write

$$\mathscr{E}(N,m,T) \ll \min(N^{1/2}m^{-1/4+\varepsilon}, N^{1/4}m^{-1/28+\varepsilon}).$$

Therefore (3.3) also holds in the range $N < m$.

To finish the proof we have to prove (3.2). It reduces to establish

$$\sqrt{m}\sup_{T>1}|\mathscr{E}'(N,m,T)| = O(\mathscr{R}) \tag{3.6}$$

where

$$\mathscr{E}'(N,m,T) = \sum_{a}\int_{T}^{2T}\eta_t(m)H(t)|E_a(w,1/2+it)|^2\,dt.$$

Using Lemma 2.4, the bound $|\eta_t(m)| \le d(m)$ and Proposition 7.2 of [**Iw**], one gets

$$\mathscr{E}'(N,m,T) \ll N^{1/2+\varepsilon}m^{-1/2}T^{1/2}\min(1,(\varDelta T)^{-3/2}) \quad \text{if } N \ge m \tag{3.7}$$

and

$$\mathscr{E}'(N,m,T) \ll NT^2m^{-1}\min(1,(NT^2/m)^{-3/4})\min(1,(\varDelta T)^{-3/2}) \quad \text{if } N < m. \tag{3.8}$$

The choice done for the discrete part forces $\varDelta = N^{-1/3}$ in (3.7) and $\varDelta = N^{-1/6}m^{-1/6}$ in (3.8). In any case (3.6) is fulfilled. □

PROOF OF THEOREM 3.2. Arguing as in the proof of Theorem 3.1 (see (3.3) and (3.6)), it is enough to prove

$$N\varDelta + N^{1/2}m^{1/2}\varDelta + \sqrt{m}\sup_{T}(|\mathscr{E}(N,m,T)| + |\mathscr{E}'(N,m,T)|) = O(N^{2/3+\varepsilon} + N^{1/3}m^{1/3+\varepsilon}). \tag{3.9}$$

Under Ramanujan-Petersson conjecture $|\lambda_j(m)| \le d(m)$, hence (use Proposition 7.2 of [**Iw**])

$$|\mathscr{E}(N,m,T)| + |\mathscr{E}'(N,m,T)| \ll T^{2+\varepsilon}\sup_{T<|t|\le 2T}|H(t)|.$$

Now, using Lemma 2.4 b), c), substituting in (3.9) and choosing $\varDelta = N^{-1/3}$ if $N \ge m$ and $\varDelta = N^{-1/6}m^{-1/6}$ if $N < m$, the result is proved. □

§4. Averaging over m.

If we average $S(N,m)$ over m, some cancellation is hoped due to the oscillation of $\lambda_j(m)$. Some of the results of [**De-Iw**] (see specially §6) and [**Lu**] quantify the cancellation in m and spectral aspect induced by this oscillation in certain sums. In this section we shall use this latter work to prove the following theorem.

THEOREM 4.1. *Let $E(N,m)$ be defined as in* Theorem 3.1 *and let α_m be arbitrary complex numbers, then for $N > M^2 > 1$ we have under* Conjecture 1.1

$$\sum_{M<m\le 2M}\alpha_m E(N,m) \ll_\varepsilon \|\alpha\|_2 N^\varepsilon(N^{2/3}M^{1/6} + M^2).$$

Choosing

$$\alpha_m = E(N,m)\left(\sum_{M<k\le 2M} |E(N,k)|^2\right)^{-1/2}$$

it is deduced at once.

Corollary 4.2. *For* $N > M^2 > 1$ *and under* Conjecture 1.1

$$\sum_{M<m\le 2M} |E(N,m)|^2 \ll_\varepsilon N^\varepsilon (N^{4/3}M^{1/3} + M^4).$$

Even without taking into account the oscillation of $\lambda_j(m)$ it is possible to improve Theorem 4.1 in some ranges using that Ramanujan-Petersson conjecture is true on average. Namely

Theorem 4.3. *Let* $E(N,m)$ *and* α_m *as before, then for* N, $M > 1$

$$\sum_{M<m\le 2M} \alpha_m E(N,m) \ll_\varepsilon \|\alpha\|_2 (N^{2/3+\varepsilon}M^{1/2} + N^{1/3}M^{5/6+\varepsilon}).$$

In the proof of Theorem 4.1 it will be important to consider the dependence on m of the function $H(t)$ defined in Proposition 2.3. The needed result is contained in the following lemma.

Lemma 4.4. *For* $N > M$, $M < m \le 2M$ *and* $1 < |t| \le T$, $t \in \boldsymbol{R}$, *we have*

$$H(t) = F(t) + F(-t)$$

with

$$F(t) = m^{-it} f_1(\Delta, t) f_2(N, \Delta, t, m)(1 + O((1 + NT^{-1}M^{-1})^{-1}))$$

where f_2 *is decreasing in* m.

Remark. Note that by Lemma 2.4

$$f_1(\Delta, t) f_2(N, \Delta, t, m) \ll T^{-3/2} N^{1/2} M^{-1/2} \min(\log N, (\Delta T)^{-3/2}).$$

Proof. As we mentioned in the proof of Lemma 2.4, $H(t)$ can be written as (see [**Ch**])

$$H(t) = (4\pi)^{-1} \sinh^{-2}(\Delta/2) h_1(t) h_2(t)$$

where $h_2 = f_1(\Delta, t)$ is even in t and h_1 is the Selberg-Harish-Chandra transform of the characteristic function of $[0, \sinh^2((X + \Delta_0)/2)]$ with $X = \operatorname{arc\,cosh}(1 + 2N/m)$. By (2.8) and (2.9) of [**Ch**]

$$h_1(t) = \sqrt{2\pi \sinh(X + \Delta_0)}(f(t) + f(-t)) \quad \text{with } f(t) = \frac{e^{it(X+\Delta_0)}\Gamma(it)}{\Gamma(it + 3/2)}(1 + O(M^2T^{-1}N^{-2})),$$

and noting that

$$e^{itX} = \left(1 + \frac{2N}{m} + \frac{2N}{m}\sqrt{1 + \frac{m}{4N}}\right)^{it} = \left(\frac{4N}{m}\right)^{it}(1 + O((1 + NT^{-1}M^{-1})^{-1})),$$

the proof is finished. □

PROOF OF THEOREM 4.1. Separating the contribution corresponding to discrete and continuous spectrum as in the proof of Theorem 3.1, it is enough to prove

$$NM^{1/2}\Delta\|\alpha\|_2 + M^{1/2}\sup_{T>1}|\mathscr{E}_\alpha(N,M,T)| + M^{1/2}\sup_{T>1}|\mathscr{E}'_\alpha(N,M,T)| = O(\mathscr{R}_\alpha) \tag{4.1}$$

where

$$\mathscr{E}_\alpha(N,M,T) = \sum_{T<|t_j|\le 2T}\sum_{M<m\le 2M}\alpha_m\lambda_j(m)H(t_j)|u_j(w)|^2,$$

$$\mathscr{E}'_\alpha(N,M,T) = \sum_a\int_T^{2T}\sum_{M<m\le 2M}\alpha_m\eta_t(m)H(t)|E_a(w,1/2+it)|^2\,dt$$

and $\mathscr{R}_\alpha$ is the allowed error, i.e.

$$\mathscr{R}_\alpha = \|\alpha\|_2N^\varepsilon(N^{2/3}M^{1/6} + M^2).$$

By Lemma 2.4 b) and the bound $\eta_t(m) \ll m^\varepsilon$

$$\mathscr{E}'_\alpha(N,M,T) \ll \|\alpha\|_2T^{-3/2}N^{1/2+\varepsilon}\min(1,(\Delta T)^{-3/2})\sum_a\int_T^{2T}|E_a(w,1/2+it)|^2\,dt.$$

Note that at the points involved in Lemma 2.1 and Lemma 2.2 ($z = i, z = (i-1)/2$) we have $E_a(z,s) = \zeta_Q(1/2+it)$ where ζ_Q is the Epstein zeta-function associated to a certain binary quadratic form. Using standard arguments (follow the steps in the proof for $k = 2$ of Theorem 4.2 of [**Iv**] replacing ζ^2 by ζ_Q and using the corresponding functional equation) the second-power moment of $\zeta_Q(1/2+it)$, $T < t < 2T$, can be bounded by $T^{1+\varepsilon}$, hence

$$\mathscr{E}'_\alpha(N,M,T) \ll \|\alpha\|_2T^{-1/2+\varepsilon}N^{1/2+\varepsilon}\min(1,(\Delta T)^{-3/2}). \tag{4.2}$$

Estimating $\mathscr{E}_\alpha(N,M,T)$, we shall use the main result of [**Lu**]. First of all, note that considering real and imaginary parts and separating them according their signs, we can always assume that $\alpha_m \ge 0$. Under this assumption we can take the same Δ_0 for every $M < m \le 2M$, because of the monotonicity in Δ of $\sum K(u(z,w))$ (see the proof of Proposition 2.3). By Lemma 4.4 and partial summation

$$\mathscr{E}_\alpha(N,M,T) \ll T^{-3/2}N^{1/2+\varepsilon}M^{-1/2}\min(1,(\Delta T)^{-3/2})\cdot(\mathscr{E}_{\alpha1} + TMN^{-1}\mathscr{E}_{\alpha2})$$

where

$$\mathscr{E}_{\alpha1} = \sum_{T<|t_j|\le 2T}|u_j(w)|^2\left|\sum_{M<m\le M'}\alpha_m\lambda_j(m)m^{-it_j}\right| \quad \mathscr{E}_{\alpha2} = \sum_{T<|t_j|\le 2T}|u_j(w)|^2\sum_{M<m\le 2M}|\alpha_m|\,|\lambda_j(m)|$$

for some $M' < 2M$. Under Conjecture 1.1, Theorem 1 of [**Lu**] (extended to $\Gamma_0(2)$) implies

$$\mathscr{E}_{\alpha 1} \ll \|\alpha\|_2 (TM)^{\varepsilon}(T^2 + T^{7/4}M^{1/4} + TM^{5/8}).$$

On the other hand, by Theorem 8.3 and Proposition 7.2 of [**Iw**]

$$\mathscr{E}_{\alpha 2} \ll \|\alpha\|_2 T^{2+\varepsilon} M^{1/2}.$$

Hence

$$\mathscr{E}_{\alpha}(N,M,T) \ll \|\alpha\|_2 (TN/M)^{1/2+\varepsilon} \min(1,(\Delta T)^{-3/2}) \times (1 + T^{-1/4}M^{1/4} + T^{-1}M^{5/8} + TM^{3/2}N^{-1}). \tag{4.3}$$

We can also bound $\mathscr{E}_{\alpha}(N,M,T)$ taking absolute values of the function under summation. With the same results used estimating $\mathscr{E}_{\alpha 2}$, we have

$$\mathscr{E}_{\alpha}(N,M,T) \ll \|\alpha\|_2 (TN)^{1/2+\varepsilon} \min(1,(\Delta T)^{-3/2}). \tag{4.4}$$

Finally, choosing $\Delta = N^{-1/3}M^{-1/3}$, by (4.3) we get for $M^{1/8} < T < N/M$

$$\mathscr{E}_{\alpha}(N,M,T) \ll \|\alpha\|_2 N^{\varepsilon}(N^{2/3}M^{-1/3} + M^{3/2}),$$

and (4.4) gives a better bound for the rest of the values of T. Substituting this bound and (4.2) in (4.1), the proof is finished. □

Proof of Theorem 4.3. As we pointed out before, this result is based on that Ramanujan-Petersson conjecture is true on average, so the proof follows in the same lines as the proof of Theorem 3.2.

In this case it is enough to prove

$$\|\alpha\|_2 (NM^{1/2}\Delta + N^{1/2}M\Delta) + M^{1/2} \sup_T (|\mathscr{E}_{\alpha}(N,M,T)| + |\mathscr{E}'_{\alpha}(N,M,T)|) = O(\mathscr{T}_{\alpha}),$$

where we have used the notation in the proof of Theorem 4.1 and

$$\mathscr{T}_{\alpha} = \|\alpha\|_2 (N^{2/3+\varepsilon}M^{1/2} + N^{1/3}M^{5/6+\varepsilon}).$$

Using Cauchy's inequality and Theorem 8.3 of [**Iw**] we have

$$\sum_{M<m\le 2M} \alpha_m \lambda_j(m) \ll \|\alpha\|_2 M^{1/2+\varepsilon}|t_j|^{\varepsilon}.$$

Hence

$$|\mathscr{E}_{\alpha}(N,m,T)| + |\mathscr{E}'_{\alpha}(N,m,T)| \ll \|\alpha\|_2 M^{1/2} T^{2+\varepsilon} \sup_{T<|t|\le 2T} |H(t)|.$$

The proof is now finished as that of Theorem 3.2, using Lemma 2.4 b), c) and choosing $\Delta = N^{-1/3}$ if $N \ge m$ and $\Delta = N^{-1/6}m^{-1/6}$ if $N < m$. □

§5. Some unconditional results.

The purpose of this section is to illustrate how unconditional (but weaker) estimates can be obtained for $E(N,m)$ in Theorem 3.1. We shall be schematic in the proof of our result because the involved ideas are the same as in Theorem 3.1 and we do not consider Conjecture 1.1 (or, more precisely, the second part of Lemma 3.3) to be unreachable.

The difference with respect to the proof of Theorem 3.1 is that Conjecture 1.1 is replaced by the following weaker unconditional result derived from a deep L^∞ bound for cusp forms due to H. Iwaniec and P. Sarnak (see [**Iw-Sa**]).

LEMMA 5.1. *With the notation of* Conjecture 1.1, *we have*

$$\sum_{T<|t_j|\le 2T} |u_j(z)|^4 = O(T^{17/6+\varepsilon}).$$

PROOF. By Proposition 7.2 of [**Iw**]

$$\sum_{T<|t_j|\le 2T} |u_j(z)|^4 \le \sup_{T<|t_j|\le 2T} |u_j(z)|^2 \sum_{T<|t_j|\le 2T} |u_j(z)|^2 \ll T^{2+\varepsilon} \sup_{T<|t_j|\le 2T} |u_j(z)|^2$$

and the result follows from Theorem 0.1 of [**Iw-Sa**]. □

THEOREM 5.2. *With the notation of* Theorem 3.1 *we have*

$$E(N,m) \ll N^{\varepsilon}\min(N^{2/3}m^{5/42}, N^{17/23}+N^{1/2}m^{47/196}) \quad \textit{if } N\ge m$$

and

$$E(N,m) \ll N^{7/24}m^{11/24+\varepsilon} \quad \textit{if } N\le m.$$

COROLLARY 5.3. *If 2^k is the greatest power of two dividing m, then for every $\varepsilon>0$ it holds uniformly in $m\le N$*

$$S(N,m) = 8|2^{k+1}-3|\sigma\Big(\frac{m}{2^k}\Big)\frac{N}{m} + O_\varepsilon(N^{145/196+\varepsilon})$$

and the asymptotic formula for large values of N

$$S(N,m) \sim 8|2^{k+1}-3|\sigma\Big(\frac{m}{2^k}\Big)\frac{N}{m}$$

is valid uniformly in the range $1\le m\le N^{17/11-\varepsilon}$.

REMARK. Note that the exponent $145/196 = 0.739\ldots$ for $m\le N$ is rather close to the conditional exponent $5/7 = 0.714\ldots$ derived in the same range from Theorem 3.1.

PROOF OF THEOREM 5.2. Lemma 5.1 and the first part of Lemma 3.3 provide the bound (see the proof of Lemma 3.3)

$$\sum_{T<|t_j|\le 2T} \lambda_j(m)|u_j(z)|^2 \ll (mT)^{\varepsilon}\min(m^{5/28}T^2, (T+m^{1/4})T^{17/12}). \tag{5.1}$$

With the notation of the proof of Theorem 3.1, the first bound of (5.1) assures for $N\ge m$

$$\mathscr{E}(N,m,T), \mathscr{E}'(N,m,T) \ll N^{1/2+\varepsilon}m^{-1/2}T^{-3/2}\min(1,(\varDelta T)^{-3/2})\cdot m^{5/28}T^{2+\varepsilon}. \tag{5.2}$$

Choosing $\Delta = N^{-1/3}m^{5/42}$ one gets

$$\mathscr{E}(N,m,T), \mathscr{E}'(N,m,T) \ll N^{2/3+\varepsilon}m^{-8/21}. \tag{5.3}$$

On the other hand, if $N < m^{23/14}$, choosing $\Delta = N^{-6/23}$ and noting that the second bound of (5.1) is better that the first one in the range $m^{6/49} \ll T \ll m^{3/7}$,

$$\mathscr{E}(N,m,T), \mathscr{E}'(N,m,T) \ll N^{1/2+\varepsilon}(m^{-51/196} + m^{-1/2}T^{-3/2}\min(1,(\Delta T)^{-3/2}) \times (T+m^{1/4})T^{17/12}),$$

hence

$$\mathscr{E}(N,m,T), \mathscr{E}'(N,m,T) \ll N^{1/2+\varepsilon}m^{-51/196} + N^{17/23+\varepsilon}m^{-1/2}. \tag{5.4}$$

Combining (5.3) and (5.4), we conclude that in any case, if $N \geq m$ the quantity

$$N\Delta + N^{1/2}m^{1/2}\Delta + \sqrt{m}\sup_T(|\mathscr{E}(N,m,T)| + |\mathscr{E}'(N,m,T)|)$$

is controlled by the bound of the first part of the theorem, and then (see the proof of Theorem 3.1) the same applies to $E(N,m)$.

The case $N \leq m \leq N^{17/11}$ is analogous. We do not pursue the best possible result in every range (but the wider uniformity) and we shall only use the second bound of (5.1). With $\Delta = N^{-5/24}m^{-1/24}$ we have that the right hand side of (5.2) is in this case

$$Nm^{-1+\varepsilon}(1 + NT^2/m)^{-3/4}\min(1, N^{15/48}m^{3/48}T^{-3/2})(T^{29/12+\varepsilon} + m^{1/4}T^{17/12+\varepsilon})$$

whose maximum order is $N^{7/24}m^{-1/24}$, reached at $T = N^{-1/2}m^{1/2}$ or $T = \Delta^{-1}$, hence

$$N\Delta + N^{1/2}m^{1/2}\Delta + \sqrt{m}\sup_T(|\mathscr{E}(N,m,T)| + |\mathscr{E}'(N,m,T)|) \ll N^{7/24}m^{11/24+\varepsilon}$$

and the second part of the theorem follows. □

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