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On commutators of foliation preserving homeomorphisms

Dedicated to Professor Hiroyasu Ishimoto on his 60th birthday

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Abstract. We consider the group of foliation preserving homeomorphisms of a foliated manifold. We compute the first homologies of the groups for codimension one foliations. Especially, we show that the group for the Reeb foliation on the 3-sphere is perfect and the groups for irrational linear foliations on the torus are not perfect.

1. Introduction.

Let M be an n-dimensional connected closed topological manifold. By $\mathscr{H}(M)$ we denote the group of all homeomorphisms of M which are isotopic to the identity by an isotopy fixed outside a compact set.

In this note we treat certain subgroups of $\mathscr{H}(M)$. Let $\boldsymbol{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \boldsymbol{R}\}$ be an n-dimensional Euclidean space and $\mathscr{F}_0$ the p-dimensional foliation of $\boldsymbol{R}^n$ whose leaves are defined by $x_{p+1} = constant, \dots, x_n = constant$ $(1 \leq p \leq n)$. A p-dimensional topological foliation $\mathscr{F}$ of M is defined to be a maximal set of C^0-charts: $\{(U_\lambda, h_\lambda), U_\lambda \text{ is open in } M, h_\lambda : U_\lambda \to \boldsymbol{R}^n, \lambda \in \Lambda\}$ of M such that $h_\lambda \circ h_\mu^{-1} : h_\mu(U_\lambda \cap U_\mu) \to h_\lambda(U_\lambda \cap U_\mu)$ preserves the leaves of foliations which are restrictions of $\mathscr{F}_0$ to $h_\mu(U_\lambda \cap U_\mu)$ and $h_\lambda(U_\lambda \cap U_\mu)$.

A homeomorphism $f : M \to M$ is called *a foliation preserving homeomorphism* (resp. *a leaf preserving homeomorphism*) if for each point x of M, the leaf through x is mapped into the leaf through $f(x)$ (resp. x), that is, $f(L_x) = L_{f(x)}$ (resp. $f(L_x) = L_x$), where L_x is the leaf of $\mathscr{F}$ which contains x. Let $F(M, \mathscr{F})$ (resp. $L(M, \mathscr{F})$) denote the group of foliation (resp. leaf) preserving homeomorphisms of $(M, \mathscr{F})$ which are isotopic to the identity by a foliation (resp. leaf) preserving isotopy fixed outside a compact set.

In §2, we consider the homologies of $L(M, \mathscr{F})$, that is, the homology groups of the group $L(M, \mathscr{F})$ and show that the homologies of $L(\boldsymbol{R}^n, \mathscr{F}_0)$ vanish in all dimension >0. This is a generalization of a result of Mather[M] to the case of foliated manifolds.

In §3, first we show that any $f \in L(M, \mathscr{F})$ can be expressed as $f = f_1 \circ f_2 \circ \cdots \circ f_r$, where each f_i is a leaf preserving homeomorphism with support in a small ball. Next we show from the above result and the result in §2 that $L(M, \mathscr{F})$ is perfect, *i.e.*, is equal to its own commutator subgroup.

In §4, we consider $F(M, \mathscr{F})$ and compute the first homology of $F(M, \mathscr{F})$ for codimension one foliations. Especially we show that for the Reeb foliation $\mathscr{F}_R$ of S^3, $F(S^3, \mathscr{F}_R)$ is perfect and for a foliation $\mathscr{F}$ of T^n defined by a 1-form $\omega = \sum a_i \, dx_i$,

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$F(T^n, \mathscr{F})$ is not perfect if one of a_i/a_j is irrational, indeed in this case the first homology of $F(T^n, \mathscr{F})$ is isomorphic to $\boldsymbol{R}/a_1\boldsymbol{Z} + \cdots + a_n\boldsymbol{Z}$.

2. Homologies of $L(\boldsymbol{R}^n, \mathscr{F}_0)$.

We recall that if G is any group, then there is a standard chain complex $C(G)$ whose homology is the homology of G.

Let $C_r(G)$ be the free abelian group on the set of all r-tuples $(g_1, \dots, g_r)$, where $g_i \in G$. The boundary operator $\partial : C_r(G) \to C_{r-1}(G)$ is defined by

$$\partial(g_1, \dots, g_r) = (g_1^{-1}g_2, \dots, g_1^{-1}g_r) + \sum_{i=1}^{r}(-1)^i(g_1, \dots, \check{g}_i, \dots, g_r).$$

Then we have $\partial^2 = 0$. The symbol $H_r(G)$ will stand for the r-th homology group of this chain complex.

Let $\boldsymbol{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \boldsymbol{R}\}$ be an n-dimensional Euclidean space and $\mathscr{F}_0$ the p-dimensional foliation of $\boldsymbol{R}^n$ whose leaves are defined by $x_{p+1} = constant, \dots, x_n = constant$ $(1 \le p \le n)$. Let $L(\boldsymbol{R}^n, \mathscr{F}_0)$ denote the group of leaf preserving homeomorphisms of $(\boldsymbol{R}^n, \mathscr{F}_0)$ which are isotopic to the identity by a leaf preserving isotopy fixed outside a compact set.

If $c = \sum k_j(g_{1j}, \dots, g_{rj})$, $(k_j \in \boldsymbol{Z})$ be an element of the chain group $C_r(L(\boldsymbol{R}^n, \mathscr{F}_0))$, we define the support of c, $supp(c)$, by $supp(c) = \bigcup_{i,j} supp(g_{ij})$.

Let $U = U' \times \boldsymbol{R}^{n-p}$, where U' is an open rectangle in $\boldsymbol{R}^p$ $(\subset \boldsymbol{R}^p \times \boldsymbol{R}^{n-p} = \boldsymbol{R}^n)$.

Then $supp(c) \subset U$ if and only if $supp(g_{ij}) \subset U$ for each i, j.

We put $L_U(\boldsymbol{R}^n, \mathscr{F}_0) = \{f \in L(\boldsymbol{R}^n, \mathscr{F}_0) \mid supp(f) \subset U\}$.

Theorem 2.1. *The homology groups* $H_r(L(\boldsymbol{R}^n, \mathscr{F}_0)) = 0$ *for* $r > 0$.

Before we prove this theorem, we need a lemma. Let $\iota : L_U(\boldsymbol{R}^n, \mathscr{F}_0) \to L(\boldsymbol{R}^n, \mathscr{F}_0)$ denote the inclusion map, and let $\iota_* : H_r(L_U(\boldsymbol{R}^n, \mathscr{F}_0)) \to H_r(L(\boldsymbol{R}^n, \mathscr{F}_0))$ denote the induced homomorphism. We have the following.

Lemma 2.2. ι_* *is an isomorphism.*

Proof. First we show that ι_* is surjective. Let $h \in H_r(L(\boldsymbol{R}^n, \mathscr{F}_0))$, and let $c \in C_r(L(\boldsymbol{R}^n, \mathscr{F}_0))$ be a cycle representing h. Choose a homeomorphism $\varphi \in L(\boldsymbol{R}^n, \mathscr{F}_0)$ which satisfies $\varphi(supp(c)) \subset U$.

Let I_φ be the inner automorphism of $L(\boldsymbol{R}^n, \mathscr{F}_0)$, given by $I_\varphi(g) = \varphi g \varphi^{-1}$. Since any inner automorphism induces the identity on homology, $(I_\varphi)_*(h) = h$. $(I_\varphi)_*(h)$ is represented by the cycle $I_\varphi(c)$ and $supp(I_\varphi(c)) = \varphi(supp(c)) \subset U$. Hence $h = \iota_* h'$, where $h' \in H_r(L_U(\boldsymbol{R}^n, \mathscr{F}_0))$ is the homology class represented by $I_\varphi(c)$.

Next we show that ι_* is injective. Suppose that $h \in H_r(L_U(\boldsymbol{R}^n, \mathscr{F}_0))$ satisfies $\iota_* h = 0$ and let c be a cycle in $C_r(L_U(\boldsymbol{R}^n, \mathscr{F}_0))$ representing h. Then there is a chain $c' \in C_{r+1}(L(R^n, \mathscr{F}_0))$ such that $\partial(c') = c$. Since $supp(c) \subset U$, it is easy to see that there is a homeomorphism $\phi \in L(\boldsymbol{R}^n, \mathscr{F}_0)$ such that ϕ is the identity in a neighborhood of $supp(c)$ and $\phi(supp(c')) \subset U$.

Then we have $\partial(I_\phi(c')) = I_\phi(\partial c') = I_\phi(c) = c$ and $I_\phi(c') \in C_{r+1}(L_U(\boldsymbol{R}^n, \mathscr{F}_0))$.

This completes the proof.

PROOF OF THEOREM 2.1. We put $U = (1,2) \times (-1,1)^{p-1} \times \boldsymbol{R}^{n-p} \subset \boldsymbol{R}^n$. Take a homeomorphism $\phi \in L(\boldsymbol{R}^n, \mathscr{F}_0)$ given by

$$\phi(x_1, \ldots, x_n) = \left(\frac{1}{3}x_1, \ldots, \frac{1}{3}x_p, x_{p+1}, \ldots, x_n\right)$$

for $(x_1, \ldots, x_p, x_{p+1}, \ldots, x_n) \in B(0, p+3) \times C(0, K)$ for some $K > 0$, where $B(0, p+3) = \{(x_1, \ldots, x_p) \in \boldsymbol{R}^p \mid (x_1)^2 + \cdots + (x_p)^2 < (p+3)^2\}$ and $C(0,K) = \{(x_{p+1}, \ldots, x_n) \mid |x_i| < K \ (i = p+1, \ldots, n)\}$. We set $U_j = \phi^j(U) = (1/3^j, 2/3^j) \times (-1/3^j, 1/3^j)^{p-1} \times \boldsymbol{R}^{n-p}$, $(j = 0, 1, 2, \ldots)$. Note that $U_0 = U$.

Then we have that $\overline{U}_j \cap \overline{U}_k = \emptyset$ if $j \neq k$ and $\{\overline{U}_j\}$ shrinks to the $(n-p)$-dimensional subspace $0 \times \boldsymbol{R}^{n-p} \subset \boldsymbol{R}^n$ as j goes to ∞.

For any $g \in L_U(\boldsymbol{R}^n, \mathscr{F}_0)$ and $i = 0, 1$, we define $\psi_i(g)$ as follow;

$$\psi_i(g)(x) = \begin{cases} \phi^j g \phi^{-j}(x) & (x \in \overline{U}_j, j \geq i) \\ x & (x \notin \bigcup_{j \geq i} \overline{U}_j). \end{cases}$$

Note that $\psi_i(g)$ is a well-defined element of $L(\boldsymbol{R}^n, \mathscr{F}_0)$ and $\psi_i : L_U(\boldsymbol{R}^n, \mathscr{F}_0) \to L(\boldsymbol{R}^n, \mathscr{F}_0)$ is a homomorphism for $i = 0, 1$.

Since $\psi_1(g) = \phi\psi_0(g)\phi^{-1}$, ψ_0 and ψ_1 are conjugate, so we have

$$(\psi_0)_* = (\psi_1)_* : H_r(L_U(\boldsymbol{R}^n, \mathscr{F}_0)) \to H_r(L(\boldsymbol{R}^n, \mathscr{F}_0)).$$

Following Mather [**M**], we define

$$\eta : L_U(\boldsymbol{R}^n, \mathscr{F}_0) \times L_U(\boldsymbol{R}^n, \mathscr{F}_0) \to L(\boldsymbol{R}^n, \mathscr{F}_0)$$

by $\eta(g, h) = g\psi_1(h)$.

As two homeomorphisms with disjoint supports commute and $supp(g) \subset U$, $supp(\psi_1(h)) \subset \bigcup_{j \geq 1} U_j$, we have $g\psi_1(h) = \psi_1(h)g$. Hence η is a group homomorphism.

Let $\Delta : L_U(\boldsymbol{R}^n, \mathscr{F}_0) \to L_U(\boldsymbol{R}^n, \mathscr{F}_0) \times L_U(\boldsymbol{R}^n, \mathscr{F}_0)$ denote the diagonal homomorphism. We have easily that $\psi_0 = \eta \circ \Delta$.

Now the proof proceeds by an induction on r. It is vacuous for $r = 0$. For the inductive step, we assume that $H_s(L(\boldsymbol{R}^n, \mathscr{F}_0)) = 0$ for $1 \leq s \leq r-1$.

By Lemma 2.2, it follows that $H_s(L_U(\boldsymbol{R}^n, \mathscr{F}_0)) = 0$ for $1 \leq s \leq r-1$.

By the Künneth formula, we have

$$H_r(L_U(\boldsymbol{R}^n, \mathscr{F}_0) \times L_U(\boldsymbol{R}^n, \mathscr{F}_0)) = H_r(L_U(\boldsymbol{R}^n, \mathscr{F}_0)) \oplus H_r(L_U(\boldsymbol{R}^n, \mathscr{F}_0)).$$

For any $h \in H_r(L_U(\boldsymbol{R}^n, \mathscr{F}_0))$, $\Delta_* h = h \oplus h$, thus $(\psi_0)_*(h) = \eta_*\Delta_*(h) = \eta_*(h \oplus h) = \iota_*(h) + (\psi_1)_*(h) = \iota_*(h) + (\psi_0)_*(h)$. Hence $\iota_*(h) = 0$. From Lemma 2.2, it follows that $h = 0$. Thus we have $H_r(L_U(\boldsymbol{R}^n, \mathscr{F}_0)) = 0$. From Lemma 2.2, it follows that $H_r(L(\boldsymbol{R}^n, \mathscr{F}_0)) = 0$, which completes the induction.

COROLLARY 2.3. *$L_U(\boldsymbol{R}^n, \mathscr{F}_0)$ and $L(\boldsymbol{R}^n, \mathscr{F}_0)$ are perfect groups.*

PROOF. This is an immediate consequence of Theorem 2.2 because that $H_1(G) = G/[G, G]$ for any group G.

3. Commutators of leaf preserving homeomorphisms.

In this section, first we show the following theorem following the proof of Lemma 4.1 in [**E-K**].

THEOREM 3.1. *Let $(M, \mathscr{F})$ be a foliated manifold. Any $f \in L(M, \mathscr{F})$ can be expressed as $f = f_1 \circ f_2 \circ \cdots \circ f_r$, where each f_i is a leaf preserving homeomorphism with support in a small ball.*

PROOF. First we prepare some notations. Let $B^p = [-1, 1]^p \subset \boldsymbol{R}^p$. In general, let $aB^p = [-a, a]^p$ for $a > 0$. We regard S^1 as the space obtained by identifying the endpoints of $[-4, 4]$ and we let $e : \boldsymbol{R} \to S^1$ denote the natural covering projection, that is, $e(x) = (x + 4)(\text{mod } 8) - 4$. Let T^p be the p-fold product of S^1. Then aB^p can be regarded as a subset of T^p for $a < 4$. Let $e = e^p : \boldsymbol{R}^p \to T^p$ denote the product covering projection.

We prove the above theorem in the following three steps.

Step 1. Let $\eta : 4B^p \to \boldsymbol{R}^p$ be the inclusion and let $I(4B^p, \boldsymbol{R}^p)$ denote the space of imbeddings of $4B^p$ into $\boldsymbol{R}^p$ with the compact open topology. Let $N(\varepsilon) = \{h \in I(4B^p, \boldsymbol{R}^p) \mid \|h(x) - x\| < \varepsilon \ (x \in 4B^p)\}$ for $\varepsilon > 0$.

In the following, for a sufficiently small ε, we will construct an isotopy $\Psi(h, t)$ of h to the identity, which satisfies $\Psi(h, t) = h$ on $\partial 4B^p$ and $\Psi(h, 1) = id$ on B^p.

Let D_1^p, D_2^p, D_3^p and D_4^p be four concentric p-cells in $T^p - 2B^p$ such that $D_j^p \subset int\, D_{j+1}^p$ for each j.

As is well-known, there exists an immersion $\alpha : T^p - D_1^p \to int\, 3B^p$. Then we can assume that $\alpha e = id$ on $2B^p$.

If $h \in N(\varepsilon)$ for a small ε, then h can be covered in a natural way by an imbedding $h_1 : T^p - D_2^p \to T^p - D_1^p$ as follows:

$$\begin{array}{ccc} T^p - D_2^p & \xrightarrow{h_1} & T^p - D_1^p \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\alpha} \\ 4B^p & \xrightarrow{h} & \boldsymbol{R}^p. \end{array}$$

Note that h_1 is an imbedding lifting h and depends continuously on h. It is an inclusion map if h is the one.

We have that $h_1(int\, D_4^p - D_2^p) \supset \partial D_3^p$ if ε is small. From the canonical Schoenflies theorem (Proposition 3.1 of [**E-K**]), we see that $h_1(\partial D_3^p)$ bounds canonically a p-ball in D_4^p. By coning, we can extend $h_1|_{T^p - D_3^p}$ to a homeomorphism $h_2 : T^p \to T^p$ canonically if ε is small.

Note that h_2 depends continuously on h and if h is the inclusion, then $h_2 = id$.

Now if h_2 is sufficiently close to id, then h_2 lifts in a natural way to a bounded homeomorphism $h_3 : \boldsymbol{R}^p \to \boldsymbol{R}^p$ so that the following diagram commutes:

$$\begin{array}{ccc} \boldsymbol{R}^p & \xrightarrow{h_3} & \boldsymbol{R}^p \\ {\scriptstyle e}\big\downarrow & & \big\downarrow{\scriptstyle e} \\ T^p & \xrightarrow{h_2} & T^p, \end{array}$$

where *bounded* means that $|h_3(x) - x| < constant$ for any $x \in \boldsymbol{R}^p$. Note that h_3 depends continuously on h_2 and if $h_2 = id$, we assume $h_3 = id$.

Let $\gamma : int\, 3B^p \to \boldsymbol{R}^p$ be a homeomorphism which is a radial expansion outside a neighborhood of $2B^p$ and is the identity on $2B^p$. Define a homeomorphism $h_4 : \boldsymbol{R}^p$

$\to \boldsymbol{R}^p$ by

$$h_4(x) = \begin{cases} \gamma^{-1} \circ h_3 \circ \gamma(x) & (x \in int\, 3B^p) \\ x & (x \in \boldsymbol{R}^p - int\, 3B^p). \end{cases}$$

The continuity of h_4 follows from the fact that h_3 is bounded. Note that (1) h_4 depends continuously on h and if $h = \eta$, then $h_4 = id$, and (2) $\alpha e \gamma h_4(x) = h \alpha e \gamma(x) = h(x)$ for $x \in 2B^p \cap h_4^{-1}(2B^p)$.

From (1), we have $h_4(B^p) \subset 2B^p$ for a small ε. Since $\alpha e \gamma = id$ on $2B^p$, we have that $h_4 = h$ on B^p.

We put $g = h_4$. Using the Alexander trick, we define an isotopy

$$g_t : \boldsymbol{R}^p \to \boldsymbol{R}^p, \quad t \in [0, 1],$$

from the identity to g,

$$g_t(x) = \begin{cases} tg\left(\frac{1}{t}x\right) & (t > 0) \\ x & (t = 0). \end{cases}$$

Define a deformation

$$\Psi : N(\varepsilon) \times [0, 1] \to I(4B^p, \boldsymbol{R}^p)$$

by $\Psi(h, t) = g_t^{-1} \circ h$.

If ε is small, we may assume that $h(\partial 4B^p) \cap 3B^p = \emptyset$ for $h \in N(\varepsilon)$. Thus we have $\Psi(h, t) = h$ on $\partial 4B^p$.

Note that (1) $\Psi(\eta, t) = \eta$ for any t, and (2) $\Psi(h, 0) = h$ and $\Psi(h, 1) = g^{-1} \circ h$, which is the identity on B^p.

Step 2. Now we consider a foliated version. Let $4B^p \times 2B^{n-p}$ be a foliated chart, that is, $\{(x, y) = (x_1, \dots, x_p, y_1, \dots, y_{n-p}) \in 4B^p \times 2B^{n-p} \mid y_1 = c_1, \dots, y_{n-p} = c_{n-p}\}$ gives a connected component of a leaf in this chart. Let $h : 4B^p \times 2B^{n-p} \to \boldsymbol{R}^p \times \boldsymbol{R}^{n-p}$ be of the form $(h(x, y), y)$ $(x \in 4B^p, y \in 2B^{n-p})$ and close to the inclusion. We put $h_y = h(\ , y)$. Note that for each $y \in 2B^{n-p}$, $h_y : 4B^p \to \boldsymbol{R}^p$ is close to the inclusion η. Then performing the procedure in Step 1, we construct g_y canonically from h_y for each $y \in 2B^{n-p}$. Using the Alexander trick again, we define

$$g_{y,t}(x) = \begin{cases} tg_y\left(\frac{1}{t}x\right) & (t > 0) \\ x & (t = 0), \text{ where } x \in 4B^p. \end{cases}$$

Note that $g_{y,t}$ is continuous on y.

Let $\lambda : 2B^{n-p} \to [0, 1]$ be a continuous function such that

$$\lambda(y) = \begin{cases} 1 & (\|y\| \le 1) \\ 0 & (\|y\| = 2). \end{cases}$$

For the deformation $\Psi(h, t)(x, y) = (g_{y,t}^{-1} \circ h_y(x), y)$, we define a new deformation Φ by

$$\Phi(h, t)(x, y) = \Psi(h, \lambda(y)t)(x, y) \quad (x \in 4B^p, y \in 2B^{n-p}).$$

This satisfies that (1) $\Phi(h,0)=h$, (2) $\Phi(h,1)(x,y)=(g_{y,\lambda(y)}^{-1}\circ h_y(x),y)$, which is the identity on $B^p\times B^{n-p}$, and (3) $\Phi(h,t)=h$ on $\partial(4B^p\times 2B^{n-p})$.

Step 3. Now we prove the theorem. Take a foliated chart $U_1(\cong int(5B^p\times 3B^{n-p})\supset 4B^p\times 2B^{n-p})$, for $(M,\mathscr{F})$. Let $f\in L(M,\mathscr{F})$. We can assume that f is close to the identity. We put $h=f|_{4B^p\times 2B^{n-p}}$. Then we can regard h as the map in Step 2. We put $\bar{f}_1=\Phi(h,1)$ and $f_1=f\circ\bar{f}_1^{-1}$. Since $\bar{f}_1=f$ on $\partial(4B^p\times 2B^{n-p})$, $\bar{f}_1$ can be extended to an element of $L(M,\mathscr{F})$ by using f. Then $supp(f_1)$ is contained in $4B^p\times 2B^{n-p}\subset U_1$.

Next taking another chart $U_2(\cong int(5B^p\times 3B^{n-p}))$, we perform the procedure in Step 2 for $f_1^{-1}\circ f$ and U_2 to get $\bar{f}_2$ and $f_2=f_1^{-1}\circ f\circ\bar{f}_2^{-1}$. Then $supp(f_2)$ is contained in U_2. Note that the identity part of $\bar{f}_2$ increases definitely than that of $\bar{f}_1$, since the deformation Φ keeps the identity part of $\bar{f}_1$ fixing. Since the support of f is compact, continuing this procedure finite times, we can get leaf preserving homeomorphisms $f_1,f_2,\dots,f_r$ such that the support of each f_i is contained in a small ball and $f=f_1\circ f_2\circ\cdots\circ f_r$.

This completes the proof.

We have the following theorem from Corollary 2.3 and Theorem 3.1.

Theorem 3.2. *$L(M,\mathscr{F})$ is perfect.*

Proof. Let $f\in L(M,\mathscr{F})$. We may assume that f is close to the identity. From Theorem 3.1, we have $f=f_1\circ f_2\circ\cdots\circ f_r$, where each f_i is a leaf preserving homeomorphism whose support is contained in a small ball.

Hence we can assume that $f_i\in L(\boldsymbol{R}^n,\mathscr{F}_0)$ for each i. From Corollary 2.3, we have that f_i is in the commutator subgroup of $L(\boldsymbol{R}^n,\mathscr{F}_0)$ and hence f is in the commutator subgroup of $L(M,\mathscr{F})$. Thus $L(M,\mathscr{F})$ is perfect.

4. $H_1(F(\mathscr{F}))$ for codimension one foliations.

In this section, we consider the first homology of $F(M,\mathscr{F})$ for a codimension one foliation $\mathscr{F}$. Let M be a compact topological manifold without boundary and $\mathscr{F}$ a codimension one foliation of M. Hereafter we simply write $F(\mathscr{F})$, $L(\mathscr{F})$ instead of $F(M,\mathscr{F})$, $L(M,\mathscr{F})$ respectively.

By Theorem 6.26 of [**S**], there exists a one dimensional foliation $\mathscr{T}$ of M transverse to $\mathscr{F}$. The following lemma is easy to prove.

Lemma 4.1. *Let f be an element of $F(\mathscr{F})$ sufficiently close to the identity. Then f is uniquely decomposed as $f=g\circ h$, where h (resp. g) is an element of $F(\mathscr{F})\cap L(\mathscr{T})$ (resp. $L(\mathscr{F})$) and h and g are also close to the identity.*

Lemma 4.2. *Let f be an element of $F(\mathscr{F})$ and L a leaf of $\mathscr{F}$. If $f(L)\neq L$, then the holonomy group of L is trivial.*

Proof. It is sufficient to prove the lemma for f close to the identity. Consider a path $\{f_t\}_{0\le t\le 1}$ in $F(\mathscr{F})$ from the identity to f. Let $f_t=g_t\circ h_t$ be the decomposition of Lemma 4.1 and C be a closed curve in L. Then $h_t(C)$ is closed for any t $(0\le t\le 1)$, hence the holonomy along C is trivial. This proves the lemma.

We define the subset S_0 of M by

$$S_0 = \{x \in M \mid \textit{there exists an element } f \textit{ of } F(\mathscr{F}) \textit{ such that } f(L_x) \neq L_x\}.$$

By definition, S_0 is an open $\mathscr{F}$-saturated set and by Lemma 4.2, all leaves in S_0 have trivial holonomy.

Theorem 4.3. *Let S be a connected component of S_0. Then clearly S is invariant under the action of $F(\mathscr{F})$ and S is one of the following three types.*

Type P: S is homeomorphic to $L \times (0,1)$ and the foliations $\mathscr{F}|_S$ and $\mathscr{T}|_S$ correspond to the product structure of $L \times (0,1)$.

Type R: There exists a closed transverse curve C in S such that C meets each leaf of $\mathscr{F}|_S$ at exactly one point and the natural map

$$p : S \to C, \quad p(x) = L_x \cap C$$

is a fibration and $\mathscr{T}|_S$ is a connection of the fibration p.

Type D: All leaves of $\mathscr{F}$ in S are dense in S and there exists a one parameter subgroup $\{\varphi_t\}$ of $F(\mathscr{F}|_S)$ whose orbits are leaves of $\mathscr{T}|_S$.

Here we make some preparations. By the results of Siebenmann [**S**], many devices used in the study of differentiable codimension one foliations are available in topological case. For example if C is a closed curve transverse to $\mathscr{F}$, then a transversal foliation $\mathscr{T}'$ is chosen so that C is a leaf of $\mathscr{T}'$. So the argument of [**I**] works for topological case. Since we are interested in the set S_0 of the leaves with trivial holonomy, we can assume that $\mathscr{T}$ is orientable. Then it is easy to construct a topological flow $\{\psi_t\}$ on M whose orbits are leaves of $\mathscr{T}$. By using $\{\psi_t\}$, we can define the notion of holonomy map and we have the following facts.

Fact I. Let x be a point of M, $y = \psi_{t_0}(x)$ and suppose that the holonomy of $L_{\psi_t(x)}$ is trivial for $0 \leq t \leq t_0$, then L_y is homeomorphic to L_x via holonomy maps. (This follows from [**I**] Corollary 3.1.)

Fact II. Let C be a closed curve transverse to $\mathscr{F}$. Suppose that any leaf in the $\mathscr{F}$-saturation of $C, S(C)$, has trivial holonomy, then a leaf in $\partial S(C)$ has a non-trivial holonomy. (This is a special case of [**I**] Lemma 3.6.)

Fact III. Let C be as above. Suppose that there exists a leaf L such that $L \cap C$ is infinite, then either (i) there exists an exceptional leaf L_0 in $S(C)$ and all leaves in $S(C) - \overline{L}_0$ are proper leaves or (ii) all leaves in $S(C)$ are dense in $S(C)$ and there exists a one parameter group $\{\varphi_t\}$ of $\mathscr{F}$-preserving homeomorphisms of $S(C)$ whose orbits are leaves of $\mathscr{T}|_{S(C)}$. (This follows from the proof of [**I**] Lemma 2.1 and Theorem 1.3.)

Proof of Theorem 4.3. Suppose that there is no closed curve transverse to $\mathscr{F}|_S$, then any leaf T of $\mathscr{T}|_S$ is homeomorphic to an open interval $(0,1)$ and $T \cap L$ is one point. So by Fact I, S is homeomorphic to $L \times (0,1)$.

If there exists a closed transverse curve C in S, then we have $S = S(C)$ by Fact II. Suppose that $C \cap L$ is finite. Then we can modify C to C' such that $C' \cap L$ is one point. Then for any leaf L' in S, we have $L' \cap C' =$ *one point*. In fact if $L' \cap C'$ has two points, then we can construct a closed transverse curve C'' such that $S(C'') \cap L = \emptyset$. But this contradicts to $S = S(C'')$. So we have a natural map $p : S \to C'$ and this is a fibration by Fact I.

Suppose that $C \cap L$ is infinite. If there exists an exceptional leaf L_0 in S, then as in the proof of Lemma 4.2, all nearby leaves of L_0 must be exceptional but this contradicts to Fact III. So by Fact III, all leaves in S are dense in S. This completes the proof.

LEMMA 4.4. *An orientation preserving homeomorphism f of $[0,1]$ is a commutator.*

PROOF. Suppose that $f(x) > x$ for any $x \in (0,1)$. Then there exists a homeomorphism h of $(0,1)$ onto $\boldsymbol{R}$ such that $h \circ f \circ h^{-1}(t) = t+1$. Let r be a rotation of $S^1 = \boldsymbol{R}/\boldsymbol{Z}$ of angle $2\pi\alpha$. If $\alpha \neq 1/2$, then by Proposition 5.1 of [**W**] we have $r = [g_1, g_2]$, where $g_\varepsilon \in \mathscr{H}(S^1)$ (the homeomorphism group of S^1), $(\varepsilon = 1,2)$. Let $\tilde{g}_\varepsilon$ be the lift of g_ε to a homeomorphism of $\boldsymbol{R}$ and define $\tilde{r} = [\tilde{g}_1, \tilde{g}_2]$. Then we have $\tilde{r}(s) = s + n + \alpha$ for some integer n. By changing the coordinate s to t by $(n+\alpha)t = s$, we have $\tilde{r}(t) = t+1$. Thus $f = h^{-1} \circ [\tilde{g}_1, \tilde{g}_2] \circ h$ is a commutator. If $f(x) < x$ for any $x \in (0,1)$, then consider f^{-1}. If f has fixed points in $(0,1)$, consider the restriction of f to each connected component of $\{x \mid f(x) \neq x\}$ and we see that any f is a commutator.

LEMMA 4.5. *Let $\mathscr{PH}(\boldsymbol{R})$ be the group of periodic homeomorphisms of $\boldsymbol{R}$ of period 1. Then any element of $\mathscr{PH}(\boldsymbol{R})$ which is close to the identity is expressed as a product of two commutators.*

PROOF. Let f be an element of $\mathscr{PH}(\boldsymbol{R})$ close to the identity. If f has a fixed point, then as in the proof of Lemma 4.4, f is a commutator. If f has no fixed points, there exists a small translation t of $\boldsymbol{R}$ such that $t \circ f$ has a fixed point. Then t is a commutator ([**W**]). Thus f ia represented by two commutators.

THEOREM 4.6. *Let $\mathscr{F}$ be a codimension one foliation of a compact manifold M. Suppose that $\mathscr{F}$ has no components of type D and has only a finite number of components of type R. Then $F(\mathscr{F})$ is perfect.*

PROOF. We can suppose that the transverse foliation $\mathscr{T}$ has a closed leaf C_i on each component S_i of type R which intersects each leaf of $\mathscr{F}|_{S_i}$ at one point. Moreover we can define an $\mathscr{F}$-preserving flow φ_t on S_i such that orbits are leaves of $\mathscr{T}|_{S_i}$ and $L_{\varphi_t(x)} = L_{\varphi_{t+1}(x)}$. Suppose that f is close to identity and let $f = g \circ h$ be the decomposition of Lemma 4.1. Then choosing a leaf L of $\mathscr{F}|_{S_i}$, h induces a periodic homeomorphism $\hat{h}^i$ of $\boldsymbol{R}$ such that $h(\varphi_t(x)) = \varphi_{\hat{h}^i(t)}(x)$ for any $x \in L$ and $t \in \boldsymbol{R}$. Since $\hat{h}^i$ is close to the identity, by Lemma 4.5 we have $\hat{h}^i = [h_1^i, h_2^i][h_3^i, h_4^i]$, where $h_\varepsilon^i \in \mathscr{PH}(\boldsymbol{R})$ $(\varepsilon = 1,2,3,4)$. For any $y \in S_i$, we choose t satisfying $\varphi_{-t}(y) \in L$ and put $\tilde{h}_\varepsilon^i(y) = \varphi_{h_\varepsilon^i(t)}(\varphi_{-t}(y))$. Then we have $h|_{S_i} = [\tilde{h}_1^i, \tilde{h}_2^i][\tilde{h}_3^i, \tilde{h}_4^i]$.

Similarly we can define $\tilde{h}_\varepsilon^i \in F(\mathscr{F}|_{S_j}) \cap L(\mathscr{T}|_{S_j})$ on each component S_j of type P such that $h|_{S_j} = [\tilde{h}_1^j, \tilde{h}_2^j][\tilde{h}_3^j, \tilde{h}_4^j]$, where $\tilde{h}_3^i$ and $\tilde{h}_4^i$ are the identity. Then we can define $h_\varepsilon \in F(\mathscr{F}) \cap L(\mathscr{T})$ by $h_\varepsilon = \tilde{h}_\varepsilon^i$ on component S_i of type R and P and by $h_\varepsilon(x) = x$ for $x \notin S_0$. Then we have $h = [h_1, h_2][h_3, h_4]$ and by Theorem 3.2, $F(\mathscr{F})$ is perfect. This completes the proof.

REMARK 4.7. From Theorem 4.6, we see that $F(S^3, \mathscr{F}_R)$ is perfect for the Reeb foliation $\mathscr{F}_R$ of S^3. In contrast with topological case, differentiable case is as follows. Let $F^r(S^3, \mathscr{F}_R)$ be the group of foliation preserving C^r-diffeomorphisms of

$(S^3, \mathscr{F}_R)$ isotopic to the identity by a foliation preserving isotopy. Then Lemma 1 of [F-U] implies that $F^r(S^3, \mathscr{F}_R)$ is not perfect for $r \geq 1$.

For a type D-component S, we define a submodule $Per(S)$ of $\boldsymbol{R}$ by

$$Per(S) = \{t \in \boldsymbol{R} \mid \varphi_t(L) = L \textit{ for one and all leaves } L \textit{ in } S\}$$

$Per(S)$ depends on the parametrization of $\{\varphi_t\}$ but the quotient group $\boldsymbol{R}/Per(S)$ is determined by $\mathscr{F}|_S$ and, as a set, this is the space of leaves of $\mathscr{F}|_S$.

THEOREM 4.8. *Let S be a type D-component. Then there exists a homomorphism* π *of* $F(\mathscr{F})$ *onto* $\boldsymbol{R}/Per(S)$ *and we have* $\ker \pi = \{f \in F(\mathscr{F}) \mid f(L) = L \textit{ for any leaf } L \textit{ in } S\}$.

PROOF. Let f be an $\mathscr{F}$-preserving homeomorphism of M and suppose that f is sufficiently close to the identity and $f = g \circ h$ be the decomposition of Lemma 4.1. Then $h(x) = \varphi_t(x)$ for some $t \in \boldsymbol{R}$ and any $x \in S$ and we define $\pi(f) = t$. For general f, f is decomposed as $f = \prod f_i$, where f_i are sufficiently close to the identity and we define $\pi(f) = \sum \pi(f_i)$. This depends on the decomposition of f but $\pi(f) \bmod Per(S)$ is uniquely determined by f and clearly π is a homomorphism. For any $t \in \boldsymbol{R}$ we define $f \in F(\mathscr{F})$ by $f(x) = \varphi_t(x)$ for $x \in S$ and $f(x) = x$ for $x \notin S$. Then $\pi(f) \equiv t \bmod Per(S)$, so π is surjective.

Let $\pi : F(\mathscr{F}) \to \prod \boldsymbol{R}/Per(S_i)$ denote the homomorphism defined by $\pi(f) = \prod \pi_i(f)$ for $f \in F(\mathscr{F})$, where π_i is a homomorphism in the above lemma for a type D-component S_i and the product is taken for all type D-components S_i of $\mathscr{F}$. Then π induces a homomorphism π_* of $H_1(F(\mathscr{F}))$ to $H_1(\prod \boldsymbol{R}/Per(S_i)) \cong \prod \boldsymbol{R}/Per(S_i)$. Then we have the following.

THEOREM 4.9. *The homomorphism* π_* *of* $H_1(F(\mathscr{F}))$ *to* $\prod \boldsymbol{R}/Per(S_i)$ *is surjective.*

This is an easy consequence of Theorem 4.8 and a non-zero element of $\ker \pi_*$ is represented by a leaf preserving homeomorphism which is not isotopic to the identity via leaf preserving homeomorphisms. For a very simple case, we have the following.

THEOREM 4.10. *Let* $\mathscr{F}$ *be a foliation of the torus* T^n *defined by a* 1*-form* $\omega = \sum a_i \, dx_i$. *If one of* a_i/a_j *is irrational, then* $H_1(F(\mathscr{F}))$ *is isomorphic to* $\boldsymbol{R}/a_1\boldsymbol{Z} + \cdots + a_n\boldsymbol{Z}$.

PROOF. In this case, T^n is the component of type D, $Per(T^n) = a_1\boldsymbol{Z} + \cdots + a_n\boldsymbol{Z}$ and $\mathscr{T}$ and φ_t can be defined by $\partial/\partial x^1$ if $a_1 \notin 0$. Let f be an element of $\ker \pi$. We can suppose that f is close to the identity and let $f = g \circ h$ be the decomposition of Lemma 4.1. Then $h(x) = \varphi_t(x)$ for some $t \in Per(T^n)$. Since φ_t is a parallel translation on each leaf of $\mathscr{F}$, f is contained in $L(\mathscr{F})$. So by Theorem 3.2, f is in the commutator subgroup of $L(\mathscr{F})$. In particular, f represents a zero element. This completes the proof.

References

[E-K] R. D. Edwards, R. C. Kirby, Deformations of spaces of imbeddings, Ann. of Math., **93** (1971), 63–88.

[F-U] K. Fukui, S. Ushiki, On the homotopy type of *F Diff* $(S^3, \mathscr{F}_R)$, J. Math. Kyoto Univ., **15-1** (1975), 201–210.

[I] H. Imanishi, On the theorem of Denjoy-Sacksteder for codimension one foliations without holonomy, J. Math. Kyoto Univ., **14** (1974), 607–634.

[M] J. N. Mather, The vanishing of the homology of certain groups of homeomorphisms, Topology, **10** (1971), 297–298.

[S] L. C. Siebenmann, Deformations of homeomorphisms on stratified sets, Comment. Math. Helv., **47** (1972), 123–163.

[W] J. W. Wood, Bundles with totally disconnected structure group, Comment. Math. Helv., **46** (1971), 257–277.

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