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Boundary behavior of positive solutions of $\Delta u = Pu$ on a Riemann surface

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Abstract. The classical Fatou limit theorem was extended to the case of positive harmonic functions on a hyperbolic Riemann surface R by Constantinescu-Cornea. They used extensively the notions of Martin's boundary and fine limit following the filter generated by the base of the subsets of R whose complements are closed and thin at a minimal boundary point of R. We shall consider such a problem for positive solutions of the Schrödinger equation on a hyperbolic Riemann surface.

1. Introduction.

J. L. Doob [**4**] and Constantinescu-Cornea [**3**] independently investigated boundary behavior of positive harmonic functions at minimal boundary points of the Martin boundary and established Fatou-type theorems on general domains. In this paper we shall concern ourselves with the same problem for positive solutions of Schrödinger's equation on a hyperbolic Riemann surface following Constantinescu-Cornea's set-up.

Throughout this paper let R be a hyperbolic Riemann surface. The Martin boundary and the set of minimal boundary points of R are denoted by Δ and Δ_1, respectively. Let K_b be the Martin kernel of a point $b \in \Delta_1$. For a closed subset E of R and a positive superharmonic function s on R the balayage of s over E is the infimum of the class of positive superharmonic functions on R majorizing s on E except for a polar subset of E and is denoted by $(s)_E$. The closed set E in R is said to be thin at a point $b \in \Delta_1$ provided that $(K_b)_E$ is a potential on R; that is, $(K_b)_E < K_b$ on some connected component of $R - E$. For a point $b \in \Delta_1$ the class of open subsets G of R whose complements are thin at the point b is denoted by $\mathscr{G}(b)$, which is a filter on R. The canonical measure of the constant harmonic function 1 on R is denoted by χ and called the harmonic measure of R. The following result is one of Fatou-type theorems due to Constantinescu-Cornea. The details of its proof can be found in their book [**3**]. *A positive harmonic function v on R has a limit following the filter $\mathscr{G}(b)$ at χ-almost every point b of Δ_1.*

We now consider Schrödinger's equation $\Delta u = Pu$ on a hyperbolic Riemann surface R, where $P(z)\,dx\,dy$ is a non-negative Hölder continuous 2-form on R and $z = x + iy$ is a local parameter of R. Let U be an open subset of R. A real-valued function $u \in C^2(U)$ is said to be a P-solution on the open set U if u satisfies the above equation. Most of the definitions concerning to harmonic functions are carried over to the

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present situation. The Martin boundary of R for this equation is denoted by Δ_P and the set of minimal boundary points of Δ_P by Δ_{P1}. Let K_a^P be the Martin kernel of a point a in Δ_{P1}. The terminology of "P-supersolution", "balayage", "P-thin", and "filter $\mathscr{G}^P(a)$ for $a \in \Delta_{P1}$" can be carried over to the context of P-solutions and play the roles of "superharmonic function", "balayage", "thin", and "filter $\mathscr{G}(b)$ for $b \in \Delta_1$" in the harmonic case, respectively. The greatest P-solution in the class of positive P-solutions on R bounded above by 1 is denoted by e^P. Its canonical measure on Δ_{P1} is denoted by χ_P and is called the P-elliptic measure of R. The P-solution e^P is either identically zero or positive on R. Throughout this paper we assume that e^P is positive on R. Thus we can show the following result in a manner quite similar to the proof of the preceding result: *If u is a positive P-solution on R, then u has a limit following the filter $\mathscr{G}^P(a)$ at χ_P-almost every point a of Δ_{P1}.*

However, this result can not be regarded as a desired Fatou-type theorem for positive P-solutions, since it contains concepts depending upon the density P: that is, the boundary Δ_{P1}, the filter $\mathscr{G}^P(a)$, and the measure χ_P on Δ_{P1}. By replacing these concepts by those independent from the density P, for example, the Martin boundary Δ, the filter $\mathscr{G}(b)$, $b \in \Delta_1$, and the measure χ on Δ_1, we shall obtain just a desired Fatou-type theorem for positive P-solutions on R.

We denote by Δ_{HP}^0 the set of points $b \in \Delta_1$ such that

$$\int_R P(w) G^P(z_1, w) K_b(w)\, du\, dv < +\infty$$

and

$$K_b(z_1) > \frac{1}{2\pi} \int_R P(w) G^P(z_1, w) K_b(w)\, du\, dv$$

for some point $z_1 \in R$, where $w = u + iv$ and $G^P(z, w)$, $(z, w) \in R \times R$, is Green's function of R relative to the equation. It will be shown in Collorary 3.5 that this subset Δ_{HP}^0 of Δ_1 has positive harmonic measure. Our main result is the following (Theorem 4.2): *A positive P-solution u on R has a limit following the filter $\mathscr{G}(b)$ at χ-almost every point b of Δ_{HP}^0.* If the density P on R satisfies the condition

$$\int_R P(w) G(z_1, w)\, du\, dv < +\infty$$

for some point $z_1 \in R$, then the set Δ_1 of minimal boundary points will be contained in the subset Δ_{HP}^0 except for a set with χ-measure zero (Corollary 3.9).

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2. Notations and preliminaries.

In this section we shall recall preliminary definitions and notations on the transformations t_{PH} and t_{HP} between the Martin boundaries Δ_P and Δ of the Riemann surface R. We refer to [**8**] for details of their definitions and related properties. And, the measurabilities of the transformations are given in this section.

Let Δ_{PH} be the set of minimal boundary points $a \in \Delta_{P1}$ such that

$$\int_R P(w) G(z_1, w) K_a^P(w)\, du\, dv < +\infty$$

for some point z_1 in R, where $G(z,w)$, $(z,w) \in R \times R$, is the harmonic Green function of R. The set Δ_{PH} is a Borel measurable subset of Δ_P. Similarly, we denote by Δ_{HP} the set of minimal boundary points $b \in \Delta_1$ such that

$$\int_R P(w) G^P(z_1, w) K_b(w)\, du\, dv < +\infty$$

for some point $z_1 \in R$. This set is also Borel measurable in Δ.

In this paragraph we shall recall the definition of the measurable transformation t_{PH} on Δ_{PH} into Δ_{HP}. To do this we need the notion of pole of a minimal positive harmonic function, which was introduced by Brelot [**1**] on any general metrizable compactification of a Green space. The reduced function of a positive P-solution u on R over a compact subset C of Δ, which is denoted by $(u)_C$, is the infimum of the class of positive P-supersolutions s majorizing u on an intersection $U \cap R$, where U is some neighborhood of C relative to the topology of $R^* = R \cup \Delta$. Let a be any boundary point in Δ_{P1} and b a boundary point in Δ. The reduced function of the minimal function K_a^P over the set $\{b\}$ $(K_a^P)_{\{b\}}$ is either constantly zero or K_a^P. If $(K_a^P)_{\{b\}} = K_a^P$, then the point b is called a pole of K_a^P on Δ. Generally speaking, the minimal function K_a^P with $a \in \Delta_{P1}$ has at least one pole on Δ and may have many poles on the boundary Δ. However, if the point a belongs to the set Δ_{PH}, then K_a^P has a unique pole on the set Δ_1, which will be contained in Δ_{HP}. Then, we can define the mapping $t_{PH} : \Delta_{PH} \to \Delta_{HP}$ by assigning the unique pole $b \in \Delta_{HP}$ of K_a^P for $a \in \Delta_{PH}$; that is, $t_{PH}(a) = b$.

Now, we shall prove that the transformation $t_{PH} : \Delta_{PH} \to \Delta_{HP}$ is measurable. To do this we need the following notation and lemma. For an open subset G of R, let

$$\Delta_P(G) = \{a \in \Delta_{P1} : G \in \mathscr{G}^P(a)\},$$

The set $\Delta_P(G)$ is measurable in Δ_{P1} (see Constantinescu-Cornea [**3**]).

Lemma 2.1. *Let C be a compact subset of the Martin boundary Δ of R. The image $t_{PH}(a)$ of $a \in \Delta_{PH}$ by t_{PH} belongs to the set C if and only if the Martin kernel K_a^P satisfies $K_a^P = (K_a^P)_C$ on R.*

Proof. Letting $b = t_{PH}(a)$ for $a \in \Delta_{PH}$, we assume that b is contained in the compact set C. Since the point b is the pole of K_a^P on Δ, we have

$$K_a^P = (K_a^P)_{\{b\}} \le (K_a^P)_C \le K_a^P \quad \text{on } R,$$

which shows that $K_a^P = (K_a^P)_C$ on R.

Suppose that the point $b = t_{PH}(a)$, $a \in \Delta_{PH}$, is not contained in the set C. Then, for each point $y \in C$ there is a closed neighborhood V_y relative to the Martin compactification R^* such that $(K_a^P)_{V_y \cap R}$ is a potential. By the compactness of C there exists a finite number of points $y_1, y_2, \dots, y_n$ in C such that $C \subset \bigcup_{i=1}^n V_{y_i}$ and $(K_a^P)_{V_{y_i} \cap R}$ is a potential. Therefore we have

$$(K_a^P)_C \le (K_a^P)_{\bigcup_{i=1}^n (V_{y_i} \cap R)} \le \sum_{i=1}^n (K_a^P)_{V_{y_i} \cap R},$$

from which it follows $(K_a^P)_C = 0$. That is, if $(K_a^P)_C = K_a^P$ on R, then b is contained in C. □

Theorem 2.2. *The transformation* $t_{PH} : \Delta_{PH} \to \Delta_1$ *is measurable.*

Proof. Let C be any compact subset of the Martin boundary Δ of R and $\{V_n\}$ be a decreasing sequence of closed neighborhoods of C converging to C with respect to the Martin topology of R^*. We denote by U_n the intersection $V_n \cap R$ and by G_n the complement $R - U_n$. Then, we have

$$(K_a^P)_C = \lim_{n \to +\infty} (K_a^P)_{U_n} \tag{1}$$

for $a \in \Delta_{P1}$.

If the image $t_{PH}(a)$ of a point $a \in \Delta_{PH}$ belongs to the compact subset C, then for every integer n we have, by the preceding lemma, $(K_a^P)_{U_n} = K_a^P$: that is, each closed subset U_n of R is not P-thin at the point $a \in \Delta_{PH}$.

Therefore we have

$$t_{PH}^{-1}(C \cap \Delta_1) = \bigcap_{n=1}^{+\infty} (\Delta_{PH} - \Delta_P(G_n)),$$

from which it follows that $t_{PH}^{-1}(C \cap \Delta_1)$ is a Borel measurable subset of Δ_{P1}, for $\Delta_P(G_n)$ is measurable in Δ_{P1} as noted before Lemma 2.1.

Since the class of sets $C \cap \Delta_1$ with compact subsets $C \subset \Delta$ generates the Borel measurable σ-ring on Δ_1. For a Borel measurable subset E of Δ_1 we have $t_{PH}^{-1}(E)$ is Borel measurable in the measurable space Δ_{PH}; that is, t_{PH} is measurable. □

We denote by Δ_{HP}^0 the set of points $b \in \Delta_{HP}$ such that

$$K_b(z_1) > \frac{1}{2\pi} \int_R P(w) G^P(z_1, w) K_b(w) \, du \, dv$$

for some point $z_1 \in R$. This set is a measurable subset of Δ. We can define the notion of pole on the boundary Δ_{P1} for each point $b \in \Delta_1$ and we can prove that for each point $b \in \Delta_{HP}^0$ there exists a unique pole a on Δ_P of b, which is contained in the set Δ_{P1}. Then, the transformation

$$t_{HP} : \Delta_{HP}^0 \to \Delta_{P1}$$

is defined by the same way as the definition of t_{PH}. In [**8**] we have proved that the composition $t_{HP} \cdot t_{PH}$ is the identity on Δ_{PH}. The following theorem may be proved by the same way as the preceding theorem.

Theorem 2.3. *The transformation* $t_{HP} : \Delta_{HP}^0 \to \Delta_{P1}$ *is measurable.*

In the following sections we shall need the next two theorems whose proofs can be found in [**8**].

Theorem 2.4. *Let a boundary point* a *be in* Δ_{PH}. *Then, a closed subset* E *of* R *is* P*-thin at* a *if and only if* E *is thin at the point* $t_{PH}(a)$.

Theorem 2.5. *Let a boundary point* b *be in* Δ_{PH}^0. *If a closed subset* E *of* R *is* P*-thin at the point* $t_{HP}(b)$, *then* E *is thin at* b.

3. Harmonic and P-elliptic measures.

In this section we shall investigate relationship between the P-elliptic measure χ_P and the harmonic measure χ by using the transformation

$$t_{PH} : \Delta_{PH} \to \Delta_1,$$

where we recall that χ_P (resp. χ) is the canonical measure of e^P on Δ_{P1} (resp. 1 on Δ_1). We denote by $\Delta(P)$ the image $t_{PH}(\Delta_{PH})$ of t_{PH}. The set Δ_1 of minimal boundary points of the Martin boundary Δ is decomposed into its four disjoint subsets:

$$\Delta_1 - \Delta_{HP}, \quad \Delta_{HP} - \Delta_{HP}^0, \quad \Delta_{HP}^0 - \Delta(P), \quad \Delta(P).$$

At first we shall show that the harmonic measure χ is supported only by two sets of them:

$$(\Delta_{HP} - \Delta_{HP}^0) \cup \Delta(P).$$

For a minimal point $b \in \Delta_1$, let V be the intersection of a neighborhood of b in the Martin compactification R^* with the Riemann surface R. Then the balayage $(K_b)_{R-V}$ of the kernel K_b over the closed set $R - V$ is potential, so that the closed set $R - V$ is thin at the point b (Hilfssatz 13.2 in Constantinescu-Cornea [**3**]). From this property of neighborhoods of a minimal boundary point we have the following lemma.

Lemma 3.1. *The subsets $t_{HP}(\Delta_{HP}^0 - \Delta(P))$ and Δ_{PH} are disjoint from each other in Δ_{P1}.*

Proof. We assume that the image t_{HP}(b) of some point $b \in \Delta_{HP}^0 - \Delta(P)$ by the mapping t_{HP} belongs to the set Δ_{PH}. Then, letting $b' = t_{PH} \cdot t_{HP}(b) \in \Delta(P)$, we have $t_{HP}(b) = t_{HP}(b')$. Let U and U' be neighborhoods of b and b' relative to the Martin topology, respectively. These neighborhoods may be assumed to be disjoint from each other. We denote by V and V' the intersections $U \cap R$ and $U' \cap R$, respectively. Then, the closed subsets $R - V$ and $R - V'$ of R are thin at the points b and b', respectively. By Theorem 2.4 and 2.5 in the preceding section the set $R - V'$ is P-thin at $t_{HP}(b')$, and hence thin at b. Therefore, we have that $(R - V) \cup (R - V') = R$ is thin at the minimal boundary point b, which is a contradiction. □

For each harmonic function v on R such that

$$\int_R P(w) G^P(z_1, w) v(w)\, du\, dv < +\infty \tag{2}$$

for some point $z_1 \in R$, let $T_{HP}v$ be the P-solution on R:

$$v(z) - \frac{1}{2\pi} \int_R P(w) G^P(z, w) v(w)\, du\, dv.$$

And, for each P-solution u on R such that

$$\int_R P(w) G(z_1, w) u(w)\, du\, dv < +\infty \tag{3}$$

for some point $z_1 \in R$, the harmonic function on R:

$$u(z) + \frac{1}{2\pi}\int_R P(w)G(z,w)u(w)\,du\,dv$$

is denoted by $T_{PH}u$. Then, for each boundary point $b \in \Delta_{HP}$ $T_{HP}K_b$ is defined and satisfies

$$T_{HP}K_b = T_{HP}K_b(z_0)K_a^P, \quad a = t_{HP}(b) \tag{4}$$

provided that $b \in \Delta_{HP}^0$, where z_0 is the origin of two Martin compactifications of R. For each $a \in \Delta_{PH}$ we can also define $T_{HP}K_a^P$ and we have

$$T_{PH}K_a^P = T_{PH}K_a^P(z_0)K_b, \quad b = t_{PH}(a). \tag{5}$$

For these relations (4) and (5) we refer to [**8**].

The following lemmas are easy consequences of Fubini's theorem.

Lemma 3.2. *Let v be a harmonic function which satisfies the condition* (2) *for some $z_1 \in R$, and ν be its canonical measure on Δ_1. Then, we have*

$$T_{HP}v = \int_{\Delta_{HP}^0} T_{HP}K_b\,d\nu(b) \quad \text{on } R.$$

Lemma 3.3. *Let u be a P-solution satisfying the condition* (3) *and μ be its canonical measure on Δ_{P1}. Then, we have*

$$T_{PH}u = \int_{\Delta_{PH}} T_{PH}K_a^P\,d\mu(a) \quad \text{on } R.$$

The next theorem gives a relation between the measures χ_P and χ.

Theorem 3.4. *The subset $\Delta_{HP}^0 - \Delta(P)$ of Δ_1 has harmonic measure zero*:

$$\chi(\Delta_{HP}^0 - \Delta(P)) = 0. \tag{6}$$

And, we have the equality, for every measurable subset E of Δ_{PH},

$$\chi_P(E) = \int_E T_{HP}K_{t_{PH}(a)}(z_0)d\chi \cdot t_{PH}(a) \tag{7}$$

$$= \int_{t_{PH}(E)} T_{HP}K_b(z_0)d\chi(b). \tag{8}$$

Proof. The constant function 1 on R is represented as the integral by χ over the subset Δ_{HP} of Δ_1, because of the inequality

$$\int_R P(w)G^P(z,w)\,du\,dv < 2\pi, \quad z \in R.$$

Since

$$\Delta_{HP} = (\Delta_{HP} - \Delta_{HP}^0) \cup (\Delta_{HP}^0 - \Delta(P)) \cup \Delta(P),$$

we have, by Lemma 3.2 and the equality (4) in this section,

$$T_{HP}1 = \int_{\Delta_{HP}^0 - \Delta(P)} T_{HP}K_b\,d\chi(b) + \int_{\Delta(P)} T_{HP}K_b\,d\chi(b) \tag{9}$$

$$= \int_{\Delta^0_{HP} - \Delta(P)} K^P_{t_{HP}(b)} T_{HP} K_b(z_0)\, d\chi(b) \tag{10}$$

$$+ \int_{\Delta(P)} K^P_{t_{HP}(b)} T_{HP} K_b(z_0)\, d\chi(b), \tag{11}$$

because of $T_{HP}K_b = 0$ for $b \in \Delta_{HP} - \Delta^0_{HP}$. Since the mapping $t_{HP} : \Delta^0_{HP} \to \Delta_{P,1}$ is measurable (Theorem 2.3), we can define the three set functions ν, ν_1 and ν_2 as follows; for every measurable subset E of Δ_{P1} we define

$$\nu(E) = \int_{t^{-1}_{HP}(E)} T_{HP} K_b(z_0)\, d\chi(b),$$

$$\nu_1(E) = \int_{t^{-1}_{HP}(E \cap (\Delta_{P1} - \Delta_{PH}))} T_{HP} K_b(z_0)\, d\chi(b),$$

$$\nu_2(E) = \int_{t^{-1}_{HP}(E \cap \Delta_{PH})} T_{HP} K_b(z_0)\, d\chi(b).$$

These set functions are measures on the Borel field of Δ_{P1} supported by the sets

$$t_{HP}(\Delta^0_{HP}), \quad t_{HP}(\Delta^0_{HP}) - \Delta_{PH}. \quad \Delta_{PH}$$

respectively, and $\nu = \nu_1 + \nu_2$. The terms (10) and (11) are written with ν, ν_1 and ν_2 as follows:

$$T_{HP}1 = \int_{\Delta_{P1}} K^P_a\, d\nu(a)$$

$$= \int_{\Delta_{P1} - \Delta_{PH}} K^P_a\, d\nu_1(a) + \int_{\Delta_{PH}} K^P_a\, d\nu_2(a).$$

On the other hand the P-solution e^P is represented as the integral over the set Δ_{PH} by its canonical measure χ_P and we have $T_{HP}1 = e^P$ on R. The uniqueness of canonical measure in the Martin integral representation theorem implies that $\nu_1 = 0$ and $\nu_2 = \chi_P$. Then it follows that

$$\chi(\Delta^0_{HP} - \Delta(P)) = 0,$$

since $T_{HP}K_b(z_0) > 0$ for $b \in \Delta^0_{HP}$ and, by Lemma 3.1,

$$t^{-1}_{HP}(\Delta_{P1} - \Delta_{PH}) = \Delta^0_{HP} - \Delta(P).$$

For a measurable subset E of Δ_{PH} we have $\chi_P(E) = \nu_2(E)$; that is, the second part of the theorem was proved. □

Corollary 3.5. *For a measurable subset E of Δ_{PH}, $\chi_P(E) = 0$ if and only if $\chi(t_{PH}(E)) = 0$.*

Let v be a positive harmonic function on R and ν its canonical measure of the Martin representation:

$$v = \int_{\Delta_1} K_b\, d\nu(b).$$

For a measurable subset B of Δ_1 the reduced function of v relative to B is denoted by

$(v)_B$. Then, we have

$$(v)_B = \int_B K_b \, dv(b)$$

by R. S. Martin [**5**]. Since the e^P satisfies the condition

$$\int_R P(w) G(z_1, w) e^P(w) \, du \, dv < +\infty$$

for each $z_1 \in R$ (T. Satō [**8**]), we can define $T_{PH} e^P$. In the following part of this section we shall show that $T_{PH} e^P$ is the reduced function of the constant function 1 relative to the set Δ^0_{HP}. To do this we need the next lemma.

Lemma 3.6. *For a point b in $\Delta(P)$ we have*

$$T_{HP} K_b(z_0) \cdot T_{PH} K^P_a(z_0) = 1, \quad a = t_{HP}(b), \tag{12}$$

where z_0 is the pole of the Martin compactifications R^* and R^*_P.

Proof. By the definitions of transformations t_{HP}, t_{PH} we have equalities (4) and (5) for $b \in \Delta(P)$ and $a = t_{HP}(b) \in \Delta_{PH}$. Since the transformation $t_{HP} \cdot t_{PH}$ is identity on Δ_{PH} and $T_{HP}(T_{PH} u) = u$ for every P-solution u on R satisfying the condition (3) for some $z_1 \in R$ (T. Satō [**8**]),
we have

$$\begin{aligned} K^P_a &= T_{HP}(T_{PH} K^P_a) \\ &= T_{PH} K^P_a(z_0) \cdot T_{HP} K_b \\ &= T_{PH} K^P_a(z_0) \cdot T_{HP} K_b(z_0) \cdot K^P_a. \end{aligned}$$

Since $K^P_a > 0$, the lemma follows. □

From these results the next theorem follows.

Theorem 3.7. *The harmonic function $T_{PH} e^P$ is the reduced function of the constant function 1 relative to the subset Δ^0_{HP} of Δ_1; that is,*

$$T_{PH} e^P = (1)_{\Delta^0_{HP}}.$$

Proof. Since by Theorem 3.4 we have

$$1 = \int_{(\Delta_{HP} - \Delta^0_{HP}) \cup \Delta(P)} K_b \, d\chi(b), \tag{13}$$

Lemma 3.2 shows that

$$\begin{aligned} e^P &= T_{HP} 1 \\ &= \int_{\Delta(P)} T_{HP} K_b \, d\chi(b) \\ &= \int_{\Delta(P)} K^P_{t_{HP}(b)} \cdot T_{HP} K_b(z_0) \, d\chi(b), \end{aligned}$$

because of $T_{HP}K_b = 0$ for $b \in \varDelta_{HP} - \varDelta_{HP}^0$. From Lemmas 3.3, 3.6 and Theorem 3.4 it follows that

$$\begin{aligned} T_{PH}e^P &= \int_{\varDelta(P)} T_{PH}K^P_{t_{HP}(b)} \cdot T_{HP}K_b(z_0)\, d\chi(b) \\ &= \int_{\varDelta(P)} K_b \cdot T_{PH}K^P_{t_{HP}(b)}(z_0) \cdot T_{HP}K_b(z_0)\, d\chi(b) \\ &= \int_{\varDelta(P)} K_b\, d\chi(b) \qquad (14) \\ &= \int_{\varDelta^0_{HP}} K_b\, d\chi(b) = (1)_{\varDelta^0_{HP}}. \qquad (15) \end{aligned}$$

Hence the proof was completed. □

Corollary 3.8. *$T_{PH}e^P = 1$ if and only if $\chi(\varDelta_1 - \varDelta^0_{HP}) = 0$.*

Proof. By the first part of Theorem 3.4 the equalities (13) and (15) in the proof of the preceding theorem show this corollary. □

Corollary 3.9. *If the density P on R satisfies the condition*

$$\int_R P(w)G(z_1, w)\, du\, dv < +\infty \qquad (16)$$

for some point z_1 in R, then

$$\chi(\varDelta_1 - \varDelta^0_{HP}) = 0.$$

Proof. By the condition (16) we have $T_{PH}e^P = 1$ and hence complete the proof by Corollary 3.8. □

4. Boundary behavior of positive solutions.

In the first place a few definitions of the boundary limit in Constantinescu-Cornea's sense are in order from their book. Let f be an extended real-valued continuous function defined on a hyperbolic Riemann surface R. The cluster set $f^\wedge(b)$ of f at each minimal boundary point $b \in \varDelta_1$ is defined as the set

$$f^\wedge(b) = \bigcap_{G \in \mathscr{G}(b)} \overline{f(G)},$$

where $\overline{f(G)}$ is the closure of the set $f(G)$ in the extended real line $[-\infty, +\infty]$ and the class $\mathscr{G}(b)$ is the filter appeared in Section 1. This cluster set is a non-empty closed connected subset of $[-\infty, +\infty]$. If $f^\wedge(b)$ reduces to a set $\{\alpha\}$ which contains only one extended real number α, then we say that the function f has a boundary limit α at $b \in \varDelta_1$ and represent this fact by $\hat{f}(b) = \alpha$. The set of all those minimal boundary points $b \in \varDelta_1$ at which the function f takes a boundary limit in the above sense is denoted by $\mathscr{F}(f)$. For details on the boundary limits $\hat{f}$ and the set $\mathscr{F}(f)$ we refer to Constantinescu-Cornea [**3**].

Now, we consider the Martin compactification R_P^* of R relative to Schrödinger's equation $\Delta u = Pu$. For a extended real-valued continuous function f on R we can also define the cluster set $f^\wedge(a)$ of f at each minimal boundary point $a \in \Delta_{P1}$ by taking the filter $\mathscr{G}^P(a)$ in place of $\mathscr{G}(b)$, $b \in \Delta_1$. $\mathscr{F}^P(f)$ is the set of points $a \in \Delta_{P1}$ at which the cluster set $f^\wedge(a)$ reduces to a one-point set. For each point $a \in \mathscr{F}^P(f)$ we can define the boundary limit $\hat{f}(a)$.

The next lemma gives a relationship between the above two cluster sets of the function f at points $a \in \Delta_{PH}$ and $b = t_{PH}(a) \in \Delta(P)$ respectively, and hence, if a point $a \in \Delta_{PH}$ belongs to $\mathscr{F}^P(f)$, then we shall obtain a relationship between two boundary limits $\hat{f}(a)$ and $\hat{f}(b)$. Its proof is based on Theorem 2.4 and 2.5.

Lemma 4.1. *Let f be an extended real-valued continuous function on R. We have $f^\wedge(a) = f^\wedge(t_{PH}(a))$ for each point $a \in \Delta_{PH}$, and $f^\wedge(t_{HP}(b)) \supset f^\wedge(b)$ for each point $b \in \Delta_{HP}^0$.*

Constantinescu and Cornea have proved the following result on existence of boundary limits of positive harmonic functions of R (Hilfssatz 14.3 in [**3**]): that is, let s be a positive superharmonic function on R and μ be a measure on Δ_1 such that

$$\int_{\Delta_1} K_b \, d\mu(b) \le s \quad \text{on } R.$$

Let f be an extended real-valued continuous function on R such that fs is a positive superharmonic function on R. Then, we have

$$\mu(\Delta_1 - \mathscr{F}(f)) = 0.$$

(In [**3**] fs was assumed to be a Wiener function on R, however we assume fs to be a positive superharmonic function on R for the sake of simplicity.) Therefore, in the particular case that $s = 1$ and μ is the harmonic measure χ the boundary limit $\hat{v}$ of a positive continuous superharmonic function v is defined a.e. on Δ_1 with respect to χ. And the quasi-bounded component of the greatest harmonic minorant of v is represented by the integral

$$\int_{\Delta_1} K_b \hat{v}(b) \, d\chi(b).$$

Accordingly, the boundary limit of a continuous potential p on R is zero a.e. on Δ_1 with respect to harmonic measure χ.

By the similar way as the case of harmonic functions Constantinescu-Cornea's result may be also proved for any continuous positive P-supersolutions on R using the Martin compactification R_P^* of R and the filter $\mathscr{G}^P(a)$, $a \in \Delta_{P1}$. Let s be a positive P-supersolution on R and μ be a measure on Δ_{P1} such that

$$\int_{\Delta_{P1}} K_a^P \, d\mu(a) \le s \quad \text{on } R.$$

For an extended real-valued continuous function f on R such that fs is a positive P-supersolution on R, then we have

$$\mu(\Delta_{P1} - \mathscr{F}^P(f)) = 0.$$

In the particular case that $s = 1$ and μ is the P-elliptic measure χ_P, a positive continuous P-supersolution u has a boundary limit $\hat{u}$ a.e. on Δ_{P1} with respect to χ_P.

If a positive P-solution u on R is bounded above by a harmonic function h on R, then the boundary limit $\hat{u}$ is defined a.e. on the set Δ_1 with respect to harmonic measure χ, for $h - u$ is a positive continuous superharmonic function on R. For any positive continuous P-supersolution u on R we can say as follows.

Theorem 4.2. *Let u be a positive continuous P-supersolution on R. The boundary limit $\hat{u}$ of u exists a.e. on Δ_{HP}^0 with respect to harmonic measure χ. And we have the relation*

$$\hat{u}(b) = \hat{u}(t_{HP}(b))$$

for almost every point $b \in \Delta_{HP}^0$ with respect to χ.

Proof. There exists a subset E of Δ_{P1} with P-elliptic measure zero such that the boundary limit $\hat{u}(a)$ is defined for each point $a \in \Delta_{P1} - E$. From Lemma 4.1 it follows that the boundary limit $\hat{u}(b)$ exists and $\hat{u}(b) = \hat{u}(t_{HP}(b))$ for $b \in \Delta(P)$ except for the set $t_{PH}(E \cap \Delta_{PH})$, where Corollary 3.5 shows $\chi(t_{PH}(E \cap \Delta_{PH})) = 0$. Since the set $\Delta_{HP}^0 - \Delta(P)$ has harmonic measure zero by Theorem 3.4, we complete the proof. □

Corollary 4.3. *Let u be a positive continuous P-supersolution on R. If a density P on R satisfies the condition*

$$\int_R P(w)G(z_1, w)\, du\, dv < +\infty \tag{17}$$

for some point $z_1 \in R$, then the boundary limit $\hat{u}$ exists a.e. on Δ_1 with respect to harmonic measure χ. And we have $\hat{u}(b) = \hat{u}(t_{HP}(b))$ for almost every point $b \in \Delta_1$ with respect to χ.

Proof. The preceding theorem gives this corollary by Corollary 3.9. □

In the remaining part of this section we shall consider boundary behavior of the P-elliptic measure e^P at minimal points $b \in \Delta_1$. Since e^P is bounded above by 1 on R, it is evident that the boundary limit of e^P is defined a.e. on the boundary Δ_1 with respect to χ. Furthermore, we can find exact values of boundary limits of e^P at minimal boundary points $b \in \Delta_1$.

Lemma 4.4. *Let f and g be real-valued continuous functions on R. For each point b in $\mathscr{F}(f) \cap \mathscr{F}(g)$ we have*

$$(\widehat{f \pm g})(b) = \hat{f}(b) \pm \hat{g}(b). \tag{18}$$

Proof. Let $\hat{f}(b) = \alpha$ and $\hat{g}(b) = \beta$. We assume that α and β are finite real numbers. For any positive number ε, we take open neighborhoods $U_\varepsilon(\alpha)$ and $U_\varepsilon(\beta)$ of α and β respectively:

$$U_\varepsilon(\alpha) = \{x \in R : |x - \alpha| < \varepsilon\},$$
$$U_\varepsilon(\beta) = \{x \in R : |x - \beta| < \varepsilon\}.$$

For $z \in f^{-1}(U_\varepsilon(\alpha)) \cap g^{-1}(U\varepsilon(\beta))$, we have

$$|\{f(z) \pm g(z)\} - (\alpha \pm \beta)| \leq 2\varepsilon.$$

From that the open subset $f^{-1}(U_\varepsilon(\alpha)) \cap g^{-1}(U\varepsilon(\beta))$ of R belongs to the class $\mathscr{G}(b)$ (Hilfssatz 14.1 in Constantinescu-Cornea [**3**]) it follows that $(f \pm g)^\wedge(b) = \{\alpha \pm \beta\}$. □

Theorem 4.5. *The boundary limit of* e^P *takes on the values* 1 *a.e. on* Δ_{HP}^0 *and* 0 *a.e. on* $\Delta_1 - \Delta_{HP}^0$ *with respect to* χ, *respectively.*

Proof. Let h be the positive harmonic function $T_{PH}e^P$. Then, $\chi(\Delta_1 - \mathscr{F}(h)) = 0$ and its boundary limit $\hat{h}$ takes on the value 1 or 0 according to $b \in \Delta_{HP}^0$ or $b \in \Delta_1 - \Delta_{HP}^0$ a.e. with respect to χ, because to Theorem 3.7, the integral representation

$$h(z) = \int_{\Delta_1} K_b(z)\hat{h}(b)\, d\chi(b)$$

and the uniqueness of the canonical measure of h. And, let p be the continuous potential

$$z \to \frac{1}{2\pi}\int_R P(w)G(z,w)e^P(w)\, du\, dv, \quad z \in R.$$

Then, $\chi(\Delta_1 - \mathscr{F}(p)) = 0$ and the boundary limit $\hat{p}$ takes on the value 0 a.e. on Δ_1 with respect to χ.

By the preceding lemma and the equality $h = e^P + p$, we have

$$\widehat{e^P} = \hat{h} - \hat{p}, \quad \text{for } b \in \mathscr{F}(h) \cap \mathscr{F}(p).$$

These complete the proof. □

Corollary 4.6. *Under the condition* (17) *in Corollary* 4.3 *the boundary limit of* e^P *takes on the value* 1 *a.e. on* Δ_1 *with respect to* χ.

Proof. This follows from Corollary 3.9. □

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