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Self homotopy groups of Hopf spaces with at most three cells

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Abstract. We prove that if X is a connected H-space with at most three cells of positive dimension, then the self homotopy set of X becomes a group relative to the binary operation induced from any multiplication on X, and we determine it's group structure in some cases.

1. Introduction.

Throughout the paper we work in the category of topological spaces of the based homotopy type of connected CW-complexes. The base point of any Hopf space is taken to be the unit. When (X, μ) is a Hopf space and A is any space, we denote by $[A, X; \mu]$ the based homotopy set $[A, X]$ with the binary operation induced from the multiplication μ. A result of James [**14**] asserts that the set $[A, X; \mu]$ forms an algebraic loop which is a group if μ is homotopy associative. O'Neill [**20**] proved that it is a group if the (normalized) Lusternik-Schnirelmann category of A, $\operatorname{cat} A$, is less than 3 (see 2.1 below). It is not in general a group. Indeed, $[X \times X \times X, X;\ \mu]$ is a group if and only if μ is homotopy associative. For example, neither $[S^7 \times S^7 \times S^7, S^7; \mu]$ nor $[S^7 \times S^7 \times S^7, S^7 \times S^7 \times S^7; \mu \times \mu \times \mu]$ is a group for every μ, since μ is not homotopy associative [**13**] (in this case $\operatorname{cat}(S^7 \times S^7 \times S^7) = 3$). Therefore the answer to the following Problem 1 is negative in general.

PROBLEM 1. *Is* $[X, X; \mu]$ *a group for every multiplication* μ?

PROBLEM 2. *If so, compute* $[X, X; \mu]$.

According to Arkowitz and Lupton [**4**, Corollary 4.4], the answer to Problem 1 is negative for exceptional simple Lie groups of rank 6 and almost all classical groups. The purpose of this paper is to give an affirmative answer to Problem 1 and partial answers to Problem 2 when X is a connected CW-complex with at most three cells. According to Browder [**7**], Hilton and Roitberg [**12**] and Zabrodsky [**25**], such a Hopf space is homotopy equivalent to one of the following fifteen complexes:

(1.1) $S^1, S^3, S^7, S^m \times S^n$ $(m, n \in \{1, 3, 7\},\ m \leq n)$, $\quad SO(3), SU(3), \quad E_n$ $(n = 1, 3, 4, 5)$,

where, for every integer m, E_m is the principal S^3-bundle over S^7 induced by $m\omega \in \pi_7(BS^3) = \boldsymbol{Z}/12\{\omega\}$ and $E_1 = Sp(2)$. Note that $SO(3) = P^3(\boldsymbol{R})$, the real projective space of dimension 3.

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If x, y are elements of an algebraic loop, then their commutator is the element $[x, y] = (xy)(yx)^{-1}$, where $(yx)^{-1}$ is the right inverse of yx. If (X, μ) is a Hopf space, then, by [**14**], it has a homotopy right inverse, say σ, and we write $xy = \mu(x, y)$, $x^{-1} = \sigma(x)$ and $[x, y] = (xy)(yx)^{-1}$. For each integer r, we define x^r, the r-th power of $x \in X$, to be $(\dots((xx)x\dots)x)$ (r-times power) if $r > 0$, the base point if $r = 0$, and $(x^{-1})^{-r}$ if $r < 0$, and we define a multiplication $\mu^{(r)}$ by

$$\mu^{(r)}(x, y) = (xy)[x, y]^r.$$

We denote by μ_0 the 'standard' multiplication if it exists. If P is a set of primes and D is a nilpotent CW-complex or a nilpotent group, then we denote by D_P the P-localization of D, and we write $n \in P$ if n is a product of primes in P. We denote by $\gcd\{k_1, \dots, k_l\}$ the greatest common divisor of integers $k_1, \dots, k_l$. For integers $m \geq 2$ and n, we denote by $\Psi(x, y, z; m, n)$ or simply by $\Psi(m, n)$ the group with generators x, y, z and relations

$$xz = zx, \quad yz = zy, \quad z^m = 1, \quad [x, y] = z^n.$$

Our first result gives an affirmative answer to Problem 1.

THEOREM 1. *Let X be one of the spaces of* (1.1) *and let A be E_n or one of the spaces of* (1.1). *If P is a set of prime numbers, then $[A, X_P; \mu]$ is a P-local group of nilpotency class ≤ 2 for every multiplication μ on X_P, and $[A, X_P; \mu'_P] \cong [A, X; \mu']_P$ for every multiplication μ' on X.*

The following four results give partial answers to Problem 2.

THEOREM 2. *Let $E'_0 = S^5 \times S^3$ and $E'_1 = SU(3)$. Then, for each integer r, m and $l = 0, 1$, we have*

$$(1) \qquad [E'_l, SU(3); \mu_0^{(r)}] \cong \Psi(12, 2r + 1),$$

$$(2) \qquad [E_m, S_p(2); \mu_0^{(r)}] \cong \Psi(120, 12(2r + 1)).$$

THEOREM 3. *There exists a multiplication μ_0 on E_5 such that $[E_m, E_5; \mu_0^{(r)}] \cong \Psi(120, 12(2r + 1))$ for every m and r.*

THEOREM 4. *Let $P_1 \cup P_2$ be a partition of the set of all prime numbers. If $n \in P_1$ and $12/\gcd\{n, 12\} \in P_2$, then E_n has a multiplication μ such that $[E_m, E_n; \mu^{(r)}] \cong \Psi(A_2 B_2, A_2(2r + 1)) \oplus (\boldsymbol{Z}/30 \oplus \boldsymbol{Z}/24)_{P_1}$ for every m and r, where A_2 and B_2 are the P_2-components of* 12 *and* 10, *respectively.*

COROLLARY 1. *There exist multiplications μ_0 on E_3 and E_4 such that*

$$(1) \qquad [E_m, E_3; \mu_0^{(r)}] \cong \Psi(40, 4(2r + 1)) \oplus \boldsymbol{Z}/3 \oplus \boldsymbol{Z}/3,$$

$$(2) \qquad [E_m, E_4; \mu_0^{(r)}] \cong \Psi(15, 3(2r + 1)) \oplus \boldsymbol{Z}/2 \oplus \boldsymbol{Z}/8$$

for every m and r.

As an application of our calculations, we have

Corollary 2. *Let* $(A,(X,\mu))$ *be one of the following*:

$$(E_m',(SU(3),\mu_0^{(r)})),\quad (E_m,(Sp(2),\mu_0^{(r)})),\quad (E_m,(E_3,\mu_0^{(r)})),\quad (E_m,(E_5,\mu_0^{(r)})).$$

Then none of the following functions is a homomorphism:

$$[\Sigma A,\Sigma X] \xleftarrow{\Sigma} [A,X;\mu] \xrightarrow{p_*} [A,S^n].$$

Here Σ *is the suspension*, n *is* 5 *or* 7 *according as* X *is* $SU(3)$ *or not*, $p : X \to S^n$ *is the bundle projection, and the abelian group structure on* $[A,S^n]$ *is given so that* $\Sigma : [A,S^n] \to [\Sigma A,\Sigma S^n]$ *is an isomorphism.*

In §2, we recall some general results for later use and give a result about a group of nilpotency class ≤ 2. In §3, we study the cases $S^m \times S^n$ for $m,n \in \{1,3,7\}$. In §4, we prove Theorem 1. In §5, we prove Theorems 2 and 3. In §6, we prove Theorem 4, Corollaries 1 and 2. In §7, we study the composition operation in $[X,X;\mu_0]$, when X is $SU(3)$ or $Sp(2)$, and prove

Corollary 3. ([**19**, Example 4.5]). *Let* $\mathscr{E}(X)$ *be the group of self homotopy equivalences of a based space* X. *Then we have*

$$\begin{aligned}\mathscr{E}(SU(3)) &= \{\alpha\beta^{-1}\gamma^{c_1},\alpha^{-1}\beta\gamma^{c_2},\beta\gamma^{c_3},\beta^{-1}\gamma^{c_4}; 0 \le c_i < 12\}\\ &= \langle x,y,z;\ x^2 = y^2 = z^{12} = 1,\ xy = yx,\ xz = zx,\ zyz = y\rangle,\\ \mathscr{E}(Sp(2)) &= \{\beta\gamma^{c_1},\beta^{-1}\gamma^{c_2}; 0 \le c_i < 120\}\\ &= \langle y,z;\ z^{120} = 1,\ y^2 = 1,\ zyz = y\rangle,\end{aligned}$$

where α, β *and* γ *are elements in Theorem 5.1 below*, x *is the complex conjugation*, $y = \beta^{-1}$, $z = \beta\gamma$, *and* $\langle x_1,\dots,x_k; r_1,\dots,r_l\rangle$ *denotes the group with generators* $x_1,\dots,x_k$ *satisfying relations* $r_1,\dots,r_l$.

In the final section, §8, we give an invariant of Hopf spaces.

We do not distinguish notationally between a map and its homotopy class. To indicate the multiplication μ considered, we denote respectively the commutator and the Samelson product by $[-,-]_\mu$ and $\langle -,-\rangle_\mu$ which are defined from the commutator map.

We thank K. Morisugi who simplified our original proof of Theorem 2.

2. General results.

Let $\operatorname{cat} A$ be the (normalized) Lusternik-Schnirelmann category of a space A such that $\operatorname{cat} A = 0$ if A is contractible. Results of O'Neill [**20**] and Whitehead [**24**, p. 464] imply the following:

Theorem 2.1. *If* (X,μ) *is a Hopf space and* A *is a space with* $\operatorname{cat} A < 3$, *then* $[A,X;\mu]$ *is a group of nilpotency class* $\le \operatorname{cat} A$.

Theorem 2.2. ([**24**, p. 465]). *Let* (X,μ) *be a Hopf space and let*

$$\{*\} = P_0 \subset P_1 \subset \cdots \subset P_c = A$$

be a sequence of subcomplexes of a CW-*complex* A *such that the boundary of each cell of* P_i *is contained in* P_{i-1} $(i = 1, \dots, c)$. *Let* Γ_i *be the set of all homotopy classes of maps* $f : A \to X$ *such that* $f|P_i$ *is null homotopic. Then* $[\Gamma_0, \Gamma_i] \subset \Gamma_{i+1}$ *for* $0 \le i \le c-1$, *where* $\Gamma_0 = [A, X; \mu]$.

Although the hypothesis in 2.1 and 2.2 are weaker than in [**24**], the proof is same. The following result is due to James and Whitehead.

THEOREM 2.3. ([**16**], [**15**]). *Let* X *be the total space of an* S^m*-bundle over* S^n *with* $n \ge 2$. *Then* X *has a cell structure* $S^m \cup_\alpha e^n \cup_\rho e^{m+n}$ *such that*

$$\Sigma\rho = \Sigma i \circ J(\chi), \tag{2.4}$$

where $i : S^m \to X$ *is the inclusion*, J *is the Hopf-Whitehead* J *homomorphism*, $\chi \in \pi_{n-1}(O(m+1))$ *is the characteristic element of the bundle*, *and* α *is the image of* χ *under the obvious homomorphism* $\pi_{n-1}(O(m+1)) \to \pi_{n-1}(S^m)$.

Let (G, μ) be a group. For simplicity, we write $xy = \mu(x, y)$ as usual. We define $\mu^{(r)}(x, y) = xy[x, y]^r$ for each integer r.

LEMMA 2.5. *If* (G, μ) *is a group of nilpotency class* ≤ 2, *then* $(G, \mu^{(r)})$ *is a group of nilpotency class* $\le \mathrm{nil}(G, \mu)$ *and* $[x, y]_r = [x, y]^{2r+1}$, *where* $[x, y]_r$ *is the commutator with respect to* $\mu^{(r)}$.

PROOF. We write $x \cdot y = \mu^{(r)}(x, y)$. Recall that in a group of nilpotency class ≤ 2 the following formulas hold:

$$[x, yz] = [x, y][x, z], \quad [xy, z] = [y, z][x, z], \quad x[y, z] = [y, z]x.$$

We then easily have

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \quad x \cdot x^{-1} = x^{-1} \cdot x = 1, \quad [x, y]_r = [x, y]^{2r+1}.$$

Hence $(G, \mu^{(r)})$ is a group and $[x, [y, z]_r]_r = [x, [y, z]^{2r+1}]^{2r+1} = [x, [y, z]]^{(2r+1)^2} = 1$. Therefore $\mathrm{nil}(G, \mu^{(r)}) \le \mathrm{nil}(G, \mu)$. □

3. Products of spheres.

Let $m, n \in \{1, 3, 7\}$. Since $\mathrm{cat}(S^m \times S^n) = 2$, Problem 1 is affirmative for $S^m \times S^n$ by 2.1. Let μ, μ' be multiplications on S^m, S^n, respectively. The product multiplication $\mu \times \mu'$ on $S^m \times S^n$ is the composition of

$$S^m \times S^n \times S^m \times S^n \xrightarrow{1 \times T \times 1} S^m \times S^m \times S^n \times S^n \xrightarrow{\mu \times \mu'} S^m \times S^n,$$

where T is the switching map. Then we have the splitting

$$[S^m \times S^n, S^m \times S^n; \mu \times \mu'] \cong [S^m \times S^n, S^m; \mu] \oplus [S^m \times S^n, S^n; \mu'].$$

Let $\mu_0 : S^n \times S^n \to S^n$ be the complex multiplication for $n = 1$, quaternionic multiplication for $n = 3$ and Cayley multiplication for $n = 7$. As usual, we write $xy = \mu_0(x, y)$ and $\mu_r = \mu_0^{(r)}$. Then $\{\mu_r; 0 \le r < l\}$ is the set of all the multiplications on S^n up to homotopy, where l is 1, 12 or 120 according as n is 1, 3, or 7. We abbreviate

$[-,-]_{\mu_r}$ to $[-,-]_r$ and $\langle -,-\rangle_{\mu_r}$ to $\langle -,-\rangle_r$. Then $\pi_6(S^3) = \boldsymbol{Z}/12\{\langle \iota_3, \iota_3\rangle_0\}$, $\pi_{14}(S^7) = \boldsymbol{Z}/120\{\langle \iota_7, \iota_7\rangle_0\}$ and $\langle \iota_n, \iota_n\rangle_r = (2r+1)\langle \iota_n, \iota_n\rangle_0$ by [**3**] (cf. [**14**]). Let $q: S^m \times S^n \to S^m \wedge S^n$ be the quotient map, and $p_1: S^m \times S^n \to S^m$, $p_2: S^m \times S^n \to S^n$ the projections.

PROPOSITION 3.1. *We have*

$$(1)\qquad [S^1 \times S^n, S^1; \mu_0] = \begin{cases} \boldsymbol{Z}\{p_1\} \oplus \boldsymbol{Z}\{p_2\} & n = 1 \\ \mathbf{Z}\{p_1\} & n \geq 2, \end{cases}$$

$$(2)\qquad [S^1 \times S^n, S^n; \mu_r] = \mathbf{Z}\{p_2\} \oplus \mathbf{Z}/2\{q^*\eta_n\} \ (n \in \{3,7\}),$$

$$(3)\qquad [S^3 \times S^3, S^3; \mu_r] = \Psi(p_1, p_2, \gamma; 12, 2r+1),$$

$$(4)\qquad [S^7 \times S^7, S^7; \mu_r] = \Psi(p_1, p_2, \gamma; 120, 2r+1),$$

$$(5)\qquad [S^7 \times S^3, S^3; \mu_r] = \mathbf{Z}\{p_2\} \oplus \mathbf{Z}/2\{\nu' \circ \eta_6 \circ p_1\} \oplus \mathbf{Z}/15\{q^*\gamma''\},$$

$$(6)\qquad [S^7 \times S^3, S^7; \mu_r] = \mathbf{Z}\{p_1\} \oplus \mathbf{Z}/24\{q^*\gamma'''\},$$

where $\gamma = q^*\gamma'$ *with* γ' *a generator of* $\pi_{2n}(S^n)$, γ'' *is a generator of* $\pi_{10}(S^3) = \mathbf{Z}/15$, $\nu' \in \pi_6(S^3)$, $\eta_n \in \pi_{n+1}(S^n)$ *are elements in* [**23**], *and* γ''' *is a generator of* $\pi_{10}(S^7) = \mathbf{Z}/24$.

PROOF. Let $k \in \{m, n\}$. By 2.1 and 2.2, we have a central extension of groups:

$$0 \to \pi_{m+n}(S^k) \xrightarrow{q^*} [S^m \times S^n, S^k; \mu_r] \to [S^m \vee S^n, S^k; \mu_r] \to 0.$$

Hence we have (1) and (2). We also have (3) and (4), since $[S^n \times S^n, S^n; \mu_r]$ is generated by p_1, p_2, $q^*\langle \iota_n, \iota_n\rangle_0$ and since $[p_1, p_2]_r = q^*\langle \iota_n, \iota_n\rangle_r$. We have (5), since the group $[S^7 \times S^3, S^3; \mu_r]$ is generated by p_2, $\nu' \circ \eta_6 \circ p_1$, $q^*\gamma''$ and since $\langle \iota_3, \nu' \circ \eta_6\rangle_r = 0$ and hence $[p_2, \nu' \circ \eta_6 \circ p_1]_r = q^*\langle \iota_3, \nu' \circ \eta_6\rangle_r = 0$. We have (6), since $[S^7 \times S^3, S^7; \mu_r]$ is generated by p_1 and $q^*\gamma'''$. □

4. Proof of Theorem 1.

For $n \in \{1, 3, 7\}$ the real projective space $P^n(\boldsymbol{R})$ has Hopf structures and satisfies the following which contains a part of Theorem 1 and is maybe well-known.

PROPOSITION 4.1. *If* P *is a set of prime numbers and* $n \in \{1, 3, 7\}$, *then*

$$\mathrm{Ind}: [P^n(\boldsymbol{R}), P^n(\boldsymbol{R})_P; \mu] \to \mathrm{Hom}(H_n(P^n(\boldsymbol{R})), H_n(P^n(\boldsymbol{R})_P)),$$

defined by $\mathrm{Ind}(f) = f_*$ *is an isomorphism for every multiplication* μ *on* $P^n(\boldsymbol{R})_P$.

PROOF. The case $n = 1$ is trivial. Let $n \in \{3, 7\}$, P any set of prime numbers, and μ any multiplication on $P^n(\boldsymbol{R})_P$. First, consider the case $2 \notin P$. We easily have $[P^{n-1}(\boldsymbol{R}), P^n(\boldsymbol{R})_P] = 0$ and $[\Sigma P^{n-2}(\boldsymbol{R}), P^n(\boldsymbol{R})_P] = 0$. Let $\gamma: S^{n-2} \to P^{n-2}(\boldsymbol{R})$ be the canonical covering map and $q': P^{n-2}(\boldsymbol{R}) \to S^{n-2}$ the quotient map. Then $q' \circ \gamma = 2\iota_{n-2}$ and so $(\Sigma^2\gamma)^* \circ (\Sigma^2 q')^*: \pi_n(P^n(\boldsymbol{R})_P) \cong \pi_n(P^n(\boldsymbol{R})_P)$. Hence $(\Sigma^2\gamma)^*: [\Sigma^2 P^{n-2}(\boldsymbol{R}), P^n(\boldsymbol{R})_P] \to \pi_n(P^n(\boldsymbol{R})_P)$ is surjective so that $[\Sigma P^{n-1}(\boldsymbol{R}), P^n(\boldsymbol{R})_P] = 0$. Thus $q^*: \pi_n(P^n(\boldsymbol{R})_P) \cong [P^n(\boldsymbol{R}), P^n(\boldsymbol{R})_P; \mu]$ and, by using the Hurewicz homomorphism, we see that Ind is an isomorphism.

Second, suppose $2 \in P$. Let $c : P^n(\boldsymbol{R}) \to P^n(\boldsymbol{R}) \vee S^n$ be a cooperation [9, p. 99]. For $f \in [P^n(\boldsymbol{R}), P^n(\boldsymbol{R})_P]$ and $\xi \in \pi_n(P^n(\boldsymbol{R})_P)$, we denote by f^{ξ} the composition of the following:

$$P^n(\boldsymbol{R}) \xrightarrow{c} P^n(\boldsymbol{R}) \vee S^n \xrightarrow{f \vee \xi} P^n(\boldsymbol{R})_P \vee P^n(\boldsymbol{R})_P \xrightarrow{\nabla} P^n(\boldsymbol{R})_P,$$

where ∇ is the folding map. Let $e : P^n(\boldsymbol{R}) \to P^n(\boldsymbol{R})_P$ be the P-localizing map. Since $[P^{n-1}(\boldsymbol{R}), P^n(\boldsymbol{R})_P; \mu] = \boldsymbol{Z}/2\{j\}$, where $j : P^{n-1}(\boldsymbol{R}) \subset P^n(\boldsymbol{R}) \xrightarrow{e} P^n(\boldsymbol{R})_P$, it follows from Puppe's theorem [9, p. 175] that $[P^n(\boldsymbol{R}), P^n(\boldsymbol{R})_P] = \{0^{\xi}, e^{\xi}; \xi \in \pi_n(P^n(\boldsymbol{R})_P)\}$, where 0 is the constant map to the base point. We then easily have that Ind is a bijective homomorphism of loops and hence of abelian groups. □

Consider the following pull-back diagram:

$$(4.2)\qquad \begin{array}{ccc} E_n & \xrightarrow{f_n} & Sp(2) \\ {\scriptstyle p_n}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ S^7 & \xrightarrow{n\iota_7} & S^7. \end{array}$$

Let $Q_n = S^3 \cup_{n\omega} e^7$, where ω is the generator of $\pi_6(S^3)$ identified with $\pi_7(BS^3) \cong \boldsymbol{Z}/12$. Then $E_n = Q_n \cup_{p_n} e^{10}$ by 2.3. Recall from [11] that $E_m \simeq E_n$ if and only if $m \equiv \pm n$ (mod 12) and from [25] that E_n admits a Hopf structure if and only if $n \not\equiv 2$ (mod 4). We denote by i_n the inclusions of S^3 into Q_n and E_n, by j_n the inclusion $Q_n \to E_n$, and by q_n the quotient maps $Q_n \to S^7$ and $E_n \to S^{10}$. Observe that $q_n = p_n \circ j_n$. Let χ_n be the characteristic element of E_n. Then $\chi_n = n\chi_1$.

LEMMA 4.3. (1) *The space Q_n is a co-Hopf space if and only if n is even. Hence* $(\operatorname{cat} Q_n, \operatorname{cat} E_n)$ *is* (1, 2) *or* (2, 3) *according as n is even or odd.*

(2) *We have* $\operatorname{cat} P^3(\boldsymbol{R}) = 3$ *and* $\operatorname{cat} X \leq 2$ *if X is one of the spaces of* (1.1) *except* $P^3(\boldsymbol{R})$ *and* E_n.

PROOF. (2) It is well-known that $\operatorname{cat} X$ is 1, 2 or 3 according as X is S^m, $S^m \times S^n$ or $P^3(\boldsymbol{R})$. Also, as is well-known, $SU(3) = Q \cup e^8$, where $Q = \Sigma(S^2 \cup e^4)$ is the suspension of the complex projective plane. It follows that $\operatorname{cat} Q = 1$ and $\operatorname{cat} SU(3) = 2$.

(1) By the method of 4.4 in [21], if n is odd, then $\operatorname{cat} Q_n = 2$ and $\operatorname{cat} E_n = 3$. Recall that a space X is a co-Hopf space if and only if $\operatorname{cat} X \leq 1$, so it follows that Q_n is not a co-Hopf space for n odd. Let $\theta : S^3 \to S^3 \vee S^3$ be the co-multiplication. Under the identification $\pi_m(S^3 \vee S^3) = \pi_m(S^3) \oplus \pi_m(S^3) \oplus \pi_{m+1}(S^3 \times S^3, S^3 \vee S^3)$, we have $\theta \circ g = g \oplus g \oplus \mathscr{H}(g)$ for every $g \in \pi_m(S^3)$, where $\mathscr{H}$ is the Hopf θ-invariant of [5]. It then follows from Theorem 3.20 of [5] that $\mathscr{H}(\omega)$ is the generator of $\pi_7(S^3 \times S^3, S^3 \vee S^3) \cong \mathbf{Z}/2$ for the generator ω of $\pi_6(S^3)$, and that Q_n is a co-Hopf space if n is even. One can construct a co-multiplication of Q_n for n even by using Theorem 15.4 of [9], although details are omitted. Since the cup length of $H^*(E_n)$ is 2, we have $\operatorname{cat} E_n \geq 2$. Since $\operatorname{cat} E_n \leq 1 + \operatorname{cat} Q_n$, we then have $\operatorname{cat} E_n = 2$ for n even. □

Notation: $c(n) = 12/\gcd\{n, 12\}$, $c(m,n) = \gcd\{n, 12\}/\gcd\{m, n, 12\}$ and $c(m, n; P)$ is the P-component of $c(m,n)$ for P a set of primes.

Recall from [**23**] that $\pi_{10}(S^7) = \mathbf{Z}/8 \oplus \mathbf{Z}/3\{\alpha_1(7)\}$, $\pi_9(S^3) = \mathbf{Z}/3\{\alpha_1(3) \circ \alpha_1(6)\}$, $\pi_9(S^7) = \mathbf{Z}/2\{\eta_7^2\}$, and $\pi_8(S^3) = \mathbf{Z}/2\{\nu' \circ \eta_6^2\}$.

Lemma 4.4. (1) *We denote by* $[\xi] \in \pi_l(E_n)$ *an element such that* $p_{n_*}[\xi] = \xi \in \pi_l(S^7)$. *We abbreviate* $[k\iota_7]$ *to* $[k]$. *Then*

(1-1) $$\pi_3(E_n) = \mathbf{Z}\{i_n\} = \pi_3(Q_n), \quad f_{n_*}(i_n) = i_1;$$

(1-2) $$\pi_4(E_n) = \mathbf{Z}/2\{i_{n_*}\eta_3\}, \quad \pi_6(E_n) = \boldsymbol{Z}/\gcd\{n, 12\}\{i_{n_*}\omega\};$$

(1-3) $$\pi_7(E_n) = \mathbf{Z}\{[c(n)]\} \oplus \begin{cases} \mathbf{Z}/2\{i_n \circ \nu' \circ \eta_6\} & n \equiv 0 \ (\mathrm{mod}\ 2) \\ 0 & n \equiv 1 \ (\mathrm{mod}\ 2), \end{cases}$$

where $f_{n_*}[c(n)] = (n/\gcd\{n, 12\})[12]$;

(1-4) $$\pi_9(E_n) = \begin{cases} 0 & n \equiv 1, 5 \ (\mathrm{mod}\ 6) \\ \boldsymbol{Z}/2\{[\eta_7^2]\} & n \equiv 2, 4 \ (\mathrm{mod}\ 6) \\ \boldsymbol{Z}/3\{i_*\alpha_1(3) \circ \alpha_1(6)\} & n \equiv 3 \ (\mathrm{mod}\ 6) \\ \boldsymbol{Z}/3\{i_*\alpha_1(3) \circ \alpha_1(6)\} \oplus \boldsymbol{Z}/2\{[\eta_7^2]\} & n \equiv 0 \ (\mathrm{mod}\ 6); \end{cases}$$

(1-5) $$\pi_{10}(E_n) = \boldsymbol{Z}/15\{i_{n_*}x\} \oplus \boldsymbol{Z}/8\{[\nu_7]\} \oplus \begin{cases} \boldsymbol{Z}/3\{[\alpha_1(7)]\} & n \equiv 0 \ (\mathrm{mod}\ 3) \\ 0 & n \not\equiv 0 \ (\mathrm{mod}\ 3), \end{cases}$$

where $\pi_{10}(S^3) = \boldsymbol{Z}/15\{x\}$;

(1-6) $$\langle [12], i_1 \rangle_{\mu_0} = 12\gamma', \quad \textit{where } \pi_{10}(Sp(2)) = \boldsymbol{Z}/120\{\gamma'\}.$$

(2) *The following is exact:*

$$0 \longrightarrow \pi_7(E_{nP}) \xrightarrow{q_m^*} [Q_m, E_{nP}] \xrightarrow{i_m^*} c(m, n; P)\pi_3(E_{nP}) \longrightarrow 0.$$

Proof. (1) By [**17**], we have (1-1), (1-2), (1-3) and (1-5) when $n = 1$. We then easily have (1-1), (1-2) and (1-3) by (4.2) and the homotopy exact sequences of the bundles. When $n \not\equiv 0$ (mod 3), (1-5) follows from the equation $\Delta_1(\alpha_1(7)) = \alpha_1(3) \circ \alpha_1(6)$ and the following commutative diagram:

$$\begin{array}{ccccccccc} 0 = \pi_{11}(S^7) & \longrightarrow & \pi_{10}(S^3) & \xrightarrow{i_*} & \pi_{10}(E_n) & \longrightarrow & \pi_{10}(S^7) & \xrightarrow{\Delta_n} & \pi_9(S^3) \\ \downarrow & & \| & & \downarrow{\scriptstyle fn_*} & & \downarrow{\scriptstyle n} & & \| \\ 0 = \pi_{11}(S^7) & \longrightarrow & \pi_{10}(S^3) & \xrightarrow{i_*} & \pi_{10}(S_p(2)) & \longrightarrow & \pi_{10}(S^7) & \xrightarrow{\Delta_1} & \pi_9(S^3). \end{array}$$

Let $n \equiv 0$ (mod 3). Then $\Delta_n(\alpha_1(7)) = 0$ by the diagram. If there exists an element $y \in \pi_{10}(E_n)$ with $3y = i_*\alpha_2(3)$, then $i_*\alpha_2(3) = f_{n_*}i_*\alpha_2(3) = 3f_{n_*}(y) = 0$. This is a contradiction. Hence (1-5) is proved. Consider the exact sequence:

$$\pi_{10}(S^7) \xrightarrow{\Delta} \pi_9(S^3) \longrightarrow \pi_9(E_n) \longrightarrow \pi_9(S^7) \xrightarrow{\Delta} \pi_8(S^3).$$

We have $\Delta\alpha_1(7) = nn'\alpha_1(3) \circ \alpha_1(6)$ and $\Delta\eta_7^2 = n(\nu' \circ \eta_6^2)$, where $\omega \equiv n'\alpha_1(3) \pmod{\nu'}$ with $n' \not\equiv 0 \pmod 3$. Hence (1-4) follows. By [**6**], we have (1-6).

(2) Consider the exact sequence of based sets:

$$\pi_4(E_n) \xrightarrow{(\Sigma m\omega)^*} \pi_7(E_n) \longrightarrow [Q_m, E_n] \xrightarrow{i_m^*} \pi_3(E_n) \xrightarrow{(m\omega)^*} \pi_6(E_n).$$

Since $\omega = x\nu' + y\alpha_1(3)$ with $x \equiv 1 \pmod 2$ and $\eta_3 \circ \Sigma\nu' = 0$ by [**23**], we have $(\Sigma m\omega)^* = 0$ by (1-2). We have $\mathrm{Im}(i_m^*) = \mathrm{Ker}(m\omega)^* = c(m,n)\pi_3(E_n)$ by (1-2). □

The proof of Theorem 1 is divided into four steps:

(Step 1) $[A, X_P; \mu]$ is a nilpotent group.

(Step 2) $[A, X_P; \mu]$ is P-local.

(Step 3) $[A, X_P; \mu'_P] \cong [A, X; \mu']_P$.

(Step 4) $\mathrm{nil}[A, X_P; \mu] \le 2$.

Proof of Theorem 1–Part I. We prove these steps here except Step 4, whose proof is postponed until the end of this section.

(Step 1) If A is S^m with $m \in \{1,3,7\}$, $S^m \times S^n$ with $m,n \in \{1,3,7\}$, $SU(3)$, or E_m with m even, then $\mathrm{cat}\, A \le 2$ so that $[A, X_P; \mu]$ is a group of nilpotency class ≤ 2 by 2.1.

Let A be $P^3(\boldsymbol{R})$ or E_m with m odd. It suffices to prove that $[A, X_P; \mu]$ is associative. Let $f_i \in [A, X_P]$ for $i = 1,2,3$ and consider the following commutative diagram:

$$\begin{array}{ccccc} & & & & T \\ & & & & \cap\big\downarrow i' \\ A & \xrightarrow{\ d\ } & A \times A \times A & \xrightarrow{f_1 \times f_2 \times f_3} & X_P \times X_P \times X_P \\ & & q\big\downarrow & & \big\downarrow q \\ & & A \wedge A \wedge A & \xrightarrow{f_1 \wedge f_2 \wedge f_3} & X_P \wedge X_P \wedge X_P, \end{array}$$

where $T = X_P \times X_P \times \{*\} \cup X_P \times \{*\} \times X_P \cup \{*\} \times X_P \times X_P$, d is the diagonal map, and q is the quotient map. Let μ be any multiplication on X_P. To simplify the notation, we denote the binary operation in $[-, X_P; \mu]$ by '+'. Since $\mu \circ (1 \times \mu) \circ i' = \mu \circ (\mu \times 1) \circ i'$, there exists a map $\sigma : X_P \wedge X_P \wedge X_P \to X_P$ such that $\mu \circ (1 \times \mu) = \mu \circ (\mu \times 1) + \sigma \circ q$. Hence we have

$$f_1 + (f_2 + f_3) = \{(f_1 + f_2) + f_3\} + \sigma \circ (f_1 \wedge f_2 \wedge f_3) \circ q \circ d.$$

Let (a,b) be (10, 3) if $A = E_m$ and (3, 1) if $A = P^3(\boldsymbol{R})$. By a cell structure of A, the map $q \circ d$ factors into

$$A \xrightarrow{q'} S^a \xrightarrow{g} S^{3b} = S^b \wedge S^b \wedge S^b \xrightarrow{i \wedge i \wedge i} A \wedge A \wedge A$$

for some g, where q' is the quotient and i is the inclusion. Hence

$$\sigma \circ (f_1 \wedge f_2 \wedge f_3) \circ q \circ d = \sigma \circ (f_1 \wedge f_2 \wedge f_3) \circ (i \wedge i \wedge i) \circ g \circ q'.$$

We prove the assertion by showing

$$\sigma \circ (f_1 \wedge f_2 \wedge f_3) \circ (i \wedge i \wedge i) \circ g = 0. \tag{4.5}$$

Let $A = E_m$ with m odd. Then $2g = 0$. If X is S^1 or $S^1 \times S^1$, then (4.5) is obviously satisfied. If X is S^3, $S^1 \times S^3$, $S^3 \times S^3$, $SU(3)$, $P^3(\boldsymbol{R})$ or E_3, then $3\pi_9(X_P) = 0$ by [**23**] and [**17**] and (1-4) of 4.4 so that $\sigma \circ (f_1 \wedge f_2 \wedge f_3) \circ (i_m \wedge i_m \wedge i_m) \circ g = 0$. If X is S^7, $S^1 \times S^7$ or $S^7 \times S^7$, then $f_k \circ i_m \in \pi_3(X_P) = 0$ so that $(f_1 \wedge f_2 \wedge f_3) \circ (i_m \wedge i_m \wedge i_m) = 0$. If X is E_1 or E_5, then $\pi_9(X_P) = 0$ by (1-4) of 4.4 so that $\sigma \circ (f_1 \wedge f_2 \wedge f_3) \circ (i_m \wedge i_m \wedge i_m) = 0$. If $X = E_4$ and $2 \notin P$, then $\pi_9(X_P) = 0$ by (1-4) of 4.4 so that $\sigma \circ (f_1 \wedge f_2 \wedge f_3) \circ (i_m \wedge i_m \wedge i_m) = 0$. If X is E_4 or E_0 and $2 \in P$, then $c(m, 4; P) \equiv 0 \pmod 4$ and hence $f_k \circ i_m = 4a_k i_n$ for some $a_k \in \boldsymbol{Z}_P$ by (2) of 4.4 so that $(f_1 \wedge f_2 \wedge f_3) \circ (i_m \wedge i_m \wedge i_m) \circ g = 0$. Let $X = E_0 = S^3 \times S^7$. If $2, 3 \notin P$, then $\pi_9(X_P) = 0$ by (1-4) of 4.4 so that $\sigma \circ (f_1 \wedge f_2 \wedge f_3) \circ (i_m \wedge i_m \wedge i_m) = 0$. If $2 \notin P$ and $3 \in P$, then $3\pi_9(X_P) = 0$ by (1-4) of 4.4 so that $\sigma \circ (f_1 \wedge f_2 \wedge f_3) \circ (i_m \wedge i_m \wedge i_m) \circ g = 0$.

Let $A = P^3(\boldsymbol{R})$. If $\pi_1(X_P) = 0$, then $f_k \circ i = 0$ so that (4.5) is satisfied. The case $X = P^3(\boldsymbol{R})$ was checked in Proposition 4.1. If X is S^1, $S^1 \times S^1$ or $S^1 \times S^7$, then $[P^3(\boldsymbol{R}), X_P] = 0$. If X is $S^1 \times S^3$, then $[P^2(\boldsymbol{R}), X_P] = 0$, whence $q^* : \pi_3(X_P) \to [P^3(\boldsymbol{R}), X_P; \mu]$ is surjective so that $[P^3(\boldsymbol{R}), X_P; \mu]$ is an abelian group.

(Step 2) We give the proof only for $A = E_m$, because other cases are similar. Consider the following exact sequence of groups:

$$\pi_7(X_P) \xrightarrow{q_m^*} [Q_m, X_P; \mu] \xrightarrow{i_m^*} \pi_3(X_P) \xrightarrow{(m\omega)^*} \pi_6(X_P).$$

The subgroup $\operatorname{Im} q_m^*$ is central by 2.2. Obviously this group and $\operatorname{Ker}(m\omega)^*$ are P-local. Hence $[Q_m, X_P; \mu]$ is P-local by 1.2 on page 4 in [**10**]. By applying this method to the following exact sequence of groups, we see that $[E_m, X_P; \mu]$ is P-local:

$$\pi_{10}(X_P) \xrightarrow{q_m^*} [E_m, X_P; \mu] \xrightarrow{j_m^*} [Q_m, X_P; \mu] \xrightarrow{\rho_m^*} \pi_9(X_P).$$

(Step 3) Let $e : X \to X_P$ be the P-localizing map. Then $e_* : [A, X; \mu'] \to [A, X_P; \mu'_P]$ is a P-isomorphism by Theorem 6.2 on page 90 in [**10**] and hence $[A, X_P; \mu'_P] \cong [A, X; \mu']_P$. □

Remark 4.6. *The above proof shows that if A is a* CW-*complex, then, for every multiplication μ, $[A, X; \mu]$ is a group provided* (i) $X = E_k$ $(k = 0, 1, 3, 4, 5)$ *and* $\dim A \le 12$ *or* (ii) $X = SU(3)$ *and* $\dim A \le 10$.

Let A be a principal S^3-bundle over S^m with $m = 5$ or 7. Its cell structure is $S^3 \cup e^m \cup e^{m+3}$ by 2.3. Let $i : S^3 \to A$ be the inclusion, $p : A \to S^m$ the projection, and $q : A \to S^{m+3}$ the quotient map. Let (X, μ) be a Hopf space.

Lemma 4.7. *For every $g \in \pi_m(X)$ and every map $h : A \to X$, the following diagram commutes:*

$$\begin{array}{ccccc} A & \xrightarrow{q} & S^m \wedge S^3 & \xrightarrow{\langle g, h \circ i \rangle_\mu} & X \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \wedge i} & & \uparrow{\scriptstyle C_\mu} \\ A \wedge A & \xrightarrow{(\varepsilon\iota_m \circ p) \wedge \mathrm{id}} & S^m \wedge A & \xrightarrow{g \wedge h} & X \wedge X, \end{array}$$

where ε is 1 or -1 and C_μ is the commutator map with respect to μ. Hence

$$[g \circ p, h]_\mu = \varepsilon \langle g, h \circ i \rangle_\mu \circ q.$$

Proof. The first square commutes for some ε with $|\varepsilon| = 1$, since $(\mathrm{id} \wedge i) \circ q$ and $(p \wedge \mathrm{id}) \circ d$ induce the same homomorphism up to sign on the integral cohomology. By definitions so does the second one. □

Lemma 4.8. (1) *For every multiplication μ on S^3 and S^7, we have*

(1-1) $$[Q_n, S^3; \mu] = \boldsymbol{Z}\{\langle c(n) \rangle\} \oplus \boldsymbol{Z}/2\{\nu' \circ \eta_6 \circ p \circ j\}, \quad i^* \langle c(n) \rangle = c(n)\iota_3;$$

(1-2) $$[E_n, S^3; \mu] = \boldsymbol{Z}\{\langle c(n) \rangle\} \oplus \boldsymbol{Z}/30, \quad i^* \langle c(n) \rangle = c(n)\iota_3;$$

(1-3) $$[Q_n, S^7; \mu] = \boldsymbol{Z}\{q_n\}, \quad [E_n, S^7; \mu] = \boldsymbol{Z}\{p_n\} \oplus \boldsymbol{Z}/24.$$

(2) *If P is a set of primes, then every multiplication μ on S^3_P and S^7_P is integral, that is, $\mu = \mu'_P$ for some multiplication μ' on S^3 and S^7.*

Proof. (1) We omit the proof of (1-3), since it is easy. By applying the functor $[-, S^3; \mu]$ to the cofibering $S^6 \xrightarrow{n\omega} S^3 \to Q_n$, (1-1) follows from 2.1, 2.2 and the equality $\eta_3 \circ \Sigma n\omega = 0$ in [**23**].

We show that the following is exact:

(4.9) $$0 \longrightarrow \pi_{10}(S^3) \xrightarrow{q^*} [E_n, S^3] \xrightarrow{j^*} [Q_n, S^3] \longrightarrow 0.$$

By 2.4, $(\Sigma\rho_n)^* : [\Sigma Q_n, S^3] \to \pi_{10}(S^3)$ is trivial, since $\pi_4(S^3) = \boldsymbol{Z}/2$ and $\pi_{10}(S^3) = \boldsymbol{Z}/15$. To prove the triviality of $\rho_n^* : [Q_n, S^3] \to \pi_9(S^3)$, we consider the following commutative square:

$$\begin{array}{ccc} [Q_n, S^3] & \xrightarrow{\rho_n^*} & \pi_9(S^3) \\ {\scriptstyle\Sigma}\downarrow & & \downarrow{\scriptstyle\Sigma} \\ [\Sigma Q_n, S^4] & \xrightarrow{(\Sigma\rho_n)^*} & \pi_{10}(S^4). \end{array}$$

We have easily

(4.10) $$\mathrm{Im}\{\Sigma i^* : [\Sigma Q_n, S^4] \to \pi_4(S^4)\} = c(n)\pi_4(S^4).$$

By 6.1 of [**18**], we have $J(\chi_1) = 2\nu_4^2 + \delta$ such that the order of δ is 3. Hence $J(\chi_n) = nJ(\chi_1) = 2n\nu_4^2 + n\delta$. By (8.10) on page 537 in [**24**], $k\iota_4 \circ \nu_4 = k^2\nu_4 + \Sigma\alpha$ for some α, whence $k\iota_4 \circ \nu_4^2 = k^2\nu_4^2$, and

$$\begin{aligned} k\iota_4 \circ \delta &= k\delta + \binom{k}{2}([\iota_4, \iota_4] \circ h_0(\delta)) \\ &= k\delta + k(k-1)(\nu_4 \circ h_0(\delta)) + \binom{k}{2}(\Sigma\alpha' \circ h_0(\delta)) \end{aligned}$$

for some α', where h_0 is the 0-th Hopf-Hilton homomorphism. Given any integer a, let $\tilde{a} = ac(n)$. We have

$$\begin{aligned} J(\chi_n)^*(\tilde{a}\iota_4) &= 2n(\tilde{a}\iota_4 \circ \nu_4^2) + n(\tilde{a}\iota_4 \circ \delta) \\ &= 2n\tilde{a}^2\nu_4^2 + n\tilde{a}\delta + \tilde{a}(\tilde{a}-1)(\nu_4 \circ h_0(\delta)) + n\binom{\tilde{a}}{2}(\Sigma\alpha' \circ h_0(\delta)) \\ &= 0, \end{aligned}$$

since $2nc(n) \equiv 0 \pmod 8$, $nc(n) \equiv 0 \pmod 3$ and the orders of $\nu_4 \circ h_0(\delta)$ and $\Sigma\alpha' \circ h_0(\delta)$ are 1 or 3. Thus $(\Sigma\rho_n)^*$ in the square is trivial by (2.4) and (4.10), whence so is ρ_n^*, since the suspension homomorphism Σ on the right hand side is injective by the EHP-sequence. Therefore (4.9) is exact.

As was proved, $[E_n, S^3; \mu]$ is a group. Hence this is generated by $\langle c(n)\rangle, q^*\gamma''$ and $\nu' \circ \eta_6 \circ p$, where γ'' is a generator of $\pi_{10}(S^3)$. By 2.2, the element $q^*\gamma''$ is in the center. We have

$$[\nu' \circ \eta_6 \circ p, \langle c(n)\rangle]_\mu = \langle \nu' \circ \eta_6, c(n) \circ \iota_3\rangle_\mu \circ q = 0,$$

since $2\langle \nu' \circ \eta_6, c(n)\iota_3\rangle_\mu = 0$ in $\pi_{10}(S^3) = \boldsymbol{Z}/15$.

(2) Let n be 3 or 7. Then $S_P^n \vee S_P^n = (S^n \vee S^n)_P$ and $S_P^n \wedge S_P^n = (S^n \wedge S^n)_P$ by 1.11 on page 58 in [**10**]. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [S^n \wedge S^n, S^n] & \longrightarrow & [S^n \times S^n, S^n] & \longrightarrow & [S^n \vee S^n, S^n] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & [S_P^n \wedge S_P^n, S_P^n] & \longrightarrow & [S_P^n \times S_P^n, S_P^n] & \longrightarrow & [S_P^n \vee S_P^n, S_P^n] & \longrightarrow & 0. \end{array}$$

Since the first vertical arrow is surjective and the third one is injective, we have (2). □

We denote by $d(m,n)$ the order of the image of $\rho_m^* : [Q_m, E_n] \to \pi_9(E_n)$.

Lemma 4.11. (1) *If* $n \not\equiv 0 \pmod 3$, $m = n \equiv 0 \pmod 3$ *or* $n \equiv 0 \pmod{12}$, *then* $d(m,n) = 1$ *and the following is exact*:

(1-1) $$0 \longrightarrow \pi_{10}(E_n) \xrightarrow{q_m^*} [E_m, E_n] \xrightarrow{j_m^*} [Q_m, E_n] \longrightarrow 0.$$

(2) *If* $n \not\equiv 2 \pmod 4$, *then, with respect to any multiplication* μ *on* E_n, *the sequence in* (2) *of* 4.4 *for* P *the set of all primes is an exact sequence of abelian groups and* (1-1) *is a central extension of groups under the hypothesis. The group structure of* $[Q_m, E_n; \mu]$ *is independent of* μ.

(3) *Let* $n \not\equiv 2 \pmod 4$. *Then there exists* $\beta \in [E_m, E_n]$ *such that* $i_m^*\beta = c(m,n)d(m,n)i_n$. *Let* $\alpha = [c(n)] \circ p_m$, $\gamma = q_m^*\gamma'$ *with* γ' *a generator of* $i_{n*}\pi_{10}(S^3) + \boldsymbol{Z}/8\{[\nu_7]\}$, $\delta = q_m^*[\alpha_1(7)]$ *for* $n \equiv 0 \pmod 3$, *and* $\varepsilon = i_n \circ \nu' \circ \eta_6 \circ p_m$ *for* $n \equiv 0 \pmod 4$. *Then, for every multiplication* μ *on* E_n, *we have the following facts*:

$\alpha, \beta, \gamma, \delta, \varepsilon$ *generate* $[E_m, E_n; \mu]$,

$\gamma, \delta, \varepsilon$ *are in the center*,

γ, δ *generate the image of* $q_m^* : \pi_{10}(E_n) \to [E_m, E_n; \mu]$,

$[\alpha, \beta]_\mu = \pm c(m,n)d(m,n)\langle [c(n)], i_n\rangle_\mu \circ q_m$.

PROOF. Before proving (1), we prove that (1) implies (2) and (3).

(2) Let $n \not\equiv 2 \pmod 4$. Then E_n is a Hopf space. Let μ be any multiplication on E_n. It follows from 2.1 that $[Q_m, E_n; \mu]$ is a group so that we have the following exact sequence of groups:

$$\pi_4(E_n) \xrightarrow{(m\Sigma\omega)^*} \pi_7(E_n) \xrightarrow{\quad q^* \quad} [Q_m, E_n; \mu] \xrightarrow{\quad i^* \quad} \pi_3(E_n) \xrightarrow{\qquad} \pi_6(E_n).$$

Since $(m\Sigma\omega)^* = 0$ by 4.4 and [**23**], q^* is injective. The image of q^* is a central subgroup of $[Q_m, E_n; \mu]$ by 2.2 and the image of i^* is isomorphic to $\boldsymbol{Z}$ by 4.4. Hence $[Q_m, E_n; \mu]$ is an abelian group whose group structure is independent of μ. Also $[E_m, E_n; \mu]$ is a group as being proved above, and (1-1) is a central extension of groups under the hypothesis by 2.2. This proves (2).

(3) Let $n \not\equiv 2 \pmod 4$. Since $j_m^*(\alpha)$, $j_m^*(\beta)$ (and $j_m^*(\varepsilon)$ if $n \equiv 0 \pmod 4$) are generators of $\mathrm{Im}(j_m^*)$, it follows that $\alpha, \beta, \gamma, \delta, \varepsilon$ generate $[E_m, E_n; \mu]$ and γ, δ generate $\mathrm{Im}(q_m^*)$. By 4.7, we have

$$[\alpha, \beta]_\mu = \pm\langle [c(n)], \beta \circ i_m \rangle_\mu \circ q_m = \pm c(m,n) d(m,n) \langle [c(n)], \mathrm{id} \rangle_\mu \circ q_m.$$

By 2.2, γ and δ are in the center. We show that so is ε. Let $C_\mu : E_n \wedge E_n \to E_n$ be the commutator map with respect to μ. Then

$$[\alpha, \varepsilon] = C_\mu \circ ([c(n)] \wedge i_n \circ \nu' \circ \eta_6) \circ (p_m \wedge p_m) \circ d = 0,$$

since $(p_m \wedge p_m) \circ d = 0$ for dimensional reasons. Consider the commutative diagram:

$$\begin{array}{ccccc}
E_m & \xrightarrow{\qquad q_m \qquad} & S^7 \wedge S^3 & \xrightarrow{\langle i_n \circ \nu' \circ \eta_6,\, c(m,n) d(m,n) i_n \rangle} & E_n \\
\Big\downarrow d & & \Big\downarrow \mathrm{id} \wedge c(m,n) d(m,n) i_n & & \Big\uparrow C_\mu \\
E_m \wedge E_m & \xrightarrow{(\pm \iota_7 \circ p_m) \wedge \beta} & S^7 \wedge E_n & \xrightarrow{\quad i_n \circ \nu' \circ \eta_6 \wedge \mathrm{id} \quad} & E_n \wedge E_n.
\end{array}$$

We have

$$\begin{aligned}
[\varepsilon, \beta]_\mu &= \pm\, c(m,n) d(m,n) \langle i_n \circ \nu' \circ \eta_6, i_n \rangle_\mu \circ q_m \\
&= \pm\, c(m,n) d(m,n) i_n \circ \langle \nu' \circ \eta_6, \mathrm{id} \rangle_{\mu'} \circ q_m \\
&= 0, \quad \text{since } \langle \nu' \circ \eta_6, \mathrm{id} \rangle_{\mu'} = 0,
\end{aligned}$$

where μ' is the multiplication on S^3 induced from μ. Hence ε is in the center. This proves (3).

In the rest of the proof we prove (1). Consider the exact sequence of based sets:

$$(4.12)_{m,n} \qquad [\Sigma Q_m, E_n] \xrightarrow{(\Sigma\rho_m)^*} \pi_{10}(E_n) \xrightarrow{\quad q_m^* \quad} [E_m, E_n] \xrightarrow{\quad j_m^* \quad} [Q_m, E_n] \xrightarrow{\quad \rho_m^* \quad} \pi_9(E_n).$$

By (2.4) and the equation $\eta_3 \circ J(\chi_m) = 0$, we have

$$(\Sigma\rho_m)^* = 0 : [\Sigma Q_m, E_n] \to \pi_{10}(E_n) \quad \text{for all } m, n.$$

Thus it remains to prove the surjectivity of j_m^* or equivalently the triviality of ρ_m^*.

When $n \equiv 1 \pmod 2$ and $n \not\equiv 0 \pmod 3$, j_m^* is surjective by 4.4 (1-4).

Let $n \equiv 0 \pmod 2$ and $n \not\equiv 0 \pmod 3$. Since $q_m \circ \rho_m = p_m \circ j_m \circ \rho_m = 0$ and $q^* : \pi_7(S^7) \to [Q_m, S^7]$ is surjective, we have $\rho_m^* = 0 : [Q_m, S^7] \to \pi_9(S^7)$. Hence $\rho_m^* = 0 : [Q_m, E_n] \to \pi_9(E_n)$ by the following commutative square:

$$\begin{array}{ccc} [Q_m, E_n] & \xrightarrow{\rho_m^*} & \pi_9(E_n) \\ {\scriptstyle p_{n*}}\downarrow & & \cong \downarrow {\scriptstyle p_{n*}} \\ [Q_m, S^7] & \xrightarrow{\rho_m^*} & \pi_9(S^7). \end{array}$$

Let $n \equiv 1 \pmod 2$ and $n \equiv 0 \pmod 3$. In this case $(4.12)_{n,n}$ is an exact sequence of algebraic loops. Take any $x \in [Q_n, E_n]$ and write $i^*(x) = ai$ with $a \in \boldsymbol{Z}$. Then $x = aj + q^*(y)$ for some $y \in \pi_7(E_n)$. Thus $j^*(\mathrm{id}^a \cdot (y \circ p)) = x$. Hence j^* is surjective.

Let $n \equiv 0 \pmod 2$ and $n \equiv 0 \pmod 3$. When $n \equiv 0 \pmod{12}$, the bundle $E_n \to S^7$ is trivial, whence j_m^* is surjective by (1-3) of 4.8 and (4.9). Let $n \equiv 6 \pmod{12}$. In this case the diagram (4.2) factors as

$$\begin{array}{ccccc} E_n & \xrightarrow{g_2} & E_{n/2} & \longrightarrow & S_p(2) \\ \downarrow & & \downarrow & & \downarrow \\ S^7 & \xrightarrow{2} & S^7 & \xrightarrow{n/2} & S^7 \end{array}$$

and

$$\begin{array}{ccccc} E_n & \xrightarrow{g_3} & E_{n/3} & \longrightarrow & S_p(2) \\ \downarrow & & \downarrow & & \downarrow \\ S^7 & \xrightarrow{3} & S^7 & \xrightarrow{n/3} & S^7. \end{array}$$

Since $n\omega = 6\omega = 2\nu' = \eta_3^3 = \Sigma(\eta_2^3)$ by [**23**], it follows that $Q_n = \Sigma(S^2 \cup_{\eta_2^3} e^6)$ and $i : S^3 \to Q_n$ and $q : Q_n \to S^7$ are suspension maps. Hence we have the following commutative diagram of exact sequences of groups:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_7(E_n) & \longrightarrow & [Q_n, E_n] & \longrightarrow & \pi_3(E_n) & \longrightarrow & 0 \\ & & {\scriptstyle g_{k*}}\downarrow & & \downarrow{\scriptstyle g_{k*}} & & \cong\downarrow{\scriptstyle g_{k*}} & & \\ 0 & \longrightarrow & \pi_7(E_{n/k}) & \longrightarrow & [Q_n, E_{n/k}] & \longrightarrow & \pi_3(E_{n/k}) & \longrightarrow & 0. \end{array}$$

As is easily seen, the first g_{k_*} is surjective for $k = 2$ and isomorphic for $k = 3$, hence so is the second g_{k_*}. Consider the commutative square:

$$\begin{array}{ccc} [Q_n, E_n] & \xrightarrow{g_{k*}} & [Q_n, E_{n/k}] \\ {\scriptstyle \rho_n^*}\downarrow & & \downarrow{\scriptstyle \rho_n^*} \\ \pi_9(E_n) & \xrightarrow{g_{k*}} & \pi_9(E_{n/k}). \end{array}$$

By 4.4 (1-4), we can show that the lower g_{k_*} is surjective. Hence the triviality of the first ρ_n^* follows from the following fact:

(4.13) the image of the second ρ_n^* does not contain an element of order 3 for $k = 2$ and of order 2 or 6 for $k = 3$.

We prove (4.13) as follows. First, let $k = 2$. In this case $E_{n/2}$ is a Hopf space. Take any $x \in [Q_n, E_n]$ and write $j^*(x) = ai \in \pi_3(E_{n/2}) = \boldsymbol{Z}\{i\}$. Then $x = a(g_2 \circ j) + q^*(y)$ for some $y \in \pi_7(E_{n/2})$ by the exact sequence

$$\pi_7(E_{n/2}) \xrightarrow{q^*} [Q_n, E_{n/2}] \xrightarrow{i^*} \pi_3(E_{n/2}).$$

Since $j^* : [E_n, E_{n/2}] \to [Q_n, E_{n/2}]$ is a homomorphism, we have $j^*(g_2^a(y \circ p)) = a(g_2 \circ j) + y \circ p \circ j = x$. Thus j^* is surjective and hence $\rho_n^* = 0 : [Q_n, E_{n/2}] \to \pi_9(E_{n/2})$.

Second, let $k = 3$. Consider the commutative square:

$$\begin{array}{ccc} [Q_n, E_{n/3}] & \xrightarrow{p_*} & [Q_n, S^7] \\ \rho_n^* \downarrow & & \downarrow \rho_n^* \\ \pi_9(E_{n/3}) & \xrightarrow{p_*} & \pi_9(S^7). \end{array}$$

Since the lower p_* is surjective by 4.4 (1-4) and since $\rho_n^*(q) = p \circ j \circ \rho_n = 0$, the image of the first ρ_n^* does not contain an element of order 2 or 6. □

Proof of Theorem 1 – Part II. We proceed to Step 4. It remains to prove that $\mathrm{nil}[A, X_P; \mu] \leq 2$ when A is $P^3(\boldsymbol{R})$ or E_m with m odd.

Let $A = P^3(\boldsymbol{R})$. If $X = P^3(\boldsymbol{R})$, then $[P^3(\boldsymbol{R}), X_P; \mu] \cong \boldsymbol{Z}_P$ by Proposition 4.1. If X is S^1, $S^1 \times S^1$ or $S^1 \times S^7$, then $[P^3(\boldsymbol{R}), X_P; \mu] = 0$ as seen in Part I. If X is S^7 or $S^7 \times S^7$, then $[P^3(\boldsymbol{R}), X_P; \mu] = 0$. If X is S^3, $S^1 \times S^3$, $S^3 \times S^3$, $S^3 \times S^7$, $SU(3)$ or E_m with $m \not\equiv 2 \pmod 4$, then $[P^2(\boldsymbol{R}), X_P] = 0$, hence $q^* : \pi_3(X_P) \to [P^3(\boldsymbol{R}), X_P; \mu]$ is surjective so that $[P^3(\boldsymbol{R}), X_P; \mu]$ is abelian.

Let $A = E_m$ with m odd. If $X = S^3$, then $\mathrm{nil}[E_m, X_P; \mu] = 1$ by 4.8. If X is S^7 or $S^7 \times S^7$, then $[E_m, X_P; \mu] \cong [E_m/S^3, X_P; \mu] = \pi_7(X_P) \oplus \pi_{10}(X_P)$ so that $\mathrm{nil}[E_m, X_P; \mu] = 1$. If $X = SU(3)$, then $[Q_m, X_P; \mu] \cong \boldsymbol{Z}_P$ so that $\mathrm{nil}[E_m, X_P; \mu] \leq 2$ by 2.2, since $\pi_7(X) = 0$ and $\pi_6(X)$ is finite. If X is S^1 or $S^1 \times S^1$, then $[E_m, X_P] = 0$. If $X = S^1 \times S^7$, then $[Q_m, X_P; \mu]$ is abelian so that $\mathrm{nil}[E_m, X_P; \mu] \leq 2$. If X is $S^1 \times S^3$, $S^3 \times S^3$, $P^3(\boldsymbol{R})$ or E_n with $n \not\equiv 2 \pmod 4$, then we can prove the assertion by the almost same method. So we give a proof only for E_n. Let $X = E_n$ with $n \not\equiv 2 \pmod 4$. By (4.12) and 2.2, it suffices to prove that $[Q_m, X_P; \mu]$ is abelian. Take any x_1, $x_2 \in [Q_m, X_P; \mu]$. By (2) of 4.4, we can write $i_m^* x_k = a_k c(m, n; P) i_n$ with $a_k \in \boldsymbol{Z}_P$. There exists a map $g : S^7 \to S^6$ which makes the following diagram commute:

$$\begin{array}{ccccc} Q_m & \xrightarrow{d} & Q_m \wedge Q_m & \xrightarrow{x_1 \wedge x_2} & X_P \wedge X_P \\ q \downarrow & & \cup \uparrow i_m \wedge i_m & & \downarrow C_\mu \\ S^7 & \xrightarrow{g} & S^3 \wedge S^3 & & X_P, \end{array}$$

where C_μ is the commutator map with respect to μ. Write $a_k = a_k''/a_k'$ with $a_k' \in P^c$ (the complement of P), $a_k'' \in \boldsymbol{Z}$, and put $h = C_\mu \circ ((1/a_1')i_n \wedge (1/a_2')i_n)$. We have

$$[x_1, x_2] = C_\mu \circ (x_1 \wedge x_2) \circ (i_m \wedge i_m) \circ g \circ q = a_1'' a_2'' c(m, n; P)^2 h \circ g \circ q,$$

which is a 2-torsion element. We show that this is trivial. If $2 \notin P$, then $\pi_7(X_P) = \boldsymbol{Z}_P$

so that $h \circ g = 0$. Assume $2 \in P$. If $n = 1, 5$, then $h \in \pi_6(X_P) = 0$. If $n = 3$, then $3\pi_6(X_P) = 0$ so that $h \circ g = 0$. If $n = 0, 4$, then $c(m, n; P) \equiv 0 \pmod 4$ so that $c(m, n; P) h \circ g = 0$. Thus $[x_1, x_2] = 0$ and hence $[Q_m, X_P; \mu]$ is abelian. □

5. Proofs of Theorems 2 and 3.

In this section, we use the notation in 4.11. We recall from [**17**] and [**6**] the following:

$$\pi_3(SU(3)) = \boldsymbol{Z}\{i\}, \quad \pi_5(SU(3)) = \boldsymbol{Z}\{[2]\} \quad \text{with } p \circ [2] = 2\iota_5,$$
$$\pi_8(SU(3)) = \boldsymbol{Z}/12\{\langle [2], i \rangle_{\mu_0}\}.$$

We define elements in $[E_l', SU(3)]$ as follows:

$$\alpha = \begin{cases} [2] \circ p_1 & l = 0 \\ [2] \circ p & l = 1, \end{cases} \qquad \beta = \begin{cases} i \circ p_2 & l = 0 \\ \mathrm{id} & l = 1, \end{cases} \qquad \gamma = \langle [2], i \rangle_{\mu_0} \circ q,$$

where p_k is the projection from $S^5 \times S^3$ to the k-th component, and $q : E_l' \to S^8$ is the quotient map. Then we have the following result which contains Theorem 2.

Theorem 5.1. *If $l = 0, 1$ and $n = 1, 4, 5$, then for every integer m, r, we have*

$$(1) \qquad [E_l', SU(3); \mu^{(r)}] = \Psi(\alpha, \beta, \gamma; 12, \pm(2r+1)k_\mu),$$

where $\langle [2], i \rangle_\mu = k_\mu \langle [2], i \rangle_{\mu_0}$, and

$$(2) \qquad [E_m, E_n; \mu^{(r)}] = \Psi(\alpha, \beta, \gamma; 120, \pm c(m,n)(2r+1)k_\mu) \oplus \begin{cases} \boldsymbol{Z}/2\{i_n \circ \nu' \circ \eta_6 \circ p_m\} & n = 4 \\ 0 & n = 1, 5, \end{cases}$$

where α, β, γ are elements defined in 4.11(3) *and $\langle [c(n)], i_n \rangle_\mu = k_\mu \gamma'$ in $\pi_{10}(E_n) = \boldsymbol{Z}/120\{\gamma'\}$.*

Proof. We have (2) by 2.5 and 4.11. By the same methods, we can prove (1) and so we omit the details. □

To prove Theorem 3, we need the following.

Lemma 5.2. *For $k \in \{0, 1, 3, 4, 5\}$, let*

$$n \equiv \begin{cases} 0 \pmod{48} & \text{if } k = 0 \\ \pm k \pmod{12} & \text{if } k = 1, 3, 5 \\ 16, 32 \pmod{48} & \text{if } k = 4. \end{cases}$$

Then there exists a multiplication μ on E_n such that the projection $f_n : (E_n, \mu) \to (Sp(2), \mu_0)$ is a Hopf map.

Proof. It follows from the S^7-version of Theorem A in [**3**] that there exist multiplications μ', μ'' on S^7 such that $n\iota_7 : (S^7, \mu') \to (S^7, \mu'')$ is a Hopf map. Then the existence of μ follows from [**1**]. □

Remark 5.3. *In the situation of* 5.2, $E_n \simeq E_k$.

Proof of Theorem 3. Let n, k, μ be as in 5.2. Let $k = 5$. The sequence (1-1) in 4.11 is a central extension of groups, and $[E_m, E_n; \mu]$ is generated by $\alpha = [12] \circ p_m$, β and γ, where $\beta \circ i_m = i_n$, $\gamma = q_m^* \gamma'$, γ' is a generator of $\pi_{10}(E_n) = \boldsymbol{Z}/120$. By 4.7, we have $[\alpha, \beta] = \pm \langle [12], i_n \rangle_\mu \circ q_m$. Also $f_{n_*} \langle [12], i_n \rangle_\mu = \langle f_{n_*}[12], f_{n_*}(i_n) \rangle_{\mu_0} = \langle n[12], i_1 \rangle_{\mu_0} = n \langle [12], i_1 \rangle_{\mu_0}$. Hence the order of $\langle [12], i_n \rangle_\mu$ is $10/\gcd\{n, 5\}$ and

$$[E_m, E_n; \mu] = \Psi(\alpha, \beta, \gamma; 120, 12 \cdot \gcd\{n, 5\}).$$

By letting $n = 7$, we obtain Theorem 3 from 2.5. □

6. Proofs of Theorem 4, Corollaries 1 and 2.

Recall $H^*(Sp(2)) = \Lambda(x_3, x_7)$.

Proposition 6.1. *Let $P_1 \cup P_2$ be a partition of the set of all primes and n an integer with $n \not\equiv 2 \pmod 4$. If $n \in P_1$ and $c(n) = 12/\gcd\{n, 12\} \in P_2$, then for any multiplications μ_1 on $S^3_{P_1} \times S^7_{P_1}$ and μ_2 on $Sp(2)_{P_2}$ there is a multiplication μ on E_n such that, for each integer r, the following is a weak pullback diagram* [**2**] *of Hopf spaces and Hopf maps*

$$\begin{array}{ccc} (E_n, \mu^{(r)}) & \xrightarrow{f_n} & (Sp(2)_{P_2}, \mu_2^{(r)}) \\ {\scriptstyle h'}\downarrow & & \downarrow{\scriptstyle h} \\ (S^3_{P_1} \times S^7_{P_1}, \mu_1^{(r)}) & \xrightarrow{i \times ni} & (K(\boldsymbol{Q}, 3) \times K(\boldsymbol{Q}, 7), \mu_0), \end{array}$$

where $h' = (1/c(n))\langle c(n) \rangle \times p$, $h = x_3 \times x_7$, i is the localization of the inclusion $S^m \to K(\boldsymbol{Z}, m)$, and μ_0 is the unique multiplication on $K(\boldsymbol{Q}, 3) \times K(\boldsymbol{Q}, 7)$. Moreover the following is the pullback diagram of algebraic loops:

$$\begin{array}{ccc} [X, E_n; \mu^{(r)}] & \xrightarrow{f_{n*}} & [X, Sp(2)_{P_2}; \mu_2^{(r)}] \\ {\scriptstyle h'_*}\downarrow & & \downarrow{\scriptstyle h_*} \\ [X, S^3_{P_1} \times S^7_{P_1}; \mu_1^{(r)}] & \xrightarrow{(i \times ni)_*} & [X, K(\boldsymbol{Q}, 3) \times K(\boldsymbol{Q}, 7); \mu_0]. \end{array}$$

Proof. It suffices to prove the assertions for $r = 0$. In fact, if $f : (X, \mu) \to (X', \mu')$ is a Hopf map, then so is $f : (X, \mu^{(r)}) \to (X', \mu'^{(r)})$. Consider the following homotopy pullback diagram:

$$\begin{array}{ccc} W & \xrightarrow{f'_n} & Sp(2)_{P_2} \\ {\scriptstyle h''}\downarrow & & \downarrow{\scriptstyle h} \\ S^3_{P_1} \times S^7_{P_1} & \xrightarrow{i \times ni} & K(\boldsymbol{Q}, 3) \times K(\boldsymbol{Q}, 7). \end{array}$$

Note that $i \times ni$ and h are Hopf maps with respect to any Hopf structures on $S^3_{P_1} \times S^7_{P_1}$ and $Sp(2)_{P_2}$, respectively. Hence by [**22**] (cf. [**2**]) there is a multiplication on W with respect to which h'' and f'_n are Hopf maps. Since $h \circ f_n = (i \times ni) \circ h'$, there is a map $g : E_n \to W$ such that $f'_n \circ g = f_n$ and $h'' \circ g = h'$. By Theorem 5.1 on page 82 in [**10**], $\pi_*(W)$ is the pullback of

$$\pi_*(S^3_{P_1} \times S^7_{P_1}) \longrightarrow \pi_*(K(\boldsymbol{Q}, 3) \times K(\boldsymbol{Q}, 7)) \longleftarrow \pi_*(Sp(2)_{P_2}).$$

By localizing g at P_i $(i = 1, 2)$, we see that g is a weak homotopy equivalence and hence a homotopy equivalence. The last assertion of 6.1 now follows from the theorem in [**10**] referred to as above. □

The following can be proved easily, so we omit its proof.

Lemma 6.2. *If (X, μ) is a Hopf space and P is a set of primes, then $(\mu_P)^{(r)} = (\mu^{(r)})_P$ for every integer r.*

Proof of Theorem 4. We use the notation in 4.11 and 6.1. For convenience, we denote by '+' the group operation in $[E_m, Sp(2); \mu_0^{(r)}]$. Let μ be a multiplication on E_n making f_n and h' Hopf maps with respect to the product multiplication $\mu'_{P_1} \times \mu''_{P_1}$ on $S^3_{P_1} \times S^7_{P_1}$ and $(\mu_0)_{P_2}$ on $Sp(2)_{P_2}$, where μ' and μ'' are any multiplications on S^3 and S^7, respectively. Then, by 6.1 and 6.2, $[E_m, E_n; \mu^{(r)}]$ is isomorphic to the pullback of

(6.3)
$$\begin{array}{ccc} & & [E_m, Sp(2)_{P_2}; (\mu_0^{(r)})_{P_2}] \\ & & \Big\downarrow h_* \\ [E_m, S^3_{P_1} \times S^7_{P_1}; (\mu'^{(r)})_{P_1} \times (\mu''^{(r)})_{P_1}] & \xrightarrow{(i \times ni)_*} & [E_m, K(\boldsymbol{Q}, 3) \times K(\boldsymbol{Q}, 7); \mu_0]. \end{array}$$

Recall that $H^*(E_m) = \Lambda(y_3, y_7)$ with $f_m^*(x_3) = y_3$ and $f_m^*(x_7) = my_7$.

By Theorem 1 (cf. [**8**]) and 4.8, we have

$$[E_m, S^3_{P_1}; (\mu'^{(r)})_{P_1}] = [E_m, S^3; \mu'^{(r)}]_{P_1} = \boldsymbol{Z}_{P_1}\{\langle c(m) \rangle\} \oplus (\boldsymbol{Z}/30)_{P_1},$$

$$[E_m, Sp(2)_{P_2}; (\mu_0^{(r)})_{P_2}] = [E_m, Sp(2); \mu_0^{(r)}]_{P_2}.$$

Also we have $[E_m, S^7_{P_1}; (\mu''^{(r)})_{P_1}] = [E_m/S^3, S^7_{P_1}; (\mu''^{(r)})_{P_1}] = \pi_7(S^7)_{P_1} \oplus \pi_{10}(S^7)_{P_1}$. Hence

$$[E_n, S^7_{P_1}; (\mu''^{(r)})_{P_1}] = \boldsymbol{Z}_{P_1}\{p\} \oplus (\boldsymbol{Z}/24)_{P_1}.$$

We have $(i \times ni)_*(a'\langle c(m) \rangle + b'p + c') = c(m)a'y_3 + nb'y_7$ for every a', $b' \in \boldsymbol{Z}_{P_1}$ and $c' \in (\boldsymbol{Z}/30 \oplus \boldsymbol{Z}/24)_{P_1}$. Since the P_2-localization preserves central extensions of nilpotent groups, every element of $[E_m, Sp(2); \mu_0^{(r)}]_{P_2}$ can be uniquely written as $a\alpha + b\beta + c\gamma$ with $a, b \in \boldsymbol{Z}_{P_2}$ and $0 \le c < A_2B_2$, where A_2 and B_2 are the P_2-components of 12 and 10 respectively. We have $(x_3 \times x_7)_*(a\alpha + b\beta + c\gamma) = by_3 + (12a + mb)y_7$. Hence $(a'\langle c(m) \rangle + b'p + c') \times (a\alpha + b\beta + c\gamma)$ is in the pullback of (6.3) if and only if

$$c(m)a' = b, \quad nb' = 12a + mb \quad \text{for } a, b \in \boldsymbol{Z}_{P_2} \quad \text{and} \quad a', b' \in \boldsymbol{Z}_{P_1}.$$

The last relations hold if and only if

(6.4) $$12a \in A_2\boldsymbol{Z}, \quad b \in c(m; P_1)\boldsymbol{Z},$$

(6.5) $$12a + mb \in n\boldsymbol{Z},$$

$$a' = b/c(m), \quad b' = (12a + mb)/n,$$

where $c(m;P_1)$ stands for the P_1-component of $c(m)$. Let

$$C = A_2/\gcd\{m, A_2\} \quad \text{and} \quad D = mA_1/\gcd\{m, 12\},$$

where $A_1 = 12/A_2$. Then C and D are prime to each other. Hence there exist integers C', D' with $CC' + DD' = 1$. Let $\Phi(m,n) = \{(x,y) \in \boldsymbol{Z} \times \boldsymbol{Z};\ xC + yD \equiv 0 \pmod n\}$. Then $\Phi(m,n) = \{(kD + ln\,C', -kC + lnD');\ k,l \in \boldsymbol{Z}\}$. If (6.4) is satisfied, then (6.5) holds if and only if $(A_1 a, b/c(m;P_1)) \in \Phi(m,n)$.

Suppose that $A_1 a = kD + ln\,C'$ and $b/c(m;P_1) = -kC + ln\,D'$ with $k,l \in \boldsymbol{Z}$. Then

$$\begin{aligned} a\alpha + b\beta &= kD\alpha/A_1 + ln\,C'\alpha/A_1 - kc(m;P_1)C\beta + ln\,c(m;P_1)D'\beta \\ &\equiv kD\alpha/A_1 - kc(m;P_1)C\beta + ln\,C'\alpha/A_1 + ln\,c(m;P_1)D'\beta \pmod{\gamma} \\ &\equiv k(D\alpha/A_1 - c(m;P_1)C\beta) + l(nC'\alpha/A_1 + nc(m;P_1)D'\beta) \pmod{\gamma}. \end{aligned}$$

Here we have used the following facts:

$$\begin{aligned} &s\alpha/A_1 + t\beta = t\beta + s\alpha/A_1 + stA_2\gamma \quad (s,t \in \boldsymbol{Z}), \\ &(xy)^n \equiv x^n y^n \pmod{[G,G]} \quad \text{in any group } G. \end{aligned}$$

Hence the pullback of (6.3) is isomorphic to the sum of $(\boldsymbol{Z}/30 \oplus \boldsymbol{Z}/24)_{P_1}$ and the subgroup of $[E_m, Sp(2); \mu_0^{(r)}]_{P_2}$ generated by $D\alpha/A_1 - c(m;P_1)C\beta$, $nC'\alpha/A_1 + nc(m;P_1)D'\beta$ and γ. As is easily seen, we have

$$\begin{aligned} [D\alpha/A_1 - c(m;P_1)C\beta, nC'\alpha/A_1 + nc(m;P_1)D'\beta] &= (nc(m;P_1)/A_1)[\alpha,\beta] \\ &= nc(m;P_1)A_2(2r+1)\gamma. \end{aligned}$$

Hence, by setting $x = D\alpha/A_1 - c(m;P_1)C\beta$, $y = nC'\alpha/A_1 + nc(m;P_1)D'\beta$ and $z = nc(m;P_1)\gamma$, we have $[E_m, E_n; \mu^{(r)}] \cong \Psi(x,y,z; A_2B_2, A_2(2r+1)) \oplus (\boldsymbol{Z}/30 \oplus \boldsymbol{Z}/24)_{P_1}$. This completes the proof of Theorem 4. □

Proof of Corollary 1. Consider the following two cases:

(i) $3 \in P_1, \quad 2 \in P_2,$

(ii) $2 \in P_1, \quad 3 \in P_2.$

By applying Theorem 4 to these cases, we obtain Corollary 1. In fact, (1) follows from (i); (2) follows from (ii). □

Proof of Corollary 2. It follows from the following commutative diagram that if $\Sigma : [A, X; \mu] \to [\Sigma A, \Sigma X]$ is a homomorphism, then so is $p_* : [A, X; \mu] \to [A, S^n]$:

$$\begin{array}{ccc} [A, X; \mu] & \xrightarrow{p_*} & [A, S^n] \\ \downarrow \Sigma & & \cong \downarrow \Sigma \\ [\Sigma A, \Sigma X] & \xrightarrow{(\Sigma p)_*} & [\Sigma A, S^{n+1}]. \end{array}$$

Hence it suffices to prove that p_* is not a homomorphism. To induce a contradiction, we assume on the contrary that p_* is a homomorphism. For simplicity, we denote by Q the spaces Q_m and Q in §4. Consider the following commutative diagram of exact sequences (cf. 4.11 and its $SU(3)$-version):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_{n+3}(X) & \xrightarrow{q^*} & [A,X] & \xrightarrow{j^*} & [Q,X] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \\ 0 & \longrightarrow & \pi_{n+3}(S^n) & \xrightarrow{q^*} & [A,S^n] & \longrightarrow & [Q,S^n] & \longrightarrow & 0. \end{array}$$

Here the lower sequence is in the stable range so that it is short exact by (2.4). Let $x, y \in [A, X; \mu]$. Then $[x, y] \in \mathrm{Ker}(j^*) = \mathrm{Im}(q^*)$, whence $[x, y] = q^*(z)$ for some $z \in \pi_{n+3}(X)$. Then $0 = [p_*x, p_*y] = p_*[x, y] = q^*p_*(z)$, whence $p_*(z) = 0$. When $X = SU(3)$, we have $z = 0$, since the first p_* in the diagram is injective by [**17**], so that $[x, y] = 1$ and $[A, SU(3); \mu]$ is commutative. This contradicts Theorem 2. When X is $Sp(2)$, E_3 or E_5, we have $15z = 0$, since $\mathrm{Ker}\{p_* : \pi_{10}(E_m) \to \pi_{10}(S^7)\} = \boldsymbol{Z}/15$ by 4.4, hence in particular $[x, y]^{15} = 1$, which contradicts Theorem 2, Theorem 3 and Corollary 1, since the order of $[x, y]$ is 2 or 10 for some x and y. In either case we have a contradiction. Therefore p_* is not a homomorphism. □

7. Composition.

In this section, (G, l, n) stands for $(SU(3), 2, 5)$ or $(Sp(2), 12, 7)$. We use the notation in Theorem 5.1 and study only the standard multiplication μ_0. We denote by '+' the group operation in $[G, G; \mu_0]$. By Theorem 5.1, every element of $[G, G]$ can be written as $a\alpha + b\beta + c\gamma$ where a, b, c are integers.

We fix generators $s_r \in H^r(S^r)$ for $r = n, 3$. Define $x_r \in H^r(G)$ by $p^*s_n = x_n$ and $i^*(x_3) = s_3$. Then $H^*(G) = \Lambda(x_3, x_n)$. Orient S^{n+3} by $q^*s_{n+3} = x_nx_3$. We need the following.

Lemma 7.1. (1) *Given $f, g, h \in [G, G]$, we have*

$$(f + g) \circ h = f \circ h + g \circ h \quad \textit{and} \quad (f + g)^*(x_r) = f^*(x_r) + g^*(x_r).$$

(2) $\alpha^*(x_3) = 0$, $\alpha^*(x_n) = lx_n$, $\beta^*(x_r) = x_r$, *and* $\gamma^*(x_r) = 0$.

Proof. Since x_r is primitive, we have the second assertion of (1). The rest is obvious by definitions. □

Thus it suffices for determining the composition operation to compute $\alpha \circ (a\alpha + b\beta + c\gamma)$ and $\gamma \circ (a\alpha + b\beta + c\gamma)$. We are able to determine only the following.

Proposition 7.2. (1) $(a'\alpha + b'\beta + c'\gamma) \circ (a\alpha + b\beta + c\gamma) \equiv (laa' + ab' + a'b)\alpha + bb'\beta \pmod{\gamma}$.

(2) $\gamma \circ (a\alpha + b\beta + c\gamma) = (la + b)b\gamma$.

(3) $\alpha \circ a\alpha = a(\alpha \circ \alpha) = la\alpha$.

(4) $\alpha \circ (\beta + c\gamma) = \alpha \circ \beta + \alpha \circ c\gamma = \alpha + 2c\gamma$ *if* $G = SU(3)$.

(5) $\alpha \circ c\gamma = c(\alpha \circ \gamma) = lcu\gamma$, *where u is 1 or an odd integer according as G is $SU(3)$ or $Sp(2)$.*

Proof. (1) We obtain (1), by looking at the induced homomorphism of the integral cohomology.

(2) Let $f = a\alpha + b\beta$ and $g = c\gamma$. We have

$$\{q \circ \mu_0 \circ (f \times g) \circ d\}^* s_{n+3} = (la + b)bx_n x_3 = \{(la + b)b\iota_{n+3} \circ q\}^* s_{n+3}.$$

Hence $q \circ \mu_0 \circ (f \times g) \circ d = (la + b)b\iota_{n+3} \circ q$ and then $\gamma \circ (f + g) = \gamma' \circ q \circ \mu_0 \circ (f \times g) \circ d = \gamma' \circ (la + b)b\iota_{n+3} \circ q = (la + b)b(\gamma' \circ q) = (la + b)b\gamma$.

(3) Let $a \geq 1$. Denote by $d^a : X \to X^a = X \times \cdots \times X$ the a-fold diagonal map, $\mu_0^a : G^a \to G$ the a-fold multiplication, and $\alpha^{\times a} = \alpha \times \cdots \times \alpha : G^a \to G^a$ for a map $\alpha : G \to G$. Then $a\alpha = \mu_0^a \circ \alpha^{\times a} \circ d^a = \mu_0^a \circ [l]^{\times a} \circ p^{\times a} \circ d^a = \mu_0^a \circ [l]^{\times a} \circ d^a \circ p = a[l] \circ p$ and hence $\alpha \circ a\alpha = [l] \circ p \circ a[l] \circ p = [l] \circ la\iota_n \circ p = la([l] \circ p) = la\alpha$. Thus $\alpha \circ a\alpha = la\alpha$, which holds also for $a = 0$. Let $I : G \to G$ be the inversion. Then $(-a)\alpha = I \circ a\alpha = I \circ a[l] \circ p = (-a)[l] \circ p$ and hence $\alpha \circ (-a)\alpha = \alpha \circ (-a)[l] \circ p = [l] \circ p \circ (-a)[l] \circ p = l(-a)\alpha$.

(4) Let $\sigma : SU(3) \to SU(3)$ be the complex conjugation. Since $\sigma^*(x_3) = x_3$ and $\sigma^*(x_5) = -x_5$, it follows that $\sigma = -\alpha + \beta + x\gamma$ for some x and the following diagram is commutative:

$$\begin{array}{ccccc} S^3 & \xrightarrow{i} & SU(3) & \xrightarrow{p} & S^5 \\ \| & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle -\iota_5} \\ S^3 & \xrightarrow{i} & SU(3) & \xrightarrow{p} & S^5. \end{array}$$

Hence $\sigma \circ \langle [2], i \rangle = \langle \sigma_*[2], \sigma_* i \rangle = \langle -[2], i \rangle = -\langle [2], i \rangle$. Since $q^*[S^8] = x_5 x_3$, we easily have $q \circ \sigma = (-\iota_8) \circ q$. Hence $\sigma \circ \gamma = \sigma \circ \langle [2], i \rangle \circ q = \langle [2], i \rangle \circ (-\iota_8) \circ q = \gamma \circ \sigma$. We have $\sigma \circ (\beta + \gamma) = (\beta + \gamma) \circ \sigma$ by the following commutative diagram:

$$\begin{array}{ccccccc} SU(3) & \xrightarrow{d} & SU(3) \times SU(3) & \xrightarrow{\beta \times \gamma} & SU(3) \times SU(3) & \xrightarrow{\mu_0} & SU(3) \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma \times \sigma} & & \downarrow{\scriptstyle\sigma \times \sigma} & & \downarrow{\scriptstyle\sigma} \\ SU(3) & \xrightarrow{d} & SU(3) \times SU(3) & \xrightarrow{\beta \times \gamma} & SU(3) \times SU(3) & \xrightarrow{\mu_0} & SU(3). \end{array}$$

Write $\alpha \circ (\beta + c\gamma) = \alpha + f(c)\gamma$. Since $(\beta + \gamma)^c = \beta + c\gamma$ for $c \geq 1$, we then have $\sigma \circ (\beta + c\gamma) = (\beta + c\gamma) \circ \sigma$. We have $-\alpha + \beta + (x - c)\gamma = (\beta + c\gamma) \circ \sigma = \sigma \circ (\beta + c\gamma) = -\alpha + \beta + \{c + x - f(c)\}\gamma$ by (2), whence $f(c) = 2c$ as desired.

(5) Let $c \geq 1$. Then $\alpha \circ c\gamma = \alpha \circ \mu_0^c \circ \gamma^{\times c} \circ d^c = \alpha \circ \mu_0^c \circ \gamma'^{\times c} \circ d^c \circ q = [l] \circ p \circ c\gamma' \circ q$. On the other hand, let l' be 2 or 3 according as G is $SU(3)$ or $Sp(2)$. Then $p \circ \gamma' = l'\nu$, where $\pi_{n+3}(S^n) = \boldsymbol{Z}/24\{\nu\}$. It follows that $\alpha \circ c\gamma = cl'\{[l] \circ \nu \circ q\} = c(\alpha \circ \gamma)$. If $G = SU(3)$, then $\gamma' = [2] \circ \nu$, since $p_*(\gamma') = 2\nu = p_*([2] \circ \nu)$, whence $\alpha \circ \gamma = 2\{[2] \circ \nu \circ q\} = 2\{\gamma' \circ q\} = 2\gamma$ and $\alpha \circ c\gamma = 2c\gamma$. Let $G = Sp(2)$. By [**17**] and [**23**], we have the following exact sequence:

$$0 \longrightarrow \pi_{10}(S^3) = \boldsymbol{Z}/15 \xrightarrow{i_*} \pi_{10}(Sp(2)) \xrightarrow{p_*} \boldsymbol{Z}/8\{3\nu\} \longrightarrow 0.$$

We have $p_*([12] \circ \nu) = 12\nu = p_*(4\gamma')$ so that $[12] \circ \nu = 4\gamma' + 8u\gamma' = 4(2u + 1)\gamma'$ for some integer u. Hence $\alpha \circ \gamma = 3([12] \circ \nu \circ q) = 3\{4(2u + 1)\gamma' \circ q\} = 12(2u + 1)\gamma$. Thus $\alpha \circ c\gamma = 12(2u + 1)c\gamma$. □

Proposition 7.3. *$\alpha_*: [SU(3), SU(3)] \to [SU(3), SU(3)]$ is not a homomorphism and hence α is not a Hopf map.*

Proof. By 7.2(1), there is a function $y: \boldsymbol{Z}/12 \to \boldsymbol{Z}/12$ such that $\alpha \circ (\alpha - \beta + c\gamma) = \alpha + y(c)\gamma$. Then $(\alpha - \beta + c'\gamma) \circ (\alpha - \beta + c\gamma) = \beta + \{y(c) + 1 - c - c'\}\gamma$ by 7.1(1) and 7.2(2). So $\alpha - \beta + \{y(c) + 1 - c\}\gamma$ is a left homotopy inverse, and hence a homotopy inverse, of $\alpha - \beta + c\gamma$. Thus $\beta = (\alpha - \beta + c\gamma) \circ (\alpha - \beta + \{y(c) + 1 - c\}\gamma) = \beta + \{y(y(c) + 1 - c) - y(c)\}\gamma$ by 7.1(1) and 7.2(2). Therefore we have

$$y(y(c) + 1 - c) = y(c). \tag{7.4}$$

On the other hand, if α_* is a homomorphism, then $\alpha \circ (\alpha - \beta + c\gamma) = \alpha \circ \alpha - \alpha + c\{\alpha \circ \gamma\} = \alpha + 2c\gamma$ by (3) and (5) of 7.2, whence $y(c) = 2c$. But this does not satisfy (7.4). Therefore α_* is not a homomorphism. □

Proof of Corollary 3. Set $f = a\alpha + b\beta + c\gamma$. Then $f^*(x_3) = bx_3$ and $f^*(x_n) = (la + b)x_n$ by 7.1. Hence f^* is an isomorphism if and only if $|la + b| = |b| = 1$. Thus by J. H. C. Whitehead's theorem, we have $\mathscr{E}(SU(3)) = \{\pm\alpha \mp \beta + c\gamma,\ \pm\beta + c\gamma; 1 \le c \le 12\}$ and $\mathscr{E}(Sp(2)) = \{\pm\beta + c\gamma; 1 \le c \le 120\}$. Let $|s| = |s'| = 1$. Then $(s\beta + c\gamma) \circ (s'\beta + c'\gamma) = ss'\beta + (c + sc')\gamma$ by 7.1(1) and 7.2(2). Hence $z^c = \beta + c\gamma$ and $z^c \circ y = -\beta + c\gamma$. The assertion for $Sp(2)$ then follows easily.

In the rest of the proof, let $G = SU(3)$. We identify $[Q, Q]$ with $[Q, SU(3)]$ by j_*. Set $\alpha_0 = [2] \circ p \circ j$ and $\beta_0 = j$. Then $\mathscr{E}(Q) = \{\pm\alpha_0 \mp \beta_0,\ \pm\beta_0\} = \boldsymbol{Z}/2\{-\alpha_0 + \beta_0\} \oplus \boldsymbol{Z}/2\{-\beta_0\}$. Hence we have an exact sequence of groups:

$$0 \longrightarrow \pi_8(SU(3)) \xrightarrow{\lambda} \mathscr{E}(SU(3)) \xrightarrow{j^*} \mathscr{E}(Q) \longrightarrow 0,$$

where $\lambda(f) = \beta + f \circ q$. A splitting $\tau: \mathscr{E}(Q) \to \mathscr{E}(SU(3))$ is defined by $\tau(-\alpha_0 + \beta_0) = x = \sigma$, the complex conjugation, and $\tau(-\beta_0) = y = -\beta$. Since, as is easily seen, x, y and $z = \beta + \gamma$ generate $\mathscr{E}(SU(3))$, and $x\lambda(f) = \lambda(f)x$ and $y\lambda(f) = \lambda(-f)y$, the assertion follows. □

8. A concluding remark.

For a Hopf space (X, μ), we define $\widetilde{\mathrm{cat}}(X, \mu)$ to be the maximum of integers n such that $[Y, X; \mu]$ is a group for every space Y with $\mathrm{cat}\, Y \le n$. We have

(1) $\widetilde{\mathrm{cat}}(X, \mu) \ge 2$ by 2.1;

(2) $\widetilde{\mathrm{cat}}(X, \mu) = \infty$ if and only if μ is homotopy associative;

(3) $\widetilde{\mathrm{cat}}(X, \mu) < \mathrm{cat}(X \times X \times X)$ if μ is not homotopy associative;

(4) $\widetilde{\mathrm{cat}}(S^3, \mu_r) = \begin{cases} \infty & r \equiv 0, 1 \pmod 3 \\ 2 & r \equiv 2 \pmod 3 \end{cases}$ by [**13**];

(5) $\widetilde{\mathrm{cat}}(S^7, \mu) = 2$, since any μ on S^7 is not homotopy associative by [**13**].

It seems that $\widetilde{\mathrm{cat}}(X, \mu)$ measures the homotopy associativity of μ. Let us propose

Problem 3. *Compute $\widetilde{\mathrm{cat}}(X, \mu)$.*

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