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On the Seifert form at infinity associated with polynomial maps

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Abstract. If a polynomial map $f : \boldsymbol{C}^n \to \boldsymbol{C}$ has a nice behaviour at infinity (e.g. it is a "good polynomial"), then the Milnor fibration at infinity exists; in particular, one can define the Seifert form at infinity $\Gamma(f)$ associated with f. In this paper we prove a Sebastiani-Thom type formula. Namely, if $f : \boldsymbol{C}^n \to \boldsymbol{C}$ and $g : \boldsymbol{C}^m \to \boldsymbol{C}$ are "good" polynomials, and we define $h = f \quad g : \boldsymbol{C}^{n+m} \to \boldsymbol{C}$ by $h(x, y) = f(x) + g(y)$, then $\Gamma(h) = (-1)^{mn}\Gamma(f) \quad \Gamma(g)$. This is the global analogue of the local result, proved independently by K. Sakamoto and P. Deligne for isolated hypersurface singularities.

The Seifert forms are unimodular bilinear forms (over $\boldsymbol{Z}$) associated with some geometrical objects: the spinnable structures.

A *spinnable structure* (or open book decomposition) on a closed manifold M is a triple $\mathscr{S} = \{F, m, g\}$ such that F is a compact manifold (with boundary), $m : F \to F$ is a diffeomorphism such that $m|_{\partial F} =$ identity, and $g : T(F, m) \to M$ is a diffeomorphism, where $T(F, m)$ is a closed manifold defined as follows. It is obtained from $F \times [0, 1]$ by identifying $(x, 1)$ with $(m(x), 0)$ for all $x \in F$, and (x, t) with (x, t') for all $x \in \partial F$ and $t, t' \in [0, 1]$. The spinnable structure $\mathscr{S}$ is called *simple* if F is a handle-body obtained from a ball by attaching handles of index $\leq [\dim M/2]$.

A closed, oriented $(2n-1)$-manifold is called Alexander manifold, if $H_n(M, \boldsymbol{Z}) = H_{n-1}(M, \boldsymbol{Z}) = 0$. If it has a simple spinnable structure, then $H_{n-1}(F, \boldsymbol{Z})$ is torsion free.

At the homological level, the geometry of a spinnable structure is coded in its Seifert form. Let $\mathscr{S} = \{F, m, g\}$ be a simple spinnable structure on an Alexander manifold M^{2n-1}. Then the bilinear form $\tilde{H}_{n-1}(F) \otimes \tilde{H}_{n-1}(F) \to \boldsymbol{Z}$, given by $(\alpha, \beta) =$ linking number $(g_{\#}(\alpha \times 0), g_{\#}(\beta \times 1/2))$ is called the Seifert form of $\mathscr{S}$.

The power of the Seifert form can be emphasized by the following result of M. Kato [**4**], and (independently) A. Durfee [**3**]: there is a one-to-one correspondence of isomorphism classes of simple spinnable structures on a 1-connected Alexander manifold M^{2n-1} with congruence classes of unimodular matrices via Seifert matrices (provided that $n \geq 4$).

We recall the following fundamental example. If $f : (\boldsymbol{C}^n, 0) \to (\boldsymbol{C}, 0)$ is the germ of an analytic function, which defines an isolated singularity, then by a result of J. Milnor [**6**], for any $(2n-1)$-dimensional sphere S_ε^{2n-1} (centered at the origin, and with sufficiently small radius ε), the map $f/|f| : S_\varepsilon^{2n-1} \backslash f^{-1}(0) \to S^1$ is a C^∞ locally trivial fibration with fiber F. Moreover, this fibration is equivalent to the fibration f :

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$f^{-1}(\partial D_r) \cap B^n_\varepsilon \to \partial D_r$, where D_r is a disc centered at the origin with radius r $(0 < r \ll \varepsilon)$, and B^n_ε is the open ball centered at the origin with radius ε.

The advantage of the first (Milnor) fibration is that it provides a simple spinnable structure of the $(2n-1)$-dimensional sphere S^{2n-1}_ε. The associated Seifert form is denoted by $\Gamma(f)$, and it is (maybe) the most powerful topological invariant of f. In general, there is no algorithm to determine it, and even in special cases its computation can be very difficult. For this reason, the following result of K. Sakamoto (or equivalently, the result of P. Deligne, which solves the problem at the variation map level) is crucial:

THEOREM. [**16**, **17**]. *Assume that* $g : (\boldsymbol{C}^m, 0) \to (\boldsymbol{C}, 0)$ *and* $h : (\boldsymbol{C}^n, 0) \to (\boldsymbol{C}, 0)$ *define isolated singularities. Define* $f : (\boldsymbol{C}^{n+m}, 0) \to (\boldsymbol{C}, 0)$ *by* $f(x, y) = g(x) + h(y)$. *Then*

$$\Gamma(f) = (-1)^{mn}\Gamma(g) \otimes \Gamma(h).$$

Actually, the Milnor fiber F of f can be identified with the join $G * H$ of the Milnor fibers G and H of g and h respectively, therefore $\tilde{H}_{n+m-1}(F) = \tilde{H}_{m-1}(G) \otimes \tilde{H}_{n-1}(H)$, hence the above formula makes sense by this identification.

The goal of this paper is to present a similar result for global polynomial maps $f : \boldsymbol{C}^n \to \boldsymbol{C}$.

We say that the *Milnor fibration at infinity* of the polynomial map $f : \boldsymbol{C}^n \to \boldsymbol{C}$ exists, if for R sufficiently big

$$f/|f| : S^{2n-1}_R \backslash f^{-1}(0) \to S^1$$

is a C^∞ locally trivial fibration ($S^{2n-1}_R = \partial B^n_R$).

The main difficulty is that, in general, the Milnor fibration at infinity does not exist (see, for example, [**13**, **14**, **12**]). On the other hand, in the last years, a large number of families of polynomial maps were constructed with nice behaviour at infinity.

In this paper, we will assume that our polynomials $f : \boldsymbol{C}^n \to \boldsymbol{C}$ satisfy the following condition:

(C) (*Regularity at infinity*)

For any $t \in \boldsymbol{C}$, the fiber $f^{-1}(t)$ is either smooth or has only isolated singularities, and there exists a sufficiently small neighbourhood D_t of t, and a sufficiently large $R(t) \gg 0$ such that for any $R \geq R(t)$

$$f : (f^{-1}(D_t) \backslash B^n_R, f^{-1}(D_t) \cap S^{2n-1}_R) \to D_t$$

is a trivial fibration over D_t.

If a polynomial f satisfies the condition (C), then it has the following properties as well:

(P1) The bifurcation set of f is exactly the set Σ_f of critical values, i.e.

$$f|_{\boldsymbol{C}^n \backslash f^{-1}(\Sigma_f)} : \boldsymbol{C}^n \backslash f^{-1}(\Sigma_f) \to \boldsymbol{C} \backslash \Sigma_f$$

is a C^∞ locally trivial fibration, with a (smooth) fiber F which has the homotopy type of a bouquet $\vee S^{n-1}$ of $(n-1)$-dimensional spheres. (For the proof of (P1), see the corresponding arguments in [**1**, **7**, **8**, **12**, **11**].)

(P2) For any disc D with the property $\Sigma_f \subset D$, there exists $R_0 \gg 0$ such that for any $R \geq R_0$

$$(\infty) \qquad f : (f^{-1}(\partial D) \cap \bar{B}_R^n, f^{-1}(\partial D) \cap S_R^{2n-1}) \to \partial D$$

is a locally trivial fibration of pair of spaces, such that its restriction on $f^{-1}(\partial D) \cap S_R^{2n-1}$ is trivial, and actually it can be extended to a trivial fibration $f^{-1}(D) \cap S_R^{2n-1} \to D$. (The proof is easy.)

The fibration $f^{-1}(\partial D) \to \partial D$ is called the *fibration of f at infinity*, and it is equivalent to the fibration $f^{-1}(\partial D) \cap \bar{B}_R^n \to \partial D$.

(P3) The Milnor fibration $f/|f| : S_R^{2n-1} \backslash f^{-1}(0) \to S^1$ exists and it is equivalent to the fibration of f at infinity. In particular, the fiber of $f/|f|$ is diffeomorphic to the generic fiber $f^{-1}(t)$ of f (See the corresponding proofs in [**14**, **13**] for $n = 2$, and notice that the dimension is not important; see also [**12**].)

Polynomial maps which satisfy the condition (C) provide simple spinnable structures of S_R^{2n-1} via their Milnor fibration at infinity.

Examples. 1. In the case of plane curves (i.e. for $n = 2$) Neumann and Rudolf [**14**] and Neumann [**13**] studied the Milnor fibration at infinity of "good curves" (i.e. polynomials with condition (C)) proving among others that the link of a *generic fiber* at infinity is fiberable if and only if f is "good". (In general, it is not true that the existence of the Milnor fibration at infinity implies the condition (C), see for example the case of "semi-tame" polynomials [**12**]; or the example $f(x, y) = x(xy - 1)$.)

2. Broughton introduced in [**1**] his "tame" polynomials, and proved that they satisfy the condition (C), and he proved properties (P1–P2) for them. The author generalized Broughton's results for the larger class of "quasitame" polynomials [**7**, **8**]. Moreover, A. Zaharia and the author extended all these results for the larger class of "M-tame" polynomials, and actually they proved also that the property (P3) is satisfied even for a larger class of (the "semi-tame") polynomials [**11**, **12**]. In [**12**] the interested reader can find some more examples and even some counter-examples.

3. Let $f = f_d + f_{d-1} + \cdots$ be the decomposition of f in its homogeneous parts. Assume that f satisfies:

$$\{\partial f_d/\partial x_1 = \cdots = \partial f_d/\partial x_n = f_{d-1} = 0\} = \{0\}.$$

Then A. Dimca [**2**] proved that f is "quasitame", in particular it satisfies (C, P1, P2, P3) (via the results of [**7**, **8**, **12**]). For more properties of these polynomials (and more motivation for the present paper), see [**5**] (and the forthcoming joint papers of R. García López and the author).

The main result of this note is the following affine analogue of Sakamoto's local result:

Theorem. *Assume that the polynomials* $g : \boldsymbol{C}^m \to \boldsymbol{C}$, $h : \boldsymbol{C}^n \to \boldsymbol{C}$ *and* $f : \boldsymbol{C}^{m+n} \to \boldsymbol{C}$ $(f(x, y) = g(x) + h(y))$ *satisfy the condition* (C). *Assume that* $\Gamma(g)$, $\Gamma(h)$ *respectively* $\Gamma(f)$ *denote their Seifert forms associated with their Milnor fibration at infinity. Then*

$$\Gamma(f) = (-1)^{mn}\Gamma(g) \otimes \Gamma(h).$$

Remarks. 1. In [**9, 10**] the author proved the following fact. For any polynomial maps g and h (and f defined as above), the generic fiber F of f is the join space $F = G * H$, where G and H are the generic fibers of g and h respectively. (For the quasi-homogeneous case, see [**15**].) Moreover, in [**9**, **10**] a global Sebastiani-Thom type result is proved for the monodromy operators at infinity. If the condition (C) is satisfied, then $\tilde{H}_{n+m-1}(F) = \tilde{H}_{m-1}(G) \otimes \tilde{H}_{n-1}(H)$, and these are the only interesting homological groups of the fibers. The formula $\Gamma(f) = (-1)^{mn}\Gamma(g) \otimes \Gamma(h)$ must be understood via this identification.

2. In general, the set of spinnable structures provided by the Milnor fibrations at infinity of polynomial maps is different from the set of spinnable structures provided by the local Milnor fibrations of isolated singularities. In the case $n = 2$, this was clarified by W. Neumann [**13**] using splice diagrams: the only spinnable structures which can be represented by both local and global construction are exactly the spinnable structures provided by quasi-homogeneous maps.

3. Sakamoto's proof cannot be extended to the global situation. His proof is based on the existence of a continuous map $[0,1] \times (B_\varepsilon \setminus \{0\}) \to B_\varepsilon$, $(r,z) \to r \circ z$, with the properties: $1 \circ z = z$, $0 \circ z = 0$, $(rs) \circ z = r \circ (s \circ z)$, $f(r \circ z) = rf(z)$, and $|r \circ z|$ is a strictly increasing function of r. Since, in the global case we can have many singular fibers with many singular points, a similar map as above does not exist. Our proof is based on a similar construction as in [**9**, **10**].

4. The following question appears naturally: what spinnable structures are provided by Milnor fibration at infinity of polynomial maps?

Even in the local case, the corresponding question is still open, however some restrictions provided by algebraic spinnable structures already appeared in the literature. We think that the global problem is even more difficult.

The proof of the theorem.

Consider the polynomial maps $g : \boldsymbol{C}^m \to \boldsymbol{C}$ and $h : \boldsymbol{C}^n \to \boldsymbol{C}$. Fix a closed disc (centered at the origin) D_g (respectively D_h) such that $\Sigma_g \subset int(D_g)$ (respectively $\Sigma_h \subset int(D_h)$). Let D_f be another closed disc such that $int(D_f) \supset D_g + D_h$. Obviously: $\Sigma_f = \Sigma_g + \Sigma_h \subset int(D_f)$.

Consider the map $u : \boldsymbol{C}^m \times \boldsymbol{C}^n \to \boldsymbol{C} \times \boldsymbol{C}$, $u(x,y) = (g(x), h(y))$. The line $L_e = \{(c,d) \in \boldsymbol{C} \times \boldsymbol{C} : c + d = e\}$ intersects $D_g \times \boldsymbol{C}$ (respectively $\boldsymbol{C} \times D_h$) in $D_g(e)$ (respectively in $D_h(e)$). If $e \in \partial D_f$, then $D_g(e) \cap D_h(e) = \varnothing$. Moreover, there exists r sufficiently large such that for any $e \in \partial D_f$ the ball $B_r^2 = \{(c,d) \in \boldsymbol{C}^2 : |c|^2 + |d|^2 \le r^2\}$ satisfies $D_g(e) \cup D_h(e) \subset L_e \cap B_r^2$.

Now, for $e \in \partial D_f$, by lemma 2.3 [**9**] one has:

$$u : u^{-1}(L_e \setminus (\boldsymbol{C} \times \Sigma_h \cup \Sigma_g \times \boldsymbol{C})) \to L_e \setminus (\boldsymbol{C} \times \Sigma_h \cup \Sigma_g \times \boldsymbol{C})$$

is a C^∞ locally trivial fibration. Therefore, $u^{-1}(L_e)$ can be identified with $u^{-1}(L_e \cap B_r^2)$. Moreover, by (C), there exists $R_g \gg 0$ (respectively $R_h \gg 0$) such that:

(1) $(u^{-1}(L_e \cap B_r^2), u^{-1}(t))$ (for $t \in L_e \cap B_r^2$) has the homotopy type of

$$(u^{-1}(L_e \cap B_r^2) \cap (B_{R_g} \times B_{R_h}), u^{-1}(t) \cap (B_{R_g} \times B_{R_h})).$$

(2) $u^{-1}(L_e \cap B_r^2 \backslash (\boldsymbol{C} \times \Sigma_h \cup \Sigma_g \times \boldsymbol{C})) \cap (B_{R_g} \times B_{R_h}) \to L_e \cap B_r^2 \backslash (\boldsymbol{C} \times \Sigma_h \cup \Sigma_g \times \boldsymbol{C})$

is a locally trivial fibration with fiber

$$G \times H = (g^{-1}(c) \cap B_{R_g}) \times (h^{-1}(d) \cap B_{R_h}), \quad ((c,d) \text{ generic}).$$

(3) The restriction of the above fibration on $\partial(B_{R_g} \times B_{R_h})$:

$$u^{-1}(L_e \cap B_r^2 \backslash (\boldsymbol{C} \times \Sigma_h \cup \Sigma_g \times \boldsymbol{C})) \cap \partial(B_{R_g} \times B_{R_h}) \to L_e \cap B_r^2 \backslash (\boldsymbol{C} \times \Sigma_h \cup \Sigma_g \times \boldsymbol{C})$$

can be extended to a trivial fibration

$$u^{-1}(L_e \cap B_r^2) \cap \partial(B_{R_g} \times B_{R_h}) \to L_e \cap B_r^2,$$

with fiber $\partial(G \times H)$.

By the above facts, for $e \in \partial D_f$, one can identify the fiber $f^{-1}(e) = u^{-1}(L_e)$ with $(u^{-1}(L_e \cap B_r^2)) \cap (B_{R_g} \times B_{R_h})$. Actually, the $(2n+2m-2)$-dimensional manifold with boundary

$$(F, \partial F) = (f^{-1}(e) \cap B_{R_f}, f^{-1}(e) \cap \partial B_{R_f}) \quad (\text{for } R_f \gg 0)$$

can be identified with the manifold (with corners and boundary):

$$(u^{-1}(L_e \cap B_r^2) \cap (B_{R_g} \times B_{R_h}), u^{-1}(L_e \cap \partial B_r^2) \cup [u^{-1}(L_e \cap B_r^2) \cap \partial(B_{R_g} \times B_{R_h})]).$$

Fix $e_0 \in \partial D_f$. Consider a diffeomorpfism $v = v_{e_0} : \boldsymbol{R}^2 \to L_{e_0}$ such that $v^{-1}(D_g(e_0)) \subset (-\infty, 0) \times \boldsymbol{R}$ and $v^{-1}(D_h(e_0)) \subset (0, \infty) \times \boldsymbol{R}$ (cf. lemma 2.1. [**9**]). Take $l := v(\{0\} \times \boldsymbol{R}) \cap B_r^2$, this is the "segment" which separates $D_g(e_0)$ and $D_h(e_0)$ in $L_{e_0} \cap B_r^2$. Set $\mathscr{G} = v((-\infty, 0] \times \boldsymbol{R}) \cap B_r^2$ and $\mathscr{H} = v([0, \infty) \times \boldsymbol{R}) \cap B_r^2$; therefore $\mathscr{G} \cap \mathscr{H} = l$ and $\mathscr{G} \cup \mathscr{H} = L_{e_0} \cap B_r^2$, $D_g(e_0) \subset \mathscr{G}$ and $D_h(e_0) \subset \mathscr{H}$.

By lemma 2.3 [**9**] one has ($\approx$ denotes a diffeomorphism):

$$\begin{aligned} &(4) & u^{-1}(\mathscr{G}) \cap (B_{R_g} \times B_{R_h}) &\approx B_{R_g} \times H \\ && u^{-1}(\mathscr{H}) \cap (B_{R_g} \times B_{R_h}) &\approx G \times B_{R_h} \\ && u^{-1}(l) \cap (B_{R_g} \times B_{R_h}) &\approx G \times H \times l. \end{aligned}$$

Let $m_g : (G, \partial G) \to (G, \partial G)$ be a geometric monodromy at infinity of g (i.e. a characteristic map of the fibration (∞) of g at infinity) with $m_g|_{\partial G} = \text{identity}$. Similarly define m_h. Then the following holds:

(5) A geometric monodromy of $u : u^{-1}(\partial D_g(e_0)) \to \partial D_g(e_0)$ is given by $m_g \times id$; a geometric monodromy of $u : u^{-1}(\partial D_h(e_0)) \to \partial D_h(e_0)$ is given by $id \times m_h$.

By (P3), (i.e. by the identification of the fibration of f at infinity and the Milnor fibration at infinity), and using Alexander duality, similarly as in the local case, the Seifert form of f can be identified (modulo a sign) with the variation map

$$Var_f : H_{n+m-1}(F, \partial F) \to H_{n+m-1}(F).$$

(Similarly for g and h.) Hence the theorem can be reformulated in terms of the variation maps: $Var_f = \pm Var_g \otimes Var_h$. In the sequel, we verify this formula. For simlicity, we will assume that $n \geq 2$, $m \geq 2$. The case $n = 1$ or $m = 1$ can be proved similarly (if we replace H_0 by $\tilde{H}_0$).

In the sequel, we will identify the fiber F with:

$$u^{-1}(L_e \cap B_r^2) \cap (B_{R_g} \times B_{R_h}).$$

Consider its decomposition in $u^{-1}(\mathcal{G}) \cap F$ and $u^{-1}(\mathcal{H}) \cap F$. Then, by (4), for $q = n + m - 2$ and $q = n + m - 1$ one has: $H_q(u^{-1}(\mathcal{G}) \cap F) = H_q(u^{-1}(\mathcal{H}) \cap F) = 0$. Therefore, by Mayer-Vietoris argument, the boundary map:

$$\partial : H_{n+m-1}(F) \to H_{n+m-2}(u^{-1}(l) \cap F)$$

is an isomorphism. Geometrically, ∂ can be described as follows: if $[\gamma] \in H_{n+m-1}(F)$ and the cycle γ is in generic position with respect to $u^{-1}(l)$, then $\partial[\gamma] = [\gamma \cap u^{-1}(l)]$.

Let $P \in l$ be an arbitrary point on l. Then by the local triviality of u over l, the natural strong deformation retract $l \mapsto P$ induces an isomorphism:

$$\begin{aligned} r : H_{n+m-2}(u^{-1}(l) \cap F) &\to H_{n+m-2}(u^{-1}(P) \cap (B_{R_g} \times B_{R_h})) \\ &= H_{n+m-2}(G \times H) = H_{m-1}(G) \otimes H_{n-1}(H). \end{aligned}$$

Notice that $u^{-1}(l) \cap F$ is a manifold (with corners) of dimension $(2n + 2m - 3)$, and with boundary:

$$[u^{-1}(\partial l) \cap (B_{R_g} \times B_{R_h})] \cup [u^{-1}(l) \cap \partial(B_{R_g} \times B_{R_h})],$$

and by duality:

$$H_{n+m-1}(u^{-1}(l) \cap F, \partial(u^{-1}(l) \cap F)) = H^{n+m-2}(u^{-1}(l) \cap F).$$

Then, by a duality argument, or by similar Mayer-Vietoris argument as above, one can prove that the natural inclusion induces an isomorphism:

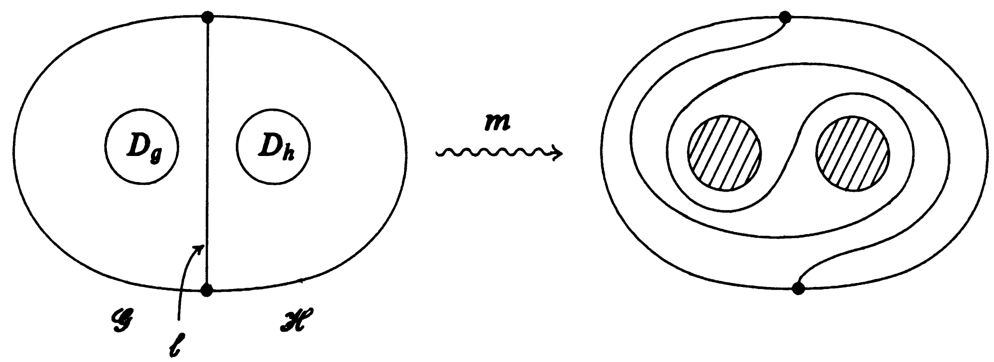
$$i : H_{n+m-1}(u^{-1}(l) \cap F, \partial(u^{-1}(l) \cap F)) \to H_{n+m-1}(F, \partial F).$$

We want to investigate the composition $V = r \circ \partial \circ Var \circ i$:

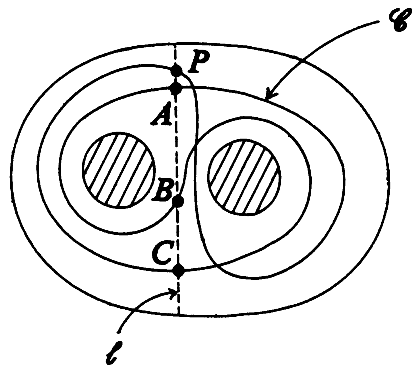
$$\begin{array}{ccc} H_{m-1}(G,\partial G) \otimes H_{n-1}(H,\partial H) \otimes H_1(l,\partial l) = H_{n+m-1}(u^{-1}(l) \cap F, \partial(u^{-1}(l) \cap F)) & \xrightarrow{i} & H_{n+m-1}(F,\partial F) \\ \Big\downarrow V & & \Big\downarrow Var_f \\ H_{m-1}(G) \otimes H_{n-1}(H) \xleftarrow{\ r\ } H_{n+m-2}(u^{-1}(l) \cap F) & \xleftarrow{\ \partial\ } & H_{n+m-1}(F) \end{array}$$

If $m_f : (F, \partial F) \to (F, \partial F)$ is the geometric monodromy of f at infinity (with $m_f|_{\partial F} =$ identify) then the variation map, by its very definition, is $Var[\gamma] = [m_f(\gamma) - \gamma]$. Let $[\alpha] \in H_{m-1}(G, \partial G)$ and $[\beta] \in H_{n-1}(H, \partial H)$ be the homology classes of the relative cycles α, respectively β. In order to find the map V, we wish to describe the intersection of the geometric cycle $m_f(\alpha \times \beta \times l) - (\alpha \times \beta \times l)$ with $u^{-1}(l)$.

Moving e along ∂D_f, we can construct a continuous family $v_e : \boldsymbol{R}^2 \to L_e$ with similar properties as v_{e_0} (cf. [**9**]). Then the monodromy action on the line $l = l_{e_0}$ is the following:



Then the cycle $\gamma = m_f(\alpha \times \beta \times l) - (\alpha \times \beta \times l)$ can be identified with the total space of a fibering p with fiber $\alpha \times \beta$ and base space the loop $\mathscr{C}$ in L_{e_0}:



The intersection $\gamma \cap u^{-1}(l)$ is the collection of the fibers of p over the (oriented) intersection $\mathscr{C} \cap \ell = P - A + B - C$. Now, (5) and the retract $r' : u^{-1}(l) \to u^{-1}(P)$ give the identifications:

$$r'(u^{-1}(P)) = \alpha \times \beta, \quad r'(u^{-1}(A)) = \alpha \times m_h(\beta),$$
$$r'(u^{-1}(B)) = m_g(\alpha) \times m_h(\beta), \quad r'(u^{-1}(C)) = m_g(\alpha) \times \beta.$$

Therefore:

$$\begin{aligned} V[\alpha \times \beta] &= \pm [\alpha \times \beta - \alpha \times m_h(\beta) + m_g(\alpha) \times m_h(\beta) - m_g(\alpha) \times \beta] \\ &= \pm [(m_g(\alpha) - \alpha) \times (m_h(\beta) - \beta)] = \pm\, Var_g(\alpha) \otimes Var_h(\beta). \end{aligned}$$

This gives $\Gamma(f) = \pm \Gamma(g) \otimes \Gamma(h)$.

The sign $\pm$ is a universal sign which depends only on n and m, and can be determined also by a careful study of the orientation of the cycles which are intersected. But, this sign does not depend on the polynomials g and h, therefore it is the same as in the case of generic homogeneous polynomials. Since for homogeneous polynomials the local and global theory agree, this sign is the same as in the local theory (i.e. as in Sakamoto's theorem) (provided that we use the same orientation conventions).

This ends the proof of the theorem.

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