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Galois covering singularities I

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Abstract. We give a necessary condition for Galois covering singularities to be log-terminal or log-canonical singularities, which is also sufficient under a certain restriction on the branch loci of the covering maps. We also give a method constructing explicitly resolutions of 2-dimensional Abel covering singularities.

Introduction.

Let Y be an open neighborhood of 0 in $\boldsymbol{C}^n$ and let $\pi : X \to Y$ be a (branched) finite Galois covering of Y, i.e., π is a proper finite holomorphic map from a normal analytic space X to Y and $\mathrm{Aut}(\pi) := \{g \in \mathrm{Aut}(X) \mid \pi \circ g = \pi\}$ acts transitively on the fiber $\pi^{-1}(y)$ of π for each point y in Y. We assume that $\pi^{-1}(0)$ consists of only one point x_0. Professor Namba proposed to call such a singularity (X, x_0) a Galois singularity and to study it. Let $B_1, B_2, \dots, B_s$ be the irreducible components of branch locus $\{y \in Y \mid \# \pi^{-1}(y) < \deg \pi\} = \pi(\{x \in X \mid \pi \text{ is not biholomorphic around } x\})$ of π and let r_j be the ramification index of π along B_j, i.e., $r_j = \deg \pi / \max\{\# \pi^{-1}(y) \mid y \in B_j\}$. Here we note that for any point x in $\pi^{-1}(B_j \backslash \mathrm{Sing}(B_1 + \cdots + B_s))$, π is expressed as $(z_1, z_2, \dots, z_n) \mapsto (z_1^{r_j}, z_2, \dots, z_n)$ by suitable local coordinate systems on neighborhoods of x and $\pi(x)$ (see [2]). Let $B_\pi = r_1 B_1 + r_2 B_2 + \cdots + r_s B_s$. We are interested in the following two problems.

PROBLEM 1. *Describe the properties and invariants of the singularity* (X, x_0) *using those of* B_π *and the covering transformation group* $\mathrm{Gal}(X/Y) := \mathrm{Aut}(\pi)$.

PROBLEM 2. *Determine all Galois coverings* $\pi : (X, x_0) \to (Y, 0)$ *with* $B_\pi = D$ *for a given divisor* D *on an open neighborhood* Y *of* 0 *in* $\boldsymbol{C}^n$.

Dimca showed that the set of all Abel coverings $\pi : X \to Y$ of Y with $B_\pi = D$ is completely described by D (Theorem 3.3 in [**1**]).

In this paper, we give a partial answer to these problems. In Section 1, we give a necessary condition for (X, x_0) to be a log-terminal or log-canonical singularity, which is also sufficient under a certain restriction on B_π. In Section 2, we give some results on Problem 2 in the non Abel covering case. In Section 3, we construct resolutions of 2-dimensional Abel covering singularities. The self intersection number, the genus of each irreducible component and the dual graphs of their exceptional sets are explicitly obtained from the data on B_π and $\mathrm{Gal}(X/Y)$. In Section 4, we give a necessary and sufficient condition for a Galois covering singularity to be a quasi-Gorenstein singularity.

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I would like to thank the refree who pointed out me the existence of [**1**].

1. On Problem 1.

Let $\pi : X \to Y$ be a finite Galois covering of an open neighborhood Y of 0 in $\boldsymbol{C}^n$ and assume that $\pi^{-1}(0) = \{x_0\}$. Let $B_\pi = r_1 B_1 + r_2 B_2 + \cdots + r_s B_s$ be as in Introduction.

PROPOSITION 1. *(X, x_0) is a $\boldsymbol{Q}$-Gorenstein singularity, i.e., there exists a nowhere vanishing holomorphic r-ple n-form on $X \backslash \mathrm{Sing}(X)$, where r is the least common multiple of $r_1, r_2, \ldots$ and r_s.*

PROOF. Let $(z_1, z_2, \ldots, z_n)$ be a coordinate system of $\boldsymbol{C}^n$ and let

$$\phi = \frac{(dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n)^r}{f_1^{r(r_1-1)/r_1} f_2^{r(r_2-1)/r_2} \cdots f_s^{r(r_s-1)/r_s}},$$

where $f_1, f_2, \ldots$ and f_s are defining equations of $B_1, B_2, \ldots$ and B_s, respectively. Then $\pi^*\phi$ is a nowhere vanishing holomorphic r-ple n-form on $X \backslash \pi^{-1}(\mathrm{Sing}(B_1 + \cdots + B_s))$. Since the codimension of $\mathrm{Sing}(B_1 + \cdots + B_s)$ is greater than 1, $\pi^*\phi$ is extended to $X \backslash \mathrm{Sing}(X)$, as a holomorphic r-ple n-form. □

By the above proposition, we can classify the singularity (X, x_0) into the following three types (see [**3**]). Let $\lambda : (\tilde{X}, E) \to (X, x_0)$ be a resolution of (X, x_0) and let ψ be a nowhere vanishing holomorphic r-ple n-form on $X \backslash \mathrm{Sing}(X)$.

I. (X, x_0) is log-terminal, i.e., the vanishing order of $\lambda^*\psi$ is greater than $-r$ along all irreducible components of the exceptional set E of λ.

II. (X, x_0) is not log-terminal and log-canonical, i.e., the vanishing order of $\lambda^*\psi$ is not smaller than $-r$ along all irreducible components of the exceptional set E of λ and equal to $-r$ along at least one irredubcible component.

III. (X, x_0) is not log-canonical and then $\lim_{m\to\infty} \sup \delta_m(X, x_0)/m^{n-1} > 0$.

For example, if a cone V over a projective manifold M is a Galois covering singularity, then it is of type I, II or III, accordingly as $\kappa(M) = -\infty$, 0 or $\dim M$. While, if $0 < \kappa(M) < \dim M$, then V never can be a Galois covering singularity.

For a holomorphic function $f = \sum_{v \in \boldsymbol{Z}^n_{\geq 0}} c_v z^v$ on Y, let $\mathrm{Supp}(f) = \{v \in \boldsymbol{Z}^n_{\geq 0} \mid c_v \neq 0\}$ and let $\Gamma_+(f)$ be the Newton polytope of f, i.e., the convex hull of $\bigcup_{v \in \mathrm{Supp}(f)} (v + \boldsymbol{R}^n_{\geq 0})$, where $z^t(v_1, v_2, \ldots, v_n) = z_1^{v_1} z_2^{v_2} \cdots z_n^{v_n}$.

DEFINITION.

$$\Gamma_+(B_\pi) = \left(1 - \frac{1}{r_1}\right)\Gamma_+(f_1) + \left(1 - \frac{1}{r_2}\right)\Gamma_+(f_2) + \cdots + \left(1 - \frac{1}{r_s}\right)\Gamma_+(f_s),$$

where $f_1, f_2, \ldots$ and f_s are defining equations of $B_1, B_2, \ldots$ and B_s, respectively.

For a face Δ of $\Gamma_+(B_\pi)$, there exists a point u in $\boldsymbol{R}^n_{\geq 0}$ such that $\Delta = \Delta(u) := \{v \in \Gamma_+(B_\pi) \mid \langle v, u \rangle = d(u)\}$, where $d(u) = \min\{\langle v, u \rangle \mid v \in \Gamma_+(B_\pi)\}$. Let $\Delta_j = \{v \in \Gamma_+(f_j) \mid \langle v, u \rangle = d_j(u)\}$, where $d_j(u) = \min\{\langle v, u \rangle \mid v \in \Gamma_+(f_j)\}$. Then $d(u) = \sum_{j=1}^{s}(1 - 1/r_j) d_j(u)$ and $\Delta = (1 - 1/r_1)\Delta_1 + (1 - 1/r_2)\Delta_2 + \cdots + (1 - 1/r_s)\Delta_s$. Here we note that Δ_j are determined uniquely by Δ, although u with $\Delta = \Delta(u)$ are not unique.

Theorem 2. *If* (X, x_0) *is log-canonical* (*resp. log-terminal*), *then* ${}^t(1,1,\dots,1) \in \Gamma_+(B_\pi)$ (*resp.* $\mathrm{Int}(\Gamma_+(B_\pi))$).

Moreover, the converse holds, if $f_1, f_2, \dots$ *and* f_s *satisfy the condition*:

$(*)$ *For each proper face* Δ *of* $\Gamma_+(B_\pi)$, *the varieties in* $(\boldsymbol{C}^\times)^n$ *defined by* $f_{j\Delta} = 0$ *are non-singular and cross transversally each other, where* $f_{j\Delta}$ *are the partial sums* $\sum_{v \in \boldsymbol{Z}^n_{\geq 0} \cap \Delta_j} c_v z^v$ *of* $f_j = \sum_{v \in \boldsymbol{Z}^n_{\geq 0}} c_v z^v$ *on* Δ_j.

Proof. Let $\Gamma^*(B_\pi)$ be the dual Newton diagram of $\Gamma_+(B_\pi)$, i.e., $\Gamma^*(B_\pi) = \{\Delta^* \mid \Delta \text{ are faces of } \Gamma_+(B_\pi)\}$, where $\Delta^* = \{u \in \boldsymbol{R}^n_{\geq 0} \mid \Delta(u) \supset \Delta\}$. Let

$$\lambda : T_{\boldsymbol{Z}^n}\,\mathrm{emb}(\Gamma^*(B_\pi)) \to T_{\boldsymbol{Z}^n}\,\mathrm{emb}(\{\text{faces of } \boldsymbol{R}^n_{\geq 0}\}) = \boldsymbol{C}^n$$

be the holomorphic map between toric varieties induced from the subdivision $\Gamma^*(B_\pi)$ of $\{\text{faces of } \boldsymbol{R}^n_{\geq 0}\}$ and let $Z = \lambda^{-1}(Y)$. Let $E_\sigma = \overline{\mathrm{orb}(\sigma)}$ for 1-dimensional cones σ in $\Gamma^*(B_\pi)$. Then the vanishing order α_σ along E_σ of the pull-back $\lambda^*\phi$ of ϕ in the proof of Proposition 1 is equal to

$$r\left(-1 + \langle {}^t(1,1,\dots,1), u\rangle - \sum_{j=1}^{s}\left(1 - \frac{1}{r_j}\right) d_j(u)\right),$$

because the vanishing order of $\lambda^*(dz_1/z_1 \wedge dz_2/z_2 \wedge \cdots \wedge dz_n/z_n)$ is equal to -1 along all E_σ, where u are the primitive elements in $\boldsymbol{Z}^n$ spanning σ. Hence $\alpha_\sigma \geq -r$ (resp. $> -r$) for all 1-dimensional cones σ in $\Gamma^*(B_\pi)$, if and only if ${}^t(1,1,\dots,1) \in \Gamma_+(B_\pi)$ (resp. $\in \mathrm{Int}(\Gamma_+(B_\pi))$).

On the other hand, let W be the normalization of $X \times_Y Z$, let $\tilde{\pi} : W \to Z$ and $\theta : W \to X$ be the projections. For any 1-dimensional cone σ in $\Gamma^*(B_\pi)$ and for any irreducible component F_σ of $\tilde{\pi}^{-1}(E_\sigma)$, the vanishing order $(\alpha_\sigma + r)r_\sigma - r$ of $(\pi \circ \theta)^*\phi = (\lambda \circ \tilde{\pi})^*\phi$ along F_σ is greater than (resp. equal to) $-r$, if and only if α_σ is so, where r_σ is the ramification index of $\tilde{\pi}$ along E_σ.

Next, assume that the condition $(*)$ is satisfied. Take a subdivision Σ of $\Gamma^*(B_\pi)$ consisting of non-singular cones and replace $\Gamma^*(B_\pi)$ with Σ in the above definition of λ. Then Z is non-singular and $\lambda^{-1}(B_1 + B_2 + \cdots + B_s)$ is normal crossing near $\lambda^{-1}(0)$. Hence for any point p in $\lambda^{-1}(0)$, there exist an open neighborhood U_p of p and a local coordinate system $(z_1, z_2, \dots, z_n)$ on U_p such that $\lambda^{-1}(B_1 + B_2 + \cdots + B_s) \cap U_p \subset \{z_1 z_2 \cdots z_n = 0\}$. Then $f = (\lambda^*\phi)_{|U_p}/(dz_1/z_1 \wedge \cdots \wedge dz_n/z_n)^r$ is a holomorphic function on U_p, if the vanishing order α_σ of $\lambda^*\phi$ is not smaller than $-r$ along all irreducible components E_σ of $\lambda^{-1}(0)$ and vanishes along $\lambda^{-1}(B_1 + B_2 + \cdots + B_s) \cap U_p$, if α_σ is greater than $-r$. Therefore, W has only toric quotient singularities and for any toric resolution $\varpi : V \to W$ of W, the vanishing order of $(\lambda \circ \tilde{\pi} \circ \varpi)^*\phi$ is greater or not smaller than $-r$ along all irreducible components of $(\theta \circ \varpi)^{-1}(x_0)$, if that of $\lambda^*\phi$ is so along those of $\lambda^{-1}(0)$, because $(\tilde{\pi} \circ \varpi)^*_{|(\tilde{\pi} \circ \varpi)^{-1}(U_p)}(dz_1/z_1 \wedge \cdots \wedge dz_n/z_n)$ has poles of order 1 along all irreducible components of $(\tilde{\pi} \circ \varpi)^{-1}(\{z_1 z_2 \cdots z_n = 0\})$. □

Example 1. If $n = 2$ and $B_\pi = r_1 B_1 + r_2 B_2 + r_3 B_3$, where B_1, B_2, B_3 are defined by $z_1 = 0$, $z_2 = 0$, $z_1^a + z_2^b = 0$ $(g.c.d.(a,b) = 1)$, respectively and r_1, r_2, r_3 are positive integers, then ${}^t(1,1) \in \mathrm{Int}(\Gamma_+(B_\pi))$ (resp. $\partial\Gamma_+(B_\pi)$), if and only if

$$\frac{1}{ar_1}+\frac{1}{br_2}+\frac{1}{r_3}>1 \quad (\text{resp.} = 1).$$

Here, the case that $r_1 = 1$ (resp. $r_1 = r_2 = 1$) implies that $B_\pi = r_2B_2 + r_3B_3$ (resp. r_3B_3).

EXAMPLE 2. If $n = 2$ and $B_\pi = 2\{z_1(z_1 + z_2^2)(z_1 + cz_2^2) = 0\}$ $(c \neq 0, 1)$, then ${}^t(1,1)$ is on a 1-dimensional face of $\Gamma_+(B_\pi)$.

EXAMPLE 3. If $n = 2$ and $B_\pi = 2\{(z_1^2 + z_2^p)(z_2^2 - z_1^q) = 0\}$ $(p, q \geq 2)$, then ${}^t(1,1) \in \partial\Gamma_+(B_\pi)$.

EXAMPLE 4. If $n = 3$ and $B_\pi = 2\{(z_1^2 + z_2^4 + z_3^4)(z_1^4 + z_2^2 + z_3^4)(z_1^4 + z_2^4 + z_3^2) = 0\}$, then ${}^t(1,1,1)$ is a vertex of $\partial\Gamma_+(B_\pi)$.

2. On Problem 2.

Let Y be a simply connected open neighborhood of 0 in $\boldsymbol{C}^n$ and let $D = r_1D_1 + r_2D_2 + \cdots + r_sD_s$ be a divisor on Y. Here, we assume that r_j are integers greater than 1 and that D_j are irreducible reduced. Let

$$\tilde{Y} = \{(w_1, w_2, \ldots, w_s, y) \in \boldsymbol{C}^s \times Y \mid w_1^{r_1} - f_1(y) = \cdots = w_s^{r_s} - f_s(y) = 0\},$$

where $f_1, f_2, \ldots$ and f_s are defining equations of $D_1, D_2, \ldots$ and D_s, respectively, let σ_j be the automorphisms of $\tilde{Y}$ defined by

$$\sigma_j : (w_1, \ldots, w_s, y) \mapsto (w_1, \ldots, w_{j-1}, \varepsilon_j w_j, w_{j+1}, \ldots, w_s, y),$$

where $\varepsilon_j = \exp(2\pi\sqrt{-1}/r_j)$ and let $\mu : \tilde{Y} \to Y$ be the projection. Then μ is an Abel covering of Y with $B_\mu = D$ and the covering transformation group $\mathrm{Gal}(\tilde{Y}/Y)$ is generated by $\sigma_1, \sigma_2, \ldots, \sigma_s$.

PROPOSITION 3. *$\tilde{Y}$ is a normal.*

PROOF. First, we note that $\tilde{Y}_0 := \mu^{-1}(Y_0)$ is non-singular, where $Y_0 = Y \backslash \mathrm{Sing}(D_{\mathrm{red}})$. Let U be an open neighborhood of $0 \in Y$, let h be a holomorphic function on $\tilde{U}_0 := \mu^{-1}(U \cap Y_0)$ and let

$$h_{c_1,\ldots,c_s} = \sum_{0 \leq \alpha_1 < r_1, \ldots, 0 \leq \alpha_s < r_s} \varepsilon_1^{-c_1\alpha_1} \cdots \varepsilon_s^{-c_s\alpha_s} (\sigma_1^{\alpha_1} \cdots \sigma_s^{\alpha_s})^* h,$$

for $0 \leq c_1 < r_1, \ldots, 0 \leq c_s < r_s$. Then

$$\sum_{0 \leq c_1 < r_1, \ldots, 0 \leq c_s < r_s} h_{c_1,\ldots,c_s} = r_1 \cdots r_s h$$

and $\sigma_j^* h_{c_1,\ldots,c_s} = \varepsilon_j^{c_j} h_{c_1,\ldots,c_s}$. Hence $h_{c_1,\ldots,c_s}/(w_1^{c_1} \cdots w_s^{c_s})$ is a $\mathrm{Gal}(\tilde{Y}/Y)$-invariant holomorphic function on $\tilde{U}_0$. Since Y is non-singular and the codimension of $Y \backslash Y_0 = \mathrm{Sing}(D_{\mathrm{red}})$ is greater than 1, there exists a holomorphic function $\overline{h_{c_1,\ldots,c_s}}$ on U the pull-back $\mu^*\overline{h_{c_1,\ldots,c_s|U_0}}$ of whose restriction to $U_0 := U \cap Y_0$ is equal to $h_{c_1,\ldots,c_s}/(w_1^{c_1} \cdots w_s^{c_s})$. Then $\bar{h} = 1/(r_1 \cdots r_s) \sum \mu^*\overline{h_{c_1,\ldots,c_s}} w_1^{c_1} \cdots w_s^{c_s}$ is a holomorphic function on $\mu^{-1}(U)$ and $\bar{h}_{|\tilde{U}_0} = h$. □

Let H be a subgroup of $\mathrm{Gal}(\tilde{Y}/Y)$ and let $\mu_H : \tilde{Y}/H \to Y$ be the natural map induced by μ. Then μ_H is an Abel covering. Moreover, $B_{\mu_H} = D$, if and only if $\sigma_j^{\alpha_j} \notin H$ for $1 \le \alpha_j < r_j$. By Theorem 3.3 in [**1**], we have:

Theorem 4. *For any Abel covering $\pi : X \to Y$ with $B_\pi = D$, there exist a subgroup H of $\mathrm{Gal}(\tilde{Y}/Y)$ and a biholomorphic map $\phi : X \to \tilde{Y}/H$ such that $\mu_H \circ \phi = \pi$.*

Next, let $Y_0 = Y \setminus \mathrm{Sing}(D_{\mathrm{red}})$ and let $\tilde{Y}_0 = \mu^{-1}(Y_0)$. Let $\lambda : \tilde{W} \to \tilde{Y}_0$ be a universal covering.

Proposition 5. *$\mu \circ \lambda : \tilde{W} \to Y_0$ is a Galois covering. The kernel of $\mathrm{Gal}(\tilde{W}/Y_0) \to \mathrm{Gal}(\tilde{Y}_0/Y_0)$ is the commutators group of $\mathrm{Gal}(\tilde{W}/Y_0)$.*

Proof. There exists an automorphism $\tilde{g}$ of $\tilde{W}$ satisfying $\lambda \circ \tilde{g} = g \circ \lambda$ for each element g in $\mathrm{Gal}(\tilde{Y}/Y)$, because λ and $g \circ \lambda$ are both universal coverings of $\tilde{Y}_0$. Hence the subgroup of $\mathrm{Aut}(\tilde{W})$ generated by $\tilde{g}$ for all $g \in \mathrm{Gal}(\tilde{Y}/Y)$ and $\pi_1(\tilde{Y}_0)$ acts transitively on the fibers of $\mu \circ \lambda$.

Next, let H be the commutators group of $\mathrm{Gal}(\tilde{W}/Y_0)$. Since $\mathrm{Gal}(\tilde{Y}_0/Y_0)$ is an abelian group, there exists a surjective homomorphism $\mathrm{Gal}(\tilde{W}/Y_0)/H \to \mathrm{Gal}(\tilde{Y}_0/Y_0)$. Suppose that this homomorphism is not isomorphic. Then the degree of the Abel covering $\tilde{W}/H \to Y_0$ induced by $\mu \circ \lambda$ is greater than $\deg(\mu)$ and the ramification index along D_j of the covering is equal to r_j. However, replacing Y in the proof of Theorem 4 with Y_0, we see that the degree of any Abel covering π' of Y_0 with $B_{\pi'} = Y_0 \cap D$ is not greater than $\deg(\mu)$, a contradiction. □

Theorem 6. *For any Galois covering $\pi : X \to Y$ with $B_\pi = D$, there exist a subgroup H of $\mathrm{Gal}(\tilde{W}/Y_0)$ and a biholomorphic map $\tau : \tilde{W}/H \to X_0 := \pi^{-1}(Y_0)$ such that $\pi \circ \tau : \tilde{W}/H \to Y_0$ is equal to the natural map induced by $\mu \circ \lambda$.*

Proof. Let W' be an irreducible component of $\tilde{W} \times_{Y_0} X_0$. Then the composite of the normalization of W' and the projection $W' \to \tilde{W}$ is an unramified covering. Hence $W' \to \tilde{W}$ is biholomorphic, because $\tilde{W}$ is simply connected. Next, let $G = \{g \in \mathrm{Gal}(\tilde{W}/Y_0) \oplus \mathrm{Gal}(X_0/Y_0) \mid gW' = W'\}$ and let $p_1 : G \to \mathrm{Gal}(\tilde{W}/Y_0)$ (resp. $p_2 : G \to \mathrm{Gal}(X_0/Y_0)$) be the restriction to G of the projection $\mathrm{Gal}(\tilde{W}/Y_0) \oplus \mathrm{Gal}(X_0/Y_0) \to \mathrm{Gal}(\tilde{W}/Y_0)$ (resp. $\mathrm{Gal}(X_0/Y_0)$). Then p_1 is an isomorphism and p_2 is a surjection. Hence the map $\tilde{W}/H \to X_0$ induced by the composite $\tilde{W} \simeq W' \to X_0$ is biholomorphic, where $H = p_1(\ker(p_2))$. □

Example 5. Let $D = B_\pi$ in Example 1 in Section 1. Assume that $1/(ar_1) + 1/(br_2) + 1/r_3 > 1$. Then $\tilde{Y}$ is log-terminal, by Theorem 2. Hence $\tilde{Y}$ is a quotient singularity because $n = 2$ (see [**4**]). Therefore, $\mathrm{Gal}(\tilde{W}/Y_0)$ is finite and $\tilde{W}$ is biholomorphic to the complement of a point of a non-singular surface. Indeed, there exists a finite subgroup G of $GL(2, \boldsymbol{C})$ isomorphic to $\mathrm{Gal}(\tilde{W}/Y_0)$ such that $\boldsymbol{C}^2/G$ is non-singular and that $B_{[\boldsymbol{C}^2 \to \boldsymbol{C}^2/G]} \simeq D$. In the table below, we show generators of the group G. Let

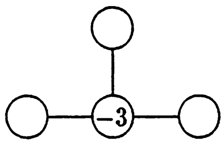
$$A_r = \begin{pmatrix} \rho_r & 0 \\ 0 & \rho_r \end{pmatrix}, \quad B_r = \begin{pmatrix} \rho_r & 0 \\ 0 & \rho_r^{-1} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$S = \frac{1}{2}\begin{pmatrix} -1+\sqrt{-1} & -1+\sqrt{-1} \\ 1+\sqrt{-1} & -1-\sqrt{-1} \end{pmatrix}, \quad V = \begin{pmatrix} \dfrac{\sqrt{-1}}{2} & \beta - \sqrt{-1}\gamma \\ -\beta - \sqrt{-1}\gamma & -\dfrac{\sqrt{-1}}{2} \end{pmatrix},$$

where $\rho_r = \exp(2\pi\sqrt{-1}/r)$, $\beta = (1-\sqrt{5})/4$ and $\gamma = (1+\sqrt{5})/4$.

a	b	r_1	r_2	r_3	Generators of G	a	b	r_1	r_2	r_3	Generators of G
1	*	2	*	2	$A_{2r_2}B_{2r_2},\ B_{2br_2},\ C$	1	4	2	1	3	$A_6,\ A_{12}B_8,\ S$
2	odd	1	*	2	$A_{2r_2}B_{2r_2},\ B_{br_2},\ C$	2	3	2	1	2	$A_4,\ B_8,\ S$
1	*	2	1	2	$B_{2b},\ C$	2	3	1	1	4	$A_4,\ A_8B_8,\ S$
2	odd	1	1	2	$B_b,\ C$	3	4	1	1	2	$A_4B_8,\ S$
1	1	2	3	3	$A_{12},\ B_4,\ S$	1	1	2	3	5	$A_{60},\ B_4,\ S,\ V$
1	2	3	1	3	$A_6,\ B_4,\ S$	1	2	3	1	5	$A_{30},\ B_4,\ S,\ V$
1	3	3	1	2	$A_4,\ B_4,\ A_{12}S$	1	3	2	1	5	$A_{20},\ B_4,\ S,\ V$
2	3	1	1	3	$B_4,\ A_6S$	1	5	2	1	3	$A_{12},\ B_4,\ S,\ V$
1	1	2	3	4	$A_{24},\ B_8,\ S$	2	3	1	1	5	$A_{10},\ B_4,\ S,\ V$
1	2	3	2	2	$A_{12},\ B_8,\ S$	2	5	1	1	3	$A_6,\ B_4,\ S,\ V$
1	2	3	1	4	$A_{12},\ A_{24}B_8,\ S$	3	5	1	1	2	$A_4,\ B_4,\ S,\ V$
1	3	2	1	4	$A_8,\ B_8,\ S$						

For subgroups H of G such that $B_{[\boldsymbol{C}^2/H \to \boldsymbol{C}^2/G]} = B_{[\boldsymbol{C}^2 \to \boldsymbol{C}^2/G]}$, i.e., H have no fixed points on $\boldsymbol{C}^2 \backslash \{0\}$, the singularities $\boldsymbol{C}^2/H$ are rational double points of type D_l, E_6, E_7, E_8, cyclic quotient singularities and a singularity with a resolution the dual graph of whose exceptional set is the following:



When $1/(ar_1) + 1/(br_2) + 1/r_3 = 1$, $\tilde{Y}$ is a simple elliptic singularity.

3. Resolutions of two-dimensional Abel covering singularities.

We keep the notations of the previous section. Let $n = 2$ and let H be a subgroup of $\mathrm{Gal}(\tilde{Y}/Y)$ satisfying the condition: $\sigma_j^{\alpha_j} \notin H$ for $1 \le \alpha_j \le r_j - 1$. Let $X = \tilde{Y}/H$ and let $\pi = \mu_H$. Then $\pi : X \to Y$ is an Abel covering with $B_\pi = D$. We may assume that $Y_0 = Y \backslash \{0\}$, by replacing Y with an open small neighborhood of 0. Let $\theta : Z \to Y$ be an embedded resolution of D, i.e., θ is a holomorphic map such that the restriction $\theta_{|\theta^{-1}(Y_0)}$ of θ to $\theta^{-1}(Y_0)$ is biholomorphic and that the reduced inverse image divisor $\theta^{-1}(D)_{\mathrm{red}} = \sum_{j=1}^{t} E_j$ of D is normal crossing. Here we may assume that E_j are the proper transformations of D_j under the the map θ for $1 \le j \le s$ and that E_j are irreducible for $s+1 \le j \le t$. Then $\theta^{-1}(0) = \sum_{j=s+1}^{t} E_j$, because $Y_0 = Y \backslash \{0\}$. Let $\tilde{\sigma}_j$ be elements in $\pi_1(Y \backslash D)$ rounding E_j once in the positive direction and let τ_j be their images $\rho(\tilde{\sigma}_j)$ under the quotient map $\rho : \pi_1(Y \backslash D) \to F := \mathrm{Gal}(X/Y)$. Then τ_j are the images of σ_j under the quotient map $\mathrm{Gal}(\tilde{Y}/Y) \to F$ for $1 \le j \le s$ and $1/(2\pi\sqrt{-1})\int_{\tilde{\sigma}_j} df_i/f_i = \delta_{ij}$ for $1 \le i, j \le s$. On the other hand, the zero divisors $[\theta^* f_i]$ of $\theta^* f_i$ are expressed as $E_i + \sum_{j=s+1}^{t} c_{ij} E_j$, where c_{ij} are positive integers. Then $1/(2\pi\sqrt{-1})\int_{\tilde{\sigma}_j} df_i/f_i = c_{ij}$. Hence $\tau_j = \sum_{i=1}^{s} c_{ij}\tau_i$ for $s+1 \le j \le t$. For each positive integer $k \le t$, let F_k be the subgroup of F generated by τ_k and all τ_j with $E_j \cap E_k \ne \emptyset$. When $k \ne l$ and $E_k \cap E_l \ne \emptyset$, let F_{kl} be the subgroup of F generated by τ_k and τ_l. Let $\nu : W \to Z$ (resp. $\lambda : W \to X$) be the composite of the normalization $W \to X \times_Y Z$ of $X \times_Y Z$ and the projection $X \times_Y Z \to Z$ (resp. X). Then F naturally acts on W and the restriction $\lambda_{|W \setminus \lambda^{-1}(x_0)}$ of λ to $W \setminus \lambda^{-1}(x_0)$ is biholomorphic, where $\{x_0\} = \pi^{-1}(0)$.

Proposition 7. *The number of the irreducible components (resp. the points) of $\nu^{-1}(E_k)$ (resp. $\nu^{-1}(E_k \cap E_l)$) is equal to $|F/F_k|$ (resp. $|F/F_{kl}|$).*

Proof. Let $k \ne l$ and assume that $E_k \cap E_l \ne \emptyset$. Then $E_k \cap E_l$ consists of one point. Let V be a small neighborhood of the point $E_k \cap E_l$ and let U be a connected component of $\nu^{-1}(V)$. Since $E_k + E_l$ is normal crossing, $\mathrm{Gal}(U/V) = F_{kl}$ and $\nu^{-1}(E_k) \cap U$ is irreducible. Hence $\tau_l \tilde{E}_k = \tilde{E}_k$ for any irreducible component $\tilde{E}_k$ of $\nu^{-1}(E_k)$. Therefore, $g\tilde{E}_k = \tilde{E}_k$ for all g in F_k. Since the covering map $\tilde{E}_k/F_k \to E_k$ is unramified and E_k is simply connected, $\{g \in F \mid g\tilde{E}_k = \tilde{E}_k\} = F_k$. □

Next, we construct the dual graph of $\tilde{E} := \lambda^{-1}(x_0) = (\theta \circ \nu)^{-1}(0)$. Let Δ be the dual graph of $E' := \sum_{j=s+1}^{t} E_j = \nu(\tilde{E})$. For a vertex (resp. an edge) α of Δ, let E'_α be the corresponding irreducible curve (resp. double point) of E' and let $F_\alpha = F_k$ (resp. F_{kl}), if $E'_\alpha = E_k$ (resp. $E_k \cap E_l$). Let $F_{\alpha 1} = F_\alpha, F_{\alpha 2}, \dots, F_{\alpha |F/F_\alpha|}$ be the conjugate classes of F_α and let $\alpha_1, \alpha_2, \dots, \alpha_{|F/F_\alpha|}$ be copies of α. Then we obtain a complex $\tilde{\Delta} = \bigcup_{\alpha \in \Delta} \bigcup_{i=1}^{|F/F_\alpha|} \alpha_i$, where α_i is a face of β_j if and only if α is a face of β and $F_{\alpha i} \supset F_{\beta j}$.

Proposition 8. *$\tilde{\Delta}$ is homeomorphic to the dual graph of $\tilde{E}$.*

Proof. Since Δ is a tree, we can choose an irreducible curve or a point $\tilde{E}_{\alpha 1}$ of $\nu^{-1}(E'_\alpha)$ for each $\alpha \in \Delta$ so that $\tilde{E}_{\beta 1} \in \tilde{E}_{\alpha 1}$, if α is a proper face of β , i.e., β is an edge of Δ and α is a vertex which is an end of β. Let $\tilde{E}_{\alpha i} = g\tilde{E}_{\alpha 1}$ for $g \in F_{\alpha i}$. Then $\tilde{E}_{\beta j} \in \tilde{E}_{\alpha i}$, if and only if α is a proper face of β and $F_{\alpha i} \supset F_{\beta j}$. □

We note that $\nu^{-1}(Z_0)$ is non-singular, where $Z_0 = Z \setminus (\bigcup_{1 \le k < l \le t} E_k \cap E_l)$. However, the inverse images of the double points $E_k \cap E_l$ of $\sum_{j=1}^{t} E_j$ under the map ν may be singular points on W. Let $k \ne l$ and assume that $E_k \cap E_l \ne \emptyset$. Let $\gamma : \boldsymbol{Z}^2 \to F$ be the homomorphism sending (a, b) to $a\tau_k + b\tau_l$ and let $N = \ker(\gamma)$.

Proposition 9. *For any point p in $\nu^{-1}(E_k \cap E_l)$, there exist an open neighborhood U_p of p and an inclusion $i : U_p \hookrightarrow T_N \,\mathrm{emb}(\{\text{faces of } \boldsymbol{R}^2_{\ge 0}\})$ such that $i(p) = \overline{orb(\boldsymbol{R}^2_{\ge 0})}$, that $i(U_p \cap \nu^{-1}(E_k)) \subset \overline{orb(\boldsymbol{R}_{\ge 0}\,{}^t(1,0))}$ and that $i(U_p \cap \nu^{-1}(E_l)) \subset \overline{orb(\boldsymbol{R}_{\ge 0}\,{}^t(0,1))}$.*

Proof. Let V be an open small neighborhood of the point $E_k \cap E_l$. Then there exists an inclusion $i_0 : V \hookrightarrow T_{\boldsymbol{Z}^2} \,\mathrm{emb}(\{\text{faces of } \boldsymbol{R}^2_{\ge 0}\})$ such that $i_0(V \cap E_k) \subset \overline{orb(\boldsymbol{R}_{\ge 0}\,{}^t(1,0))}$ and that $i_0(V \cap E_l) \subset \overline{orb(\boldsymbol{R}_{\ge 0}{}^t(0,1))}$. Let U_p be a connected component of $\nu^{-1}(V)$ containning p. Then $B_{\nu_{|U_p}} = |\tau_k| E_k + |\tau_l| E_l$. Let $\tilde{V} \to V$ be the Abel covering constructed as in Section 2 for $Y = V$ and $D = B_{\nu_{|U_p}}$. Then we have an inclusion $i_1 : \tilde{V} \hookrightarrow T_{N'}\,\mathrm{emb}(\{\text{faces of } \boldsymbol{R}^2_{\geq 0}\})$, where $N' = \boldsymbol{Z}\,{}^t(|\tau_k|, 0) \oplus \boldsymbol{Z}\,{}^t(0, |\tau_l|)$ and an isomorphism $h : \mathrm{Gal}(\tilde{V}/V) \simeq \boldsymbol{Z}^2/N'$ such that $i_1 \circ g = h(g) \circ i_1$ for all $g \in \mathrm{Gal}(\tilde{V}/V)$. Let H be the kernel of the homomorphism $\mathrm{Gal}(\tilde{V}/V) \to F$ sending $h^{-1}((1,0))$ and $h^{-1}((0,1))$ to τ_k and τ_l, respectively. Then $U_p \simeq \tilde{V}/H$ and $T_{N'}\,\mathrm{emb}(\{\text{faces of } \boldsymbol{R}^2_{\geq 0}\})/h(H) \simeq T_N\,\mathrm{emb}(\{\text{faces of } \boldsymbol{R}^2_{\geq 0}\})$. □

Let Θ be the convex hull of $(\boldsymbol{R}^2_{\geq 0} \backslash \{0\}) \cap N$, let $\{v_1, \dots, v_q\} = \partial\Theta \cap \boldsymbol{R}^2_{>0} \cap N$, let v_0 and v_{q+1} be the primitive elements in $\boldsymbol{R}_{>0}\,{}^t(1,0) \cap N$ and $\boldsymbol{R}_{>0}\,{}^t(0,1) \cap N$, respectively. Here we may assume that v_j and v_{j+1} are adjacent on $\partial\Theta$ for $0 \leq j \leq q$. Then $\overline{0 v_j v_{j+1}} \cap N = \{0, v_j, v_{j+1}\}$. Hence $\{v_j, v_{j+1}\}$ are bases of N. Therefore, there exist integers c_j $(1 \leq j \leq q)$ such that $v_{j-1} + c_j v_j + v_{j+1} = 0$, if $q > 0$. When $q = 0$, U_p is non-singular.

Proposition 10. *When $q > 0$, there exists a resolution $\varpi : \tilde{U}_p \to U_p$ such that each irreducible component C_j of the exceptional set $\varpi^{-1}(p) = \sum_{j=1}^q C_j$ is a non-singular rational curve, that $C_j^2 = c_j$ for $1 \leq j \leq q$, that C_j and C_{j+1} intersect at a point for $0 \leq j \leq q$ and that $C_j \cap C_k = \varnothing$, if $j < k-1$, where C_0 and C_{q+1} are the proper transformation of $U_p \cap \nu^{-1}(E_k)$ and $U_p \cap \nu^{-1}(E_l)$, respectively. Moreover, $(\nu \circ \varpi)^* E_k = \sum_{j=0}^q \langle v_j, (1,0) \rangle C_j$.*

Proof. Let $\Sigma = \{\{0\}, \boldsymbol{R}_{\geq 0} v_i, \boldsymbol{R}_{\geq 0} v_j + \boldsymbol{R}_{\geq 0} v_{j+1} \mid 0 \leq i \leq q+1, 0 \leq j \leq q\}$, let $W_1 = T_N\,\mathrm{emb}(\Sigma)$ and let $\varpi' : W_1 \to T_N\,\mathrm{emb}(\{\text{faces of } \boldsymbol{R}^2_{\geq 0}\})$ be the holomorphic map induced by the subdivision Σ of $\{\text{faces of } \boldsymbol{R}^2_{\geq 0}\}$. Then W_1 is non-singular. Let $C'_j = \overline{orb(\boldsymbol{R}_{\geq 0} v_j)}$ for $0 \leq j \leq q+1$. Then C'_j are non-singular rational curves with $(C'_j)^2 = c_j$ for $1 \leq j \leq q$, $(\varpi')^{-1}(i(p)) = \sum_{j=1}^q C'_j$ and C'_j intersect C'_{j+1} at a point for $0 \leq j \leq q$. Let f_0 and f_1 be the holomorphic functions on $W_0 := T_{\boldsymbol{Z}^2}\,\mathrm{emb}(\{\text{faces of } \boldsymbol{R}^2_{\geq 0}\}) \simeq \boldsymbol{C}^2$ and W_1, respectively, corresponding to $(1,0) \in N^* \cap (\boldsymbol{R}^2_{\geq 0})^*$. Then $[f_0] = \overline{orb(\boldsymbol{R}_{\geq 0}\,{}^t(1,0))}$ and $[f_1] = \sum_{j=0}^q \langle v_j, (1,0) \rangle C'_j$. □

Let $C_p = \sum_{j=1}^q \langle v_j, (1,0) \rangle C_j$ and let $d_{kl} = \langle v_1, (1,0) \rangle / \langle v_0, (1,0) \rangle$. Then $C_0 \cdot C_p = |\tau_k| d_{kl}$, because $v_0 = (|\tau_k|, 0)$. Let

$$\tilde{W} = \left(W \Big\backslash \bigcup_{0 \leq k < l \leq t} \nu^{-1}(E_k \cap E_l) \right) \cup \bigcup_{0 \leq k < l \leq t} \left(\bigcup_{p \in \nu^{-1}(E_k \cap E_l)} \tilde{U}_p \right),$$

where $\tilde{U}_p = U_p$ if U_p is non-singular and let $\psi : \tilde{W} \to W$ be the natural projection. Then $\lambda \circ \psi : \tilde{W} \to X$ is a resolution of X.

Proposition 11. *For each $k \geq s+1$ and for each irreducible component $\tilde{E}_k$ of the proper transformation of E_k under the map $\nu \circ \psi$,*

$$\tilde{E}_k^2 = |F_k| \left(\frac{E_k^2}{|\tau_k|^2} - \sum_{k \neq l,\, E_k \cap E_l \neq \varnothing} \frac{d_{kl}}{|F_{kl}|} \right)$$

and

$$g(\tilde{E}_k) = 1 - \frac{|F_k|}{|\tau_k|} + \frac{|F_k|}{2} \sum_{k \neq l,\, E_k \cap E_l \neq \varnothing} \left(\frac{1}{|\tau_k|} - \frac{1}{|F_{kl}|} \right).$$

PROOF. First, we note that the degree of the covering $\tilde{E}_k \to E_k$ is equal to $|F_k|/|\tau_k|$, because $F_k = \{g \in F \mid g\tilde{E}_k = \tilde{E}_k\}$ and $\langle \tau_k \rangle = \{g \in F \mid gx = x \text{ for all } x \in \tilde{E}_k\}$. Let U_k be a small neighborhood of E_k. Then there exists a holomorphic function f on U_k such that the zero divisor $[f]$ of f is $E_k + E'$, that $E' \cap E_l = \varnothing$ for $l \neq k$ and that E' intersects E_k transversally at $-E_k^2$ points, because E_k is a non-singular rational curve and $E_k^2 < 0$. Let $\tilde{U}_k$ be the connected component of $(\nu \circ \psi)^{-1}(U_k)$ containing $\tilde{E}_k$. Then

$$[(\nu \circ \psi)^*_{|\tilde{U}_k} f] = |\tau_k|\tilde{E}_k + \tilde{E}' + \sum_{E_k \cap E_l \neq \varnothing} \sum_{p \in \nu^{-1}_{|\psi(\tilde{U}_k)}(E_k \cap E_l)} C_p$$

and $\tilde{E}' = (\nu \circ \psi)^{-1}_{|\tilde{U}_k}(E')$ intersects $\tilde{E}_k$ transversally at $-E_k^2|F_k|/|\tau_k|$ points. Note that $\nu^{-1}_{|\psi(\tilde{U}_k)}(E_k \cap E_l)$ consists of $|F_k|/|F_{kl}|$ points. The first equality follows from these facts.

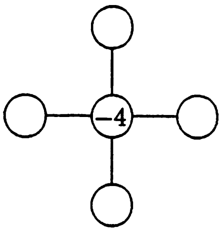
On the other hand, by Riemann-Hurwitz formula, we have

$$2g(\tilde{E}_k) - 2 = -2\frac{|F_k|}{|\tau_k|} + \sum_{E_k \cap E_l \neq \varnothing} \frac{|F_k|}{|F_{kl}|}\left(\frac{|F_{kl}|}{|\tau_k|} - 1\right).$$

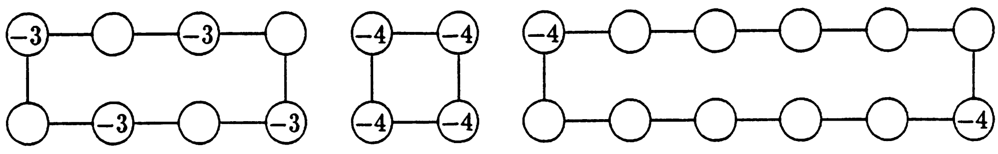
This implies the second equality. □

We can obtain the weighted dual graph of the exceptional set of the resolution $\lambda \circ \psi$: $\tilde{W} \to X$, by Propositions 8, 10 and 11.

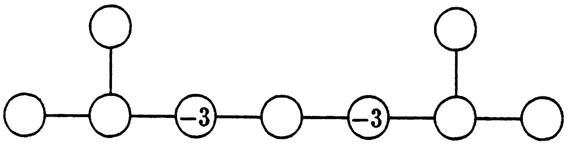
EXAMPLE 6. Let $D = B_\pi$ in Example 2 in Section 1. If $H = \{id\}$, $\langle \sigma_1\sigma_2 \rangle$ or $\langle \sigma_1\sigma_2, \sigma_2\sigma_3 \rangle$, then X is a simple elliptic singularity of multiplicity 4, 2 or 1. If $H = \langle \sigma_1\sigma_2\sigma_3 \rangle$, then X is a log-canonical singularity with a resolution the dual graph of whose exceptional set is the following:

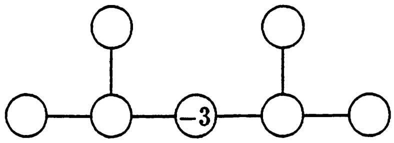


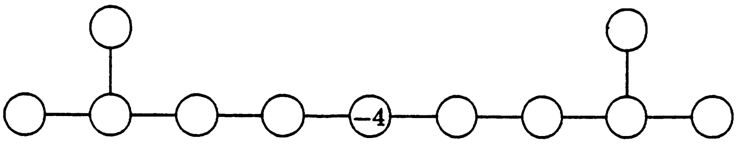
EXAMPLE 7. Let $n = 2$ and let $D = 2D_1 + 2D_2 + 2D_3 + 2D_4$, where D_1, D_2, D_3 and D_4 are the divisors on $Y = \{(z_1, z_2) \in \boldsymbol{C}^2 \mid |z_1|, |z_2| < 1\}$ defined by $z_1 = 0$, $z_1 + z_2^2 = 0$, $z_2 = 0$ and $z_2 + z_1^2 = 0$, respectively. If $H = \{0\}$, $\langle \sigma_1\sigma_2\sigma_3\sigma_4 \rangle$ or $\langle \sigma_1\sigma_2 \rangle$, then X is a cusp singularity with a resolution the dual graph of whose exceptional set is the following:



If $H = \langle \sigma_1\sigma_2\sigma_3 \rangle$, $\langle \sigma_1\sigma_2\sigma_3, \sigma_2\sigma_3\sigma_4 \rangle$ or $\langle \sigma_1\sigma_2\sigma_3, \sigma_1\sigma_2\sigma_4 \rangle$, then X is a log-canonical singularity with a resolution the dual graph of whose exceptional set is the following:







4. Quasi-Gorensteiness.

Let Y be an open neighborhood of 0 in $\boldsymbol{C}^n$, let D be a divisor on Y and let $\mu : \tilde{Y} \to Y$ be the Abel covering constructed as in Section 2. Since $\tilde{Y}$ is the analytic subset in $\boldsymbol{C}^s \times Y$ defined by $w_1^{r_1} - f_1 = \cdots = w_s^{r_s} - f_s = 0$,

$$\phi = \frac{\mu^*(dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n)}{w_1^{r_1-1} w_2^{r_2-1} \cdots w_s^{r_s-1}}$$

is a nowhere vanishing holomorphic n-form on $\tilde{Y}_0 = \mu^{-1}(Y_0)$ and $\sigma_j^* \phi = \exp(2\pi\sqrt{-1}/r_j)\phi$, where $(z_1, z_2, \ldots, z_n)$ is a local coordinate system of Y and $Y_0 = Y \backslash \mathrm{Sing}(D_{red})$. Let $\lambda : \tilde{W} \to \tilde{Y}_0$ be a universal covering and let $\chi : \mathrm{Gal}(\tilde{W}/Y_0) \to \boldsymbol{C}^\times$ be the composite of the quotient map $\mathrm{Gal}(\tilde{W}/Y_0) \to \mathrm{Gal}(\tilde{Y}_0/Y_0)$ and the homomorphism $\mathrm{Gal}(\tilde{Y}_0/Y_0) \to \boldsymbol{C}^\times$ sending σ_j to $\exp(2\pi\sqrt{-1}/r_j)$. Then $g^*(\lambda^*\phi) = \chi(g)(\lambda^*\phi)$ for $g \in \mathrm{Gal}(\tilde{W}/Y_0)$. On the other hand, for any Galois covering $\pi : X \to Y$ with $B_\pi = D$, there exists a subgroup H of $\mathrm{Gal}(\tilde{W}/Y_0)$ with $\pi^{-1}(Y_0) \simeq \tilde{W}/H$, by Theorem 6. Then we have:

PROPOSITION 12. *X is a quasi-Gorenstein singularity, if and only if* $H \subset \ker(\chi)$.

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