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The Gaussian curvature of Alexandrov surfaces

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Abstract. We generalize the Gauss-Bonnet theorem for Alexandrov surfaces and show that we can define the Gaussian curvature almost everywhere on an Alexandrov surface.

§0. Introduction.

A classical theorem in the theory of surfaces states that if Δ is a sufficiently small geodesic triangle bounding a disk on a smooth Riemannian 2-manifold M, and if A, B and C are the inner angles of Δ, then

$$\int_{\Delta} G\,dM = A + B + C - \pi, \tag{0.1}$$

where G is the Gaussian curvature and dM the Lebesgue measure induced from the Riemannian metric of M. The Gaussian curvature $G(p)$ at a point $p \in M$ is interpreted as follows. If $\{\Delta_i\}_{i=1,2,\dots}$ is a decreasing sequence of sufficiently small geodesic triangles such that p is contained within each Δ_i and such that $\lim_{i\to\infty} \Delta_i = \{p\}$, then

$$\lim_{i\to\infty} \frac{e(\Delta_i)}{\mathrm{Area}(\Delta_i)} = G(p). \tag{0.2}$$

The quantity $e(\Delta_i)$ is known as the *excess*. The notion of the excess of a sufficiently small geodesic triangle Δ on a Busemann G-surface was introduced by Busemann ([**B**]). It is defined as follows.

$$e(\Delta) := A + B + C - \pi$$

Although Gaussian curvature is not defined on a Busemann G-surface, Busemann introduced the notion of the angular measure between two geodesics emanating from a point on a G-surface X and proved the fundamental relation between the total excess $e(X)$ and the Euler characteristic $\chi(X)$ of X (see 43.3 in [**B**]). This result may be considered as the G-surface version of the Gauss-Bonnet theorem. As stated in the Basic theorem, we obtain the corresponding result for Alexandrov surfaces.

By an Alexandrov space X, we mean a complete locally compact length space of finite Hausdorff dimension n with curvature bounded below by a real number k. Such spaces have been studied recently by Burago, Gromov and Perelman ([**BGP**]). For a point $x \in X$, we can define a space of directions Σ_x at x, which generalizes the unit tangent sphere. Technically this space is defined as the completion of the set of certain equivalence classes of geodesics emanating from x. If Σ_x is not isometric to the canonical $(n-1)$-sphere, then we call x a *metrically singular point*. In [**OS**], [**P**], Otsu and Shioya, and Perelman showed that the set of metrically singular points in an Alexandrov space of dimension n is a measure zero set with respect to the n-dimensional Hausdorff measure. Moreover, Otsu and Shioya showed that the space X has $C^{0,1/2}$-Riemannian structure almost everywhere. However, since in general, an Alexandrov space X has no C^2-Riemannian structure, it is not necessary that the sectional curvature is definable. The purpose of this article is to investigate 2-dimensional Alexandrov spaces by using the concept of the Busemann total excess of geodesic triangles and show that we can define the Gaussian curvature almost everywhere with respect to the 2-dimensional Hausdorff measure.

From this point let X be an Alexandrov surface, that is, a 2-dimensional Alexandrov space of curvature $\geq k$ without boundary. Then, as shown by Burago, Gromov and Perelman, for an arbitrary point $x \in X$, Σ_x is a circle of circumference not greater than 2π, and X is a topological 2-dimensional manifold ([**BGP**], [**P**]). Thus a small geodesic triangle encloses an open disk domain. A geodesic triangle is said to be non-degenerate if all of its angles are positive. It is said to be degenerate if it is not non-degenerate. A degenerate triangle has zero excess and zero area as well. Throughout this article, Δ also represents the open disk domain and $\mathscr{H}^2$ represents the 2-dimensional Hausdorff measure. Let $x \in X$ be a metrically singular points. Then there exists a sequence $\{\Delta_i\}$ of non-degenerate geodesic triangles such that $x \in \Delta_i$ for all i and $\lim_{i\to\infty} \Delta_i = \{x\}$ satisfying $\lim_{i\to\infty} e(\Delta_i) = 2\pi - L(\Sigma_x) > 0$, where $L(\Sigma_x)$ is the length of Σ_x (see Corollary 2.2). If $x \in X$ is not metrically singular, then we might expect the limit in (0.2) to exist. Thus it will be natural to call this limit the Gaussian curvature $G(x)$ at x.

By means of (0.2) and the Toponogov comparison theorem, we define a natural (curvature) measure e on an Alexandrov surface. Once a natural measure e has been established on X, the Radon-Nikodym theorem implies that there exist a unique absolutely continuous set function ψ and a singular set function ϕ such that for $E \subset X$,

$$e(E) = \psi(E) + \phi(E)$$

(see Proposition 4.5). Moreover, there exists an $\mathscr{H}^2$-measurable function f defined almost everywhere on X such that

$$\psi(E) = \int_E f \, d\mathscr{H}^2$$

(see Theorem 4.6). The function f may be understood as the Gaussian curvature of X.

The Vitali covering theorem is employed for the proof of the existence of the limit in (0.2) at almost all points in X. In fact, we consider for a small positive number d the family U_d of all small geodesic triangles

$$U_d := \{\Delta \subset X \mid \text{the interior angle at each of the three vertices is greater than } d\}.$$

For a point $x \in X$ and $d > 0$, we define

$$\underline{G_d}(x) := \liminf_{\Delta \to \{x\}, x \in \Delta, \Delta \in U_d} \frac{e(\Delta)}{\mathrm{Area}(\Delta)}$$

and

$$\overline{G_d}(x) := \limsup_{\Delta \to \{x\}, x \in \Delta, \Delta \in U_d} \frac{e(\Delta)}{\mathrm{Area}(\Delta)}.$$

Moreover we put

$$\underline{G}(x) := \inf_{d>0} \underline{G_d}(x) \quad \text{and} \quad \bar{G}(x) := \sup_{d>0} \overline{G_d}(x).$$

A point $x \in X$ is called a *curvature regular point* if $\underline{G}(x) = \bar{G}(x) < \infty$. If $x \in X$ is not a curvature regular point, we call x a *curvature singular point*. In Lemma 2.7, we will show that if x is a metrically singular point, then $\underline{G}(x) = \infty$ and x is a curvature singular point. On the other hand, the edges of the double of a flat 2-disk are not metrically singular points, but they are nevertheless curvature singular points.

We now state our main theorem.

MAIN THEOREM. *Almost all points on X are curvature regular points, where 'almost all' is with respect to the 2-dimensional Hausdorff measure $\mathscr{H}^2$. Moreover, if X is compact, then*

$$\int_X \underline{G}(x)\, d\mathscr{H}^2 \leq 2\pi\chi(X),$$

where $\chi(X)$ is the Euler characteristic of X.

In general, the set of metrically singular points of an n-dimensional Alexandrov space is of dimension $\leq n-1$. In the case of an Alexandrov surface, we see in Lemma 1.3 that the set of metrically singular points is countable. However, the following example shows that the set of curvature singular points forms a fractal set with dimension greater than 1.

EXAMPLE. The primitive function of the Cantor function is convex. Using this function, we can construct an Alexandrov surface X of revolution of curvature bounded below by 0 such that the set of curvature singular points is a fractal set.

To prove the Main theorem, we first observe the following basic theorem.

BASIC THEOREM ([**A1**], [**B**]). *Let $D \subset X$ be a bounded open domain with boundary consisting of finitely many closed geodesic polygons and $\beta_1, \beta_2, \ldots, \beta_J$ be the angles of the boundary ∂D measured with respect to the interior of D. Let $\Delta_1, \Delta_2, \ldots, \Delta_N$ be a simplicial decomposition of D into geodesic triangles. Then*

$$\sum_{n=1}^{N} e(\Delta_n) + \sum_{i=1}^{I} (2\pi - L(\Sigma_{p_i})) = 2\pi\chi(D) - \sum_{j=1}^{J} (\pi - \beta_j),$$

where $p_1, p_2, \ldots, p_I$ are the vertices of $\Delta_1, \Delta_2, \ldots, \Delta_N$ lying in the interior of D.

The fact that $L(\Sigma_x) \leq 2\pi$ for any $x \in X$ implies that the first term on the left-hand side of the equation in the Basic theorem is bounded above by the right-hand side. Moreover, in Theorem 2.0, we will show that it is bounded below by $k\,\mathrm{Area}(D)$.

By rewriting $\sum_{n=1}^{N} e(\Delta_n) = \sum_{n=1}^{N}(e(\Delta_n)/\mathrm{Area}(\Delta_n) \times \mathrm{Area}(\Delta_n))$, and tending to finer and finer triangulation, we obtain

$$k\,\mathrm{Area}(D) \le \int_D \underline{G}(x)\,d\mathscr{H}^2 \le e(D),$$

where $e(D) := 2\pi\chi(D) - \sum_{j=1}^{J}(\pi - \beta_j)$ and is called the total excess of D. Thus $\underline{G}(x)$ is integrable on D. This fact and the Vitali covering theorem of measure theory (Proposition 2.12) play important roles in the proof of the Main theorem.

In Section 4, we will show that the total excess $e(\cdot)$ can be extended to a completely additive set function. For example, let X be the double of a flat disk of radius 1 and O the interior of one of the copies of the disk. Then $e(O) = 0$ and $e(\bar{O}) = 4\pi$, where $\bar{O}$ is the closure of O. Thus by the Radon-Nikodym theorem, $e(\cdot)$ is divided into two set functions. One of these is an absolutely continuous set function and the other a singular set function. Moreover, the absolutely continuous set function can be represented as the integral of a function f defined on almost all points in X. In Theorem 4.6, we show that f coincides with $\underline{G}$ almost everywhere.

REMARK. The assumption that X has no boundary is not essential. In fact, the Basic theorem and Theorem 2.0 hold for $D \subset X$ if D has no boundary points of X. Thus, the first assertion of the Main theorem holds because the 2-dimensional Hausdorff measure of the boundary is zero, while for the second assertion, we need more delicate argument about the shape of the boundary of X.

We refer the basic tools of Alexandrov geometry to **[BGP]** and [S]. A curve with length equal to the distance between the extremal points is called a geodesic. For two geodesics γ and σ in X starting from the same point, $\angle(\gamma,\sigma)$ denotes the angle between γ and σ at the starting point. For two points x and y in X, let xy denote a geodesic joining x to y and $|xy|$ the distance between x and y. For three points x, y, and z in X, let $\angle(x,y,z)$ denote the angle $\angle(yx,yz)$ and $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ be three points in the k-plane such that $|\tilde{x}\tilde{y}| = |xy|$, $|\tilde{y}\tilde{z}| = |yz|$, and $|\tilde{z}\tilde{x}| = |zx|$. We denote by $\tilde{\angle}(x,y,z)$ the angle at $\tilde{y}$ of the geodesic triangle whose vertices are $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$. We always assume that geodesics are parameterized by arc-length.

§1. The basic concepts.

Since X has no boundary, for a point $x \in X$, the space of directions Σ_x is a circle of circumference not greater than 2π. Let $C(\Sigma_x)$ be the cone over Σ_x, that is, in this case, the two-dimensional flat cone of the angle at whose vertex equaling the circumference of Σ_x. We first recall:

LEMMA 1.1 (**[BGP]**). *For an arbitrary point $x \in X$ and a positive number δ, there are finitely many geodesics starting from x such that their tangent vectors are maximal δ-net in Σ_x. Moreover the pointed Hausdorff limit* $\lim_{r\to 0}(1/rX, x)$ *of the scaling of the metric of X is $C(\Sigma_x)$.*

A point $x \in X$ is called a metrically singular point if the length $L(\Sigma_x)$ of Σ_x is less than 2π. A point which is not metrically singular is called a metrically regular point. It should be noted that an interior point of a geodesic is a metrically regular point.

LEMMA 1.2 ([**BGP**]). *If $x \in X$ is a point such that $L(\Sigma_x) > 2\pi - 2\delta$ for a small positive number δ, then for sufficiently small $r > 0$, there is a positive number $\varepsilon_1 = \varepsilon_1(\delta, r)$ with $\lim_{\delta,r\to 0} \varepsilon_1(\delta, r) = 0$ which satisfies the condition that the r-distance ball $B(x, r)$ around x is bi-Lipschitz homeomorphic to the r-distance ball in $\boldsymbol{R}^2$ of the Lipschitz constant in* $(1 - \varepsilon_1, 1 + \varepsilon_1)$.

Otsu and Shioya showed that in general, the set of metrically singular points of an n-dimensional Alexandrov space is a measure zero set with respect to the n-dimensional Hausdorff measure. Since the dimension of X is two, we can show a stronger result. We call a point $x \in X$ a δ-singular point if $L(\Sigma_x) \le 2\pi - 2\delta$, or equivalently, the diameter of Σ_x is not greater than $\pi - \delta$. We designate by $\mathrm{Sing}_\delta X$ the set of δ-singular points in X and define $\mathrm{Sing}\, X := \bigcup_{\delta > 0} \mathrm{Sing}_\delta X = \{\text{metrically singular points in } X\}$.

LEMMA 1.3 ([**BGP**]). *For an arbitrary positive number δ, δ-singular points are isolated. In particular,* $\mathrm{Sing}\, X$ *is a countable set.*

PROOF. Suppose not. Then there is a point $p \in X$ and is a sequence $\{p_i\} \subset \mathrm{Sing}_\delta X$ such that $\lim_{i\to\infty} p_i = p$. Taking a subsequence if necessary, we may assume that if $i > j$, then $|pp_i|$ is sufficiently less than $|pp_j|$. Since $\tilde{\angle}(pp_ip_j) \le \angle(pp_ip_j) \le \pi - \delta$, there is a positive number τ depending only k and δ such that $\angle(p_ipp_j) \ge \tilde{\angle}(p_ipp_j) \ge \tau$ for arbitrary i, j with $i \ne j$. However this is a contradiction because Σ_p is compact. This completes the proof of the first half. The latter half follows easily as a consequence of the first half. □

REMARK 1.4. In [**OS**], Otsu and Shioya gave an example of an Alexandrov surface X obtained as the Hausdorff limit of convex polyhedra. The surface X is such that $\mathrm{Sing}\, X$ is a dense subset of X. We call this space X Otsu and Shioya's example.

DEFINITION 1.5. Let γ and σ be geodesics starting from the same point $x \in X$. Since X is a topological manifold, for sufficiently small $r > 0$, γ and σ divide $B(x, r) \backslash \{\gamma, \sigma\}$ into two domains A and B such that any geodesic joining a point in A and a point in B must lie across γ or σ. We denote interior angles $\measuredangle_x^A$ of A and $\measuredangle_x^B$ at x as follows. Since Σ_x is a circle, the tangent vectors corresponding to γ and σ divide Σ_x into two intervals I_A and I_B such that the vector tangent to any geodesic joining x to a point in A is included in I_A, and the vector tangent to any geodesic joining x to a point in B is included in I_B. We put $\measuredangle_x^A := L(I_A)$ and $\measuredangle_x^B := L(I_B)$.

We obtain the following proposition directly from Lemma 1.1.

PROPOSITION 1.6. (1) *In the above situation, if $\measuredangle_x^A < \measuredangle_x^B$ then for sufficiently small $s > 0$ and $t > 0$, a geodesic joining $\gamma(s)$ and $\sigma(t)$ is contained completely in the domain A, with the exception of its endpoints.*

(2) *Assume that $\measuredangle_x^A \le \measuredangle_x^B < \pi$. Then for sufficiently small $s > 0$ and $t > 0$ there is a curve $\alpha_{s,t}$ joining $\gamma(s)$ to $\sigma(t)$ which excluding their endpoints, lie entirely in the domain B except their end points such that*

$$\lim_{s,t\to 0} \arccos\left(\frac{s^2 + t^2 - L(\alpha_{s,t})^2}{2st}\right) = \measuredangle_x^B.$$

COROLLARY 1.7. *In the same context,*

$$\angle(\gamma,\sigma) = \min\{ \measuredangle_p^A, \measuredangle_p^B\}.$$

Since X is a topological manifold, if three points x, y, and z of X are sufficiently close to each other, then the geodesic triangle consisting of the three geodesics xy, yz, and zx encloses an open disk domain which we will also call a geodesic triangle, hoping this will cause no confusion. By $\Delta(xyz)$, or simply Δ, we denote this open disk domain. For such a triangle, let $\measuredangle(xyz)$ denote the interior angle of $\Delta(xyz)$ at y. We define the *total excess* of $\Delta(xyz)$ by

$$e(\Delta(xyz)) := \measuredangle(xyz) + \measuredangle(yzx) + \measuredangle(zxy) - \pi.$$

Let $\mathscr{D}$ be the family of all the bounded open sets in X whose boundaries consist of finitely many geodesic polygons. If $D \in \mathscr{D}$, then the closure $\bar{D}$ is compact and D can be decomposed into geodesic triangles $\Delta(x_1y_1z_1), \dots, \Delta(x_ny_nz_n)$. Let $\chi(D)$ be the Euler characteristic of D. We define $\{p_1,\dots,p_J\} := (\{x_1,\dots,x_n\} \cup \{y_1,\dots,y_n\} \cup \{z_1,\dots,z_n\}) \cap D$. Let $\beta_1,\dots,\beta_K$ be the interior angles of D at vertices of the boundary ∂D. The following is the basic theorem for Alexandrov surfaces providing the fundamental relation between the total excess and the Euler characteristic.

THEOREM 1.8 ([**A1**], [**B**]).

$$\sum_{i=1}^{n} e(\Delta(x_iy_iz_i)) + \sum_{j=1}^{J}(2\pi - L(\Sigma_{p_j})) = 2\pi\chi(D) - \sum_{k=1}^{K}(\pi - \beta_k). \tag{1.9}$$

The proof of this theorem is similar to that of formula (43.3) in [**B**]. If p_i for $i \in \{1,2,\dots,J\}$ is a singular point, then the sum of the interior angles at p_i of all triangles meeting at p_i is less than 2π. This is the reason that the second term on the left-hand side is added. We omit the proof.

Since the right-hand side of (1.9) is independent of the triangulation, so too is the left-hand side. Thus the total excess of D is naturally defined as

$$e(D) := \sum_{i=1}^{n} e(\Delta(x_iy_iz_i)) + \sum_{j=1}^{J}(2\pi - L(\Sigma_{p_j})).$$

Note that this definition is compatible with the definition of $e(\Delta(xyz))$.

COROLLARY TO THEOREM 1.8. *If X is compact, then*

$$e(X) = 2\pi\chi(X).$$

§2. The estimate of total excess.

In this section we estimate the total excess of $D \in \mathscr{D}$.

The second term on the left-hand side of (1.9) depends on the triangulation of D. Therefore it is not certain whether or not its limit exists for a sequence of progressively finer geodesic triangulations of D. The following Theorem 2.0 plays an important role in justifying the existence of the limit of the first (and hence the second) term of (1.9).

THEOREM 2.0. *If* $D \in \mathscr{D}$ *is decomposed into finitely many geodesic triangles* $\{\Delta(x_i y_i z_i)\}_{i=1,2,\dots,n}$, *then*

$$\sum_{i=1}^{n} e(\Delta(x_i y_i z_i)) \geq k\,\mathrm{Area}(D),$$

where $\mathrm{Area}(D)$ *is the 2-dimensional Hausdorff measure of* D. *Furthermore, we have*

$$e(D) \geq k\,\mathrm{Area}(D).$$

First we prepare some lemmas.

LEMMA 2.1. *Let* γ *and* σ *be geodesics starting from* $x \in X$. *Then*

$$\lim_{t\to 0} \angle(x, \gamma(t), \sigma(t)) = \lim_{t\to 0} \angle(x, \sigma(t), \gamma(t)) = \frac{\pi - \angle(\gamma, \sigma)}{2}$$

and

$$\lim_{t\to 0} e(\Delta(x, \gamma(t), \sigma(t))) = 0.$$

OUTLINE OF THE PROOF. We may assume that the lower bound of curvature X is 0. Put $\theta := \angle(\gamma, \sigma)$. Since $(\pi - \theta)/2 \lesssim \angle(x, \gamma(t), \sigma(t))$ by the definition of the angle and the Alexandrov convexity, it suffices to show $\angle(x, \gamma(t), \sigma(t)) \lesssim (\pi - \theta)/2$. Here $A \lesssim B$ means $\lim_{t\to 0}(A/B) \leq 1$ and moreover $A \approx B$ means that $\lim_{t\to 0}(A/B) = 1$. Then by the definition of the angle of two geodesics, we see that $|\gamma(t)\sigma(t)| \approx 2t\sin(\theta/2)$ and $|\gamma(2t)\sigma(t)| \approx t\sqrt{2(1-\cos\theta)}$. Then this and the Alexandrov convexity implies that $(\pi + \theta)/2 \lesssim \angle(\sigma(t), \gamma(t), \gamma(2t))$ and consequently $\angle(x, \gamma(t), \sigma(t)) = \pi - \angle(\sigma(t), \gamma(t), \gamma(2t)) \lesssim (\pi - \theta)/2$. This completes the proof of the first half. The second half is in consequence of Proposition 1.6.

COROLLARY 2.2. *Let* γ_i $(i = 1, 2, 3)$ *be geodesics starting from* x *such that* $\angle(\gamma_i, \gamma_{i+1}) + \angle(\gamma_{i+1}, \gamma_{i+2})$ *is strictly greater than* $\angle(\gamma_i, \gamma_{i+2})$, *where the indices are calculated* mod 3, *then* x *is contained in the interior of* $\Delta(\gamma_1(t), \gamma_2(t), \gamma_3(t))$ *for all small* $t > 0$, *and*

$$\lim_{t\to 0} e(\Delta(\gamma_1(t), \gamma_2(t), \gamma_3(t))) = 2\pi - L(\Sigma_x).$$

The following useful lemma is obtained from the cosine formula for triangles in $\boldsymbol{R}^2$ (cf. §3 in [**OSY**]).

LEMMA 2.3. *For any positive number* ε, *there are positive numbers* $\varepsilon_1 = \varepsilon_1(\varepsilon)$ *and* $\varepsilon_1' = \varepsilon_1'(\varepsilon)$ *satisfying* $\lim_{\varepsilon\to 0}\varepsilon_1 = 0$ *and* $\lim_{\varepsilon\to 0}\varepsilon_1' = 0$ *with the following properties. Let* $O_\varepsilon \subset X$ *be an open set bi-Lipschitz homeomorphic to an open set in* $\boldsymbol{R}^2$ *with a bi-Lipschitz homeomorphism* f *such that the Lipschitz constants of* f *and* f^{-1} *are in* $(1-\varepsilon, 1+\varepsilon)$. *Let* γ *be a geodesic in* O_ε *of length* l. *If* $t/l > \varepsilon_1'$ *and* $t/l < 1 - \varepsilon_1'$, *then the angle* $\angle(f(\gamma(0)), f(\gamma(t)), f(\gamma(l))) > \pi - \varepsilon_1$ *and thus* $\angle(f(\gamma(t)), f(\gamma(0)), f(\gamma(l))) < \varepsilon_1$.

Let $\tilde{U}_d$ for small $d > 0$ be the set of all geodesic triangles in X such that if $\Delta \in \tilde{U}_d$ then all angles of the corresponding triangle sketched in the k-plane are greater than d.

Let O_ε and f be as in Lemma 2.3. Then if ε is sufficiently smaller than d, for $\Delta(xyz) \subset O_\varepsilon$ in $\tilde{U}_d$, the two triangles $\tilde{\Delta}(xyz) \subset k$-plane and $\Delta(f(x), f(y), f(z)) \subset \boldsymbol{R}^2$ are almost congruent to each other. Additionally, Lemma 2.3 implies that the proportion

$$\mathrm{Area}(f(\Delta(xyz)))/\mathrm{Area}(\Delta(f(x), f(y), f(z)))$$

is close to 1. Thus we have obtained

LEMMA 2.4. *There is a positive constant* $\varepsilon_2 = \varepsilon_2(d, \varepsilon)$ *with* $\lim_{\varepsilon \to 0} \varepsilon_2 = 0$ *which satisfies the condition that for* $\Delta(xyz) \subset O_\varepsilon$ *in* $\tilde{U}_d$,

$$(*) \qquad (1 - \varepsilon_2)\,\mathrm{Area}(\tilde{\Delta}(xyz)) < \mathrm{Area}(\Delta(xyz)) < (1 + \varepsilon_2)\,\mathrm{Area}(\tilde{\Delta}(xyz)).$$

In particular, $(\mathrm{Diam}(\Delta(xyz)))^2/\mathrm{Area}(\Delta(xyz))$ *is bounded above by a positive constant depending only on* d *and* ε, *where* $\mathrm{Diam}(\Delta(xyz))$ *is the diameter of* $\Delta(xyz)$.

The Gauss-Bonnet theorem for the k-plane implies that $e(\tilde{\Delta}(xyz)) = k\,\mathrm{Area}(\tilde{\Delta}(xyz))$. Moreover, since $e(\Delta(xyz)) \geq e(\tilde{\Delta}(xyz))$, we see from Lemma 2.4 that if k is a negative number $-\kappa^2$, then

$$e(\Delta(xyz)) > \frac{-\kappa^2}{1 - \varepsilon_2}\mathrm{Area}(\Delta(xyz)).$$

for $\Delta(xyz) \in \tilde{U}_d$ in O_ε. For a sufficiently small fixed $d > 0$, we consider a geodesic triangle $\Delta(xyz) \subset O_\varepsilon$ which does not belong to $\tilde{U}_d$. Assume that $\tilde{\angle}(yxz) \leq d$. We choose points $y' \in xy$ and $z' \in xz$ with $0 < |xy'| = |xz'|$ sufficiently small and then subtract $\Delta(xy'z')$ from $\Delta(xyz)$. By cutting off all such triangles at vertices with angles not greater than d, we obtain a domain Π bounded by a geodesic polygon. All of the angle at vertices of Π are greater than d. Thus Π can be triangulated by small triangles belonging $\tilde{U}_d$, where $\mathrm{Area}(\Delta(xyz)) - \mathrm{Area}(\Pi)$ is arbitrarily small. Therefore from Lemma 2.1 and Theorem 1.8, we have

$$e(\Delta) > \frac{-\kappa^2}{1 - \varepsilon_2/2}\mathrm{Area}(\Delta)$$

for all $\Delta \subset O_\varepsilon$. Thus from Lemma 1.2, we obtain:

LEMMA 2.5. *Let X be an Alexandrov surface with curvature bounded from below by a negative number $k = -\kappa^2$. For an arbitrary positive number ε, there exists a positive number $\delta_1 = \delta_1(\varepsilon)$ with $\lim_{\varepsilon \to 0} \delta_1(\varepsilon) = 0$ which satisfies the following. If $p \in X$ is not a δ_1-singular point, then for an arbitrary triangle $\Delta(xyz)$ in a sufficiently small neighborhood of p,*

$$e(\Delta(xyz)) > (-\kappa^2 - \varepsilon)\,\mathrm{Area}(\Delta(xyz)).$$

COROLLARY 2.6. *With the assumptions of Lemma 2.5, if we further assume that $D \in \mathscr{D}$ has no δ_1-singular points, then*

$$e(D) > (-\kappa^2 - 2\varepsilon)\,\mathrm{Area}(D).$$

PROOF OF COROLLARY 2.6. In the case where the closure $\bar{D}$ of D has no δ_1-singular points, this follows immediately from Lemma 2.5. If ∂D contains δ_1-singular points, then Lemma 1.3 implies that there are only finitely many such singular points. Then Lemma 2.1 concludes the proof. □

LEMMA 2.7. *Let* $\{\Delta_i\}_{i=1,2,\dots}$ *be a sequence of geodesic triangles and* x *a point such that each* Δ_i *contains* x *in its interior. If* $\lim_{i\to\infty}\Delta_i = \{x\}$, *then*

$$\lim_{i\to\infty} e(\Delta_i) = 2\pi - L(\Sigma_x).$$

PROOF. It suffices to prove that there is a subsequence $\{\Delta_{i(i')}\} \subset \{\Delta_i\}$ such that

$$\lim_{i'\to\infty} e(\Delta_{i(i')}) = 2\pi - L(\Sigma_x).$$

From Corollary 2.2, we construct a decreasing sequence of geodesic triangles $\{\hat{\Delta}_j\}$ such that $x \in \hat{\Delta}_j$ and $\lim_{j\to\infty}\hat{\Delta}_j = \{x\}$. Here each vertex of $\hat{\Delta}_j$ lies in the interior of a geodesic from x, and hence is a metrically regular point. We choose subsequences $\{\Delta_{i(i')}\}_{i'=1,2,\dots} \subset \{\Delta_i\}$ and $\{\hat{\Delta}_{j(i')}\}_{i'=1,2,\dots} \subset \{\hat{\Delta}_j\}$ such that

$$\Delta_{i(i')} \supset \overline{\hat{\Delta}_{j(i')}} \quad \text{and } \hat{\Delta}_{j(i')} \supset \overline{\Delta_{i(i'+1)}} \quad \text{for } i' = 1,2,\dots.$$

For simplicity, we rewrite $\Delta_{i'} := \Delta_{i(i')}$ and $\hat{\Delta}_{i'} := \hat{\Delta}_{j(i')}$. Then writing $D_{i'} := \Delta_{i'}\backslash\overline{\hat{\Delta}_{i'}}$, $\hat{D}_{i'} := \hat{\Delta}_{i'}\backslash\overline{\Delta_{i'+1}}$, and denoting by $p_{i'}^1, p_{i'}^2$, $p_{i'}^3$ the vertices of $\Delta_{i'}$, we have from the construction of $\{\hat{\Delta}_j\}$,

$$e(\Delta_{i'}) = e(D_{i'}) + e(\hat{\Delta}_{i'}) \tag{2.8}$$

and

$$\begin{aligned} e(\hat{\Delta}_{i'}) &= e(\hat{D}_{i'}) + e(\Delta_{i'+1}) + \sum_{l=1}^{3}(2\pi - L(\Sigma_{p_{i'+1}^l})) \\ &\geq e(\hat{D}_{i'}) + e(\Delta_{i'+1}). \end{aligned} \tag{2.9}$$

Since we may assume that X is of curvature bounded below by a negative number $-\kappa^2$, Lemma 1.3, Corollary 2.6, and formulas (2.8) and (2.9) imply that for arbitrarily small $\varepsilon > 0$, if i' is sufficiently large,

$$e(\Delta_{i'}) \geq (-\kappa^2 - \varepsilon)\,\mathrm{Area}(D_{i'}) + e(\hat{\Delta}_{i'})$$

and

$$e(\Delta_{i'}) \leq (\kappa^2 + \varepsilon)\,\mathrm{Area}(\hat{D}_{i'-1}) + e(\hat{\Delta}_{i'-1}).$$

This proves $\lim_{i'\to\infty} e(\Delta_{i'}) = 2\pi - L(\Sigma_x)$. □

Collecting the above lemmas, we have proved

PROPOSITION 2.10. *Let* p *be a point of an Alexandrov surface with curvature bounded below by a negative number* $k = -\kappa^2$. *For any* $\varepsilon > 0$ *there exists a neighborhood* V_ε *of* p *which satisfies the following.*

(1) *If* p *is a metrically regular point, then for any triangle* $\Delta \subset V_\varepsilon$,

$$e(\Delta) > (-\kappa^2 - \varepsilon)\,\mathrm{Area}(\Delta).$$

(2) *If p is a metrically singular point, and if p is in the interior of $\Delta \subset V_\varepsilon$, then*

$$e(\Delta) > 0.$$

In particular

$$e(\Delta) > -\kappa^2 \operatorname{Area}(\Delta).$$

If p is not contained in the interior of Δ, then

$$e(\Delta) > (-\kappa^2 - \varepsilon) \operatorname{Area}(\Delta).$$

PROOF OF THEOREM 2.0 FOR THE CASE $k \leq 0$.

The proof in the case that $k = 0$ is clear because every triangle then has non-negative excess. So assume that $k = -\kappa^2 < 0$. We need only prove for each i that

$$e(\Delta(a_i b_i c_i)) \geq -\kappa^2 \operatorname{Area}(\Delta(a_i b_i c_i)).$$

For an arbitrarily fixed $\varepsilon > 0$, we decompose the geodesic triangle $\Delta(abc) := \Delta(a_i b_i c_i)$ into finitely many triangles $\Delta(x_1 y_1 z_1), \ldots, \Delta(x_m y_m z_m)$ such that each is contained in some neighborhood V_ε obtained in Proposition 2.10. Then Proposition 2.10 implies that the excess of each triangle is greater than $(-\kappa^2 - \varepsilon)$ times its area. Thus if we put $\{p_1, p_2, \ldots, p_J\} := (\bigcup_{i=1}^m \{x_i, y_i, z_i\}) \cap D$,

$$\begin{aligned}
e(\Delta(abc)) &= \sum_{\nu=1}^{m} e(\Delta(x_\nu y_\nu z_\nu)) + \sum_{\mu=1}^{J} (2\pi - L(\Sigma_{p_\mu})) \\
&\geq \sum_{\nu=1}^{m} e(\Delta(x_\nu y_\nu z_\nu)) \\
&> (-\kappa^2 - \varepsilon) \sum_{\nu=1}^{m} \operatorname{Area}(\Delta(x_\nu y_\nu z_\nu)) \\
&= (-\kappa^2 - \varepsilon) \operatorname{Area}(\Delta(abc)).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the proof for $k \leq 0$ is concluded. □

The proof of Theorem 2.0 in the case $k > 0$ requires more delicate discussion. The Vitali covering theorem is employed for the proof of this case as well as the proof of the Main theorem.

We now recall the Vitali covering theorem, a well-known theorem in measure theory. Let Y be a metric space. By definition, a family $\mathscr{V}$ of subsets in Y is called a Vitali class of a given set $E \subset Y$ if for arbitrary point $x \in E$ and arbitrary positive number δ, there is a subset $U \in \mathscr{V}$ such that $x \in U$ and $\operatorname{Diam}(U) \leq \delta$. Let $\mathscr{H}^s$ be the s-dimensional Hausdorff measure for $s > 0$.

THE VITALI COVERING THEOREM. *Let E be a subset of $\boldsymbol{R}^n$ and $\mathscr{V}$ be a Vitali class of E. Then for any $s > 0$, we can choose a sequence $\{U_i\}_{i=1,2,\ldots} \subset \mathscr{V}$ with $\overline{U_i} \cap \overline{U_j} = \emptyset$ for $i \neq j$ which satisfies that one of the following.*

(1) $\mathscr{H}^s(E \setminus \cup_i \overline{U_i}) = 0$.

(2) $\sum_{i=1}^\infty (\operatorname{Diam}(U_i))^s = \infty$.

We apply the above theorem to Alexandrov surfaces. We see from Lemma 1.2 and Lemma 1.3 that the above theorem holds for an Alexandrov surface. For $d > 0$, let U_d be the family of all triangles $\varDelta$ for which the angle at each of the three vertices of $\varDelta$ is greater than d. We observe from Lemma 2.7 that if $x \in X$ is a metrically regular point, then any $\varDelta \in U_d$ in a sufficiently small neighborhood of x belongs to $\tilde{U}_{d/2}$. Thus we see from Lemma 2.4 that there is a constant $c > 0$ depending only on d and k which satisfies the following. If $\mathscr{V} \subset U_d$ is a Vitali class of a set $E \subset X$, then

$$\mathscr{V}_c := \{\varDelta \in \mathscr{V} \mid (\mathrm{Diam}(\varDelta))^2/\mathrm{Area}(\varDelta) < c\}$$

is a Vitali class of $E \backslash \mathrm{Sing}\, X$. Then if E is bounded, there is no subsequence $\{\varDelta_i\} \subset \mathscr{V}_c$ with $\overline{\varDelta_i} \cap \overline{\varDelta_j} = \emptyset$ for $i \neq j$ such that $\sum_{i=1}^{\infty}(\mathrm{Diam}(\varDelta_i))^2 = \infty$. Thus we have obtained

PROPOSITION 2.11. *Let E be a subset of an Alexandrov surface. Let d be an arbitrary small positive number and $\mathscr{V}$ a Vitali class of E consisting of geodesic triangles in U_d. Then we can choose a sequence $\{\varDelta_i\}_{i=1,2,} \subset \mathscr{V}$ with $\overline{\varDelta_i} \cap \overline{\varDelta_j} = \emptyset$ for $i \neq j$ such that*

$$\mathrm{Area}(E \backslash \cup_i \overline{\varDelta_i}) = 0.$$

REMARK. The conclusion of Proposition 2.11 holds for a family $\mathscr{V}$ of geodesic triangles in U_d such that $\overline{\mathscr{V}} := \{\overline{\varDelta} \mid \varDelta \in \mathscr{V}\}$ is a Vitali class of E.

PROOF OF THEOREM 2.0 FOR THE CASE $k = \kappa^2 > 0$. We need only prove for an arbitrary geodesic triangle $\varDelta_0$ and for any fixed positive number $\varepsilon \ll \kappa^2$,

$$e(\varDelta_0) > (\kappa^2 - \varepsilon)\,\mathrm{Area}(\varDelta_0).$$

Fix a small positive number d. By Lemma 2.4, the area formula of triangles in κ^2-plane, and the inequality $e(\varDelta) \geq e(\tilde{\varDelta})$, we see that the family

$$\mathscr{V} := \{\varDelta \in U_d \mid \overline{\varDelta} \subset \varDelta_0, (\kappa^2 - \varepsilon/2)\,\mathrm{Area}(\varDelta) < e(\varDelta)\}$$

is a Vitali class of $\varDelta_0 \cap \{\text{metrically regular points}\}$. Thus from Proposition 2.11 we can choose a finite number of geodesic triangles $\varDelta_1, \ldots, \varDelta_m \in \mathscr{V}$ with $\overline{\varDelta_i} \cap \overline{\varDelta_j} = \emptyset$ for $i \neq j$ such that

$$\mathrm{Area}\left(\varDelta_0 \backslash \bigcup_{i=1}^{m} \overline{\varDelta_i}\right) < \frac{\varepsilon}{2\kappa^2 - \varepsilon}\mathrm{Area}(\varDelta_0).$$

Noting that $e(\varDelta_0 \backslash \bigcup_{i=1}^{m} \overline{\varDelta_i}) \geq 0$ since X is of curvature bounded below by a positive number, we then obtain

$$\begin{aligned} e(\varDelta_0) &\geq e\left(\varDelta_0 \backslash \bigcup_{i=1}^{m} \overline{\varDelta_i}\right) + \sum_{i=1}^{m} e(\varDelta_i) \\ &\geq \sum_{i=1}^{m} e(\varDelta_i) \\ &> (\kappa^2 - \varepsilon/2) \sum_{i=1}^{m} \mathrm{Area}(\varDelta_i) \end{aligned}$$

$$> (\kappa^2 - \varepsilon/2)\Big(1 - \frac{\varepsilon}{2\kappa^2 - \varepsilon}\Big)\operatorname{Area}(\varDelta_0)$$
$$= (\kappa^2 - \varepsilon)\operatorname{Area}(\varDelta_0). \qquad \square$$

The following corollary is stronger than Lemma 1.3.

COROLLARY TO THEOREM 2.0.

(1) *Let $x \in X$ be an arbitrary point. For any $\varepsilon > 0$, there is a positive number r such that if $\{p_1, p_2, \cdots\} := B(x,r) \cap \operatorname{Sing} X$, then*

$$2\pi - L(\Sigma_x) \leq \sum_{i=1}^{\infty}(2\pi - L(\Sigma_{p_i})) < (2\pi - L(\Sigma_x)) + \varepsilon.$$

(2) *Let $E \subset X$ be a compact set and $\{p_1, p_2, \cdots\} := \operatorname{Sing} X \cap E$. Then*

$$\sum_{i=1}^{\infty}(2\pi - L(\Sigma_{p_i})) < \infty.$$

PROOF. We show only (1). The first inequality is clear. Let $\varDelta$ be a geodesic triangle containing x and $\{p_1', p_2', \cdots\} := \varDelta \cap \operatorname{Sing} X$. For arbitrary $m \in \boldsymbol{N}$, take a triangulation of $\varDelta$ such that each p_i' $(1 \leq i \leq m)$ is a vertex of some triangle. Then from Theorem 1.8 and Theorem 2.0 we obtain

$$e(\varDelta) \geq k \operatorname{Area}(\varDelta) + \sum_{i=1}^{m}(2\pi - L(\Sigma_{p_i'})). \tag{2.13}$$

This implies that $\sum_{i=1}^{\infty}(2\pi - L(\Sigma_{p_i'})) \leq e(\varDelta) - k\operatorname{Area}(\varDelta)$ and in particular if $B(x,r) \subset \varDelta$ then $\sum_{i=1}^{\infty}(2\pi - L(\Sigma_{p_i})) \leq e(\varDelta) - k\operatorname{Area}(\varDelta)$. By Lemma 2.7, $e(\varDelta)$ tends to $2\pi - L(\Sigma_x)$ as $\varDelta \to x$. This completes the proof. $\square$

§3. Proof of the Main theorem.

First, we recall that for $x \in X$ and $d > 0$,

$$\underline{G_d}(x) := \liminf_{\varDelta \to \{x\}, x \in \varDelta, \varDelta \in U_d} \frac{e(\varDelta)}{\operatorname{Area}(\varDelta)}$$

and

$$\overline{G_d}(x) := \limsup_{\varDelta \to \{x\}, x \in \varDelta, \varDelta \in U_d} \frac{e(\varDelta)}{\operatorname{Area}(\varDelta)}.$$

Then, we define

$$\underline{G}(x) := \inf_{d>0} \underline{G_d}(x) \quad \text{and} \quad \overline{G}(x) := \sup_{d>0} \overline{G_d}(x).$$

If $\underline{G}(x) = \overline{G}(x) < \infty$, then we call this quantity the Gaussian curvature $G(x)$ at x and call x a curvature regular point.

Since $\underline{G_d}(x)$ is monotone non-decreasing and $\overline{G_d}(x)$ is monotone non-increasing with respect to d, the Main theorem is equivalent to the following theorem.

THEOREM 3.1. *For arbitrarily fixed $D \in \mathscr{D}$ and for sufficiently small $d > 0$, we have $\underline{G_d}(x) = \overline{G_d}(x) < \infty$ almost everywhere in D.*

We first make the following claim.

CLAIM. *The function $\underline{G_d}$ is measurable and satisfies the condition*:

$$k\,\mathrm{Area}(D) \le \int_D \underline{G_d}(x)\,d\mathscr{H}^2 \le e(D). \tag{3.2}$$

PROOF. For any $a \in \boldsymbol{R}$,

$$\{x \in D \,|\, \underline{G_d}(x) < a\} = \bigcup_{m\in \boldsymbol{N}} \bigcap_{n\in \boldsymbol{N}} \bigcup_{\varDelta \in U(a,m,n)} \varDelta,$$

where

$$U(a,m,n) := \left\{\varDelta \in U_d \,|\, \varDelta \subset D, \mathrm{Diam}(\varDelta) < \frac{1}{n}, \frac{e(\varDelta)}{\mathrm{Area}(\varDelta)} < a - \frac{1}{m}\right\}.$$

The set $\bigcup_{\varDelta \in U(a,m,n)} \varDelta$ is open and consequently measurable. Thus $\underline{G}(x)$ is a measurable function.

The first inequality is clear from Theorem 2.0. Let $\{T_i\}_{i=1,2,\dots}$ be a sequence of geodesic triangulations in U_d of D such that T_{i+1} is a subdivision of T_i and such that the diameter of each triangle of T_i is less than $1/i$ for all i. Let $\{\varDelta_j^i\}_{j=1,2,\dots J(i)}$ for $i \in \boldsymbol{N}$ be triangles of T_i and define a function $f_i : D\backslash \bigcup_{j=1}^{J(i)} \partial\varDelta_j^i \to \boldsymbol{R}$ by

$$f_i(x) := \frac{e(\varDelta_{j(x)}^i)}{\mathrm{Area}(\varDelta_{j(x)}^i)},$$

where $\varDelta_{j(x)}^i$ is the triangle of T_i containing x. Note that

$$\int_{D\backslash \bigcup_j \partial\varDelta_j^i} f_i\,d\mathscr{H}^2 = \sum_{j=1}^{J(i)} e(\varDelta_j^i) \le e(D).$$

With $\hat{D} := D\backslash \bigcap_{i=1}^{\infty}(\bigcup_{j=1}^{J(i)} \partial\varDelta_j^i)$, clearly $\mathrm{Area}(D) = \mathrm{Area}(\hat{D})$. Thus we obtain from Fatou's Lemma

$$\begin{aligned}\int_D \underline{G_d}(x)\,d\mathscr{H}^2 &= \int_{\hat{D}} \underline{G_d}(x)\,d\mathscr{H}^2 \\ &\le \int_{\hat{D}} \varliminf_{i\to\infty} f_i(x)\,d\mathscr{H}^2 \\ &\le \varliminf_{i\to\infty} \int_{\hat{D}} f_i(x)\,d\mathscr{H}^2 \\ &\le e(D).\end{aligned}$$

□

PROOF OF THEOREM 3.1. For a small geodesic triangle $\varDelta$ we put

$$\tilde{e}(\varDelta) := \int_{\varDelta} \underline{G_d}(x)\,d\mathscr{H}^2.$$

Then, as is well known in measure theory, using the Vitali covering theorem, we see that there is a measure zero set $N \subset D$ such that for all $x \in D \backslash N$,

$$\lim_{\varDelta \to x, x \in \varDelta, \varDelta \in U_d} \frac{\tilde{e}(\varDelta)}{\mathrm{Area}(\varDelta)} = \underline{G_d}(x). \tag{3.3}$$

Here we note that (3.2) and (3.3) imply that

$$\liminf_{\varDelta \to x, x \in \varDelta, \varDelta \in U_d} \left(\frac{e(\varDelta) - \tilde{e}(\varDelta)}{\mathrm{Area}(\varDelta)} \right) = 0 \tag{3.4}$$

for any $x \in D \backslash N$. Thus if for $x \in D \backslash N$,

$$\limsup_{\varDelta \to x, x \in \varDelta, \varDelta \in U_d} \left(\frac{e(\varDelta) - \tilde{e}(\varDelta)}{\mathrm{Area}(\varDelta)} \right) = 0, \tag{3.5}$$

then

$$\underline{G_d}(x) = \overline{G_d}(x).$$

Therefore it suffices for the proof of Theorem 3.1 to show that (3.5) holds for almost all $x \in D \backslash N$. Suppose not. Then there is a positive number $a > 0$ such that, defining

$$E_a := \left\{ x \in D \,\middle|\, \limsup_{\varDelta \to x, x \in \varDelta, \varDelta \in U_d} \left(\frac{e(\varDelta) - \tilde{e}(\varDelta)}{\mathrm{Area}(\varDelta)} \right) > a \right\} \Big\backslash N,$$

$\mathrm{Area}(E_a) > 0$. From this point, we omit the subscript a for convenience. Since $0 < \mathrm{Area}(E) < \infty$ and the Hausdorff measure is regular, there is an open set $O \subset D$ such that $E \subset O$ and $\mathrm{Area}(O) < 2\mathrm{Area}(E)$. Let ε be a positive number such that $\varepsilon < a/3$. By (3.4), the family of those geodesic triangles $\varDelta \in U_d$ satisfying

$$\frac{e(\varDelta) - \tilde{e}(\varDelta)}{\mathrm{Area}(\varDelta)} < \varepsilon$$

forms a Vitali class of E. From Proposition 2.11 we can choose a sequence $\{\varDelta_i\}_{i=1,2,\ldots}$ of such triangles with $\overline{\varDelta_i} \cap \overline{\varDelta_{i'}} = \emptyset$ for $i \neq i'$ that satisfy

$$\mathrm{Area}\left(E \Big\backslash E \cap \bigcup_{i=1}^{\infty} \overline{\varDelta_i} \right) = 0.$$

Further, we may assume that $\varDelta_i \subset O$. Then we obtain

$$\sum_{i=1}^{\infty} (e(\varDelta_i) - \tilde{e}(\varDelta_i)) < \varepsilon \sum_{i=1}^{\infty} \mathrm{Area}(\varDelta_i) \leq \varepsilon \,\mathrm{Area}(O) < 2\varepsilon \,\mathrm{Area}(E). \tag{3.6}$$

From the definition of E, the family of those geodesic triangles $\varDelta \in U_d$ satisfying

$$\frac{e(\varDelta) - \tilde{e}(\varDelta)}{\mathrm{Area}(\varDelta)} > a \tag{3.7}$$

also forms a Vitali class of E. Thus from Proposition 2.11, for arbitrarily fixed $\delta > 0$ and for each $i \in N$ there is a finite number of these triangles $\{\varDelta_i^j\}_{j=1}^{J(i)}$ with $\varDelta_i^j \subset \varDelta_i$ and

$\overline{\Delta_i^j} \cap \overline{\Delta_i^{j'}} = \emptyset$ for $j \neq j'$ such that

$$\mathrm{Area}\left(E \cap \Delta_i \setminus \bigcup_{j=1}^{J(i)} \overline{\Delta_i^j}\right) < \delta/2^i.$$

Then from the formula (3.7),

$$\begin{aligned}
\mathrm{Area}(E \cap \Delta_i) &< \sum_{j=1}^{J(i)} (\mathrm{Area}(\Delta_i^j)) + \delta/2^i \\
&< \frac{1}{a} \sum_{j=1}^{J(i)} (e(\Delta_i^j) - \tilde{e}(\Delta_i^j)) + \delta/2^i \\
&\leq \frac{1}{a} (e(\Delta_i) - \tilde{e}(\Delta_i)) + \delta/2^i.
\end{aligned} \tag{3.8}$$

Here, the last inequality follows from the inequalities

$$e\left(\Delta_i \setminus \overline{\bigcup_{j=1}^{J(i)} \Delta_i^j}\right) \geq \int_{\Delta_i \setminus \overline{\bigcup_{j=1}^{J(i)} \Delta_i^j}} \underline{G_d}(x)\, d\mathscr{H}^2$$

and

$$e(\Delta_i) \geq \sum_{j=1}^{J} e(\Delta_i^j) + e\left(\Delta_i \setminus \overline{\bigcup_{j=1}^{J(i)} \Delta_i^j}\right).$$

Taking the summation of (3.8) for $i = 1, 2, \ldots$ and using (3.6), we obtain at last that

$$\begin{aligned}
\mathrm{Area}(E) &= \sum_{i=1}^{\infty} \mathrm{Area}(E \cap \Delta_i) \\
&\leq \frac{1}{a} \sum_{i=1}^{\infty} (e(\Delta_i) - \tilde{e}(\Delta_i)) + \delta \\
&< \frac{1}{a} 2\varepsilon\, \mathrm{Area}(E) + \delta \\
&< \frac{2}{3} \mathrm{Area}(E) + \delta.
\end{aligned}$$

Thus if we choose δ as $\delta < 1/3\mathrm{Area}(E)$, then a contradiction is derived. This completes the proof of Theorem 3.1. □

REMARKS.

(1) For $d > 0$ and for every point $x \in X$, we write

$$\underline{G_d'}(x) := \liminf_{\Delta \to \{x\}, x \in \bar{\Delta}, \Delta \in U_d} \frac{e(\Delta)}{\mathrm{Area}(\Delta)}$$

and

$$\overline{G_d'}(x) := \limsup_{\Delta \to \{x\}, x \in \bar{\Delta}, \Delta \in U_d} \frac{e(\Delta)}{\mathrm{Area}(\Delta)}.$$

Here $\underline{G_d'}(x) \le \underline{G_d}(x) \le \overline{G_d}(x) \le \overline{G_d'}(x)$. Consider the case in which p is the vertex of a flat cone. Then $\underline{G_d'}(p) = 0$, but $\underline{G_d}(p) > 0$. Then noting the remark following Proposition 2.11, we have the same result as in the Main theorem for $\underline{G_d'}(x)$ and $\overline{G_d'}(x)$, that is, for almost all $x \in X$, $\inf_{d>0}\underline{G_d'}(x) = \sup_{d>0}\overline{G_d'}(x) < \infty$.

(2) We put for every $x \in X$,

$$\underline{G_0}(x) := \liminf_{\Delta \to \{x\}, x \in \Delta} \frac{e(\Delta)}{\mathrm{Area}(\Delta)}$$

and

$$\overline{G_0}(x) := \limsup_{\Delta \to \{x\}, x \in \Delta} \frac{e(\Delta)}{\mathrm{Area}(\Delta)}.$$

It is not certain if the main results hold for $\underline{G_0}(x)$ and $\overline{G_0}(x)$.

(3) Let X be Otsu and Shioya's example introduced in Remark 1.4. Then for any $x \in X$ and any $\varepsilon > 0$, there is a biangle of diameter less than ε for which x is a vertex. Thus if we set for $x \in X$,

$$\overline{G_0'}(x) := \limsup_{\Delta \to \{x\}, x \in \bar{\Delta}} \frac{e(\Delta)}{\mathrm{Area}(\Delta)},$$

then for any $x \in X$, $\overline{G_0'}(x) = \infty$.

§4. Curvature measure.

In this section we discuss the total excess of *good* subsets of an Alexandrov surface. We agree that X is of curvature bounded below by a negative number $-\kappa^2$. Let E be any compact proper subset of X. Let $\{D_i\}_{i=1,2,\dots}$ be a sequence of open sets such that for any i, $D_i \in \mathscr{D}$, $E \subset D_i$ and $\overline{D_{i+1}} \subset D_i$, and such that $\bigcap_i D_i = E$.

LEMMA 4.1. *The sequence $\{e(D_i)\}_{i=1,2,\dots}$ converges. Moreover letting $\{D_j'\}_{j=1,2,\dots}$ be another sequence of domains in $\mathscr{D}$ such that $\bigcap_j D_j' = E$ (without the assumption that $\overline{D_{j+1}'} \subset D_j'$), the limit $\lim_{j\to\infty}(e(D_j'))$ exists and equals $\lim_{i\to\infty}(e(D_i))$.*

PROOF. The former statement is easily checked by using the fact, obtained in Theorem 2.0, that the sequence $\{e(D_i) + \kappa^2 \mathrm{Area}(D_i)\}$ is monotone non-increasing and bounded below by 0. The latter is also clear. (cf. Lemma 2.7) □

Therefore $e(E) := \lim_{i\to\infty} e(D_i)$ is well-defined. We call this the total excess of E. Then we see that $e(E) \ge -\kappa^2 \mathrm{Area}(E)$. For example, by Lemma 2.7, we see that the total excess $e(x)$ of a point $x \in X$ equals $2\pi - L(\Sigma_x)$.

Now we recall the first variation formula shown in [**BGP**]. Let $\gamma : [0, l] \to X$ be a minimal geodesic. Then, for any geodesic σ with $\sigma(0) = \gamma(0)$, if $t_0 \in (0, l)$, then $\lim_{s\to 0} \tilde{\angle}(\gamma(t_0), \gamma(0), \sigma(s)) = \angle(\gamma, \sigma)$. Moreover we can show similarly as Lemma 2.1 that $\lim_{s\to 0} \angle(\gamma(t_0), \sigma(s), \gamma(0)) = \pi - \angle(\gamma, \sigma)$. This follows the following proposition. Its proof is omitted.

PROPOSITION 4.2. *Let $D \in \mathscr{D}$ and $p_1, p_2, \dots, p_J$ be the vertices of ∂D. Then*

$$e(\bar{D}) = e(D) + \sum_{j=1}^{J}(2\pi - L(\Sigma_{p_j})).$$

REMARK 4.3. Conversely, let $D \in \mathscr{D}$ and $\{D_i\}_{i=1,2,\dots}$ be a sequence of subsets in $\mathscr{D}$ such that for any i, $\overline{D_i} \subset D$ and $\overline{D_i} \subset D_{i+1}$, and such that $\cup_i D_i = D$. Then $\lim_{i\to\infty} e(D_i) = e(D)$.

Proposition 4.2 and the above remark imply that if $D \in \mathscr{D}$ then

$$e(\partial D) = \sum_{j=1}^{J}(2\pi - L(\Sigma_{p_i})).$$

Thus we see that for any geodesic γ,

$$e(\gamma) = (2\pi - L(\Sigma_x)) + (2\pi - L(\Sigma_y)),$$

where x and y are the end points of γ.

Now we discuss the σ-algebra of X. Let X be compact. Let $\mathscr{E}$ be the family of all subsets E that can be obtained by taking a finite disjoint union of members of $\mathscr{D}$ and adding to and removing from this union a finite number of points and geodesics. Here, for convenience, we agree that geodesics are parameterized by open intervals and hence do not contain endpoints. Note that $\mathscr{E}$ is a finitely additive class. Let $E \in \mathscr{E}$. Then E is expressed by

$$\left(\bigcup_{i=1}^{I} \overline{D_i}\backslash\{\gamma_1, \dots, \gamma_m, x_1, \dots, x_n\}\right) \cup \{\sigma_1, \dots, \sigma_\mu\} \cup \{y_1, \dots, y_\nu\},$$

where $D_1, D_2, \dots D_I$ are disjoint subsets in $\mathscr{D}$, γ_j and σ_j are geodesics, and x_j and y_j are points for each j. Then it is natural to define the total excess $e(E)$ of E as

$$e(E) := \sum_{i=1}^{I} e(D_i) - \sum_{j=1}^{n}(2\pi - L(\Sigma_{x_j})) + \sum_{k=1}^{\nu}(2\pi - L(\Sigma_{y_k})).$$

Note that $e(E) \geq -\kappa^2 \operatorname{Area}(E)$. Therefore we see that if we put $m(E) := e(E) + \kappa^2 \operatorname{Area}(E)$, then $m(\cdot)$ is a finitely additive measure on a finitely additive class $\mathscr{E}$. In analogy to Proposition 4.2 and Remark 4.3, we obtain the following Lemma.

LEMMA 4.4. *Let $E \in \mathscr{E}$. For any $\varepsilon > 0$, there is an open set $O \in \mathscr{E}$ and is a compact set $P \in \mathscr{E}$ with $E \subset O$ and $P \subset E$ such that*

$$m(E) > m(O) - \varepsilon \quad \textit{and} \quad m(E) < m(P) + \varepsilon.$$

It should be noted that in general, we cannot choose O so as to satisfy $\bar{E} \subset O$. For example, let $E \in \mathscr{E}$ be such that there is an open subset $D \in \mathscr{D}$ with $D \subset E \subset \bar{D}$. Let $p \in \partial E$ be a metrically singular point. If $p \notin E$, we should choose the open set O of Lemma 4.4 such that $p \notin O$. Conversely if $p \in E$, we should choose the compact set P such that $p \in P$. The proof of Lemma 4.4 is omitted.

Let $\{E_i\}_{i=1,2,\ldots}$ be an infinite sequence of subsets contained in $\mathscr{E}$ with $E_i \cap E_j = \emptyset$ for $i \neq j$ such that $\bigcup_i E_i \in \mathscr{E}$. Then, using Lemma 4.4 for each E_i and $\bigcup_i E_i$, we obtain $m(\bigcup_i E_i) = \sum_{i=1}^{\infty} m(E_i)$. Thus $m(\cdot)$ is a completely additive measure on $\mathscr{E}$.

Now we define the total excess $e(F)$ of an arbitrary subset $F \subset X$ as follows. Define

$$m^*(F) := \inf \sum_{i=1}^{\infty} m(E_i),$$

where the infimum is taken over all $E_1, E_2, \ldots \in \mathscr{E}$, and $F \subset \bigcup_{i=1}^{\infty} E_i$. Also define

$$e(F) := m^*(F) - \kappa^2 \operatorname{Area}(F).$$

The total excess $e(F)$ is well-defined. In fact, since $m(\cdot)$ is a completely additive measure on $\mathscr{E}$, $m^*(F) = m(F)$ for $F \in \mathscr{E}$.

EXAMPLE.

(1) Let X be the double of a flat disk of radius 1 and O the interior of one of the copies of the disk. Then we see that $e(O) = 0$ and $e(\bar{O}) = 4\pi$.

(2) Let X be a flat cone with vertex p. Then for any $r > 0$, $e(\overline{B(p,r)}) = 2\pi - L(\Sigma_p)$ and $e(B(p,r)\backslash\{p\}) = 0$.

We write

$$\tilde{\mathscr{E}} := \{m^*\text{-measurable set}\} \cap \{\mathscr{H}^2\text{-measurable set}\}.$$

Note that $\mathscr{E} \subset \tilde{\mathscr{E}}$. Then the total excess is a completely additive set function on $\tilde{\mathscr{E}}$. Thus by the Radon-Nikodym theorem, we obtain the following proposition.

PROPOSITION 4.5. *There are a unique $\mathscr{H}^2$-absolutely continuous set function ψ and a unique $\mathscr{H}^2$-singular set function ϕ on $\tilde{\mathscr{E}}$ such that*

$$e(E) = \psi(E) + \phi(E), \quad E \in \tilde{\mathscr{E}}.$$

Moreover, there is a $\mathscr{H}^2$-measurable function f defined on almost all points of X with respect to $\mathscr{H}^2$-measure such that

$$\psi(E) = \int_E f(x)\, d\mathscr{H}^2.$$

Since $\psi(\cdot)$ is an absolutely continuous set function and $\phi(\cdot)$ is a singular set function, the formula $e(\cdot) \geq -\kappa^2 \operatorname{Area}(\cdot)$ implies that ϕ is positive. In addition, by using Proposition 2.11, we observe that for any $d > 0$ and for almost all $x \in X$,

$$\limsup_{\Delta \to \{x\}, x \in \Delta, \Delta \in U_d} \frac{\phi(\Delta)}{\operatorname{Area}(\Delta)} = 0.$$

Thus we obtain the following theorem.

THEOREM 4.6. *The function $f(x)$ coincides with the Gaussian curvature $G(x)$ almost everywhere.*

Before ending this article, we would like to propose a problem. For simplicity, let X be compact. We see from Proposition 4.5 and Theorem 4.6 that for almost all

$x \in X$,

$$\lim_{r \to 0} \frac{e(B(x,r))}{\pi r^2} = G(x).$$

Make the definitions

$$E_\alpha := \left\{ x \in X \,\middle|\, \lim_{r \to 0} \frac{e(B(x,r))}{r^\alpha} \text{exists and is finite.} \right\}, \quad \alpha \in [0,2]$$

and

$$E'_\alpha := \left\{ x \in E_\alpha \,\middle|\, \lim_{r \to 0} \frac{e(B(x,r))}{r^\alpha} = 0 \right\}.$$

Then, with $G_\alpha(x) := \lim_{r \to 0} \dfrac{e(B(x,r))}{r^\alpha}$ for $x \in E_\alpha$, $G_\alpha(x) \geq 0$ holds for $\alpha < 2$.

PROBLEM. *Is it true that $\mathscr{H}^\alpha(X \backslash E_\alpha) = 0$ for $\alpha \in [0,2]$, and that*

$$I := \left\{ \alpha \in [0,2] \,\middle|\, \int_{E_\alpha \backslash E'_\alpha} G_\alpha(x) \, d\mathscr{H}^\alpha > 0 \right\}$$

is a countable set? Moreover does it hold that

$$2\pi\chi(X) = \sum_{\alpha \in I} \frac{1}{\lambda(\alpha)} \int_{E_\alpha \backslash E'_\alpha} G_\alpha(x) \, d\mathscr{H}^\alpha,$$

where $\lambda(\alpha)$ is a constant depending on α? For example $\lambda(2) = \pi$ and $\lambda(1) = 2$.

EXAMPLE. Let X be the double of a flat disk of radius 1. If x is a point on the edge C of X, then a direct calculation shows that $G_1(x) = 4$. Thus

$$\frac{1}{2} \int_C G_1(x) \, d\mathscr{H}^1 = 4\pi = 2\pi\chi(X).$$

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