

On the C^∞ -Goursat problem for some second order equations with variable coefficients

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§ 1. Introduction.

In this paper we study C^∞ -Goursat problem for the following L :

$$(1.1) \quad L = \partial_t \partial_x + a(t, x) \partial_y^2 + b(t, x) \partial_y + c(t, x),$$

where $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$, for

$$(t, x, y) \in [0, \infty) \times \mathbf{R}^2 \quad \text{or} \quad (t, x, y) \in (-\infty, 0] \times \mathbf{R}^2.$$

The coefficients $a(t, x)$, $b(t, x)$ and $c(t, x)$ are real valued C^∞ -functions, which are independent of y . For given C^∞ -functions $f(t, x, y)$, $g(x, y)$ and $h(t, y)$ with the compatibility condition $g(0, y) = h(0, y)$, the Goursat problem is to find a function $u(t, x, y)$ which satisfies

$$(P) \quad \begin{cases} Lu = f(t, x, y) \in C^\infty_{(t,x,y)}, \\ u(0, x, y) = g(x, y) \in C^\infty_{(x,y)}, \\ u(t, 0, y) = h(t, y) \in C^\infty_{(t,y)}, \quad \text{for } t \geq 0 \text{ or } t \leq 0. \end{cases}$$

We say that the Goursat problem (P) is \mathcal{E} -wellposed for $t \geq 0$ (resp. for $t \leq 0$) if for any data $\{f, g, h\} \in \mathcal{E}_{(t,x,y)} \times \mathcal{E}_{(x,y)} \times \mathcal{E}_{(t,y)}$ there exists a unique solution $u(t, x, y)$ of (P) belonging to $\mathcal{E}_{(t,x,y)}$ with $t \geq 0$ (resp. $t \leq 0$). In this case we also say that the Goursat problem for L is \mathcal{E} -wellposed for $t \geq 0$ (resp. for $t \leq 0$). If (P) is \mathcal{E} -wellposed for $t \geq 0$ (resp. for $t \leq 0$) then it follows from Banach's closed graph theorem that the linear mapping $\{f, g, h\} \rightarrow u(t, x, y)$ is continuous from $\mathcal{E}_{(t,x,y)} \times \mathcal{E}_{(x,y)} \times \mathcal{E}_{(t,y)}$ to $\mathcal{E}_{(t,x,y)}$ for $t \geq 0$ (resp. for $t \leq 0$).

The C^∞ -Goursat problem with constant coefficients has been treated by several authors, for instance [4], [5], [6], and [8]. When the coefficients a , b and c are constant, we know that the necessary and sufficient condition for (P) to be \mathcal{E} -wellposed for both $t \geq 0$ and $t \leq 0$, is $a = b = 0$. In the case of variable coefficients what is the necessary condition for (P) to be \mathcal{E} -wellposed? It is the main problem that we study in this paper. On the other hand Nishitani [9] and Mandai [7] had also studied C^∞ -Goursat problem for general operators with variable coefficients. However our operator is excluded from their concern.

§ 2. Results.

In this section, we show our main results in this paper; proof is carried out in the following sections.

By using the Taylor expansion of coefficients $a(t, x)$, $b(t, x)$ (the size of N is fixed later):

$$(2.1) \quad a(t, x) = \sum_{j+k \leq N} a_{jk} t^j x^k / j! k! + A_N(t, x),$$

$$(2.2) \quad b(t, x) = \sum_{j+k \leq N} b_{jk} t^j x^k / j! k! + B_N(t, x),$$

we introduce the following conditions:

(C-1) There exists (j_0, k_0) such that $a_{j_0, k_0} \neq 0$ and $a_{jk} = 0$ for $(j, k) \in \Omega$, where $\Omega = \{(j, k); j + k < j_0 + k_0\} \cup \{(j, k); j + k = j_0 + k_0 \text{ and } j > \max\{j_0, k_0\}\} \cup \{(j, k); j + k = j_0 + k_0 \text{ and } k > \max\{j_0, k_0\}\}$.

$$(C-2) \quad b_{jk} = 0 \text{ for } 2j + 2k < j_0 + k_0 - 1.$$

Our main result is the following theorem.

THEOREM 2.1. *Assume that the condition (C-1) and (C-2) hold. If $a_{j_0, k_0} > 0$, then (P) is not \mathcal{E} -wellposed for $t \geq 0$.*

If $j_0 + k_0 \leq 1$, then (C-2) is free. Hence we get the following corollary:

COROLLARY 2.1. *If (P) is \mathcal{E} -wellposed both for $t \geq 0$ and for $t \leq 0$, then $a_{00} = a_{10} = a_{01} = 0$.*

In fact, when $j_0 k_0$ is even, by making the change of variables $x' = -x$ or $t' = -t$, this case can be reduced to that of $a_{j_0, k_0} > 0$.

Now we consider the case both j_0 and k_0 are odd and $a_{j_0, k_0} < 0$. A simple example is the following:

$$(2.3) \quad L_1 = \partial_t \partial_x - atx \partial_y^2, \quad a > 0.$$

Let us consider the following Goursat problem:

$$(P_1) \quad \begin{cases} L_1 u = f \in C_{(t, x, y)}^\infty, \\ u(0, x, y) = u(t, 0, y) = 0. \end{cases}$$

THEOREM 2.2. *The Goursat problem (P₁) has a unique solution $u \in C^\infty$ of the following form:*

$$(2.4) \quad u(t, x, y) = \int_0^t \int_0^x \int_{-1}^1 \frac{f(\tau, \xi, y + \sigma \sqrt{a(t^2 - \tau^2)(x^2 - \xi^2)})}{\pi \sqrt{1 - \sigma^2}} d\sigma d\xi d\tau.$$

We can generalize Theorem 2.2 as follows. Let us consider the Goursat problem (P'₁):

$$(P'_1) \quad \begin{cases} \partial_t \partial_x u - at^p x^q \partial_y^2 u = f(t, x, y) \in C^\infty, \\ u(0, x, y) = u(t, 0, y) = 0, \\ \text{where } a > 0, \text{ and } p \text{ and } q \text{ are odd numbers.} \end{cases}$$

THEOREM 2.2'. *The Goursat problem (P'_1) has a unique solution $u \in C^\infty$ of the form:*

$$(2.4') \quad u(t, x, y) = \int_0^t \int_0^x \int_{-1}^1 f(\tau, \xi, y + \sigma\theta) / (\pi\sqrt{1 - \sigma^2}) d\sigma d\xi d\tau$$

$$\theta = 2\sqrt{a(t^{p+1} - \tau^{p+1})(x^{q+1} - \xi^{q+1}) / (p+1)(q+1)}.$$

Next we study the role of lower order terms. Let L_2 be an operator of the form:

$$(2.5) \quad L_2 = \partial_t \partial_x - bt^r x^s \partial_y,$$

where b is nonzero real constant, and r and s are nonnegative integers. We notice that the operator L_2 dose not include the term ∂_y^2 . Let us consider the Goursat problem:

$$(P_2) \quad \begin{cases} L_2 u = f \in C^\infty_{(t,x,y)}, \\ u(0, x, y) = g(x, y) \in C^\infty_{(x,y)}, \\ u(t, 0, y) = h(t, y) \in C^\infty_{(t,y)}, \quad g(0, y) = h(0, y). \end{cases}$$

THEOREM 2.3. *The Goursat problem (P_2) is not \mathcal{E} -wellposed for $t \geq 0$ nor $t \leq 0$.*

Finally we consider the following operator L_3 and the classical Goursat problem (P_3).

$$(2.6) \quad L_3 = \partial_t \partial_x - c(t, x, y),$$

where $c(t, x, y)$ is complex valued C^∞ -function,

$$(P_3) \quad \begin{cases} L_3 u = f(t, x, y) \in C^\infty_{(t,x,y)}, \\ u(t, 0, y) = u(0, x, y) = 0, \quad (t, x, y) \in \mathbf{R}_+^1 \times \mathbf{R}^2 \text{ or } (t, x, y) \in \mathbf{R}_-^1 \times \mathbf{R}^2. \end{cases}$$

THEOREM 2.4. *The Goursat problem (P_3) has a unique solution $u \in C^\infty$.*

In particular when $c(t, x, y)$ is constant A , the solution u of (P_3) has the following expression which will be used in §4.

THEOREM 2.4' (cf. Hadamard [2]). *The Goursat problem*

$$(P'_3) \quad \begin{cases} \partial_t \partial_x u - Au = f(t, x, y) \in C^\infty, \\ u(t, 0, y) = u(0, x, y) = 0, \end{cases}$$

has a unique solution which has the following integral representation:

$$(2.7) \quad u = \int_0^t \int_0^x f(\tau, \xi, y) \Psi(t - \tau, x - \xi) d\xi d\tau,$$

where Ψ is the solution of the following Goursat problem:

$$(2.8) \quad \begin{cases} \partial_t \partial_x \Psi = A \Psi, \\ \Psi(0, x) = \Psi(t, 0) = 1. \end{cases}$$

REMARK 2.1. The solution Ψ of (2.8) is expressed as $\Psi = I_0(2\sqrt{Atx})$ with modified Bessel function of order zero, namely

$$(2.9) \quad I_0(z) = \sum_{k=0}^{\infty} \left(\frac{z^2}{4}\right)^k / (k!)^2.$$

§3. Proof of Theorem 2.1.

We prove this theorem by contradiction. We assume (P) to be \mathcal{E} -wellposed for $t \geq 0$.

First we consider the Goursat problem with oscillatory data:

$$(3.1) \quad \begin{cases} \partial_t \partial_x u + a(t, x) \partial_y^2 u + b(t, x) \partial_y u + c(t, x) u = e^{i\eta} f(t, x), \\ u(0, x, y) = e^{i\eta} g(x), \\ u(t, 0, y) = e^{i\eta} h(t), \quad g(0) = h(0). \end{cases}$$

PROPOSITION 3.1. *If we assume that (P) is \mathcal{E} -wellposed, then the solution u of (3.1) has the form;*

$$(3.2) \quad u = e^{i\eta} v(t, x; \eta).$$

PROOF. We set

$$(3.3) \quad u = e^{i\eta} v(t, x).$$

Then v is the solution of the following problem:

$$(3.1') \quad \begin{cases} \partial_t \partial_x v - A(t, x, i\eta) v = f(t, x), \\ v(0, x) = g(x), \\ v(t, 0) = h(t), \quad g(0) = h(0), \\ \text{where } A(t, x, i\eta) = -\{a(t, x)(i\eta)^2 + b(t, x)i\eta + c(t, x)\}. \end{cases}$$

By Theorem 2.4, (3.1') has a unique solution $v \in C_{(t,x)}^\infty$, and u defined by (3.3) is a solution of (3.1). By the assumption of \mathcal{E} -wellposedness, this u is the unique solution of (3.1). q.e.d.

Second, we consider the domain of dependence of (3.1'). The following Proposition is obvious.

PROPOSITION 3.2. *Let us consider the Goursat problem (3.1'). The domain of dependence of (t_0, x_0) is included in $D(t_0, x_0)$ defined by*

$$D(t_0, x_0) = \{(t, x); 0 \leq t \leq t_0, 0 \leq x \leq x_0\} \text{ for } x_0 \geq 0,$$

$$D(t_0, x_0) = \{(t, x); 0 \leq t \leq t_0, 0 \geq x \geq x_0\} \text{ for } x_0 < 0.$$

More precisely, let v be a solution of (3.1'), and assume that $f(t, x) = 0$ ($t, x \in D(t_0, x_0)$), $g(x) = 0$ $x \in D(0, x_0)$ and $h(t) = 0$ $t \in D(t_0, 0)$. Then $v(t_0, x_0) = 0$.

Now we consider the following Goursat problem.

$$(3.4) \quad \begin{cases} \partial_t \partial_x v = A(t, x, i\eta)v, \\ v(0, x) = v(t, 0) = 1. \end{cases}$$

Let us consider the Taylor expansion of $A(t, x, i\eta)$. For simplicity, we use the notations;

$$(3.5) \quad A(t, x, i\eta) = \sum_{m=0}^2 \eta^m a_m(t, x),$$

$$(3.6) \quad a_m(t, x) = \sum_{r+s \leq N} a_{m,r,s} t^r x^s + B_{m,N}(t, x),$$

$$(3.7) \quad B_{m,N} = \sum_{r+s=N+1} (t^r x^s / r!s!) \partial_t^r \partial_x^s a_m(\theta t, \theta' x), \quad 0 < \theta, \theta' < 1.$$

By (3.4), (3.5) and (3.6) we have

$$(3.8) \quad \begin{aligned} \partial_t \partial_x v &= \sum_{m=0}^2 \sum_{r+s \leq N} \eta^m a_{m,r,s} t^r x^s v + \sum_{m=0}^2 \eta^m B_{m,N}(t, x)v \\ &\equiv L_N v + R_N v. \end{aligned}$$

Therefore (3.4) becomes

$$(3.9) \quad \begin{cases} \partial_t \partial_x v = L_N v + R_N v, \\ v(0, x) = v(t, 0) = 1. \end{cases}$$

We recall the assumption of Theorem 2.1 and (C-1). We single out the term $\eta^2 a_{2,j_0,k_0} t^{j_0} x^{k_0} v$ in the right-hand side of (3.8) and consider the following Goursat problem:

$$(3.10) \quad \begin{cases} \partial_t \partial_x \phi = a\eta^2 t^{j_0} x^{k_0} \phi, \quad (a > 0), \\ \phi(0, x) = \phi(t, 0) = 1. \end{cases}$$

PROPOSITION 3.3. The solution of the Goursat problem (3.10) is expressed as

$$(3.11) \quad \phi(t, x; \eta) = \sum_{n=0}^{\infty} \{a\eta^2 t^{j_0+1} x^{k_0+1} / (j_0 + 1)(k_0 + 1)\}^n / (n!)^2.$$

PROOF. For simplicity we denote $j_0 = p$ and $k_0 = q$. Then (3.10) becomes

$$(3.12) \quad \begin{cases} \partial_t \partial_x \phi = a\eta^2 t^p x^q \phi, \\ \phi(0, x) = \phi(t, 0) = 1. \end{cases}$$

We set the solution of (3.12) to be

$$(3.13) \quad \phi = \sum_{j,k} \phi_{jk} t^j x^k / j! k!.$$

Putting (3.13) into (3.12), and comparing the coefficients of $t^j x^k$, we obtain the formula

$$(3.14) \quad \phi_{j+1,k+1} = \{a\eta^2 j! k! / (j-p)!(k-q)!\} \phi_{j-p,k-q}, \quad j \geq 0, k \geq 0.$$

It follows from the Goursat data that

$$(3.15) \quad \phi_{0,0} = 1, \quad \phi_{j,0} = 0, \quad \phi_{0,k} = 0, \quad j, k \geq 1.$$

By using (3.14) and (3.15), $\phi_{j,k}$ are determined as the following.

$$(3.16) \quad \begin{cases} \phi_{n(p+1),n(q+1)} = (a\eta^2)^n \{(p+1)n\}! \{(q+1)n\}! / (p+1)^n (q+1)^n (n!)^2, \\ \phi_{j,k} = 0 \quad \text{for } (j,k) \neq (n(p+1), n(q+1)), \quad n \geq 0. \end{cases}$$

Therefore we obtain

$$(3.17) \quad \begin{aligned} \phi(t, x) &= \sum_n \frac{t^{n(p+1)} x^{n(q+1)} (a\eta^2)^n \{n(p+1)\}! \{n(q+1)\}!}{\{n(p+1)\}! \{n(q+1)\}! (p+1)^n (q+1)^n (n!)^2} \\ &= \sum_n \left\{ \frac{t^{p+1} x^{q+1} a\eta^2}{(p+1)(q+1)} \right\}^n / (n!)^2. \end{aligned}$$

The right-hand side of (3.17) converge uniformly on every compact set. Thus (3.17) is the solution of (3.10). q.e.d.

Furthermore, we have the following estimate for $\phi(t, x)$.

PROPOSITION 3.4. *Let*

$$(3.18) \quad t_\eta = \eta^{-\varepsilon_1}, \quad x_\eta = \eta^{-\varepsilon_2}.$$

If $a > 0$ and

$$(3.19) \quad 2\alpha = 2 - (j_0 + 1)\varepsilon_1 - (h_0 + 1)\varepsilon_2 > 0$$

holds then there exist positive constant C_1 and C_2 such that

$$(3.20) \quad \phi(t_\eta, x_\eta; \eta) > C_1 \exp(C_2 \eta^\alpha).$$

PROOF. Putting

$$(3.21) \quad t^{p+1} x^{q+1} a\eta^2 / (p+1)(q+1) = z^2 / 4,$$

then

$$(3.22) \quad \phi(t, x; \eta) = I_0(z),$$

where

$$(3.23) \quad I_0(z) = \sum \left(\frac{z}{2}\right)^{2n} / (n!)^2.$$

When z is real, we have

$$(3.24) \quad I_0(z) > \frac{1}{2} \left\{ \exp\left(\frac{z}{2}\right) + \exp\left(-\frac{z}{2}\right) \right\}.$$

By (3.21), we have

$$(3.25) \quad z = 2\eta \sqrt{t^{p+1} x^{q+1} a / (p+1)(q+1)}.$$

Hence

$$\begin{aligned} z(t_\eta, x_\eta; \eta) &= \{2\sqrt{a/(p+1)(q+1)}\} \eta^{1-\varepsilon_1(p+1)/2-\varepsilon_2(q+1)/2} \\ &= 2C_2\eta^\alpha, \quad \alpha > 0. \end{aligned}$$

We have from (3.24)

$$(3.26) \quad \begin{aligned} \phi(t_\eta, x_\eta; \eta) &> \frac{1}{2} \exp(C_2\eta^\alpha), \\ C_2 &= \sqrt{a/(p+1)(q+1)}. \end{aligned}$$

Thus we complete the proof of Proposition 3.4.

Without loss of generality we can assume $j_0 \geq k_0$. Let us consider the following Goursat problem:

$$(3.27) \quad \begin{cases} \partial_t \partial_x \psi = L_N \psi, \\ \psi(0, x) = \psi(t, 0) = 1, \end{cases}$$

where $L_N = L_N(t, x; \eta) = \sum_{m=0}^2 \sum_{r+s \leq N} \eta^m a_{m,r,s} t^r x^s$.

By virtue of Theorem 2.4, (3.27) has a unique solution $\psi(t, x; \eta)$. The value of $\psi(t_\eta, x_\eta; \eta)$ is approximated by $\phi(t_\eta, x_\eta; \eta)$ in the following sense.

PROPOSITION 3.5. *Let ψ be a solution of (3.27). If we define*

$$(3.28) \quad \begin{aligned} \varepsilon_1 &= (4k_0 + 4)\lambda, \quad \varepsilon_2 = (4k_0 + 6)\lambda, \\ \lambda &= 2\{j_0(4k_0 + 4) + k_0(4j_0 + 6) + 8k_0 + 11\}^{-1}, \end{aligned}$$

then we have

$$(3.29) \quad \begin{aligned} \psi(t_\eta, x_\eta; \eta) &= \phi(t_\eta, x_\eta; \eta)(1 + o(\eta)) + o(\eta), \quad \eta \rightarrow +\infty, \\ \text{where } t_\eta &= \eta^{-\varepsilon_1}, \quad x_\eta = \eta^{-\varepsilon_2}. \end{aligned}$$

Since the proof of Proposition 3.5 is rather complicated, we prove this later in §6.

Lastly we consider the following Goursat problem:

$$(3.30) \quad \begin{cases} \partial_t \partial_x v - L_N v - R_N v = -R_N \psi, \\ v(0, x) = v(t, 0) = 1. \end{cases}$$

By (3.27), ψ is the solution of (3.30). Rewriting (3.30), we have

$$(3.31) \quad \begin{cases} \partial_t \partial_x u - A(t, x, \partial_y)u = -R_N \psi \exp(iy\eta), \\ u(0, x, y) = u(t, 0, y) = \exp(iy\eta). \end{cases}$$

Obviously $u = \psi \exp(iy\eta)$ is the solution of (3.31). Because of the continuity from data to solutions and the existence of the dependence domain, there exists positive number h such that the absolute value of $u(t_\eta, x_\eta, \eta)$ ($= |\psi(t_\eta, x_\eta; \eta)|$) is estimated by the value of $f(= -R_N \psi \exp(iy\eta))$, $\exp(iy\eta)$ and their α -th derivatives ($|\alpha| \leq h$) at $D(t_\eta, x_\eta)$. Putting

$$(3.32) \quad \max_{(t,x) \in D(t_\eta, x_\eta)} |\psi| = M(\eta),$$

we have the following estimates:

PROPOSITION 3.6.

$$(3.33) \quad \begin{aligned} |\partial_t^{h_1} \partial_x^{h_2} \partial_y^{h_3} (R_N \psi(t, x; \eta) e^{iy\eta})| &\leq CM(\eta) \eta^{-\varepsilon(N+1)+2(h_1+h_2+h_3)+2}, \\ \text{for } (t, x) \in D(t_\eta, x_\eta), \quad h_1 + h_2 + h_3 &\leq h, \quad \text{where } \min\{\varepsilon_1, \varepsilon_2\} = \varepsilon. \end{aligned}$$

We prove this Proposition in §7. Now, we choose N (which appeared in (2.1) and (2.2)) so that the following inequality holds;

$$(3.34) \quad -\varepsilon(N+1) + 2h + 2 = -\rho < 0.$$

Then we have

$$(3.35) \quad |u| = |\psi| \leq C\eta^h + CM(\eta)\eta^{-\rho}, \quad \text{where } C \text{ does not depend on } \eta.$$

Therefore

$$\begin{aligned} M(\eta) &\leq C\eta^h + CM(\eta)\eta^{-\rho}, \\ M(\eta)(1 - C\eta^{-\rho}) &\leq C\eta^h. \end{aligned}$$

Since

$$|\psi(t_\eta, x_\eta; \eta)| \leq M(\eta)$$

then

$$(3.36) \quad |\psi(t_\eta, x_\eta; \eta)| \leq C'\eta^h.$$

On the other hand, by Proposition 3.4 and Proposition 3.5, $\psi(t_\eta, x_\eta; \eta)$ increase with exponential order as $\eta \rightarrow +\infty$, which cannot be compatible to (3.36). Thus we complete the proof of Theorem 2.1.

§4. Proof of Theorem 2.2.

Let us consider the following equation.

$$(4.1) \quad \partial_t \partial_x u - atx \partial_y^2 u = f(t, x, y) \in C^\infty, \quad a > 0.$$

By setting

$$(4.2) \quad t^2/2 = T, \quad x^2/2 = X,$$

(4.1) becomes

$$(4.3) \quad \partial_T \partial_X u - a \partial_y^2 u = f(\sqrt{2T}, \sqrt{2X}, y)/2\sqrt{TX}.$$

Now we set $a \partial_y^2 = A$, Theorem 2.4' suggests that the solution of (4.3) is expressed as a formal integral:

$$w = \int_0^T \int_0^X I_0(2\sqrt{A(T-\tau)(X-\xi)}) f(\sqrt{2\tau}, \sqrt{2\xi}, y)/2\sqrt{\tau\xi} d\xi d\tau.$$

Using the formula

$$(4.4) \quad I_0(z) = \sum_{k=0}^{\infty} (z^2/4)^k / (k!)^2 = \int_{-1}^1 e^{z\sigma} / \pi \sqrt{1-\sigma^2} d\sigma,$$

we have

$$(4.5) \quad I_0(2\sqrt{A(T-\tau)(X-\xi)}) = \int_{-1}^1 \exp\{2\sigma\sqrt{a(T-\tau)(X-\xi)}\partial_y\} / \pi \sqrt{1-\sigma^2} d\sigma.$$

If $g(T, X, y)$ is analytic function with respect to y , then

$$(4.6) \quad \begin{aligned} \exp(\theta \partial_y) g(T, X, y) &= g(T, X, y + \theta), \\ \theta &= 2\sigma\sqrt{a(T-\tau)(x-\xi)}. \end{aligned}$$

Therefore

$$(4.7) \quad w = \int_0^T \int_0^X \int_{-1}^1 f(\sqrt{2\tau}, \sqrt{2\xi}, y + \theta) / (2\pi\sqrt{\tau\xi} \sqrt{1-\sigma^2}) d\sigma d\xi d\tau.$$

By (4.2) we obtain

$$(4.8) \quad \begin{aligned} w &= \int_0^t \int_0^x \int_{-1}^1 f(\tau, \xi, y + \sigma\theta_1) / (\pi\sqrt{1-\sigma^2}) d\sigma d\xi d\tau, \\ \theta_1 &= \sqrt{a(t^2 - \tau^2)(x^2 - \xi^2)}. \end{aligned}$$

Next proposition says that w is actually the solution of the problem (P_1) .

PROPOSITION 4.1. *Let an operator K be*

$$(4.9) \quad \begin{aligned} (Kf)(t, x, y) &= \int_0^t \int_0^x \int_{-1}^1 f(\tau, \xi, y + \sigma\theta_1) / (\pi\sqrt{1-\sigma^2}) d\sigma d\xi d\tau \\ \theta_1 &= \sqrt{a(t^2 - \tau^2)(x^2 - \xi^2)}, \quad f \in C^\infty_{(t,x,y)}. \end{aligned}$$

Then $Kf \in C^\infty$ and Kf satisfies (P_1) .

PROOF.

$$(4.10) \quad L_1(Kf) - f = (\partial_t \partial_x - atx \partial_y^2)(Kf) - f(t, x, y) \\ = - \int_0^t \int_0^x \int_{-1}^1 atx(1 - \sigma^2) f_{yy}(\tau, \xi, y + \sigma \theta_1) / \pi \sqrt{1 - \sigma^2} d\sigma d\xi d\tau \\ + \int_0^t \int_0^x \int_{-1}^1 atx \sigma f_y(\tau, \xi, y + \sigma \theta_1) / \pi \sqrt{1 - \sigma^2} \theta_1 d\sigma d\xi d\tau.$$

Since

$$(4.11) \quad \partial_\sigma f_y(\tau, \xi, y + \sigma \theta_1) = \theta_1 f_{yy},$$

we have

$$(4.12) \quad L_1(Kf) - f = - \int_0^t \int_0^x \int_{-1}^1 atx \sqrt{(1 - \sigma^2)} \partial_\sigma f_y(\tau, \xi, y + \sigma \theta_1) / (\pi \theta_1) d\sigma d\xi d\tau \\ + \int_0^t \int_0^x \int_{-1}^1 atx \sigma f_y(\tau, \xi, y + \sigma \theta_1) / (\pi \sqrt{1 - \sigma^2} \theta_1) d\sigma d\xi d\tau.$$

Using the integration by parts, we find that

$$(4.13) \quad \int_{-1}^1 \sqrt{1 - \sigma^2} \partial_\sigma f_y d\sigma = \int_{-1}^1 \sigma f_y / \sqrt{1 - \sigma^2} d\sigma.$$

Thus by (4.12) and (4.13) we conclude

$$(4.14) \quad L_1(Kf) - f = 0. \quad \text{q.e.d.}$$

Next we will show the uniqueness.

PROPOSITION 4.2. *If $L_1 u = 0$ and $u(0, x, y) = u(t, 0, y) = 0$, then $u = 0$.*

PROOF. If $L_1 u = 0$, then

$$(4.15) \quad 0 = K(L_1 u) = \int_0^t \int_0^x \int_{-1}^1 (L_1 u)(\tau, \xi, y + \sigma \theta_1) / \pi \sqrt{1 - \sigma^2} d\sigma d\xi d\tau.$$

For simplicity, here we introduce new notations,

$$T(t, \tau) = \sqrt{t^2 - \tau^2}, \quad X(x, \xi) = \sqrt{x^2 - \xi^2},$$

hence $\theta_1 = \sqrt{a} TX$. (c.f. (4.8))

By simple computations,

$$(4.16) \quad \partial_\tau u(\tau, \xi, y + \sigma \theta_1) \\ = u_t(\tau, \xi, y + \sigma \theta_1) + u_y(\tau, \xi, y + \sigma \theta_1) (-\tau \sigma \sqrt{a} X(x, \xi) / T(t, \tau)),$$

$$(4.17) \quad \partial_\xi \partial_\tau u(\tau, \xi, y + \sigma \theta_1) = u_{tx}(\tau, \xi, y + \sigma \theta_1) + u_{yy} \tau \xi \sigma^2 a + u_y \sigma \tau \xi \sqrt{a} / T(t, \tau) X(x, \xi) \\ - u_{yx}(\tau, \xi, y + \sigma \theta_1) \tau \sigma \sqrt{a} X(x, \xi) / T(t, \tau) \\ - u_{yt}(\tau, \xi, y + \sigma \theta_1) \xi \sigma \sqrt{a} T(t, \tau) / X(x, \xi),$$

$$(4.18) \quad \partial_\xi u_y = u_{yx} - u_{yy} \xi \sigma \sqrt{a} T(t, \tau) / X(x, \xi).$$

Hence

$$(4.19) \quad \begin{aligned} u_{tx} - a\tau \xi u_{yy} &= \partial_\xi \partial_\tau u + (\partial_\xi u_y)(\tau \sigma \sqrt{a} X / T) \\ &\quad + (\partial_\tau u_y)(\xi \sigma \sqrt{a} T / X) - u_{yy} \tau \xi a(1 - \sigma^2) \\ &= \partial_\xi \partial_\tau u + Q. \end{aligned}$$

We will show $\int_0^t \int_0^x \int_{-1}^1 Q / \pi \sqrt{1 - \sigma^2} d\sigma d\xi d\tau = 0$. Since $u(t, 0, y) = 0$, we have $u_y(t, 0, y) = 0$. Then it follows that

$$(4.20) \quad \int_0^x (\partial_\xi u_y) X(x, \xi) d\xi = \int_0^x \xi u_y / X(x, \xi) d\xi.$$

In the same way

$$(4.21) \quad \int_0^t (\partial_\tau u_y) T(t, \tau) d\tau = \int_0^t \tau u_y / T(t, \tau) d\tau.$$

Notice that

$$(4.22) \quad \partial_\sigma u_y = \sqrt{a} T(t, \tau) X(x, \xi) u_{yy}.$$

By (4.20), (4.21) and (4.22) we have

$$(4.23) \quad \int_0^t \int_0^x \int_{-1}^1 Q / \pi \sqrt{1 - \sigma^2} d\sigma d\xi d\tau = 0.$$

Therefore we conclude

$$(4.24) \quad 0 = K(L_1 u) = \int_0^t \int_0^x \int_{-1}^1 \partial_\tau \partial_\xi u / \pi \sqrt{1 - \sigma^2} d\sigma d\xi d\tau = u.$$

Thus we complete the proof of Proposition 4.2.

By Proposition 4.1 and Proposition 4.2 we obtain Theorem 2.2.

We can prove Theorem 2.2' in the same way as the above proof.

§5. Proof of Theorem 2.3.

We prove the theorem by contradiction. Recall that

$$(2.5) \quad L_2 = \partial_t \partial_x - b t^r x^s \partial_y, \quad b \neq 0, \quad \text{real.}$$

Let $u = \{\exp(iny)\} v(t, x)$. $L_2 u = 0$ becomes

$$(5.1) \quad \partial_t \partial_x v = A t^r x^s v, \quad A = inb.$$

Let us consider the following Goursat data:

$$(5.2) \quad v(0, x) = v(t, 0) = 1.$$

Let the formal solution of (5.1)–(5.2) be (5.3).

$$(5.3) \quad v = \sum_{j,k} v_{jk} t^j x^k / j! k!.$$

Putting (5.3) into (5.1), and comparing the coefficient of $t^j x^k$, we have

$$(5.4) \quad v_{j+1,k+1} = j! k! A v_{j-r,k-s} / (j-r)! (k-s)!.$$

It follows from (5.2) that

$$(5.5) \quad v_{0,0} = 1, \quad v_{0,k} = 0, \quad v_{j,0} = 0, \quad j, k \geq 1.$$

By (5.4) and (5.5), we have

$$(5.6) \quad v_{k(r+1),k(s+1)} = \{k(r+1)\}! \{k(s+1)\}! A^k / \{(r+1)(s+1)\}^k (k!)^2,$$

$$(5.7) \quad v_{p,q} = 0, \quad (p,q) \neq (k(r+1), k(s+1)), \quad k \geq 0.$$

Therefore

$$(5.8) \quad \begin{aligned} v &= \sum_k [t^{k(r+1)} x^{k(s+1)} / \{k(r+1)\}! \{k(s+1)\}!] v_{k(r+1),k(s+1)} \\ &= \sum_k \{t^{r+1} x^{s+1} A / (r+1)(s+1)\}^k / (k!)^2. \end{aligned}$$

Here we recall the property of Bessel functions:

LEMMA 5.1. *The Bessel function*

$$(5.9) \quad J_0(z) = \sum_k (-z^2/4)^k / (k!)^2$$

has the following representation for large $|z|$ with

$$-\pi/2 + \delta < \arg z < \pi/2 - \delta, \quad \delta > 0,$$

$$(5.10) \quad J_0(z) = \{\exp i(z - \pi/4) + \exp(-i(z - \pi/4))\} / \sqrt{2\pi z} + O(|z|^{-2/3}).$$

Putting $A = inb$ in (5.8), we have

$$(5.11) \quad v = \sum_k \{t^{r+1} x^{s+1} inb / (r+1)(s+1)\}^k / (k!)^2.$$

Let

$$(5.12) \quad B = t^{r+1} x^{s+1} b / (r+1)(s+1),$$

and

$$(5.13) \quad z^2 = -i4nB.$$

We define z in the following way;

$$\begin{aligned} z &= \sqrt{-4Bn} \exp(in/4) \quad \text{when } B < 0, \\ z &= \sqrt{4Bn} \exp(-in/4) \quad \text{when } B > 0. \end{aligned}$$

Then it follows that

$$-\pi/2 + \delta < \arg z < \pi/2 - \delta, \quad \delta > 0,$$

and

$$\operatorname{Im} z = \sqrt{-4Bn}/\sqrt{2} \quad \text{when } B < 0,$$

$$\operatorname{Im} z = -\sqrt{4Bn}/\sqrt{2} \quad \text{when } B > 0.$$

By the Lemma 5.1 we have

$$v = J_0(z) \sim \{C_1/n^{1/4}\} \exp(C_2\sqrt{n}), \quad C_1, C_2 > 0.$$

Since the Goursat data are $u(0, x) = u(t, 0) = \exp(iny)$, this shows that the mapping from data to solutions is not continuous. q.e.d.

§ 6. Proof of Proposition 3.5.

In this section we assume $t, x \geq 0$. Recall the following (3.27), (3.17) and (3.18);

$$(3.27) \quad \begin{cases} \partial_t \partial_x \psi = L_N \psi, \\ \psi(0, x) = \psi(t, 0) = 1. \end{cases}$$

$$(3.17) \quad \phi(t, x) = \sum_n \{t^{j_0+1} x^{k_0+1} a \eta^2 / (j_0 + 1)(k_0 + 1)\}^n / (n!)^2,$$

$$\text{where } a = a_{2, j_0, k_0} > 0.$$

$$(3.18) \quad t_\eta = \eta^{-\varepsilon_1}, \quad x_\eta = \eta^{-\varepsilon_2}.$$

We will show that there exists a pair $\{\varepsilon_1, \varepsilon_2\}$ which satisfies the following two conditions:

- (i) $\phi(t_\eta, x_\eta)$ grows up with exponential order of η ,
- (ii) the value $\psi(t_\eta, x_\eta)$ is approximated by $\phi(t_\eta, x_\eta)$, more precisely (3.29) holds.

SKETCH OF THE PROOF. First we obtain the solution ψ of (3.17) by the successive approximation. We define sequence of functions $\{\psi_n\}$ as follows;

$$(6.1) \quad \begin{cases} \psi_0 = 1, \\ \psi_n = \int_0^x \int_0^t L_N(\tau, \xi; \eta) \psi_{n-1}(\tau, \xi; \eta) d\tau d\xi, \quad n \geq 1. \end{cases}$$

We have the following estimate;

LEMMA 6.1.

$$(6.2) \quad |\psi_n(t, x; \eta)| < M \eta^{2n} t^n x^n / (n!)^2, \quad n \geq 0,$$

where $|L_N(t, x; \eta)| < M \eta^2$ for $0 \leq t, x \leq 1$.

Let ψ denote

$$(6.3) \quad \psi(t, x; \eta) = \sum_{n=0}^{\infty} \psi_n(t, x; \eta).$$

By Lemma 6.1, this series ψ converge uniformly on every compact subset, and this ψ is the solution of (3.27).

Let us divide the series ψ into two parts;

$$(6.4) \quad \psi = \sum_{n \leq \eta_3} \psi_n + \sum_{n > \eta_3} \psi_n,$$

where $\eta_3 = \eta^{\varepsilon_3}$, ε_3 is positive constant.

We divide the following proof into 3 parts;

PART 1. In this part we will show the following:

If $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ satisfies some conditions, then it follows that

$$(6.5) \quad \sum_{n \leq \eta_3} \psi_n(t_\eta, x_\eta; \eta) = \left\{ \sum_{n \leq \eta_3} \phi_n(t_\eta, x_\eta; \eta) \right\} \{1 + o(\eta)\}, \quad \eta \rightarrow +\infty,$$

where

$$\phi_n(t, x; \eta) = \{t^{j_0+1} x^{k_0+1} a\eta^2 / (j_0 + 1)(k_0 + 1)\}^n / (n!)^2.$$

PART 2. Here, we will show the following:

If $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ satisfies some conditions, then

$$\sum_{n > \eta_3} \psi_n(t_\eta, x_\eta; \eta) = o(\eta), \quad \eta \rightarrow +\infty.$$

PART 3. We define $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ which satisfies (3.19) and all conditions in Part 1 and Part 2.

REMARK 6.1. Part 1 is the most important. Since the estimate is delicate, we divide it into some cases.

REMARK 6.2. The condition (C-2) (in §2) is used in Part 2, but is not used in Part 1.

PART 1. By (6.1), we have

$$(6.6) \quad \psi_n = \sum_{\alpha_1} \sum_{\alpha_2} \cdots \sum_{\alpha_n} a_{\alpha_1} a_{\alpha_2} \cdots a_{\alpha_n} \eta^{m_1+m_2+\cdots+m_n} \\ \times \frac{t^{r_1+r_2+\cdots+r_n+n} x^{s_1+s_2+\cdots+s_n+n}}{(r_1+1)(r_1+r_2+2) \cdots (r_1+\cdots+r_n+n)(s_1+1)(s_1+s_2+2) \cdots (s_1+\cdots+s_n+n)},$$

where $\alpha_i = (m_i, r_i, s_i)$, and \sum_{α_i} is the summation with

$$0 \leq m_i \leq 2, \quad 0 \leq r_i + s_i \leq N.$$

We denote above expression simply by

$$(6.7) \quad \psi_n = \sum_{\alpha} \psi_n(\alpha),$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$,

$$(6.8) \quad \psi_n(\alpha) = \left(\prod_{p=1}^n a_{\alpha_p} \right) \eta^{m_1+m_2+\dots+m_n} \times \frac{t^{r_1+r_2+\dots+r_n+n} x^{s_1+s_2+\dots+s_n+n}}{(r_1+1)(r_1+r_2+2) \cdots (r_1+r_2+\dots+r_n+n)(s_1+1)(s_1+s_2+2) \cdots (s_1+s_2+\dots+s_n+n)}.$$

When $\alpha_1 = \alpha_2 = \dots = \alpha_n = (2, j_0, k_0) \equiv \hat{\alpha}$,

$$(6.9) \quad \psi_n(\hat{\alpha}, \hat{\alpha}, \dots, \hat{\alpha}) = a^n \eta^{2n} \frac{t^{n(j_0+1)} x^{n(k_0+1)}}{(j_0+1)^n (k_0+1)^n (n!)^2} = \phi_n,$$

where $a = a_{2, j_0, k_0} = a_{\hat{\alpha}}$.

We rewrite (6.7) as follows:

$$(6.10) \quad \psi_n = \phi_n \left\{ 1 + \sum'_\alpha \psi_n(\alpha) / \phi_n \right\},$$

where \sum'_α means the summation of the terms with $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ $\alpha_i = (m_i, r_i, s_i)$ satisfying $0 \leq m_i \leq 2$, $r_i + s_i \leq N$ except $\alpha = (\hat{\alpha}, \hat{\alpha}, \dots, \hat{\alpha})$.

Hence we have

$$(6.11) \quad \psi_n(\alpha) / \phi_n = \prod_{p=1}^n \left\{ (a_{\alpha_p} / a) \frac{(j_0+1)(k_0+1)p^2}{(r_1+r_2+\dots+r_p+p)(s_1+s_2+\dots+s_p+p)} \right\} \times \eta^{m_1+m_2+\dots+m_n-2n} t^{r_1+r_2+\dots+r_n-nj_0} x^{s_1+s_2+\dots+s_n-nk_0}.$$

To estimate (6.11), we use the following fact:

LEMMA 6.2. *Let A and B are real and non-negative numbers. If $A + B = A' + B'$ and $|A - B| > |A' - B'|$, then $AB < A'B'$.*

Now we divide the summation \sum'_α in (6.10) as follows;

$$(6.12) \quad \sum'_\alpha = \sum_1 + \sum_{2,0} + \dots + \sum_{2,n} + \sum_3,$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_i = (m_i, r_i, s_i)$.

\sum_1 means the summation of terms with $m_i = 2$ and $r_i + s_i = j_0 + k_0$, ($i = 1, 2, \dots, n$).

$\sum_{2,p}$ means the summation of the terms with $m_i = 2$ ($i = 1, 2, \dots, n$), and the number of elements of $\{i; r_i + s_i > j_0 + k_0\}$ is p .

\sum_3 means the summation of the remainder terms.

ESTIMATES OF \sum_1 . We will estimate (6.11). Without loss of generality we can assume $j_0 \geq k_0$. In this case $r_i + s_i = j_0 + k_0$ and by the condition (C-1), we have

$$(6.13) \quad k_0 \leq r_i, s_i \leq j_0.$$

Thus

$$(6.14) \quad \sum_{i=1}^p (r_i + 1) + \sum_{i=1}^p (s_i + 1) = p(j_0 + 1) + p(k_0 + 1),$$

$$(6.15) \quad \left| \sum_{i=1}^p (r_i + 1) - \sum_{i=1}^p (s_i + 1) \right| \leq p(j_0 + 1) - p(k_0 + 1).$$

Hence, by Lemma 6.2, we have

$$(6.16) \quad (j_0 + 1)(k_0 + 1)p^2 / \left\{ \sum_{i=1}^p (r_i + 1) \right\} \left\{ \sum_{i=1}^p (s_i + 1) \right\} \leq 1, \quad \text{for } 1 \leq p \leq n.$$

We define ω as follows;

$$(6.17) \quad \sum_{i=1}^n s_i - nk_0 = - \sum_{i=1}^n r_i + nj_0 = \omega.$$

In this case $\omega \geq 1$, and the number of p with $a_{\alpha_p} \neq a$ is at most ω . Therefore we have

$$(6.18) \quad |\psi_n(\alpha)/\phi_n| \leq \prod_{p=1}^n |a_{\alpha_p}/a|(x/t)^\omega \leq (Kx/t)^\omega,$$

where $|a_{\alpha_p}/a| \leq K$.

The number of pairs $\{r_i, s_i\}$ satisfying (6.17) is at most $(nN)^\omega$. Thus we obtain

$$(6.19) \quad \sum_1 |\psi_n(\alpha)/\phi_n| \leq \sum_{\omega \geq 1} (nNKx/t)^\omega.$$

So the value of (6.19) at $(t, x) = (t_\eta, x_\eta)$ for $n \leq \eta_3$ is estimated as follows:

$$(6.20) \quad \sum_1 |(\psi_n(\alpha)/\phi_n)(t_\eta, x_\eta)| \leq \sum_{\omega \geq 1} (NK\eta^{\varepsilon_1 - \varepsilon_2 + \varepsilon_3})^\omega.$$

If we assume

$$(6.21) \quad \varepsilon_1 - \varepsilon_2 + \varepsilon_3 < 0,$$

then we have

$$(6.22) \quad \sum_1 |(\psi_n(\alpha)/\phi_n)(t_\eta, x_\eta)| \leq C\eta^{-\sigma},$$

where C and σ are positive constants independent of n and η . Hereafter we assume (6.21).

ESTIMATES OF $\sum_{2,1}$. In this case there exists a unique α_i with $r_i + s_i > j_0 + k_0$. Let (r_i, s_i) with $r_i + s_i > j_0 + k_0$ be (\bar{r}, \bar{s}) . Let us consider the right-hand side of (6.11). It follows that

$$(6.23) \quad \left\{ \sum_{i=1}^p (r_i + 1) \right\} \left\{ \sum_{i=1}^p (s_i + 1) \right\} > \left\{ \sum' (r_i + 1) \right\} \left\{ \sum' (s_i + 1) \right\},$$

where \sum' is the sum of terms with $r_i + s_i = j_0 + k_0$. The number of the terms with $r_i + s_i = j_0 + k_0$ is at least $p - 1$. So by Lemma 6.2, we have

$$(6.24) \quad \begin{aligned} \{\sum'(r_i + 1)\}\{\sum'(s_i + 1)\} &> (p - 1)^2(j_0 + 1)(k_0 + 1) \quad \text{for } p \geq 2 \\ (r_1 + 1)(s_1 + 1) &> 0 \quad \text{for } p = 1. \end{aligned}$$

Therefore we have

$$(6.25) \quad \begin{aligned} \prod_{p=1}^n \left[(j_0 + 1)(k_0 + 1)p^2 / \left\{ \sum_{i=1}^p (r_i + 1) \right\} \left\{ \sum_{i=1}^p (s_i + 1) \right\} \right] \\ > \frac{(j_0 + 1)(k_0 + 1)}{1} \frac{2^2}{1^2} \frac{3^2}{2^2} \cdots \frac{n^2}{(n-1)^2} = n^2(j_0 + 1)(k_0 + 1). \end{aligned}$$

Hence

$$(6.26) \quad \begin{aligned} |\psi_n(\alpha)/\phi_n| &\leq \left\{ \prod_{p=1}^n |a_{\alpha_p}/a| \right\} (j_0 + 1)(k_0 + 1)n^2 \\ &\times t^{\sum'(r_i - j_0)} x^{\sum'(s_i - k_0)} t^{\bar{r} - j_0} x^{\bar{s} - k_0}. \end{aligned}$$

Estimates of \sum' can be carried out in the same way as \sum_1 .

In this case $(r_1, s_1) = (\bar{r}, \bar{s})$ or $(r_2, s_2) = (\bar{r}, \bar{s})$ or \cdots or $(r_n, s_n) = (\bar{r}, \bar{s})$. So we want to show

$$(6.27) \quad Cn^3 t_n^{\bar{r} - j_0} x_n^{\bar{s} - k_0} = o(\eta) \quad \eta \rightarrow +\infty, \quad \text{for } n < \eta_3.$$

If (6.21) holds, then $\varepsilon_2 > \varepsilon_1$. Therefore the most delicate case is $\bar{r} = j_0 + k_0 + 1$, $\bar{s} = 0$. Here we can estimate as follows;

$$(6.28) \quad \begin{aligned} n^3 t_n^{\bar{r} - j_0} x_n^{\bar{s} - k_0} &< \eta^{3\varepsilon_3 - \varepsilon_1(\bar{r} - j_0) - \varepsilon_2(\bar{s} - k_0)} \\ &< \eta^{3\varepsilon_3 - \varepsilon_1(k_0 + 1) + \varepsilon_2 k_0}. \end{aligned}$$

Because of $\bar{r} + \bar{s} \leq N$, the number of elements of $\{(\bar{r}, \bar{s}); \bar{r} + \bar{s} \leq N\}$ is at most $(N + 1)^2$. Then we have

$$(6.29) \quad \sum_{2,1} |(\psi_n(\alpha)/\phi_n)(t_\eta, x_\eta)| < CN^2 \eta^{3\varepsilon_3 - \varepsilon_1(k_0 + 1) + \varepsilon_2 k_0}.$$

Therefore, if we assume

$$(6.30) \quad 3\varepsilon_3 - \varepsilon_1(k_0 + 1) + \varepsilon_2 k_0 < 0,$$

then we obtain

$$(6.31) \quad \sum_{2,1} |(\psi_n(\alpha)/\phi_n)(t_\eta, x_\eta)| < C\eta^{-\sigma},$$

where C and σ are positive constants independent of η and n . Here after we assume (6.30).

ESTIMATES OF $\sum_{2,2}$. In this case we have

$$\begin{aligned}
 (6.32) \quad & \left\{ \sum_{i=1}^p (r_i + 1) \right\} \left\{ \sum_{i=1}^p (s_i + 1) \right\} \\
 & = \left\{ \sum' (r_i + 1) + \sum'' (r_i + 1) \right\} \left\{ \sum' (s_i + 1) + \sum'' (s_i + 1) \right\} \\
 & \geq \left\{ \sum' (r_i + 1) \right\} \left\{ \sum' (s_i + 1) \right\},
 \end{aligned}$$

where \sum' is the sum of terms with $r_i + s_i = j_0 + k_0$, the number of terms in \sum' is at least $p - 2$, and \sum'' is the sum of terms with $r_i + s_i > j_0 + k_0$. In the same way as $\sum_{2,1}$, we have

$$(6.33) \quad \left\{ \sum_{i=1}^p (r_i + 1) \right\} \left\{ \sum_{i=1}^p (s_i + 1) \right\} > (p - 2)^2 (j_0 + 1)(k_0 + 1) \quad \text{for } p \geq 3.$$

Therefore

$$\begin{aligned}
 (6.34) \quad & \prod_{p=1}^n \left[(j_0 + 1)(k_0 + 1)p^2 / \left\{ \sum_{i=1}^p (r_i + 1) \right\} \left\{ \sum_{i=1}^p (s_i + 1) \right\} \right] \\
 & > (j_0 + 1)^2 (k_0 + 1)^2 \frac{3^2 4^2 \dots n^2}{1^2 2^2 \dots (n - 2)^2} \\
 & = \{n(n - 1)(j_0 + 1)(k_0 + 1)\}^2 / (2!)^2.
 \end{aligned}$$

Hence we obtain

$$\begin{aligned}
 (6.35) \quad & |\psi_n(\alpha) / \phi_n| \leq \left\{ \prod_{p=1}^n |a_{\alpha_p} / a| \right\} (Cn(n - 1) / 2!)^2 \\
 & \quad \times t^{\sum' (r_i - j_0)} x^{\sum' (s_i - k_0)} t^{\sum'' (r_i - j_0)} x^{\sum'' (s_i - k_0)}.
 \end{aligned}$$

The estimates of \sum' can be carried out in the same way as \sum_1 . In \sum'' , the most delicate term is $s_i = 0, r_i = j_0 + k_0 + 1$. Considering the number of terms in $\sum_{2,2}$, we have

$$(6.36) \quad \sum_{2,2} |(\psi_n(\alpha) / \phi_n)(t_\eta, x_\eta)| \leq [C\eta^{\{k_0 e_2 - (k_0 + 1)e_1 + 3e_3\}}]^2.$$

ESTIMATES OF $\sum_{2,k}$. In a similar way as the estimates of $\sum_{2,2}$, we have the following;

$$(6.37) \quad \sum_{2,k} |(\psi_n(\alpha) / \phi_n)(t_\eta, x_\eta)| \leq [C\eta^{\{k_0 e_2 - (k_0 + 1)e_1 + 3e_3\}}]^k.$$

After all by (6.30), we have

$$(6.38) \quad \sum_{k=1}^n \sum_{2,k} |(\psi_n(\alpha) / \phi_n)(t_\eta, x_\eta)| \leq C\eta^{-\sigma},$$

where C and σ are positive constants independent of η and n .

ESTIMATES OF \sum_3 . \sum_3 means the sum of terms in which there exists α_i with $m_i = 1$ or 0. Here we divide the terms in \sum_3 as follows;

$$(6.39) \quad \sum_3 = \sum_{3,1} + \sum_{3,2} + \cdots + \sum_{3,n}.$$

Where $\sum_{3,p}$ means the summation of terms where the number of element of $\{i; m_i \leq 1\}$ is p . First we consider $\sum_{3,1}$. Recall the expression (6.11). Concerning the part of $m_i = 2$, we can treat in the same way as the preceding cases. The most delicate part is $m_i = 1, r_i = s_i = 0$. Considering the number of terms in $\sum_{3,1}$ and (6.26), we have the following:

$$(6.40) \quad \begin{aligned} \sum_{3,1} |(\psi_n(\alpha)/\phi_n)(t_\eta, x_\eta)| &< C\eta^{1-2} t_\eta^{-j_0} x_\eta^{-k_0} n^2 n \\ &\leq C\eta^{-1-\varepsilon_1 j_0 - \varepsilon_2 k_0 + 3\varepsilon_3} \quad \text{for } n \leq \eta_3 = \eta^{-\varepsilon_3}. \end{aligned}$$

Next we consider $\sum_{3,p}$. We can obtain the following estimate in a similar way as the above.

$$(6.41) \quad \sum_{3,p} |(\psi_n(\alpha)/\phi_n)(t_\eta, x_\eta)| \leq [C\eta^{-1-\varepsilon_1 j_0 - \varepsilon_2 k_0 + 3\varepsilon_3}]^p.$$

Therefore if we assume

$$(6.42) \quad -1 - \varepsilon_1 j_0 - \varepsilon_2 k_0 + 3\varepsilon_3 < 0,$$

then we have

$$(6.43) \quad \sum_3 |(\psi_n(\alpha)/\phi_n)(t_\eta, x_\eta)| \leq C\eta^{-\sigma}.$$

Thus we complete the part 1.

PART 2. Recall the expression (6.7) and (6.8), the number of terms in \sum_α is at most $(3N^2)^n$. Therefore we have

$$(6.44) \quad \left| \sum_\alpha \psi_n(\alpha)(t, x) \right| < K_1^n (3N^2)^n \eta^{H(n)} t^{T(n)} x^{X(n)} / (n!)^2$$

where $H(n) = \sum_{i=1}^n m_i, T(n) = \sum_{i=1}^n (r_i + 1), X(n) = \sum_{i=1}^n (s_i + 1),$

$$|a_{\alpha_p}| < K_1.$$

Let us denote by q_1 the number of the terms in \sum_α with $m_i = 1$ and q_2 with $m_i = 0$. By Stirling's formula

$$(6.45) \quad n! = \sqrt{2\pi n} n^{n+1/2} e^{-n+\theta/12n} \quad 0 \leq \theta \leq 1,$$

we have

$$(6.46) \quad \left| \sum_\alpha \psi_n(\alpha)(t_\eta, x_\eta) \right| < C^n \eta^{F(n)}.$$

Where C is constant and $F(n)$ is defined by

$$(6.47) \quad F(n) = 2n - q_1 - 2q_2 - \varepsilon_1 \left\{ \sum_{i=1}^n (r_i + 1) \right\} - \varepsilon_2 \left\{ \sum_{i=1}^n (s_i + 1) \right\} - 2n\varepsilon_3.$$

Since

$$(6.48) \quad \sum_{i=1}^n (r_i + 1) = \sum_{i=1}^n (j_0 + 1) - \sum_{i=1}^n (j_0 - r_i) = n(j_0 + 1) - \sum_{i=1}^n (j_0 - r_i),$$

$$(6.48') \quad \sum_{i=1}^n (s_i + 1) = \sum_{i=1}^n (k_0 + 1) - \sum_{i=1}^n (k_0 - s_i) = n(k_0 + 1) - \sum_{i=1}^n (k_0 - s_i),$$

we have

$$(6.49) \quad \begin{aligned} F(n) &= n\{2 - 2\varepsilon_3 - \varepsilon_1(j_0 + 1) - \varepsilon_2(k_0 + 1)\} - q_1 - 2q_2 \\ &\quad + \varepsilon_1 \sum_{i=1}^n (j_0 - r_i) + \varepsilon_2 \sum_{i=1}^n (k_0 - s_i) \\ &\equiv nf_1 + f_2. \end{aligned}$$

Terms f_1 and f_2 are defined by above.

First we assume that

$$(6.50) \quad f_1 = 2 - 2\varepsilon_3 - \varepsilon_1(j_0 + 1) - \varepsilon_2(k_0 + 1) < 0.$$

Next we consider f_2 ;

$$(6.51) \quad f_2 = \sum_{i=1}^n \{\varepsilon_1(j_0 - r_i) + \varepsilon_2(k_0 - s_i)\} - q_1 - 2q_2.$$

Dividing the terms in $\sum_{i=1}^n$ of (6.51) into 4, we have

$$(6.52) \quad \begin{aligned} f_2 &= \sum_{i=1}^n f_{2,i} - q_1 - 2q_2 \\ &= \sum_{(1)} f_{2,i} + \sum_{(2)} f_{2,i} + \sum_{(3)} f_{2,i} + \sum_{(4)} f_{2,i} - q_1 - 2q_2, \end{aligned}$$

where $f_{2,i} = \varepsilon_1(j_0 - r_i) + \varepsilon_2(k_0 - s_i)$,

where $\sum_{(1)}$ is the sum of terms with $m_i = 2$ and $r_i + s_i = j_0 + k_0$, $\sum_{(2)}$ with $m_i = 2$ and $r_i + s_i > j_0 + k_0$, $\sum_{(3)}$ with $m_i = 1$, and $\sum_{(4)}$ with $m_i = 0$.

ESTIMATES OF $\sum_{(1)}$.

$$(6.53) \quad \sum_{(1)} f_{2,i} = \sum_{(1)} (\varepsilon_2 - \varepsilon_1)(k_0 - s_i).$$

In this case $k_0 \leq s_i$. By (6.21) we have

$$(6.54) \quad \sum_{(1)} f_{2,i} \leq 0.$$

ESTIMATES OF $\sum_{(2)}$. In this case $r_i + s_i \geq j_0 + k_0 + 1$. Hence

$$(6.55) \quad \begin{aligned} \sum_{(2)} f_{2,i} &\leq \sum_{(2)} \{\varepsilon_1(s_i - k_0 - 1) + \varepsilon_2(k_0 - s_i)\} \\ &= \sum_{(2)} \{k_0(\varepsilon_2 - \varepsilon_1) - s_i(\varepsilon_2 - \varepsilon_1) - \varepsilon_1\} \\ &\leq \sum_{(2)} \{k_0(\varepsilon_2 - \varepsilon_1) - \varepsilon_1\}. \end{aligned}$$

If we assume

$$(6.56) \quad k_0(\varepsilon_2 - \varepsilon_1) - \varepsilon_1 < 0,$$

then we have

$$(6.57) \quad \sum_{(2)} f_{2,i} \leq 0.$$

ESTIMATES OF $\sum_{(3)}$. Here we use condition (C-2) in §2. Then we have

$$(6.58) \quad \begin{aligned} \sum_{(3)} f_{2,i} - q_1 &= \sum_{(3)} (\varepsilon_1 j_0 - \varepsilon_1 r_i + \varepsilon_2 k_0 - \varepsilon_2 s_i - 1) \\ &\leq \sum_{(3)} [\varepsilon_1 j_0 + \varepsilon_2 k_0 - \varepsilon_1 \{(j_0 + k_0)/2 - 1/2 - s_i\} - \varepsilon_2 s_i - 1] \\ &= \sum_{(3)} \{\varepsilon_1 (j_0 - k_0 + 1) + 2\varepsilon_2 k_0 - 2 + 2(\varepsilon_1 - \varepsilon_2) s_i\} / 2 \\ &\leq (1/2) \sum_{(3)} \{\varepsilon_1 (j_0 - k_0 + 1) + 2\varepsilon_2 k_0 - 2\}. \end{aligned}$$

If we assume

$$(6.59) \quad \varepsilon_1 (j_0 - k_0 + 1) + 2\varepsilon_2 k_0 - 2 < 0,$$

then we have

$$(6.60) \quad \sum_{(3)} f_{2,i} - q_1 \leq 0.$$

ESTIMATES OF $\sum_{(4)}$. It follows that

$$(6.61) \quad \begin{aligned} \sum_{(4)} f_{2,i} - 2q_2 &= \sum_{(4)} \{\varepsilon_1 (j_0 - r_i) + \varepsilon_2 (k_0 - s_i) - 2\} \\ &\leq \sum_{(4)} (\varepsilon_1 j_0 + \varepsilon_2 k_0 - 2). \end{aligned}$$

If we assume

$$(6.62) \quad \varepsilon_1 j_0 + \varepsilon_2 k_0 - 2 < 0,$$

then we have

$$(6.63) \quad \sum_{(4)} f_{2,i} - 2q_2 \leq 0.$$

Thus we proved that if $\{\varepsilon_i\}$ satisfy (6.21), (6.50), (6.56), (6.59) and (6.62), then we have the following:

$$(6.64) \quad \sum_{n>\eta_3} \sum_{\alpha} \psi_n(\alpha)(t_\eta, x_\eta) = o(\eta), \quad \eta \rightarrow +\infty.$$

Finally we remark that if (6.50) holds, then we have

$$(6.65) \quad \sum_{n>\eta_3} \sum_{\alpha} \phi_n(\alpha)(t_\eta, x_\eta) = o(\eta), \quad \eta \rightarrow +\infty,$$

where $\phi_n(t, x) = \{t^{j_0+1} x^{k_0+1} a \eta^2 / (j_0 + 1)(k_0 + 1)\}^n / (n!)^2$.

PART 3. Until now we impose many assumptions on $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$.

We will show that there exists $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ which satisfies these assumptions.

LEMMA 6.3. *Let*

$$\begin{aligned}\varepsilon_1 &= (4k_0 + 4)\lambda, & \varepsilon_2 &= (4k_0 + 6)\lambda, & \varepsilon_3 &= \lambda, \\ \lambda &= 2/\{j_0(4k_0 + 4) + k_0(4k_0 + 6) + 8k_0 + 11\},\end{aligned}$$

then $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ satisfies (3.19), (6.21), (6.30), (6.42), (6.50), (6.56), (6.59) and (6.62).

Thus we complete the proof of Proposition 3.5.

§7. Proof of Proposition 3.6.

Recall that ψ is the solution of the equation:

$$(3.27) \quad \partial_t \partial_x \psi = L_N \psi, \quad \psi(0, x) = \psi(t, 0) = 1.$$

Hence

$$(7.1) \quad \partial_t \psi(t, x; \eta) = \int_0^x L_N(t, \xi; \eta) \psi(t, \xi; \eta) d\xi.$$

Moreover L_N has the estimates

$$(7.2) \quad |\partial_t^{h_1} \partial_x^{h_2} (L_N(t, x; \eta))| < \eta^2 A, \quad \text{for } (t, x) \in D(t_\eta, x_\eta), h_1 + h_2 \leq h,$$

where A is a constant independent of η .

First we show the following:

LEMMA 7.1.

$$(7.3) \quad \begin{aligned}|\partial_t^{h_1} \psi| &\leq (2A\eta^2)^{h_1} M(\eta), \\ \text{for } 0 \leq h_1 \leq h, \quad (t, x) \in D(t_\eta, x_\eta), \quad \eta : \text{large.}\end{aligned}$$

PROOF. We prove this lemma by induction.

By (3.32), (7.3) holds for $h_1 = 0$. Assume that (7.3) holds for $0 \leq h_1 \leq k$. Since

$$(7.4) \quad \partial_x \partial_t^{k+1} \psi = \partial_t^k (\partial_t \partial_x \psi) = \partial_t^k (L_N \psi).$$

It follows that

$$\begin{aligned}\partial_t^{k+1} \psi(t, x; \eta) &= \int_0^x \partial_t^k \{L_N(t, \xi; \eta) \psi(t, \xi; \eta)\} d\xi \\ &= \int_0^x \sum_{j=0}^k \binom{k}{j} (\partial_t^j L_N) (\partial_t^{k-j} \psi) d\xi.\end{aligned}$$

Using the above expression, we estimate $\partial_t^{k+1} \psi$ in the following:

$$\begin{aligned}
 (7.5) \quad |\partial_t^{k+1}\psi| &\leq |x| \sum_{j=0}^k \binom{k}{j} A\eta^2 (2A\eta^2)^{k-j} M(\eta) \\
 &< (A\eta^2)^{k+1} 2^k \sum_{j=0}^k \binom{k}{j} (2A\eta^2)^{-j} M(\eta) \\
 &= (A\eta^2)^{k+1} 2^k (1 + 1/2A\eta^2)^k M(\eta) \\
 &< (2A\eta^2)^{k+1} M(\eta), \quad \text{for large } \eta \text{ with } (1 + 1/2A\eta^2)^k < 2.
 \end{aligned}$$

This shows that (7.3) holds for $h_1 = k + 1$.

q.e.d.

In the same way, we can prove

LEMMA 7.1'.

$$\begin{aligned}
 (7.3') \quad |\partial_x^{h_2}\psi| &\leq (2A\eta^2)^{h_2} M(\eta) \\
 &\text{for } 0 \leq h_2 \leq h, \quad (t, x) \in D(t_\eta, x_\eta), \quad \eta : \text{large}.
 \end{aligned}$$

In a similar manner, we have

LEMMA 7.2.

$$\begin{aligned}
 (7.6) \quad |\partial_t^{h_1} \partial_x^{h_2}\psi| &\leq (2A\eta^2)^{h_1+h_2} M(\eta) \\
 &\text{for } h_1 + h_2 \leq h, \quad h_1, h_2 \geq 1, \quad (t, x) \in D(t_\eta, x_\eta), \quad \eta : \text{large}.
 \end{aligned}$$

Now, let us prove Proposition 3.7. Recall the following;

$$\begin{aligned}
 (7.7) \quad R_N &= \sum_{m=0}^2 \eta^m B_{m,N}(t, x), \\
 B_{m,N} &= \sum_{r+s=N+1} (t^r x^s / r!s!) \partial_t^r \partial_x^s a_m(\theta t, x\theta'), \quad 0 < \theta, \theta' < 1.
 \end{aligned}$$

There exists a positive constant A' (independent of η) such that

$$\begin{aligned}
 (7.8) \quad |\partial_t^{h_1} \partial_x^{h_2} R_N(t, x; \eta)| &< (\eta^{-\varepsilon})^{N+1-(h_1+h_2)} A' \eta^2, \\
 &\text{for } (t, x) \in D(t_\eta, x_\eta), \quad h_1 + h_2 \leq h.
 \end{aligned}$$

Therefore by Lemma 7.1, Lemma 7.1', Lemma 7.2 and (7.8), we estimate $R_N\psi$ as follows;

$$\begin{aligned}
 |\partial_t^{h_1} \partial_x^{h_2} (R_N\psi)| &= \left| \sum_{j=0}^{h_1} \sum_{k=0}^{h_2} \binom{h_1}{j} \binom{h_2}{k} (\partial_t^j \partial_x^k R_N) (\partial_t^{h_1-j} \partial_x^{h_2-k} \psi) \right| \\
 &\leq \sum_j \sum_k \binom{h_1}{j} \binom{h_2}{k} (\eta^{-\varepsilon})^{N+1-(j+k)} (2A''\eta^2)^{h_1+h_2-(j+k)+1} M(\eta) \\
 &= (2A''\eta^2)^{h_1+h_2} (\eta^{-\varepsilon})^{N+1} M(\eta) \sum_j \sum_k \binom{h_1}{j} \binom{h_2}{k} (2A''\eta^{2-\varepsilon})^{-(j+k)} \\
 &= (2A''\eta^2)^{h_1+h_2} (\eta^{-\varepsilon})^{N+1} M(\eta) (1 + 1/2A''\eta^{2-\varepsilon})^{h_1+h_2},
 \end{aligned}$$

where $A'' > A, A'$.

Then it follows that

$$(7.9) \quad |\partial_t^{h_1} \partial_x^{h_2} \partial_y^{h_3} (R_N \psi e^{i\gamma\eta})| < C(2A''\eta^2)^{h_1+h_2+1} (\eta^{-\varepsilon})^{N+1} \eta^{h_3} M(\eta).$$

Thus we proved the Proposition 3.6.

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References

- [1] R. Courant and D. Hilbert; *Methods of Mathematical Physics 1*. Inter science publishers. New York (1953).
- [2] L. Hadamard; *Le Problème de Cauchy*. Paris (1932).
- [3] Y. Hasegawa; On the C^∞ -Goursat problem for 2nd order Equations with real constant coefficients. *Proc. Japan acad.* **51** (1975) 516–519.
- [4] Y. Hasegawa; On the C^∞ -Goursat problem for equations with constant coefficients. *J. Math. Kyoto Univ.* **19** (1979) 125–151.
- [5] Y. Hasegawa; On the Levi condition for Goursat Problem. *J. Math. Kyoto Univ.* **27** (1987) 1–25.
- [6] Y. Hasegawa; A Remark on the C^∞ -Goursat Problem II. *Publ. RIMS, Kyoto Univ.* **28** (1992) 207–222.
- [7] T. Mandai; On the C^∞ -well-posedness of Goursat Problems. *Publ. RIMS, Kyoto Univ.* **21** (1985) 843–852.
- [8] T. Nishitani; On the \mathcal{E} -well-posedness for the Goursat problem with constant coefficients. *J. Math. Kyoto Univ.* **20** (1980) 179–190.
- [9] T. Nishitani; Une remarque sur le problème de Goursat. *Séminaire sur les équations aux dérivées partielles hyperboliques et holomorphes (1981–82)*, J. Vaillant, Université de Paris VI.

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