The Sylvester's law of inertia in simple graded Lie algebras

Dedicated to Professor Ichiro Satake on the occasion of his seventieth anniversary birthday

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Introduction.

Let $H_n(R)$ be the vector space of $n \times n$ real symmetric matrices. The group $GL(n, R)^0$ (= the identity component of $GL(n, R)$) acts on $H_n(R)$ by the rule: $X \mapsto AXA^t$, $X \in H_n(R)$, $A \in GL(n, R)^0$. The Sylvester's law of inertia asserts that, by this action of $GL(n, R)^0$, $X$ is transformed into the canonical form diag$(1, \ldots, 1, -1, \ldots, -1, 0, \ldots, 0)$, which is uniquely determined by $X$. The simple Lie algebra $sp(n, R)$ has a unique gradation $sp(n, R) = g_{-1} + g_0 + g_1$, where $g_{-1} = H_n(R)$ and $g_0 \simeq gl(n, R)$. The $GL(n, R)^0$-module $H_n(R)$ is imbedded in $sp(n, R)$ as the $G_0^0$-module $g_{-1}$, where $G_0^0$ is the analytic subgroup of Aut $g$ generated by $g_0$. The Sylvester's law of inertia for $H_n(R)$ is no other than obtaining the complete representatives of $G_0^0$-orbits in $g_{-1}$. As a generalization of this situation, one can pose:

**Problem.** Let $g = \sum_{k=-v}^{v} g_k$ be a real simple graded Lie algebra, $G_0$ the group of grade-preserving automorphisms of $g$ and let $G_0^0$ be the identity component of $G_0$. Find the $G_0^0$-orbit decomposition and the $G_0$-orbit decomposition of $g_{-1}$.

When $v = 1$, this problem is equivalent to the problem of finding the orbits in a compact simple Jordan triple system under the structure group or the identity component of the structure group. Also it is equivalent to finding the orbit decomposition of a tangent space by the linear isotropy group for a symmetric $R$-space.

The purpose of this paper is to settle the above problem for the case $v = 1$ by a unified method. Partial answers have been obtained by Satake [22, 23], Kaneyuki [9, 10] and Takeuchi [27] in the following we shall describe briefly how to get the two kinds of orbit decompositions of $g_{-1}$. The sections 1 and 2 are preliminary sections. We give a quick review for the followings: classification and construction of gradations in semisimple Lie algebras [13, 12], the root theory in simple graded Lie algebras $g = g_{-1} + g_0 + g_1$ ([13]), the Jordan triple system $\mathcal{B}$ on $g_{-1}$ (Loos [18]) and the root-theoretic version of a frame (= a maximal system of pairwise orthogonal idempotents) $\{e_1, \ldots, e_r\}$ in $g_{-1}$, and the Jordan algebra structure $\mathfrak{U}_{p}$ $(0 \leq p \leq r)$ in $g_{-1}$. In §3, applying a result of Matsumoto [19], we get a set of good representatives of $G_0$ mod $G_0^0$, which allows us to get the $G_0$-orbit decomposition from the $G_0^0$-orbit decomposition. We consider the root system $\Delta^\ast$ corresponding to a certain symmetric real flag domain $M^\ast$. It turns out that the Weyl group $\mathcal{W}(\Delta^\ast)$ of $\Delta^\ast$, viewed as a subgroup of $G_0^0$, acts on the frame $\{e_1, \ldots, e_r\}$ as signed permutations. Then we can choose the candidates $o_{p,q}$ $(0 \leq p, q \leq r, p + q \leq r)$ of representatives of the $G_0^0$-orbits, which are defined in
terms of the frame. Let \( V_k \) (\( 0 \leq k \leq r \)) be the union of the \( G_0 \)-orbits through the points \( o_{p,q} \) with \( p + q = k \). The sets \( V_k \) were introduced by Takeuchi \[28\] in a different way. Theorem 3.3 (Gindikin-Kaneyuki \[6\]) shows that each \( V_k \) is \( G_0 \)-stable and that it consists of equi-dimensional \( G_0 \)-orbits. Therefore, in order to find the orbit decomposition, we have only to separate the \( G_0 \)-orbits in \( V_k \) (\( 0 \leq k \leq r \)). In the sections 4 and 5, we carry out this procedure, by using the action of \( W(\Delta^\ast) \) and the reduced norm of the Jordan algebra \( \mathfrak{A} \). The main results are Theorems 4.1, 4.2, 5.1, 5.2 and 5.5–5.7. In §6, we give a list of all open \( G_0 \)-orbits whose ambient spaces \( g_{-1} \) are simple Jordan algebras. (Partial results have been obtained by D’Atri-Gindikin \[4\] and Kaneyuki \[9\].) This provides a classification of \( \omega \)-domains in the sense of Koecher \[16\] in simple Jordan algebras.

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Notation and Convention: \( G^0 \) or \((G)^0 \) denotes the identity component of a Lie group \( G \). \( G_\theta \) or \((G)_\theta \) denotes the subgroup of a group \( G \) consisting of elements left fixed by an involutive automorphism \( \theta \). GLA (resp. JTS) is an abbreviation for “graded Lie algebra” (resp. Jordan triple system). \( E \) denotes a unit matrix.

### §1. Semisimple graded Lie algebras.

Let

\[
g = \sum_{k=-v}^v g_k
\]

be a real semisimple GLA of the \( v \)-th kind (we are assuming that the subspace \( g_{-1} \) is not zero). We assume further that the gradation (1.1) is of type \( a_0 \), that is, \( g^- := \sum_{k<0} g_k \) is generated by \( g_{-1} \). Let \((g,Z,\tau)\) be the associated graded triple; more precisely, \( Z \in g \) is the characteristic element of the gradation (1.1), i.e., each subspace \( g_k \) is the eigenspace of \( \text{ad} Z \) for the eigenvalue \( k \), and \( \tau \) is a grade-reversing Cartan involution of \( g \). Let

\[
h = \sum_{k \text{ even}} g_k, \quad m = \sum_{k \text{ odd}} g_k.
\]

Then \( g \) is expressed as a \( Z_2 \)-GLA

\[
g = h + m,
\]

which is also the decomposition by the involution \( \sigma := \text{Ad} \exp \pi i Z \), in which case we have \( \sigma|_h = 1 \) and \( \sigma|_m = -1 \). Consider the Cartan decomposition by \( \tau \):

\[
g = f + p,
\]

where \( \tau|_f = 1 \) and \( \tau|_p = -1 \). Since \( \sigma \) and \( \tau \) commute, we have the \((\sigma, \tau)\)-decomposition

\[
g = f_0 + m_\tau + p_0 + m_p,
\]

where \( f_0 = h \cap f, p_0 = h \cap p, m_\tau = m \cap f \) and \( m_p = m \cap p \). Note that \( Z \in p_0 \). Choose a
maximal abelian subspace $a$ of $p$ containing $Z$. Then $a$ is contained in $g_0 \cap p \subset p_0$. Let $A$ be the root system for the pair $(g,a)$, which is called a root system of $g$ compatible with the gradation. Let $(,)$ denote the Killing form of $g$. Then we have a partition of $A$:

\begin{equation}
A = \prod_{k=\nu}^{\nu} A_k,
\end{equation}

where $A_k = \{\alpha \in A : (\alpha, Z) = k\}$, and each graded subspace $g_k$ can be written as

\begin{equation}
g_0 = c(a) + \sum_{\alpha \in \Delta_0} g^\alpha,
\end{equation}

\begin{equation}
g_k = \sum_{\alpha \in \Delta_k} g^\alpha, \quad k \neq 0,
\end{equation}

where $c(a)$ is the centralizer of $a$ in $g$, and $g^\alpha$ denotes the root space for a root $\alpha \in A$. Choose a linear order in $A$ in such a way that

\begin{equation}
\prod_{k=1}^{\nu} A_k \subset A^+ \subset \prod_{k=0}^{\nu} A_k,
\end{equation}

where $A^+$ denotes the set of positive roots with respect to this order. Let $\Pi$ be the fundamental system for $A$. Since the gradation is of type $\alpha_0$, it is known [13] that $\Pi_k := \Pi \cap A_k = \emptyset$ for $k \geq 2$, and hence we have a partition of $\Pi$:

\begin{equation}
\Pi = \Pi_0 \prod \Pi_1, \quad \Pi_1 \neq \emptyset.
\end{equation}

Let us consider the reverse process. Let $g$ be a semisimple Lie algebra and $a$ be a maximal $R$-split abelian subalgebra of $g$, and let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ be a fundamental system of the root system $A$ for the pair $(g,a)$. A root $\alpha \in A$ can be written as

\begin{equation}
\alpha = \sum_{i=1}^{r} m_i(\alpha) \alpha_i.
\end{equation}

Suppose that we are given a partition $\Pi = \Pi_0 \prod \Pi_1$ with $\Pi_1 \neq \emptyset$. For a root $\alpha \in A$, we define the height $h_{\Pi_1}(\alpha)$ of $\alpha$ relative to $\Pi_1$ by putting

\begin{equation}
h_{\Pi_1}(\alpha) = \sum_{\alpha_i \in \Pi_1} m_i(\alpha).
\end{equation}

If we put

\begin{equation}
\Delta_k = \{\alpha \in A : h_{\Pi_1}(\alpha) = k\},
\end{equation}

then we have a partition $A = \bigsqcup_{k=\nu}^{\nu} A_k$, where $\nu$ is equal to the the height $h_{\Pi_1}(\mathcal{G})$ of the highest root $\mathcal{G} \in A$. Let us define the subspaces $(g_k)_{-\nu \leq k \leq \nu}$ by the equalities (1.7). Then we have a GLA $g = \sum_{k=\nu}^{\nu} g_k$ of type $\alpha_0$ (cf. [13]).

**Theorem 1.1** [13]. *Let $g$ be a real semisimple Lie algebra, and $\Delta$ be a restricted root system of $g$. Let $\Pi$ be a fundamental system of $\Delta$ and $\mathcal{G}$ be the highest root of $\Delta$. Then there exists a bijection between the set of gradations of the $v$-th kind of type $\alpha_0$ in*
$g$ and the set of subsets $\Pi_{1}$ of $\Pi$ satisfying $h_{\Pi_{1}}(g) = \nu$. The bijection is compatible with the respective isomorphisms.

A gradation of the first kind in $g$ is trivially of type $\alpha_{0}$; any gradation of the second kind in $g$ is of type $\alpha_{0}$, provided that $g$ is simple (Tanaka [28]).

\section{Jordan triple systems on $g_{-1}$}

We retain the notation in § 1. Let

\begin{equation}
\begin{aligned}
g = g_{-1} + g_{0} + g_{1}
\end{aligned}
\end{equation}

be a simple GLA (of the first kind), and $(g, Z, \tau)$ be the associated graded triple. Let $\Delta$ be a root system of $g$ compatible with the gradation. As a special case of (1.6), we have a partition $\Delta = \Delta_{-1} \coprod \Delta_{0} \coprod \Delta_{1}$. Choose a linear order in $\Delta$ satisfying (1.8). As is known in Takeuchi [26], one can choose a maximal system of strongly orthogonal roots $\Gamma = \{\beta_{1}, \ldots, \beta_{r}\}$ in $\Delta_{1}$ in such a way that $(\beta_{1}, \beta_{1}) = \cdots = (\beta_{r}, \beta_{r})$. The number $r$ is equal to the split rank of the symmetric triple $(g, g_{0}, \sigma)$. Choose a root vector $E_{i} \in g^{\beta_{i}} \subset g_{1} \ (1 \leq i \leq r)$ in such a way that

\begin{equation}
\begin{aligned}
[E_{i}, E_{-i}] = \hat{\beta}_{i} = \frac{2}{(\beta_{i}, \beta_{i})} \beta_{i},
\end{aligned}
\end{equation}

where $E_{-i} = -\tau E_{i} \in g^{-\beta_{i}} \subset g_{-1}$. Let

\begin{equation}
\begin{aligned}
X_{i} = E_{i} + E_{-i} \in m_{p}.
\end{aligned}
\end{equation}

Then the real span $c$ of $X_{1}, \ldots, X_{r}$ is a maximal abelian subspace of $m_{p}$. The root system $\Delta(g, c)$ for the pair $(g, c)$ is the split root system for the symmetric triple $(g, g_{0}, \sigma)$. It is known (Oshima-Sekiguchi [20]) that $\Delta(g, c)$ is either of type $C$ or of type BC. Let $\alpha_{0}$ be the subspace of $a$ spanned by $\beta_{1}, \ldots, \beta_{r}$, and $\varpi$ be the orthogonal projection of $a$ onto $\alpha_{0}$ with respect to $(\ , \ )$. Then, by considering the inverse Cayley transformation ([8]) of $c$ onto $\alpha_{0}$ and by taking the inner products with $Z$, we have

\begin{equation}
\begin{aligned}
\varpi((\Delta_{0})^{+}) - (0) = & \left\{ \frac{1}{2}(\beta_{i} - \beta_{j}) : 1 \leq i < j \leq r \right\}, \\
\varpi(\Delta_{1}) = & \left\{ \frac{1}{2}(\beta_{i} + \beta_{j}) : 1 \leq i < j \leq r \right\},
\end{aligned}
\end{equation}

provided that $\Delta(g, c)$ is of type $C$, or

\begin{equation}
\begin{aligned}
\varpi((\Delta_{0})^{+}) - (0) = & \left\{ \frac{1}{2}(\beta_{i} - \beta_{j}) (1 \leq i < j \leq r); \frac{1}{2} \beta_{i} (1 \leq i \leq r) \right\}, \\
\varpi(\Delta_{1}) = & \left\{ \frac{1}{2}(\beta_{i} + \beta_{j}) (1 \leq i \leq j \leq r); \frac{1}{2} \beta_{i} (1 \leq i \leq r) \right\},
\end{aligned}
\end{equation}

provided that $\Delta(g, c)$ is of type BC, where $(\Delta_{0})^{+} = \Delta_{0} \cap \Delta^{+}$. We put

\begin{equation}
\begin{aligned}
a_{ij} = & \sum_{\alpha \in \Delta_{1}} g^{-\alpha} \quad i \leq j, \\
\varpi(\alpha) = & \frac{1}{2}(\beta_{i} + \beta_{j}), \\
c_{i} = & \sum_{\alpha \in \Delta_{1}} g^{-\alpha} \quad \varpi(\alpha) = \frac{1}{2} \beta_{i},
\end{aligned}
\end{equation}
Then \( g_{-1} \) can be expressed as
\[
(2.7) \quad g = \sum_{1 \leq i < j \leq r} a_{ij} + \sum_{1 \leq i \leq r} c_{i}.
\]
If \( \mathcal{A}(g, c) \) is of type \( C \), then the second term of the right-hand side of (2.7) does not appear. The dimensions \( \dim a_{ij} (i < j) \), \( \dim a_{ii} \) and \( \dim c_{i} \) do not depend on the choice of \( i \) and \( j \) ([7]).

Let us consider a triple product \( B_{\tau} \) on \( g_{-1} \):
\[
(2.8) \quad B_{\tau}(X, Y, U) = \frac{1}{2} \{ \tau Y, X \}, \quad X, Y, U \in g_{-1}.
\]
It is known (Loos [17], Satake [21]) that the pair \( \mathfrak{B} = (g_{-1}, B_{\tau}) \) is a compact simple JTS and that \( g \) is isomorphic to the Kantor-Tits-Koecher construction for \( \mathfrak{B} \) (These two facts can be obtained in more general setting of a simple GLA of the second kind and the corresponding compact generalized JTS; see [1, 13]). For simplicity we write \( e_{i} \) for \( E_{-i}(1 \leq i \leq r) \) and \( (X Y U) \) for \( B_{\tau}(X, Y, U) \). As usual, we define the linear operator \( L(X, Y) \) on \( g_{-1} \) by
\[
(2.9) \quad L(X, Y) U = (XYU), \quad U \in g_{-1}.
\]
Let
\[
(2.10) \quad o_{p,q} = \sum_{i=1}^{p} e_{i} - \sum_{j=p+1}^{p+q} e_{j}, \quad 0 \leq p, q \leq r, \quad p + q \leq r.
\]
By using the facts [6] that \( e_{i} (1 \leq i \leq r) \) is an idempotent of the JTS \( \mathfrak{B} \) and that \( L(e_{i}, e_{j}) = 0 (i \neq j) \), we see that \( o_{p,q} \) is an idempotent of \( \mathfrak{B} \) and that
\[
(2.11) \quad L(o_{p,r-p}, o_{p,r-p}) = L(o_{r,0}, o_{r,0}), \quad 0 \leq p \leq r.
\]

**Lemma 2.1.** Let \( g_{-1}(\lambda) \) be the eigenspace of \( L(o_{r,0}, o_{r,0}) \) corresponding to the eigenvalue \( \lambda \). Then we have \( g_{-1} = g_{-1}(1) + g_{-1}(\frac{1}{2}) \), and
\[
(2.12) \quad g_{-1}(1) = \sum_{1 \leq i < j \leq r} a_{ij},
\]
\[
(2.13) \quad g_{-1}(\frac{1}{2}) = \sum_{1 \leq i \leq r} c_{i}.
\]

**Proof.** Consider the Peirce decomposition (Satake [21]) of \( g_{-1} \) with respect to the operator \( L(o_{r-\rho}, o_{r-\rho}) = L(o_{r,0}, o_{r,0}) \):
\[
(2.14) \quad g_{-1} = g_{-1}(1) + g_{-1}(\frac{1}{2}) + g_{-1}(0).
\]
Choose a root \( \alpha \in A_{1} \) such that \( \varpi(\alpha) = \frac{1}{2}(\beta_{i} + \beta_{j}), \quad i \leq j \). We have \( \sum_{k=1}^{r} (\beta_{k}, \alpha) = \frac{1}{2} \sum_{k=1}^{r} (\beta_{k}, \beta_{i} + \beta_{j}) = 2 \). Let \( X \in g^{-\alpha} \). Then it follows that
\[
L(o_{r,0}, o_{r,0})(X) = B_{\tau}(o_{r,0}, o_{r,0}, X) = \frac{1}{2}[[\tau(o_{r,0}), o_{r,0}], X]
\]
\[
= \frac{1}{2} \sum_{k=1}^{r} [-E_{k}, E_{-k}], X] = -\frac{1}{2} \sum_{k=1}^{r} [-\hat{\beta}_{k}, X] = \frac{1}{2} \left( \sum_{k=1}^{r} (\hat{\beta}_{k}, \alpha) \right) X = X,
\]
which implies that the right-hand side of (2.12) is contained in $g_{-1}(1)$. Similarly we have that the right-hand side of (2.13) is contained in $g_{-1}(\frac{1}{2})$. Consequently the lemma follows from (2.14) and (2.7).

We introduce a multiplication $\square_p$ in $g_{-1}$:

\begin{equation}
(2.15) \quad X \square_p Y = B_t(X, o_{p,r-p}, Y), \quad X, Y \in g_{-1}, \quad 0 \leq p \leq r.
\end{equation}

As a property of the Peirce decomposition of a JTS ([21]), we know that $g_{-1}(1)$ become a Jordan algebra with unit element $o_{p,r-p}$ with respect to the multiplication $\square_p$.

**Proposition 2.2.** Let $g = g_{-1} + g_0 + g_1$ be a real simple GLA. Then the pair $(g_{-1}, \square_p), 0 \leq p \leq r$, is a Jordan algebra with $o_{p,r-p}$ as unit element, if and only if the split root system $\Delta(g, c)$ is of type $C$. In this case the Jordan algebra $(g_{-1}, \square_p)$ is simple.

**Proof.** Suppose first that $\Delta(g, c)$ is of type $C$. Then we have (2.4). Therefore there are no roots $\alpha \in \Delta$ such that $\varpi(\alpha) = \frac{1}{2} \beta_i$ $(1 \leq i \leq r)$, and so we have $g_{-1}(\frac{1}{2}) = \{0\}$. By Lemma 2.1, we have $g_{-1}(1) = g_{-1}$. Conversely, suppose that $(g_{-1}, \square_p)$ is a Jordan algebra with unit element $o_{p,r-p}$. Then, for any $X \in g_{-1}$, we have $X = o_{p,r-p} \square_p X = B_t(o_{p,r-p}, o_{p,r-p}, X) = L(o_{p,0}, o_{r,0})X$, which implies that $g_{-1}(1) = g_{-1}$ and $g_{-1}(\frac{1}{2}) = \{0\}$. Consequently $\Delta(g, c)$ is of type $C$, by (2.4) and (2.5). To prove the second assertion, consider the involution $^*$ of the Jordan algebra $g_{-1} = g_{-1}(1)$:

\begin{equation}
(2.16) \quad X^* = B_t(o_{p,r-p}, X, o_{p,r-p}), \quad X \in g_{-1}.
\end{equation}

Then $B_t$ can be reconstructed as follows ([21]):

\begin{equation}
(2.17) \quad B_t(X, Y, U) = (X \square_p Y^*) \square_p U + X \square_p (Y^* \square_p U) - Y^* \square_p (X \square_p U).
\end{equation}

Let $W$ be an ideal of the Jordan algebra $g_{-1}$. Then, by using (2.17), we have that $B_t(W, g_{-1}, g_{-1}) + B_t(g_{-1}, g_{-1}, W) \subset W$. This means that $W$ is a $K$-ideal (cf. [13]) of the JTS $\clubsuit$. $\clubsuit$ is compact simple, and hence by a result of [1], it is $K$-simple. Therefore $W = \{0\}$ or $W = g_{-1}$. Thus the Jordan algebra $g_{-1}$ is simple. □

The simple Jordan algebra $(g_{-1}, \square_p)$ is denoted by $\mathcal{Y}_p$.

§ 3. Generalities on the orbit decomposition of $g_{-1}$.

We retain the notation in the previous sections. We will consider exclusively a simple GLA (2.1) : $g = g_{-1} + g_0 + g_1$. We denote by $\text{Aut} g$ the automorphism group of the Lie algebra $g$, and denote by $G^0$ the identity component of $\text{Aut} g$. Let $G_0$ be the subgroup of $\text{Aut} g$ consisting of all grade-preserving automorphisms of the GLA $g$. We need the following subgroups of $\text{Aut} g$:

$G := G_0 G^0$, which is an open subgroup of $\text{Aut} g$,

$G'$ the Zariski connected component of $\text{Aut} g$, which is a subgroup of $G$,

$G'_0 := G_0 \cap G'$, which is the Zariski connected component of $G_0$,

$G'_0$ the (topological) identity component of $G_0$.

$K := \{g \in G : g \tau = \tau g\}$, which is the maximal compact subgroup of $G$ with Lie $K = I$.

$K_0 = G_0 \cap K$,

$K'_0$ the identity component of $K_0$. 

Let $\Delta$ be a root system of $\mathfrak{g}$ compatible with the gradation and $\Pi = \{ \alpha_1, \ldots, \alpha_\ell \}$ be a fundamental system of $\Delta$ with respect to an order satisfying (1.8). Let $\{Z_1, \ldots, Z_\ell\}$ be the basis of a dual to $\Pi$ with respect to $(\cdot, \cdot)$. Consider the involutive automorphisms of $\mathfrak{g}$:

$$
\epsilon_k = \text{Ad} \exp \pi i Z_k, \quad 1 \leq k \leq \ell.
$$

**Lemma 3.1 (Matsumoto [19]).** Let $Q_1$ be the free abelian subgroup of $\text{Aut} \mathfrak{g}$ generated by $e_1, \ldots, e_\ell$, and let $Q_0 := Q_1 \cap G^0$. Then $Q_1$ is a subgroup of $G'$, and

$$
G'/G^0 \simeq Q_1/Q_0,
$$

in particular,

$$
G' = Q_1G^0.
$$

Since $\epsilon_k$ is $+1$ or $-1$ on each root space $\mathfrak{g}^\alpha, \alpha \in \Delta \cup \{0\}$, it follows from (1.7) that $\epsilon_k$ is grade-preserving for any gradation of $\mathfrak{g}$. This implies, in particular, that $Q_1$ is a subgroup of $G_0$, and hence we have

$$
Q_1G_0^0 \subset G'.
$$

Look at the $(\sigma, \tau)$-decomposition (1.5) for the GLA $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$. It is easy to see that $\mathfrak{g}^* := \mathfrak{f}_0 + \mathfrak{m}_p$ is a reductive subalgebra of $\mathfrak{g}$. The center of $\mathfrak{g}^*$ is at most one-dimensional and the semisimple part of $\mathfrak{g}^*$ is simple ([7]). The triple $(\mathfrak{g}^*, \mathfrak{f}_0, \sigma)$ is a Riemannian symmetric triple, the noncompact dual of $(\mathfrak{f}, \mathfrak{f}_0, \sigma)$. Let $G^*$ be the connected Lie subgroup of $G$ corresponding to $\mathfrak{g}^*$. Then $K^0_0$ is a maximal compact subgroup of $G^*$. $M^* = G^*/K^0_0$ is the symmetric space corresponding to $(\mathfrak{g}^*, \mathfrak{f}_0, \tau)$. We have the Cartan decomposition

$$
G^* = K^0_0 \exp \mathfrak{m}_p.
$$

Since $\mathfrak{c}$ is a maximal abelian subspace of $\mathfrak{m}_p$, one can consider the root system $\Delta^*$ for the pair $(\mathfrak{g}^*, \mathfrak{c})$ (or for the symmetric space $M^*$). In Table I, we give a list of real simple GLA’s of the first kind and the corresponding subset $\Pi_1$ of $\Pi$ ([13, 12, 14, 18]). In Table II, we give the root systems $\Delta(\mathfrak{g}, \mathfrak{c})$ and $\Delta^*$ for each simple GLA’s of the first kind ([20, 25, 18]). The following notations are used in Table I: $H$ the quaternion algebra over $\mathbf{R}, \mathbf{O}$ (resp. $\mathbf{O}'$) the Cayley (resp. the split Cayley) algebra over $\mathbf{R}$, and $O^C = O \otimes_R C$. $M_{p,q}(K)$ the vector space of $p \times q$ matrices with entries in $K$, where $K = R, C, H, O, O'$ or $O^C$; $H_n(K)$ the vector space of hermitian matrices of degree $n$ with entries in $K$; $SH_n(H)$ the vector space of skew-hermitian quaternion matrices of degree $n$; $\text{Alt}_n(K)$ the vector space of skew-symmetric matrices of degree $n$ with entries in $K$; $\text{Sym}_n(C)$ the vector space of complex symmetric matrices of degree $n$. We employ the numbering of simple roots used in Bourbaki [2].

By the property $[\mathfrak{f}_0, \mathfrak{m}] \subset [\mathfrak{g}_0, \mathfrak{m}] \subset \mathfrak{m}$, the group $K^0_0$ acts on $\mathfrak{m}$ by the adjoint representation. Moreover, since $[\mathfrak{f}_0, \mathfrak{m}_p] \subset \mathfrak{m}_p$ and $[\mathfrak{f}_0, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{-1}$, it follows that this $K^0_0$-action on $\mathfrak{m}$ leaves both $\mathfrak{m}_p$ and $\mathfrak{g}_{-1}$ stable.
Table I

<table>
<thead>
<tr>
<th>$g,g_0,g_{-1}$</th>
<th>$\Pi$</th>
<th>$\Pi_1$</th>
</tr>
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<tbody>
<tr>
<td>11 $\langle\mathfrak{sl}(n,R),\mathfrak{sl}(p,R)+\mathfrak{sl}(n-p,R)+R,M_{p,n-p}(R)\rangle$, $n \geq 3, 1 \leq p \leq [n/2]$</td>
<td>$A_{n-1}$</td>
<td>${s_p}$</td>
</tr>
<tr>
<td>12 $\langle\mathfrak{sl}(n,H),\mathfrak{sl}(p,H)+\mathfrak{sl}(n-p,H)+R, M_{p,n-p}(H)\rangle$, $n \geq 3, 1 \leq p \leq [n/2]$</td>
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<td>${s_p}$</td>
</tr>
<tr>
<td>13 $\langle\mathfrak{su}(n,n),\mathfrak{sl}(n,C)+R,H_n(C)\rangle$, $n \geq 3$</td>
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<td>${s_n}$</td>
</tr>
<tr>
<td>14 $\langle\mathfrak{sp}(n,R),\mathfrak{sl}(n,R)+R,H_n(R)\rangle$, $n \geq 3$</td>
<td>$C_n$</td>
<td>${s_n}$</td>
</tr>
<tr>
<td>15 $\langle\mathfrak{sp}(n,n),\mathfrak{sl}(n,H)+R,SH_n(H)\rangle$, $n \geq 2$</td>
<td>$C_n$</td>
<td>${s_n}$</td>
</tr>
<tr>
<td>16 $\langle\mathfrak{so}(p+1,q+1),\mathfrak{so}(p,q)+R,M_{p,q}(R)\rangle$, $0 \leq p &lt; q$ or $3 \leq p = q$</td>
<td>$B_{p+1}(p &lt; q)$</td>
<td>${a}$</td>
</tr>
<tr>
<td>$\quad$</td>
<td>$D_{p+1}(p = q)$</td>
<td>${a}$</td>
</tr>
<tr>
<td>17 $\langle\mathfrak{so}(4n),\mathfrak{sl}(n,H)+R,H_n(H)\rangle$, $n \geq 3$</td>
<td>$C_n$</td>
<td>${s_n}$</td>
</tr>
<tr>
<td>18 $\langle\mathfrak{so}(n,n),\mathfrak{sl}(n,R)+R,\mathfrak{Alt}_n(R)\rangle$, $n \geq 4$</td>
<td>$D_n$</td>
<td>${s_n}$</td>
</tr>
<tr>
<td>19 $\langle\mathfrak{E}<em>6(6),\mathfrak{so}(5,5)+R,M</em>{1,2}(O')\rangle$</td>
<td>$E_6$</td>
<td>${a}$</td>
</tr>
<tr>
<td>20 $\langle\mathfrak{E}<em>6(-26),\mathfrak{so}(1,9)+R,M</em>{1,2}(O)\rangle$</td>
<td>$A_2$</td>
<td>${a}$</td>
</tr>
<tr>
<td>21 $\langle\mathfrak{E}_7(-25),\mathfrak{E}_6(-26)+R,H_3(O)\rangle$</td>
<td>$E_7$</td>
<td>${a}$</td>
</tr>
<tr>
<td>22 $\langle\mathfrak{E}_7(7),\mathfrak{E}_6(6)+R,H_3(O')\rangle$</td>
<td>$C_3$</td>
<td>${a}$</td>
</tr>
<tr>
<td>23 $\langle\mathfrak{E}_8(8),\mathfrak{so}(3,1)+R,H_3(J)\rangle$</td>
<td>$C_3$</td>
<td>${a}$</td>
</tr>
<tr>
<td>24 $\langle\mathfrak{E}<em>7(n),\mathfrak{so}(n-p,C)+C,M</em>{p,n-p}(C)\rangle$, $n \geq 3, 1 \leq p \leq [n/2]$</td>
<td>$A_{n-1}$</td>
<td>${s_p}$</td>
</tr>
<tr>
<td>25 $\langle\mathfrak{sp}(n,C),\mathfrak{sl}(n,C)+C,\mathfrak{Sym}_n(C)\rangle$, $n \geq 3$</td>
<td>$C_n$</td>
<td>${s_n}$</td>
</tr>
<tr>
<td>26 $\langle\mathfrak{so}(n+2,C),\mathfrak{so}(n,C)+C,M_{1,1}(C)\rangle$, $n \geq 3, n \neq 4$</td>
<td>$B_{(n+2)/2}$</td>
<td>${a}$</td>
</tr>
<tr>
<td>$\quad$</td>
<td>$D_{(n+2)/2}$</td>
<td>${a}$</td>
</tr>
<tr>
<td>27 $\langle\mathfrak{so}(2n,C),\mathfrak{sl}(n,C)+C,\mathfrak{Alt}_n(C)\rangle$, $n \geq 4$</td>
<td>$D_n$</td>
<td>${s_n}$</td>
</tr>
<tr>
<td>28 $\langle\mathfrak{E}<em>7^c,\mathfrak{so}(10,C)+C,M</em>{1,2}(O^c)\rangle$</td>
<td>$E_8$</td>
<td>${a}$</td>
</tr>
<tr>
<td>29 $\langle\mathfrak{E}_8^c,\mathfrak{E}_6^c+C,H_3(O^c)\rangle$</td>
<td>$E_7$</td>
<td>${a}$</td>
</tr>
</tbody>
</table>

**Lemma 3.2.** Let us define a linear endomorphism $\varphi$ on $m$ by

$$\varphi(X) = \frac{1}{2}(X - IX), \quad X \in m,$$

where $I = \text{ad}_m Z$. Then $\varphi$ is a $K_0^0$-isomorphism of $m_p$ onto $g_{-1}$.

**Proof.** The inclusion $\varphi(m_p) \subset g_{-1}$ follows from the fact $I^2 = 1$. Since $I$ interchanges $m_p$ with $m_1$, $\varphi$ sends $m_p$ to $g_{-1}$ isomorphically. Since $K_0^0$ acts on $g$ as grade-preserving automorphisms, the element $Z$ is left fixed by $K_0^0$. Hence we have $[\text{Ad}_m K^0_0, I] = 0$, which implies that $\varphi$ commutes with the $K^0_0$-action. \hfill $\square$

Let $a_{-1} := \varphi(c) \subset g_{-1}$. Then $a_{-1}$ is spanned by $e_1, \ldots, e_r$, since $\varphi(X_i) = e_i$. Let $W(A^*)$ be the Weyl group for the root system $A^*$ (or, for the symmetric space $M^*$). Then we have

$$W(A^*) \cong N_{K_0^0}(c)/C_{K_0^0}(c),$$

where $N_{K_0^0}(c)$ (resp. $C_{K_0^0}(c)$) is the normalizer (resp. centralizer) of $c$ in $K_0^0$. $W(A^*)$ acts on $c$ as signed permutations:

$$X_i \mapsto \pm X_{\rho(i)}, \quad \rho \in \Sigma_r,$$

where $\Sigma_r$ is the permutation group of $\{1, \ldots, r\}$. By Lemma 3.2, this action of $W(A^*)$ is transferred onto $a_{-1}$ via $\varphi$ as the signed permutations:

$$e_i \mapsto \pm e_{\rho(i)}, \quad \rho \in \Sigma_r.$$
Recall the quadratic representation $P$ of the compact simple JTS $\mathfrak{B} = (g_{-1}, B_{\tau})$:

\[(3.10) \quad P(X)Y = (XYX), \quad X, Y \in g_{-1}.\]

The structure group $\text{Str} \mathfrak{B}$ of the JTS $\mathfrak{B}$ is, by definition, the totality of the elements $g \in \text{GL}(g_{-1})$ satisfying the condition:

\[(3.11) \quad g(XYU) = ((gX)(g^{-1}Y)(gU)), \quad X, Y, U \in g_{-1},\]

where $g^*$ is the adjoint operator of $g$ with respect to the trace form of $\mathfrak{B}$. A computation shows that

\[(3.12) \quad \text{Str} \mathfrak{B} = \{ g \in \text{GL}(g_{-1}) : P(gX) = gP(X)g^*, X \in g_{-1} \}.\]

Noting that the GLA $g$ is isomorphic to the Kantor-Tits-Koecher construction for $B_{\tau}$, we conclude from Satake [21] that the group $G_0$ is isomorphic to $\text{Str} \mathfrak{B}$ and that this isomorphism is given by taking the restriction of the $G_0$-action on $g$ to $g_{-1}$. As a result, the rank of the operator $P(X)$ is constant on each $G_0$-orbit in $g_{-1}$, when $X$ varies through that orbit. Let $V_k$ ($0 \leq k \leq r$) be the union of $G_0$-orbits through the points $o_{p,q}$.
with \( p + q = k \), that is,

\[
V_k = \bigcup_{p+q=k} G_0^0 \cdot 0_{pq} \subset \mathfrak{g}_{-1}, \quad 0 \leq k \leq r.
\]

**Theorem 3.3** (Gindikin-Kaneyuki [6]). Let \( \mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 \) be a real simple GLA and \( r \) be the split rank of the symmetric pair \( \mathfrak{g}_r, \mathfrak{g}_0 \). Then (1) \( V_k \) is expressed as

\[
V_k = \{ X \in \mathfrak{g}_{-1} : \text{rk} P(X) = i_k \}, \quad 0 \leq k \leq r,
\]

where \( \text{rk} \) denotes the rank and \( i_k = \text{rk} P(0_{k,0}) \). The closure \( \overline{V}_k \) of \( V_k \) is given by

\[
\overline{V}_k = \{ X \in \mathfrak{g}_{-1} : \text{rk} P(X) \leq i_k \}, \quad 0 \leq k \leq r.
\]

(2) Each \( V_k \) is \( G_0 \)-stable and

\[
\mathfrak{g}_{-1} = V_0 \prod V_1 \prod \cdots \prod V_r.
\]

(3) An orbit \( G_0^0 \cdot 0_{p,q} \) is open if and only if it is contained in \( V_r \), or equivalently, \( p + q = r \).

The assertion (2) was obtained also by Takeuchi [27] by a different method.

**Lemma 3.4.** Let \( \text{Aut} \mathfrak{B} \) denote the automorphism group of the JTS \( \mathfrak{B} \). Then

\[
\text{Aut} \mathfrak{B} = K_0.
\]

**Proof.** The trace form \( \gamma_{\mathfrak{B}} \) of \( \mathfrak{B} \) is positive definite, since \( \mathfrak{B} \) is compact. \( \text{Aut} \mathfrak{B} \) is, by definition, the subgroup of \( \text{Str} \mathfrak{B} = G_0 \) consisting of all elements \( g \in \text{Str} \mathfrak{B} \) satisfying the condition

\[
\gamma_{\mathfrak{B}}(gX, gY) = \gamma_{\mathfrak{B}}(X, Y), \quad X, Y \in \mathfrak{g}_{-1}.
\]

On the other hand, we have (cf. [1] and Lemma 3.10 [13])

\[
\gamma_{\mathfrak{B}}(X, Y) = -\frac{1}{2}(X, \tau Y), \quad X, Y \in \mathfrak{g}_{-1}.
\]

Now let \( g \in K_0 \). Then, since \( g \) commutes with \( \tau \), we have that \( g \) satisfies (3.18), which implies that \( K_0 \subset \text{Aut} \mathfrak{B} \). By the definition, \( \text{Aut} \mathfrak{B} \) is a compact subgroup of \( \text{Str} \mathfrak{B} \). But \( K_0 \) is a maximal compact subgroup of \( G_0 \). Hence we have that \( K_0 = \text{Aut} \mathfrak{B} \). \( \square \)

**§ 4. The orbit decompositions of \( \mathfrak{g}_{-1} \).**

**Theorem 4.1.** Let \( \mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 \) be a real simple GLA, and \( r \) be the split rank of the symmetric pair \( \mathfrak{g}_r, \mathfrak{g}_0 \). Suppose that \( \Delta^* \) is of type \( A \). Then the orbit decompositions of \( \mathfrak{g}_{-1} \) under the groups \( G_0^0 \) and \( G_0 \) are given by

\[
\mathfrak{g}_{-1} = \bigcup_{p+q=r} G_0^0 \cdot 0_{p,q} = \bigcup_{p+q=r} G_0 \cdot 0_{p,q}.
\]

**Proof.** Since \( \Delta^* \) is of type \( A \), it follows (Tables I and II) that \( \mathfrak{U}_r = (\mathfrak{g}_{-1}, \square_r) \) is a compact simple Jordan algebra. In this case, the JTS \( \mathfrak{B} \) comes from the Jordan algebra \( \mathfrak{U}_r \). As a result, \( G_0 \), identified with the structure group \( \text{Str} \mathfrak{B} \), coincides with the struc-
tude group of $\mathfrak{U}_r$. Therefore the first equality in (4.1) is the one proved by Kaneyuki [9, 10] and Satake [23]. Since $\mathfrak{U}_r$ is compact simple, it is known (Koecher [15], Vinberg [29]) that $V_{r,0} := G_0^0 \cdot o_{r,0}$ is a homogeneous irreducible self-dual convex cone in $\mathfrak{g}_{-1}$. Let $G(V_{r,0})$ be the automorphism group of the cone $V_{r,0}$. By Satake [21], we have

\begin{equation}
G_0|_{\mathfrak{g}_{-1}} = \text{Str} \mathfrak{B} = G(V_{r,0}) \times \{ \pm 1 \}.
\end{equation}

As was shown in [10], any $G(V_{r,0})$-orbit in $\mathfrak{g}_{-1}$ coincides with a $G_0^0$-orbit in $\mathfrak{g}_{-1}$. Therefore the second equality in (4.1) follows from (4.2).

Now let

\begin{equation}
\Gamma_k = \left\{ \sum_{i=1}^k \delta_{i_1} e_{i_1} \in \mathfrak{a}_{-1} : \delta_{i_1}, \ldots, \delta_{i_k} = \pm 1, \ 1 \leq i_1, \ldots, i_k \leq r \right\}, \quad 1 \leq k \leq r,
\end{equation}

\begin{equation}
\Gamma_0 = \{0\}.
\end{equation}

Then the Weyl group $W(\Delta^*)$ acts on $\Gamma_k$ by (3.9) and we have

\begin{equation}
\Gamma_k = \bigcup_{p+q=k} W(\Delta^*) \cdot o_{p,q}, \quad 0 \leq k \leq r.
\end{equation}

Therefore it follows from (3.7) and (3.13) that

\begin{equation}
V_k = G_0^0 \Gamma_k, \quad 0 \leq k \leq r.
\end{equation}

**Theorem 4.2.** Let $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ and $r$ be the same as in Theorem 4.1. Suppose that $\Delta^*$ is of type $B$, $BC$ or $C$. Then the orbit decompositions of $\mathfrak{g}_{-1}$ under $G_0^0$ and $G_0$ are given by

\begin{equation}
\mathfrak{g}_{-1} = \prod_{k=0}^r G_0^0 \cdot o_{k,0} = \prod_{k=0}^r G_0 \cdot o_{k,0}.
\end{equation}

In particular, there is a single open orbit $G_0^0 \cdot o_{r,0} = G_0 \cdot o_{r,0}$.

**Proof.** In view of (3.16), it suffices to show that

\begin{equation}
V_k = G_0^0 \cdot o_{k,0} = G_0 \cdot o_{k,0}, \quad 0 \leq k \leq r.
\end{equation}

By the assumption for $\Delta^*$, the Weyl group $W(\Delta^*)$ consists of all signed permutations of the form (3.9). Consequently, $W(\Delta^*)$ acts on $\Gamma_k$ transitively, i.e., $\Gamma_k = W(\Delta^*) \cdot o_{k,0}$. Hence (4.5) implies the first equality in (4.7). The second equality in (4.7) follows from the fact that $V_k$ is $G_0$-stable (Theorem 3.3).

**Remark.** The second equality in (4.7) was obtained also by Takeuchi [27].

In the following we will be concerned exclusively with the case where $\Delta^*$ is of type $D$.

**Lemma 4.3.** Suppose that $\Delta^*$ is of type $D_r$. Then

\begin{equation}
V_r = G_0^0 \cdot o_{r,0} \cup G_0^0 \cdot o_{r-1,1}.
\end{equation}

\begin{equation}
V_k = G_0^0 \cdot o_{k,0} = G_0 \cdot o_{k,0}, \quad 0 \leq k \leq r - 1.
\end{equation}
PROOF. In view of (4.5), it suffices to prove that
\[ \Gamma_r = W(A^*) \cdot o_{r,0} \prod W(A^*) \cdot o_{r-1,1}, \]
(4.10)
\[ \Gamma_k = W(A^*) \cdot o_{k,0}, \quad 0 \leq k \leq r - 1. \]

By the assumption for $A^*$, a signed permutation $e_i \mapsto \delta_i e_i$, $\delta_i = \pm 1$ ($1 \leq i \leq r$) lies in $W(A^*)$ if and only if $\prod_{i=1}^{r} \delta_i = 1$. Therefore $o_{p,q}$ with $q$ even (resp. odd) is conjugate to $o_{r,0}$ (resp. $o_{r-1,1}$) under $W(A^*)$. Hence (4.10) follows from (4.4). Let us next consider $o_{p,q}$ with $p + q = k$, $0 \leq k \leq r - 1$. If $q$ is even, then $o_{p,q}$ is conjugate to $o_{k,0}$ under $W(A^*)$. Suppose $q$ is odd. Let $\mu$ be the signed permutation defined by $\mu(e \swarrow) = \delta_{l} e_{l}$ ($1 \leq l \leq r$), where $\delta_{l} = -1$ for $p + 1 \leq l \leq p + q + 1$, otherwise $\delta_{l} = 1$. Then $\mu$ belongs to $W(A^*)$ and $\mu(o_{p,q}) = o_{k,0}$. This implies (4.10)$_{2}$.

Back to the situation in §2, suppose that $A(g, c)$ is of type $C$, and consider the Jordan algebra $\mathfrak{U}_p = (\mathfrak{g}_{-1}, \Pi_p)$, $0 \leq p \leq r$. Let $P_p : \mathfrak{g}_{-1} \rightarrow \text{End} \mathfrak{g}_{-1}$ be the quadratic representation of $\mathfrak{U}_p$. Then we have

**Lemma 4.4.** Let $0 \leq p \leq r$. Then
\[ P(X) = P_p(X)P(o_{p,r-p}), \quad X \in \mathfrak{g}_{-1}. \]

Moreover the operator $P(o_{p,r-p})$ is nondegenerate on $\mathfrak{g}_{-1}$.

**Proof.** Let $Y \in \mathfrak{g}_{-1}$. By using (2.16) and (2.17), we have
\[ P(X)Y = (XYX) = (X \square_p Y^* \square_p X + X \square_p (Y^* \square_p X) - Y^* \square_p (X \square_p X)
= 2X \square_p (X \square_p Y^*) - (X \square_p X) \square_p Y^*
= P_p(X)Y^* = P_p(X)P(o_{p,r-p})Y. \]

Since $A(g, c)$ is of type $C$, we have that $\mathfrak{g}_{-1}(1) = \mathfrak{g}_{-1}$ (cf. §2). On the other hand, by Satake [21], $\pm 1$ are the only eigenvalues of $P(o_{p,r-p})$ on $\mathfrak{g}_{-1}(1)$, which yields the second assertion.

Consider the JTS $(.)_p$ coming from $\mathfrak{U}_p$ ($0 \leq p \leq r$):
\[ (XYU)_p = (X \square_p Y) \square_p U + X \square_p (Y \square_p U) - Y \square_p (X \square_p U), \]
where $X, Y, U \in \mathfrak{g}_{-1}$, and define the linear operator $L_p(X, Y)$ by
\[ L_p(X, Y)U = (XYU)_p. \]

**Lemma 4.5.** Let $X, Y \in \mathfrak{g}_{-1}$. Then
\[ L_p(X, Y) = L(X, P(o_{p,r-p})Y). \]

**Proof.** For simplicity we write $f_p$ for $o_{p,r-p}$. By the definition of a JTS, we have
\[ L(X, P(f_p)Y)U = (X(f_p Yf_p)U)
= ((Yf_p X)f_p U) + (Xf_p (Yf_p U)) - (Yf_p (Xf_p U))
= (X \square_p Y) \square_p U + X \square_p (Y \square_p U) - Y \square_p (X \square_p U)
= (XYU)_p = L_p(X, Y)U. \]
Proposition 4.6. Suppose that $\Delta(g, c)$ is of type $C$. Let $(\text{Str} \mathfrak{U}_p)^0$ and $(\text{Str} \mathfrak{B})^0$ denote the identity components of the structure groups $\text{Str} \mathfrak{U}_p$ and $\text{Str} \mathfrak{B}$, respectively. Then we have

\begin{equation}
(\text{Str} \mathfrak{U}_p)^0 = (\text{Str} \mathfrak{B})^0 = G_0^0.
\end{equation}

Proof. Lie $\text{Str} \mathfrak{U}_p$ (resp. Lie $\text{Str} \mathfrak{B}$) is generated by $L_p(X, Y)$ (resp. $L(X, Y)$), when $X$ and $Y$ vary through $\mathfrak{g}_{-1}$. Therefore the proposition follows from Lemma 4.5 and the non-degeneracy of $P(o_{p, -p})$. \hfill \square

Table II tells us that if $\Delta^*$ is of type $D_r$, then $\Delta(g, c)$ is of type $C$. In this case one has the Jordan algebra $\mathfrak{U}_r = (\mathfrak{g}_{-1}, \coprod_{r})$ (Proposition 2.2).

Proposition 4.7. Let $g = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ be a real simple GLA. Suppose that $\Delta^*$ is of type $D_r$. Let $N$ be the reduced norm of the Jordan algebra $\mathfrak{U}_r = (\mathfrak{g}_{-1}, \coprod_{r})$. Suppose $N(o_{r,0})N(o_{r-1,1}) < 0$. Then

\begin{equation}
V_r = G_0^0 \cdot o_{r,0} \prod G_0^0 \cdot o_{r-1,1}.
\end{equation}

In particular, there are exactly two open $G_0^0$-orbits in $\mathfrak{g}_{-1}$.

Proof. By the assumption, $\Delta(g, c)$ is of type $C$. Therefore, by Corollary 2.11 [6], we have that $V_r = \{ X \in \mathfrak{g}_{-1} : \det P(X) \neq 0 \}$. Lemma 4.4 implies that $X \in V_r$ if and only if $\det P_r(X) \neq 0$ if and only if $N(X) \neq 0$. We have thus

\begin{equation}
V_r = \{ X \in \mathfrak{g}_{-1} : N(X) \neq 0 \}.
\end{equation}

Let $V_r^+$ (resp. $V_r^-$) be the totality of elements $X \in \mathfrak{g}_{-1}$ satisfying $N(X) > 0$ (resp. $< 0$). Then

\begin{equation}
V_r = V_r^+ \prod V_r^-.
\end{equation}

Suppose for simplicity that $N(o_{r,0}) > 0$. Then $N(o_{r-1,1}) < 0$. We have $o_{r,0} \in V_r^+$ and $o_{r-1,1} \in V_r^-$. The reduced norm $N$ is a relative invariant polynomial on $\mathfrak{g}_{-1}$, that is,

\begin{equation}
N(gX) = \chi(g)N(X), \quad X \in \mathfrak{g}_{-1}, \quad g \in \text{Str} \mathfrak{U}_r,
\end{equation}

where $\chi$ is an $R^*$-valued character of $\text{Str} \mathfrak{U}_r$. Suppose now that $g \in G_0^0 = (\text{Str} \mathfrak{U}_r)^0$ (cf. Proposition 4.6). Then we have $N(go_{r,0}) = \chi(g)N(o_{r,0}) > 0$, and hence $G_0^0 \cdot o_{r,0} \subset V_r^+$. Similarly $G_0^0 \cdot o_{r-1,1} \subset V_r^-$. These two imply (4.18). \hfill \square

Corollary 4.8. Under the situation in Proposition 4.7, suppose that $N(o_{r,0}) > 0$ (resp. $< 0$) and $N(o_{r-1,1}) < 0$ (resp. $> 0$). Then

\begin{equation}
G_0^0 \cdot o_{r,0} = \{ X \in \mathfrak{g}_{-1} : N(X) > 0 \text{ (resp. } < 0) \},
\end{equation}

\begin{equation}
G_0^0 \cdot o_{r-1,1} = \{ X \in \mathfrak{g}_{-1} : N(X) < 0 \text{ (resp. } > 0) \}.
\end{equation}

§ 5. The orbit decompositions of $\mathfrak{g}_{-1}$ (continued).

In this section we consider the case where $\Delta^*$ is of type $D$. 
5.1.

**Theorem 5.1.** Let \((g, g_0, g_{-1}) = (\mathfrak{sl}(2p, \mathbb{R}), \mathfrak{sl}(p, \mathbb{R}) + \mathfrak{sl}(p, \mathbb{R}) + \mathbb{R}, M_p(\mathbb{R}))\). Then the orbit decompositions of \(g_{-1}\) under the groups \(G_0^0\) and \(G_0\) are given by

\[ g_{-1} = \prod_{k=0}^{p-1} G_0^0 \cdot o_{k,0} \prod_{k=0}^{p} G_0 \cdot 0_{k,0} \]

(5.1)

\[ g_{-1} = \prod_{k=0}^{p} G_0 \cdot o_{k,0}. \]

(5.2)

There are exactly two open orbits \(G_0^0 \cdot o_{p,0}\) and \(G_0^0 \cdot o_{p-1,1}\) which are mutually diffeomorphic.

**Proof.** In this case, \(\Delta\) is of type \(A_{2p-1}\) and is given by

\[ \Delta = \{ \pm (\lambda_i - \lambda_j) : 1 \leq i < j \leq 2p \}. \]

(5.3)

The simple root system \(\Pi\) is given by

\[ \Pi = \{ \alpha_i = \lambda_i - \lambda_i+1 : 1 \leq i \leq 2p - 1 \}. \]

(5.4)

Since \(\Pi_1 = \{ \alpha_p \}\) (cf. Table I), we have

\[ \Delta_1 = \{ \lambda_i - \lambda_{p+i} : 1 \leq i,j \leq p \}. \]

(5.5)

The corresponding gradation of \(g = \mathfrak{sl}(2p, \mathbb{R})\) is

\[ g = g_{-1} + g_0 + g_1 \]

(5.6)

\[ \rightarrow p \rightarrow \]

\[ \left\{ \begin{array}{c} (0 \; | \; 0) \uparrow \in \mathbb{R} \\ * \; | \; 0 \end{array} \right\} + \left\{ \begin{array}{c} * \; | \; 0 \\ 0 \; | \; * \end{array} \right\} + \left\{ \begin{array}{c} 0 \; | \; * \\ 0 \; | \; 0 \end{array} \right\}. \]

Let

\[ \Gamma = \{ \beta_i = \lambda_i - \lambda_{p+i} : 1 \leq i \leq p \}. \]

(5.7)

Then \(\Gamma\) is a maximal system of strongly orthogonal roots in \(\Delta_1\). Let \(E_{ij} \in g_{-1} = M_p(\mathbb{R}) (1 \leq i,j \leq p)\) be the matrix whose \((k,\ell)\)-entry is \(\delta_{ik}\delta_{j\ell}\). It can be seen that the root vector \(E_{-i} \in g_{-\beta_i} (1 \leq i \leq p)\) is given by the matrix \(E_{ii} \in M_p(\mathbb{R}) = g_{-1}\). Therefore

\[ o_{k,0} = \sum_{i=1}^{k} E_{ii} \in M_p(\mathbb{R}), \quad 1 \leq k \leq p, \]

(5.8)

\[ o_{p-1,1} = \sum_{i=1}^{p-1} E_{ii} - E_{pp}. \]

The reduced norm \(N\) of the Jordan algebra \(\mathfrak{U}_p = M_p(\mathbb{R})\) is given by \(N(X) = \det X, X \in M_p(\mathbb{R})\). Hence \(N(o_{p,0}) = 1\) and \(N(o_{p-1,1}) = -1\). Consequently, by Proposition 4.7, we have that \(V_p = G_0^0 \cdot o_{p,0} \prod G_0 \cdot o_{p-1,1}\). Combining this with (4.9) and (3.16), we get (5.1).

Let us next consider the \(G_0\)-orbit decomposition of \(g_{-1}\). For \(g = \mathfrak{sl}(2p, \mathbb{R})\), it is known (Matsumoto [19]) that \(Q_1 \text{ mod } Q_0\) is generated by \(\varepsilon_1\). Since \(\varepsilon_1\) is not in \(G_0^0\), we have \(\varepsilon_1 \in G_0 - G_0^0\) (cf. (3.4)). Choose the subset \(\Pi_1' = \{ \alpha_1 \}\) of \(\Pi\). Then
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$h_{II}'(g) = 1$. Let

\[ (5.9) \quad g = g_{-1}' + g_{0}' + g_{1}' \]

be the gradation of $g$ corresponding to $\Pi_1'$ (cf. §1), and let

\[ (5.10) \quad \Delta = \bigsqcup_{k=-1}^{1} \Delta_k' \]

be the corresponding partition of $\Delta$. Since $\beta_1 \in \Delta_1'$ and $\beta_k \in \Delta_0'$ for $k \geq 2$, we have that $E_{-1}$ lies in $g_{-1}'$ and $E_{-k}$ (for $k \geq 2$) lies in $g_{0}'$ (cf. (3.1), (1.7), (1.11), (1.12)). Hence $\epsilon_1$ sends $E_{-1}$ to $-E_{-1}$ and leaves $E_{k}$ (for $k \geq 2$) fixed. Consequently $\epsilon_1(0_{p,0}) = -E_{-1} + \sum_{2}^{p} E_{-i}$.

Let $a \in W(\Delta^{*})$ be the element interchanging $E_{-1}$ with $E_{-p}$ and leaving all other $E_{-k}$ (for $k \neq 1, p$) fixed. Then it follows that $a \epsilon_1(0_{p,0}) = 0_{p-1,1}$.

5.2. THEOREM 5.2. Let $(\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_{-1}) = (\mathfrak{so}(2n, 2n), \mathfrak{gl}(2n, R), \text{Alt}_{2n}(R))$. Then the orbit decompositions of $\mathfrak{g}_{-1}$ under the groups $G_0^{0}$ and $G_0$ are given by (5.1) and (5.2) with $p$ replaced by $n$.

PROOF. The Lie algebra $g = \mathfrak{so}(2n, 2n)$ is realized as

\[ (5.11) \quad \mathfrak{so}(2n, 2n) = \{ A \in \mathfrak{gl}(4n, R) : AS + SA = 0 \} = \left\{ \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} : A_1 + A_4 = 0, \quad A_2, A_3 \in \text{Alt}_{2n}(R) \right\}, \]

where $S = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$. The root system $\Delta$ is of type $D_{2n}$.

\[ \Delta = \{ \pm (\lambda_i \pm \lambda_j) : 1 \leq i < j \leq 2n \}, \]

\[ \Pi = \{ x_i = \lambda_i - \lambda_{i+1} (1 \leq i \leq 2n - 1), x_{2n} = \lambda_{2n-1} + \lambda_{2n} \}. \]

Since $\Pi_1 = \{ x_{2n} \}$ (cf. Table I), we have

\[ (5.13) \quad \Delta_1 = \{ \lambda_i + \lambda_j : 1 \leq i < j \leq 2n \}. \]

The gradation $g = g_{-1} + g_{0} + g_{1}$ corresponding to $\Pi_1$ is given by (5.6) with $p$ replaced by $2n$.

Put

\[ (5.14) \quad \Gamma = \{ \beta_i = \lambda_{2i-1} + \lambda_{2i} : 1 \leq i \leq n \}. \]

Then $\Gamma$ is a maximal system of strongly orthogonal roots in $\Delta_1$. It can be seen that the root vector $E_{-i} \in g^{-\beta_i}$ (for $1 \leq i \leq n$) is given by the matrix $-E_{2i-1,2i} + E_{2i,2i-1} \in \text{Alt}_{2n}(R) = g_{-1}$. If we denote by $\text{Pf}(X)$ the Pfaffian of an alternating matrix $X$, then the above matrix realization of $E_{-i}$ shows that $\text{Pf}(o_{n,0}) = (-1)^n$ and $\text{Pf}(o_{n-1,1}) = (-1)^{n-1}$. Since the Pfaffian is the reduced norm of the Jordan algebra $\mathfrak{U}_n = \text{Alt}_{2n}(R)$, it follows from Proposition 4.7 that $V_n = G_0^{0} \cdot o_{n,0} \bigsqcup G_0^{0} \cdot o_{n-1,1}$. Therefore we get (5.1) with $p$ replaced by $n$. 
Let us next study the open $G_0$-orbits. For $g = \text{so}(2n, 2n)$, it is known (Matsumoto [19]) that $e_1$ is one of representatives of $Q_1 \mod Q_0$. Similarly as before, we have $e_1 \in G_0 - G_0^0$. Choose a subset $\Pi'_1 = \{\alpha_1\}$ of $\Pi$. Then $h_{\Pi'_1}(\partial) = 1$.

Consider the gradation (5.9) of $g = \text{so}(2n, 2n)$ corresponding to $\Pi'_1$ and the partition (5.10) of $A$. Since $h_{\Pi'_1}(\beta_i) = 1 \neq 0$ and $h_{\Pi'_1}(\beta_k) = 0$ for $k \geq 2$, we have that $E_{-1} \in g'_{-1}$ and $E_{-k} \in g'_0$ for $k \geq 2$. On the other hand $e_1 = 1$ on $g'_0$ and $= -1$ on $g'_{-1} + g'_1$. Hence $e_1$ sends $E_{-1}$ to $-E_{-1}$ and leaves $E_{-k}$ ($k \geq 2$) fixed. Let $a \in W(A^*)$ be the element interchanging $E_{-1}$ with $E_{-n}$ and leaving all other elements $E_{-k}$ ($k \neq 1, n$) fixed. Then we have that $ae_1(o_{n,0}) = o_{n-1,1}$, and hence $G_0 \cdot o_{n-1,1} = G_0 \cdot o_{n,0}$, which proves (5.2) with $p$ replaced by $n$. Since $e_1$ normalizes $G_0^0$, we see that $e_1(G_0^0 \cdot o_{n,0}) = G_0^0 \cdot o_{n-1,1}$. \hfill $\square$

5.3 Let us now consider the case $(g, g_0, g_{-1}) = (E_7(7), E_6(6) + R, H_3(O'))$. There is only one possibility of gradations of the first kind for $g = E_7(7)$. That gradation corresponds to $\Pi_1 = \{\alpha_7\}$. Let $\Gamma = \{\beta_1, \beta_2, \beta_3\}$, where

$$
\begin{align*}
\beta_1 &= 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \\
\beta_2 &= \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \\
\beta_3 &= \alpha_7.
\end{align*}
$$

(5.15)

It can be checked that $\Gamma$ is a maximal system of strongly orthogonal roots in $A_1$. As was shown in [6], $\{e_1, e_2, e_3\}, e_i = E_{-i}$, is a frame (= a maximal system of orthogonal primitive idempotents) of $B$. In the present case, the triple product $B_t$ of $B$ comes from the natural Jordan algebra structure $\mathfrak{A}$ of $g_{-1} = H_3(O')$ (cf. Loos [18]), that is,

$$
B_t(X, U, Y) = X \circ (U \circ Y) + (X \circ U) \circ Y - U \circ (X \circ Y),
$$

(5.16)

where $\circ$ denotes the Jordan multiplication in $\mathfrak{A}$. Therefore the two structure groups coincide:

$$
\text{Str } \mathfrak{A} = \text{Str } B.
$$

(5.17)

Let $e_{ii}$ ($i = 1, 2, 3$) be the diagonal matrix $\text{diag}(\delta_{ii}, \delta_{2i}, \delta_{3i}) \in H_3(O')$. Then $\{e_{11}, e_{22}, e_{33}\}$ is a frame in $H_3(O')$.

**Lemma 5.3.** $o_{3,0}$ is an invertible element in the Jordan algebra $\mathfrak{A} := H_3(O')$.

**Proof.** Let $P_{\mathfrak{A}}$ be the quadratic representation of $\mathfrak{A}$. Then (5.16) implies that $P_{\mathfrak{A}}(X) = P(X)$ for $X \in g_{-1} = H_3(O')$, and hence $P_{\mathfrak{A}}(o_{3,0}) = P(o_{3,0})$. The operator $P(o_{3,0})$ is nondegenerate, by Lemma 4.4. Therefore $o_{3,0}$ is an invertible element in $\mathfrak{A}$. \hfill $\square$

Recall the Jordan algebra $\mathfrak{A}_3 = (g_{-1}, \Box_3)$ in §2. By (5.16), $\mathfrak{A}_3$ is a mutant of $\mathfrak{A}$ by the invertible element $o_{3,0}$.

**Lemma 5.4.** $N(o_{3,0})N(o_{2,1}) < 0$.

**Proof.** Let $N_{\mathfrak{A}}$ be the reduced norm of $\mathfrak{A}$. Then we have (Braun-Koecher [3])

$$
N(X) = N_{\mathfrak{A}}(X)N_{\mathfrak{A}}(o_{3,0}), \quad X \in g_{-1}.
$$

(5.18)
Since \(o_{3,0}\) is invertible in \(\mathfrak{W}\), we have \(N_{\mathfrak{W}}(o_{3,0}) \neq 0\). Now consider the two frames \(\{e_{1}, e_{2}, e_{3}\}\) and \(\{e_{11}, e_{22}, e_{33}\}\) in \(\mathfrak{B}\). By Proposition 11.8 in Loos [18] and Lemma 3.4 here, there exists an element \(k \in K_{0}\) such that

\[
ke_{3,0} = \sum_{i=1}^{3} \delta_i e_{ii},
\]

where \(\delta_i = \pm 1\). \(N_{\mathfrak{W}}\) is a relative invariant polynomial for the group \(\text{Str} \mathfrak{W}\). Therefore there exists an \(R^{*}\)-valued character \(\chi\) of \(\text{Str} \mathfrak{W} = \text{Str} \mathfrak{B} = G_{0}\) such that

\[
N_{\mathfrak{W}}(gX) = \chi(g)N_{\mathfrak{W}}(X), \quad X \in g_{-1}, g \in G_{0}.
\]

Since \(K_{0}\) is contained in the commutator subgroup \([G_{0}, G_{0}]\), \(\chi(K_{0}) = 1\). Therefore we have

\[
N_{\mathfrak{W}}(o_{3,0}) = N_{\mathfrak{W}}(ko_{3,0}) = N_{\mathfrak{W}} \left( \sum_{i=1}^{3} \delta_i e_{ii} \right) = \delta_1 \delta_2 \delta_3.
\]

Similarly we have \(N_{\mathfrak{W}}(o_{2,1}) = -\delta_1 \delta_2 \delta_3\). Therefore, in view of (5.18), we have \(N(o_{3,0})N(o_{2,1}) < 0\).

**Theorem 5.5.** Let \((\mathfrak{g}, \mathfrak{g}_{0}, \mathfrak{g}_{-1}) = (E_{7(7)}, E_{6(6)} + R, H_{3}(O'))\). Then the orbit decompositions of \(\mathfrak{g}_{-1}\) under the groups \(G_{0}\) and \(G_{0}\) are given by

\[
g_{-1} = \prod_{k=0}^{2} G_{0} \cdot o_{k,0} \prod G_{0} \cdot o_{3,0} \prod G_{0} \cdot o_{2,1},
\]

(5.22)

\[
g_{-1} = \prod_{k=0}^{3} G_{0} \cdot o_{k,0}.
\]

(5.23)

There are exactly two open orbits \(G_{0} \cdot o_{3,0}\) and \(G_{0} \cdot o_{2,1}\) which are mutually diffeomorphic. There is a single open \(G_{0}\)-orbit in \(g_{-1}\).

**Proof.** (5.22) follows from Lemmas 4.3 and 5.4 and Proposition 4.7. Let us consider the \(G_{0}\)-orbit decomposition of \(g_{-1}\). In the present case \(g = E_{7(7)}, Q_{1} \mod Q_{0}\) is generated by \(e_{2}\) (Matsumoto [19]), and hence \(e_{2} \in G_{0} - G_{0}^{0}\). Consider the subset \(\Pi'_{1} = \{\alpha_{2}\}\) of \(\Pi_{1}\). Then \(h_{\Pi'_{1}}(g) = 2\). Let \(g = \sum_{k=-2}^{2} g_{k}'\) be the gradation of \(g\) corresponding to \(\Pi'_{1}\) and let \(A = \bigoplus_{k=-2}^{2} A_{k}'\) be the corresponding partition of \(\Delta\). By the same reason as for \(g = \text{sl}(2p, R)\), we have that \(e_{2} = 1\) on \(g_{-2} + g_{0}' + g_{2}'\) and \(e_{2} = -1\) on \(g_{-1}' + g_{1}'\). On the other hand, we have \(\beta_{1} \in A_{2}', \beta_{2} \in A_{1}'\) and \(\beta_{3} \in A_{0}'\) (cf. (5.15)). Consequently \(e_{2}(o_{3,0}) = e_{1} - e_{2} + e_{3}\). Let \(a \in W(A')\) be the element interchanging \(e_{2}\) with \(e_{3}\) and leaving \(e_{1}\) fixed. Then it follows that \(ae_{2}(o_{3,0}) = o_{2,1}\), which implies \(e_{2}(G_{0}^{0} \cdot o_{3,0}) = G_{0}^{0} \cdot o_{2,1}\). This proves (5.23).

5.4. Let us consider the final case \((\mathfrak{g}, \mathfrak{g}_{0}, \mathfrak{g}_{-1}) = (\text{so}(p+1, q+1), \text{so}(p, q) + R, R^{p+q}), 2 \leq p \leq q\), in which case \(r = 2\) (cf. Table II). The root system \(\Delta\) of \(\mathfrak{g}\) is of type \(B_{p+1}\) or \(D_{p+1}\) according as \(p < q\) or \(p = q\), respectively. \(\Delta\) is given by

\[
\Delta = \{ \pm (\lambda_{i} \pm \lambda_{j}) \mid 1 \leq i < j \leq p + 1; \lambda_{i} \mid 1 \leq i \leq p + 1\}, \quad p < q,
\]

(5.24)
The gradation of \( g \) corresponds to the subset \( \Pi_1 = \{ \alpha_1 = \lambda_1 - \lambda_2 \} \) of \( \Pi \). \( \Delta_1 \) is given by
\[
\Delta_1 = \{ \lambda_1 \pm \lambda_i (2 \leq i \leq p+1); \lambda_1 \},
\]
where \( \lambda_1 \) occurs only when \( p < q \). The subset of \( \Delta_1 \)
\[
\Gamma = \{ \beta_1 = \lambda_1 + \lambda_2, \beta_2 = \lambda_1 - \lambda_2 \}
\]
is a maximal system of strongly orthogonal roots in \( \Delta_1 \). In this situation we get the simple Jordan algebra \( \mathfrak{U}_2 = (\mathfrak{g}_{-1}, \coprod_{2}) \) of rank 2 with unit element \( e := 0_{2,0} \) (cf. \( \S \)). We need some results on simple Jordan algebras of rank 2 due to Braun-Koecher [3]: The reduced norm \( N \) of \( \mathfrak{U}_2 \) is of signature \( (p, q) \), and the multiplication \( \Pi_2 \) can be expressed as
\[
N(e_1, e_1) = N(e_2, e_2) = 0, \quad N(e_1, e_2) = \frac{1}{2}.
\]
**Theorem 5.6.** Let \( (\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_{-1}) = (\mathfrak{so}(p+1, q+1), \mathfrak{so}(p, q) + \mathfrak{R}, \mathfrak{R}^{p+q}), \) \( 2 \leq p \leq q \). Then the \( G_0 \)-orbit decomposition of \( \mathfrak{g}_{-1} \) is given by
\[
\mathfrak{g}_{-1} = \prod_{k=0}^{1} G_0 \cdot 0_{k,0} \prod G_0 \cdot 0_{2,0} \prod G_0 \cdot 0_{1,1}.
\]
**Proof.** By using (5.28), we see that \( N(0_{2,0}) = 1 \) and \( N(0_{1,1}) = -1 \). Therefore, from Lemma 4.3 and Proposition 4.7, the assertion follows. \( \square \)

**Theorem 5.7.** Under the same assumption as in Theorem 5.6, the \( G_0 \)-orbit decomposition of \( \mathfrak{g}_{-1} \) is given as follows:
\[
\mathfrak{g}_{-1} = \prod_{k=0}^{2} G_0 \cdot 0_{k,0} \quad \text{for } p = q,
\]
\[
\mathfrak{g}_{-1} = \prod_{k=0}^{1} G_0 \cdot 0_{k,0} \prod G_0 \cdot 0_{2,0} \prod G_0 \cdot 0_{1,1} \quad \text{for } p < q.
\]
**Proof.** Suppose first \( p = q \). In this case, one of generators of \( Q_1 \mod Q_0 \) is \( \epsilon_{p+1} \) (Matsumoto [19]). Note that \( \epsilon_{p+1} \in G_0 - G_0^0 \). Choose the subset \( \Pi_1' = \{ \alpha_{p+1} \} \) of \( \Pi \). Then \( h_{\Pi_1'}(\beta) = 1 \). Let \( g = \sum_{k=-1}^{1} g_k \) be the gradation of \( g \) corresponding to \( \Pi_1' \), and let \( A = \prod_{k=-1}^{1} A_k \) be the corresponding partition of \( \Delta \). We have \( \epsilon_{p+1} = 1 \) on \( g_0 \), and \( \epsilon_{p+1} = -1 \) on \( g_{-1} + g_1' \). We also have \( \beta_1 \in A_1 \) and \( \beta_2 \in A_0 \), since \( h_{\Pi_1'}(\beta_1) = 1 \) and \( h_{\Pi_1'}(\beta_2) = 0 \). As a result, \( \epsilon_{p+1}(\alpha_{2,0}) = -e_1 + e_2 \). Choose an element \( a \in W(\Delta^*) \) interchanging \( e_1 \) with \( e_2 \). Then \( a\epsilon_{p+1}(\alpha_{2,0}) = \alpha_{1,1} \), which implies that \( \epsilon_{p+1}(G_0 \cdot \alpha_{2,0}) = G_0 \cdot \alpha_{1,1} \). This, together with Lemma 4.3, proves (5.30).
Next consider the case $p < q$. Put $C_{pq}^+ = G_0^0 \cdot o_{2,0}$ and $C_{pq}^- = G_0^0 \cdot o_{1,1}$ for simplicity. Choose a coordinate system $(x_i)$ in $\mathfrak{g}_{-1} = \mathbb{R}^{p+q}$ such that the reduced norm $N(X)$ is expressed as the canonical form $x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$. Then

$$(5.32) \quad C_{pq}^\pm = \left\{ (x_i) \in \mathbb{R}^{p+q} : \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2 \geq 0 \right\}.$$ 

Let $S_{pq}^\pm$ be the level surfaces of $N$, that is,

$$(5.33) \quad S_{pq}^\pm = \left\{ (x_i) \in C_{pq}^\pm : N(X) = \pm 1 \right\}.$$ 

Then $C_{pq}^\pm$ are diffeomorphic to $S_{pq}^\pm \times \mathbb{R}^+$, respectively. An easy argument shows that $S_{pq}^+$ (resp. $S_{pq}^-$) is diffeomorphic to $S^{p-1} \times \mathbb{R}^q$ (resp. $S^{q-1} \times \mathbb{R}^p$), where $S^k$ denotes a $k$-sphere. Consider the $i$-th homology groups $H_i(C_{pq}^\pm, Z), 0 \leq i \leq p + q$. Then the above argument shows that $H_i(C_{pq}^+; Z) \simeq H_i(S^{p-1}, Z)$ and $H_i(C_{pq}^-; Z) \simeq H_i(S^{q-1}, Z)$. Suppose that $C_{pq}^\pm$ are homeomorphic to each other. Then we have $H_i(S^{p-1}, Z) \simeq H_i(S^{q-1}, Z)$ for any $i, 0 \leq i \leq p + q$, which implies $p = q$. This contradicts the hypothesis $p < q$. Therefore $C_{pq}^+$ is not homeomorphic to $C_{pq}^-$. Suppose now that there exists only one open $G_0^0$-orbit in $\mathfrak{g}_{-1}$. Then there exists $a \in G_0 - G_0^0$ such that $ao_{2,0} = o_{1,1}$. We then have $a(C_{pq}^+) = C_{pq}^-$, and hence $C_{pq}^+$ is homeomorphic to $C_{pq}^-$, which is a contradiction. Therefore there are exactly two open $G_0$-orbits. \hfill \Box

6. Open $G_0^0$-orbits

Let $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ be a real simple GLA. Suppose that the split root system $\Delta(\mathfrak{g}, c)$ of the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$ is of type $C_r$. Then we have the simple Jordan algebras $\mathfrak{U}_p = (\mathfrak{g}_{-1}, \Box_p)$ with unit element $o_{p,r-p}(0 \leq p \leq r)$ \hspace{1em} (cf. §2). For an element $g \in \text{Str} \mathfrak{U}_p$, we define

$$(6.1) \quad \theta(g) := (g^*)^{-1},$$

where $g^*$ is the adjoint operator of $g$ with respect to the trace form $\gamma_p$ of $\mathfrak{U}_p$. Then $\theta$ is an involutive automorphism of Str $\mathfrak{U}_p$. We denote by $\text{Aut}_{\text{JTS}} \mathfrak{U}_p$ the automorphism group of the JTS $(4.13)$ coming from the Jordan algebra $\mathfrak{U}_p$, and we denote by $(\text{Str} \mathfrak{U}_p)_{\theta}$ the subgroup of $\theta$-fixed elements of Str $\mathfrak{U}_p$. Then, by the definition of $\text{Aut}_{\text{JTS}} \mathfrak{U}_p$, we have

$$(6.2) \quad (\text{Str} \mathfrak{U}_p)_{\theta} = \text{Aut}_{\text{JTS}} \mathfrak{U}_p,$$

\hspace{1em} \text{Proposition 6.1.} Suppose that the split root system $\Delta(\mathfrak{g}, c)$ of the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$ is of type $C_r$. Then the open orbit $G_0^0 \cdot o_{p,r-p} \hspace{1em} (0 \leq p \leq r)$ is expressed as a symmetric coset space:

$$(6.3) \quad G_0^0 \cdot o_{p,r-p} = (\text{Str} \mathfrak{U}_p)^0 / (\text{Str} \mathfrak{U}_p)^0 \cap \text{Aut} \mathfrak{U}_p,$$

where $\text{Aut} \mathfrak{U}_p$ denotes the automorphism group of the Jordan algebra $\mathfrak{U}_p$. \hspace{1em} (Note that $G_0^0 = (\text{Str} \mathfrak{U}_p)^0$ by $(4.17)$).
\textbf{Proof.} \(\text{Aut} \mathfrak{U}_p\) is an open subgroup of \(\text{Aut}_{\text{JTS}} \mathfrak{U}_p\) (cf. Satake [21]). Consequently, noting (6.2), we have the inclusions
\begin{equation}
((\text{Str} \mathfrak{U}_p)_\theta)^0 \subset \text{Aut} \mathfrak{U}_p \subset (\text{Str} \mathfrak{U}_p)_\theta.
\end{equation}
By taking the intersection of each term in (6.4) with \((\text{Str} \mathfrak{U}_p)^0\), it follows that
\begin{equation}
(((\text{Str} \mathfrak{U}_p)^0)_\theta)^0 \subset (\text{Str} \mathfrak{U}_p)^0 \cap \text{Aut} \mathfrak{U}_p \subset (((\text{Str} \mathfrak{U}_p)^0)_\theta)^0,
\end{equation}
which implies that the coset space in the right-hand side of (6.3) is a symmetric coset space. Since \(\text{Aut} \mathfrak{U}_p\) is the isotropy subgroup of \(\text{Str} \mathfrak{U}_p\) at the unit element \(o_{p,r-p}\), \(G^0 \cdot o_{p,r-p}\) has the coset space expression (6.3).
\[\square\]

Every open orbit \(G^0 \cdot o_{p,r-p}\) is an \(\omega\)-domain in the sense of Koecher [16], since that orbit is a connected component of \(V_r\) (note that \(V_r\) coincides with the totality of invertible elements in \(\mathfrak{U}_p\), by Lemma 4.4). As a result, open \(G^0\)-orbits exhaust all \(\omega\)-domains in real simple Jordan algebras. The results similar to Proposition 6.1 were obtained also by Faraut-Gindikin [5] and Vinberg [29].

\textbf{Remark 6.2.} Assuming that \(\Delta(g, c)\) is of type \(C\), let us consider the quadratic representation \(P(X)\) of the JTS \(\mathfrak{B}\). Then \(P(X)\) is nondegenerate for \(X \in V_r\) \((6.6)\). \(\det P(X)\) has a constant sign on each connected component of \(V_r\). Put
\begin{equation}
\Phi(X) = \log|\det P(X)|, \quad X \in V_r.
\end{equation}
Then, by Koecher [16] together with Lemma 4.4, the Hessian \(\text{Hess}(\Phi(X))\) is non-degenerate on each \(G^0\)-orbit. Hence \(\text{Hess}(\Phi(X))\) is a \(G^0\)-invariant pseudo-riemannian metric on it. As a conclusion, an open \(G^0\)-orbit provides with an example of pseudo-Hessian symmetric space (For the definition of a Hessian symmetric space, see Shima [24]).

In the following, we give the explicit forms of open \(G^0\)-orbits and their coset space expression (6.3) for each simple \(\text{GLA}(g, g_0, g_{-1})\) with split root system of type \(C\). Partial results have been obtained by Kaneyuki [11] and d’Atri-Gindikin [4].

(I1) with \(p = n/2\),
\[\{X \in M_p(R) : \det X > 0\}, \quad \{X \in M_p(R) : \det X < 0\}.\]
Both are expressed as \(GL(p, R)^0 \times GL(p, R)^0/\text{diagonal}\).

(I2) with \(p = n/2\),
\[\{X \in M_p(H) : \det X \neq 0\} = GL(p, H) \times GL(p, H)/\text{diagonal}.\]

(I3) \(H_{n-i,i}(C) = GL(n, C)/U(n-i,i), \quad 0 \leq i \leq n\).

(I4) \(H_{n-i,i}(R) = GL(n, R)^0/\text{SO}(n-i,i), \quad 0 \leq i \leq n\).

(I5) \(\{X \in SH_n(H) : \det X \neq 0\} = GL(n, H)/\text{SO}^*(2n).\)

(I6) i) \(p = 0\),
\[\{(x_i) \in \mathbb{R}^q : x_1^2 + \cdots + x_q^2 \neq 0\} = \mathbb{R}^+ \times \text{SO}(q)/\text{SO}(q-1).\]
ii) $p = 1$,

\[
\{(x_i) \in \mathbb{R}^{q+1} : x_1^2 - x_2^2 - \cdots - x_{q+1}^2 > 0, x_1 > 0\},
\]

\[
\{(x_i) \in \mathbb{R}^{q+1} : x_1^2 - x_2^2 - \cdots - x_{q+1}^2 > 0, x_1 < 0\},
\]

\[
\{(x_i) \in \mathbb{R}^{q+1} : x_1^2 - x_2^2 - \cdots - x_{q+1}^2 < 0\},
\]

The first two are expressed as $\mathbb{R}^+ \times SO(1,q)^0 / SO(q)$. The third one is expressed as $\mathbb{R}^+ \times SO(1,q)^0 / SO(1,q-1)^0$.

iii) $p \geq 2$,

\[
\left\{ (x_i) \in \mathbb{R}^{q+p} : \sum_{i=1}^{p}x_i^2 - \sum_{j=p+1}^{p+q}x_j^2 > 0 \right\} = \mathbb{R}^+ \times SO(p,q)^0 / SO(p-1,q)^0,
\]

\[
\left\{ (x_i) \in \mathbb{R}^{q+p} : \sum_{i=1}^{p}x_i^2 - \sum_{j=p+1}^{p+q}x_j^2 < 0 \right\} = \mathbb{R}^+ \times SO(p,q)^0 / SO(p,q-1)^0.
\]

(I7) $H_{n-i,i}(H) = GL(n,H)/Sp(n-i,i), \quad 0 \leq i \leq n.$

(I8) $\{X \in \text{Alt}_{2n}(\mathbb{R}) : \text{Pff}(X) > 0\}, \quad \{X \in \text{Alt}_{2n}(\mathbb{R}) : \text{Pff}(X) < 0\}.$

Both are expressed as $GL(2n,\mathbb{R})^0 / Sp(n,\mathbb{R}).$

(I11) $\{X \in H_3(O') : N(X) > 0\}, \quad \{X \in H_3(O') : N(X) < 0\},$

where $N$ denotes the reduced norm of $H_3(O')$. Both are expressed as $\mathbb{R}^+ \times E_{6(6)}/F_{4(4)}$.

(I12) $H_{3-i,i}(O), \quad i = 0,1,2,3.$

$H_{3,0}(O)$ and $H_{0,3}(O)$ are expressed as $\mathbb{R}^+ \times E_{6(-26)}/F_{4}.$

$H_{2,1}(O)$ and $H_{1,2}(O)$ are expressed as $\mathbb{R}^+ \times E_{6(-26)}/F_{4(-20)}.$

(I13) with $p = n/2$,

$\{X \in M_p(\mathbb{C}) : \det X \neq 0\} = GL(p,\mathbb{C}) \times GL(p,\mathbb{C})/\text{diagonal}.$

(I14) $\{X \in \text{Sym}_n(\mathbb{C}) : \det X \neq 0\} = GL(p,\mathbb{C})/SO(n,\mathbb{C}).$

(I15) $\{(z_i) \in \mathbb{C}^n : z_1^2 + \cdots + z_n^2 \neq 0\} = \mathbb{C}^* \times SO(n,\mathbb{C})/SO(n-1,\mathbb{C}).$

(I16) $\{X \in \text{Alt}_{2n}(\mathbb{C}) : \text{Pff}(X) \neq 0\} = GL(2n,\mathbb{C})/Sp(n,\mathbb{C}).$

(I18) $\{X \in H_3(O^C) : N(X) \neq 0\} = \mathbb{C}^* \times E_6^C / F_4^C,$

where $N$ denotes the reduced norm of the Jordan algebra $H_3(O^C)$.

In the above list, $H_{n-i,i}(K)$ denotes the set of $n \times n K$-hermitian matrices of signature $(n - i, i)$, where $K = \mathbb{R}, \mathbb{C}, H, O$.

References