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The Sylvester's law of inertia in simple graded Lie algebras

Dedicated to Professor Ichiro Satake on the occasion of his seventieth anniversally birthday

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Introduction.

Let $H_n(\boldsymbol{R})$ be the vector space of $n \times n$ real symmetric matrices. The group $GL(n,\boldsymbol{R})^0$ (= the identity component of $GL(n,\boldsymbol{R})$) acts on $H_n(\boldsymbol{R})$ by the rule: $X \mapsto AX\,{}^t\!A$, $X \in H_n(\boldsymbol{R})$, $A \in GL(n,\boldsymbol{R})^0$. The Sylvester's law of inertia asserts that, by this action of $GL(n,\boldsymbol{R})^0$, X is transformed into the canonical form $\operatorname{diag}(1,\dots,1,-1,\dots,-1,0,\dots,0)$, which is uniquely determined by X. The simple Lie algebra $\mathfrak{sp}(n,\boldsymbol{R})$ has a unique gradation $\mathfrak{sp}(n,\boldsymbol{R}) = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, where $\mathfrak{g}_{-1} = H_n(\boldsymbol{R})$ and $\mathfrak{g}_0 \simeq \mathfrak{gl}(n,\boldsymbol{R})$. The $GL(n,\boldsymbol{R})^0$-module $H_n(\boldsymbol{R})$ is imbedded in $\mathfrak{sp}(n,\boldsymbol{R})$ as the G_0^0-module $\mathfrak{g}_{-1}$, where G_0^0 is the analytic subgroup of Aut $\mathfrak{g}$ generated by $\mathfrak{g}_0$. The Sylvester's law of inertia for $H_n(\boldsymbol{R})$ is no other than obtaining the complete representatives of G_0^0-orbits in $\mathfrak{g}_{-1}$. As a generalization of this situation, one can pose:

PROBLEM. Let $\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k$ be a real simple graded Lie algebra, G_0 the group of grade-preserving automorphisms of $\mathfrak{g}$ and let G_0^0 be the identity component of G_0. Find the G_0^0-orbit decomposition and the G_0-orbit decomposition of $\mathfrak{g}_{-1}$.

When $\nu = 1$, this problem is equivalent to the problem of finding the orbits in a compact simple Jordan triple system under the structure group or the identity component of the structure group. Also it is equivalent to finding the orbit decomposition of a tangent space by the linear isotropy group for a symmetric R-space.

The purpose of this paper is to settle the above problem for the case $\nu = 1$ by a unified method. Partial answers have been obtained by Satake [**22, 23**], Kaneyuki [**9, 10**] and Takeuchi [**27**]. In the following we shall describe briefly how to get the two kinds of orbit decompositions of $\mathfrak{g}_{-1}$. The sections 1 and 2 are preliminary sections. We give a quick review for the followings: classification and construction of gradations in semisimple Lie algebras [**13, 12**], the root theory in simple graded Lie algebras $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ ([**13**]), the Jordan triple system $\mathfrak{B}$ on $\mathfrak{g}_{-1}$ (Loos [**18**]) and the root-theoretic version of a frame (= a maximal system of pairwise orthogonal idempotents) $\{e_1,\dots,e_r\}$ in $\mathfrak{g}_{-1}$, and the Jordan algebra structure $\mathfrak{A}_p$ $(0 \le p \le r)$ in $\mathfrak{g}_{-1}$. In § 3, applying a result of Matsumoto [**19**], we get a set of good representatives of G_0 mod G_0^0, which allows us to get the G_0-orbit decomposition from the G_0^0-orbit decomposition. We consider the root system Δ^* corresponding to a certain symmetric real flag domain M^*. It turns out that the Weyl group $W(\Delta^*)$ of Δ^*, viewed as a subgroup of G_0^0, acts on the frame $\{e_1,\dots,e_r\}$ as signed permutations. Then we can choose the candidates $o_{p,q}$ $(0 \le p,q \le r, p+q \le r)$ of representatives of the G_0^0-orbits, which are defined in terms of the frame. Let V_k $(0 \leq k \leq r)$ be the union of the G_0^0-orbits through the points $o_{p,q}$ with $p+q=k$. The sets V_k were introduced by Takeuchi [28] in a different way. Theorem 3.3 (Gindikin-Kaneyuki [6]) shows that each V_k is G_0-stable and that it consists of equi-dimensional G_0^0-orbits. Therefore, in order to find the orbit decomposition, we have only to separate the G_0^0-orbits in V_k $(0 \leq k \leq r)$. In the sections 4 and 5, we carry out this procedure, by using the action of $W(\Delta^*)$ and the reduced norm of the Jordan algebra $\mathfrak{A}_r$. The main results are Theorems 4.1, 4.2, 5.1, 5.2 and 5.5–5.7. In § 6, we give a list of all open G_0^0-orbits whose ambient spaces $\mathfrak{g}_{-1}$ are simple Jordan algebras. (Partial results have been obtained by D'Atri-Gindikin [4] and Kaneyuki [9].) This provides a classification of ω-domains in the sense of Koecher [16] in simple Jordan algebras.

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Notation and Convention: G^0 or $(G)^0$ denotes the identity component of a Lie group G. G_θ or $(G)_\theta$ denotes the subgroup of a group G consisting of elements left fixed by an involutive automorphism θ. GLA (resp. JTS) is an abbreviation for "graded Lie algebra" (resp. Jordan triple system). E denotes a unit matrix.

§ 1. Semisimple graded Lie algebras.

Let

$$\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k \tag{1.1}$$

be a real semisimple GLA of the ν-th kind (we are assuming that the subspace $\mathfrak{g}_{-1}$ is not zero). We assume further that the gradation (1.1) is of type α_0, that is, $\mathfrak{g}^- := \sum_{k<0} \mathfrak{g}_k$ is generated by $\mathfrak{g}_{-1}$. Let $(\mathfrak{g}, Z, \tau)$ be the associated graded triple; more precisely, $Z \in \mathfrak{g}$ is the characteristic element of the gradation (1.1), i.e., each subspace $\mathfrak{g}_k$ is the eigenspace of ad Z for the eigenvalue k, and τ is a grade-reversing Cartan involution of $\mathfrak{g}$. Let

$$\mathfrak{h} = \sum_{k \text{ even}} \mathfrak{g}_k, \quad \mathfrak{m} = \sum_{k \text{ odd}} \mathfrak{g}_k. \tag{1.2}$$

Then $\mathfrak{g}$ is expressed as a Z_2-GLA

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \tag{1.3}$$

which is also the decomposition by the involution $\sigma := \mathrm{Ad}\exp \pi i Z$, in which case we have $\sigma|_{\mathfrak{h}} = 1$ and $\sigma|_{\mathfrak{m}} = -1$. Consider the Cartan decomposition by τ:

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \tag{1.4}$$

where $\tau|_{\mathfrak{k}} = 1$ and $\tau|_{\mathfrak{p}} = -1$. Since σ and τ commutes, we have the (σ, τ)-decomposition

$$\mathfrak{g} = \mathfrak{k}_0 + \mathfrak{m}_{\mathfrak{k}} + \mathfrak{p}_0 + \mathfrak{m}_{\mathfrak{p}}, \tag{1.5}$$

where $\mathfrak{k}_0 = \mathfrak{h} \cap \mathfrak{k}, \mathfrak{p}_0 = \mathfrak{h} \cap \mathfrak{p}, \mathfrak{m}_{\mathfrak{k}} = \mathfrak{m} \cap \mathfrak{k}$ and $\mathfrak{m}_{\mathfrak{p}} = \mathfrak{m} \cap \mathfrak{p}$. Note that $Z \in \mathfrak{p}_0$. Choose a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ containing Z. Then $\mathfrak{a}$ is contained in $\mathfrak{g}_0 \cap \mathfrak{p} \subset \mathfrak{p}_0$. Let Δ be the root system for the pair $(\mathfrak{g}, \mathfrak{a})$, which is called a root system of $\mathfrak{g}$ compatible with the gradation. Let $(,)$ denote the Killing form of $\mathfrak{g}$. Then we have a partition of Δ:

$$\Delta = \coprod_{k=-\nu}^{\nu} \Delta_k, \tag{1.6}$$

where $\Delta_k = \{\alpha \in \Delta : (\alpha, Z) = k\}$, and each graded subspace $\mathfrak{g}_k$ can be written as

$$\begin{aligned} \mathfrak{g}_0 &= \mathfrak{c}(\mathfrak{a}) + \sum_{\alpha \in \Delta_0} \mathfrak{g}^{\alpha}, \\ \mathfrak{g}_k &= \sum_{\alpha \in \Delta_k} \mathfrak{g}^{\alpha}, \quad k \neq 0, \end{aligned} \tag{1.7}$$

where $\mathfrak{c}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$, and $\mathfrak{g}^{\alpha}$ denotes the root space for a root $\alpha \in \Delta$. Choose a linear order in Δ in such a way that

$$\coprod_{k=1}^{\nu} \Delta_k \subset \Delta^+ \subset \coprod_{k=0}^{\nu} \Delta_k, \tag{1.8}$$

where Δ^+ denotes the set of positive roots with respect to this order. Let Π be the fundamental system for Δ. Since the gradation is of type α_0, it is known [13] that $\Pi_k := \Pi \cap \Delta_k = \emptyset$ for $k \geq 2$, and hence we have a partition of Π:

$$\Pi = \Pi_0 \coprod \Pi_1, \quad \Pi_1 \neq \emptyset. \tag{1.9}$$

Let us consider the reverse process. Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{a}$ be a maximal $\boldsymbol{R}$-split abelian subalgebra of $\mathfrak{g}$, and let $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ be a fundamental system of the root system Δ for the pair $(\mathfrak{g}, \mathfrak{a})$. A root $\alpha \in \Delta$ can be written as

$$\alpha = \sum_{i=1}^{\ell} m_i(\alpha)\alpha_i. \tag{1.10}$$

Suppose that we are given a partition $\Pi = \Pi_0 \coprod \Pi_1$ with $\Pi_1 \neq \emptyset$. For a root $\alpha \in \Delta$, we define the height $h_{\Pi_1}(\alpha)$ of α relative to Π_1 by putting

$$h_{\Pi_1}(\alpha) = \sum_{\alpha_i \in \Pi_1} m_i(\alpha). \tag{1.11}$$

If we put

$$\Delta_k = \{\alpha \in \Delta : h_{\Pi_1}(\alpha) = k\}, \tag{1.12}$$

then we have a partition $\Delta = \coprod_{k=-\nu}^{\nu} \Delta_k$, where ν is equal to the the height $h_{\Pi_1}(\vartheta)$ of the highest root $\vartheta \in \Delta$. Let us define the subspaces $(\mathfrak{g}_k)_{-\nu \leq k \leq \nu}$ by the equalities (1.7). Then we have a GLA $\mathfrak{g} = \sum_{k=-\nu}^{\nu} \mathfrak{g}_k$ of type α_0 (cf. [13]).

THEOREM 1.1 ([13]). *Let $\mathfrak{g}$ be a real semisimple Lie algebra, and Δ be a restricted root system of $\mathfrak{g}$. Let Π be a fundamental system of Δ and ϑ be the highest root of Δ. Then there exists a bijection between the set of gradations of the ν-th kind of type α_0 in*

$\mathfrak{g}$ *and the set of subsets* Π_1 *of* Π *satisfying* $h_{\Pi_1}(\vartheta) = \nu$. *The bijection is compatible with the respective isomorphisms.*

A gradation of the first kind in $\mathfrak{g}$ is trivially of type α_0; any gradation of the second kind in $\mathfrak{g}$ is of type α_0, provided that $\mathfrak{g}$ is simple (Tanaka [**28**]).

§ 2. Jordan triple systems on $\mathfrak{g}_{-1}$.

We retain the notation in § 1. Let

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 \tag{2.1}$$

be a simple GLA (of the first kind), and $(\mathfrak{g}, Z, \tau)$ be the associated graded triple. Let Δ be a root system of $\mathfrak{g}$ compatible with the gradation. As a special case of (1.6), we have a partition $\Delta = \Delta_{-1} \coprod \Delta_0 \coprod \Delta_1$. Choose a linear order in Δ satisfying (1.8). As is known in Takeuchi [**26**], one can choose a maximal system of strongly orthogonal roots $\Gamma = \{\beta_1, \ldots, \beta_r\}$ in Δ_1 in such a way that $(\beta_1, \beta_1) = \cdots = (\beta_r, \beta_r)$. The number r is equal to the split rank of the symmetric triple $(\mathfrak{g}, \mathfrak{g}_0, \sigma)$. Choose a root vector $E_i \in \mathfrak{g}^{\beta_i} \subset \mathfrak{g}_1$ $(1 \leq i \leq r)$ in such a way that

$$[E_i, E_{-i}] = \check{\beta}_i = \frac{2}{(\beta_i, \beta_i)} \beta_i, \tag{2.2}$$

where $E_{-i} = -\tau E_i \in \mathfrak{g}^{-\beta_i} \subset \mathfrak{g}_{-1}$. Let

$$X_i = E_i + E_{-i} \in \mathfrak{m}_{\mathfrak{p}}. \tag{2.3}$$

Then the real span $\mathfrak{c}$ of $X_1, \ldots, X_r$ is a maximal abelian subspace of $\mathfrak{m}_{\mathfrak{p}}$. The root system $\Delta(\mathfrak{g}, \mathfrak{c})$ for the pair $(\mathfrak{g}, \mathfrak{c})$ is the split root system for the symmetric triple $(\mathfrak{g}, \mathfrak{g}_0, \sigma)$. It is known (Oshima-Sekiguchi [**20**]) that $\Delta(\mathfrak{g}, \mathfrak{c})$ is either of type C or of type BC. Let $\mathfrak{a}_0$ be the subspace of $\mathfrak{a}$ spanned by $\beta_1, \ldots, \beta_r$, and ϖ be the orthogonal projection of $\mathfrak{a}$ onto $\mathfrak{a}_0$ with respect to (,). Then, by considering the inverse Cayley transformation ([**8**]) of $\mathfrak{c}$ onto $\mathfrak{a}_0$ and by taking the inner products with Z, we have

$$\begin{cases} \varpi((\Delta_0)^+) - (0) = \left\{ \frac{1}{2}(\beta_i - \beta_j) : 1 \leq i < j \leq r \right\}, \\ \varpi(\Delta_1) = \left\{ \frac{1}{2}(\beta_i + \beta_j) : 1 \leq i \leq j \leq r \right\}, \end{cases} \tag{2.4}$$

provided that $\Delta(\mathfrak{g}, \mathfrak{c})$ is of type C, or

$$\begin{cases} \varpi((\Delta_0)^+) - (0) = \left\{ \frac{1}{2}(\beta_i - \beta_j) \ (1 \leq i < j \leq r);\ \frac{1}{2}\beta_i \ (1 \leq i \leq r) \right\}, \\ \varpi(\Delta_1) = \left\{ \frac{1}{2}(\beta_i + \beta_j) \ (1 \leq i \leq j \leq r);\ \frac{1}{2}\beta_i \ (1 \leq i \leq r) \right\}, \end{cases} \tag{2.5}$$

provided that $\Delta(\mathfrak{g}, \mathfrak{c})$ is of type BC, where $(\Delta_0)^+ = \Delta_0 \cap \Delta^+$. We put

$$\begin{aligned} \mathfrak{a}_{ij} &= \sum_{\substack{\alpha \in \Delta_1 \\ \varpi(\alpha) = \frac{1}{2}(\beta_i + \beta_j)}} \mathfrak{g}^{-\alpha}, \quad i \leq j, \\ \mathfrak{c}_i &= \sum_{\substack{\alpha \in \Delta_1 \\ \varpi(\alpha) = \frac{1}{2}\beta_i}} \mathfrak{g}^{-\alpha}. \end{aligned} \tag{2.6}$$

Then $\mathfrak{g}_{-1}$ can be expressed as

(2.7) $$\mathfrak{g} = \sum_{1\le i\le j\le r} \mathfrak{a}_{ij} + \sum_{1\le i\le r} \mathfrak{c}_i.$$

If $\Delta(\mathfrak{g},\mathfrak{c})$ is of type C, then the second term of the right-hand side of (2.7) does not appear. The dimensions $\dim \mathfrak{a}_{ij}$ $(i<j)$, $\dim \mathfrak{a}_{ii}$ and $\dim \mathfrak{c}_i$ do not depend on the choice of i and j ([**7**]).

Let us consider a triple product B_τ on $\mathfrak{g}_{-1}$:

(2.8) $$B_\tau(X,Y,U) = \frac{1}{2}[[\tau Y, X], U], \quad X, Y, U \in \mathfrak{g}_{-1}.$$

It is known (Loos [**17**], Satake [**21**]) that the pair $\mathfrak{B} = (\mathfrak{g}_{-1}, B_\tau)$ is a compact simple JTS and that $\mathfrak{g}$ is isomorphic to the Kantor-Tits-Koecher construction for $\mathfrak{B}$ (These two facts can be obtained in more general setting of a simple GLA of the second kind and the corresponding compact generalized JTS; see [**1**, **13**]). For simplicity we write e_i for E_{-i} $(1\le i\le r)$ and (XYU) for $B_\tau(X,Y,U)$. As usual, we define the linear operator $L(X,Y)$ on $\mathfrak{g}_{-1}$ by

(2.9) $$L(X,Y)U = (XYU), \quad U \in \mathfrak{g}_{-1}.$$

Let

(2.10) $$o_{p,q} = \sum_{i=1}^{p} e_i - \sum_{j=p+1}^{p+q} e_j, \quad 0\le p,q\le r, \quad p+q\le r.$$

By using the facts [**6**] that e_i $(1\le i\le r)$ is an idempotent of the JTS $\mathfrak{B}$ and that $L(e_i,e_j)=0$ $(i\ne j)$, we see that $o_{p,q}$ is an idempotent of $\mathfrak{B}$ and that

(2.11) $$L(o_{p,r-p}, o_{p,r-p}) = L(o_{r,0}, o_{r,0}), \quad 0\le p\le r.$$

LEMMA 2.1. *Let $\mathfrak{g}_{-1}(\lambda)$ be the eigenspace of $L(o_{r,0},o_{r,0})$ corresponding to the eigenvalue λ. Then we have $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}(1) + \mathfrak{g}_{-1}(\frac{1}{2})$, and*

(2.12) $$\mathfrak{g}_{-1}(1) = \sum_{1\le i\le j\le r} \mathfrak{a}_{ij},$$

(2.13) $$\mathfrak{g}_{-1}\left(\frac{1}{2}\right) = \sum_{1\le i\le r} \mathfrak{c}_i.$$

PROOF. Consider the Peirce decomposition (Satake [**21**]) of $\mathfrak{g}_{-1}$ with respect to the operator $L(o_{p,r-p}, o_{p,r-p}) = L(o_{r,0}, o_{r,0})$:

(2.14) $$\mathfrak{g}_{-1} = \mathfrak{g}_{-1}(1) + \mathfrak{g}_{-1}\left(\frac{1}{2}\right) + \mathfrak{g}_{-1}(0).$$

Choose a root $\alpha\in\Delta_1$ such that $\varpi(\alpha) = \frac{1}{2}(\beta_i+\beta_j)$, $i\le j$. We have $\sum_{k=1}^{r}(\check{\beta}_k,\alpha) = \sum_{k=1}^{r}(\check{\beta}_k,\varpi(\alpha)) = \frac{1}{2}\sum_{k=1}^{r}(\check{\beta}_k,\beta_i+\beta_j) = 2$. Let $X\in\mathfrak{g}^{-\alpha}$. Then it follows that

$$\begin{aligned} L(o_{r,0},o_{r,0})(X) &= B_\tau(o_{r,0},o_{r,0},X) = \frac{1}{2}[[\tau(o_{r,0}),o_{r,0}],X] \\ &= \frac{1}{2}\sum_{k=1}^{r}[[-E_k,E_{-k}],X] = -\frac{1}{2}\sum_{k=1}^{r}[\check{\beta}_k,X] = \frac{1}{2}\left(\sum_{k=1}^{r}(\check{\beta}_k,\alpha)\right)X = X, \end{aligned}$$

which implies that the right-hand side of (2.12) is contained in $\mathfrak{g}_{-1}(1)$. Similarly we have that the right-hand side of (2.13) is contained in $\mathfrak{g}_{-1}(\frac{1}{2})$. Consequently the lemma follows from (2.14) and (2.7). $\square$

We introduce a multiplication $\square_p$ in $\mathfrak{g}_{-1}$:

$$X\square_p Y = B_\tau(X, o_{p,r-p}, Y), \quad X, Y \in \mathfrak{g}_{-1}, \quad 0 \le p \le r. \tag{2.15}$$

As a property of the Peirce decomposition of a JTS ([**21**]), we know that $\mathfrak{g}_{-1}(1)$ become a Jordan algebra with unit element $o_{p,r-p}$ with respect to the multiplication $\square_p$.

PROPOSITION 2.2. *Let $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ be a real simple GLA. Then the pair $(\mathfrak{g}_{-1}, \square_p), 0 \le p \le r$, is a Jordan algebra with $o_{p,r-p}$ as unit element, if and only if the split root system $\Delta(\mathfrak{g}, \mathfrak{c})$ is of type C. In this case the Jordan algebra $(\mathfrak{g}_{-1}, \square_p)$ is simple.*

PROOF. Suppose first that $\Delta(\mathfrak{g}, \mathfrak{c})$ is of type C. Then we have (2.4). Therefore there are no roots $\alpha \in \Delta$ such that $\varpi(\alpha) = \frac{1}{2}\beta_i$ $(1 \le i \le r)$, and so we have $\mathfrak{g}_{-1}(\frac{1}{2}) = (0)$. By Lemma 2.1, we have $\mathfrak{g}_{-1}(1) = \mathfrak{g}_{-1}$. Conversely, suppose that $(\mathfrak{g}_{-1}, \square_p)$ is a Jordan algebra with unit element $o_{p,r-p}$. Then, for any $X \in \mathfrak{g}_{-1}$, we have $X = o_{p,r-p}\square_p X = B_\tau(o_{p,r-p}, o_{p,r-p}, X) = L(o_{r,0}, o_{r,0})X$, which implies that $\mathfrak{g}_{-1}(1) = \mathfrak{g}_{-1}$ and $\mathfrak{g}_{-1}(\frac{1}{2}) = (0)$. Consequently $\Delta(\mathfrak{g}, \mathfrak{c})$ is of type C, by (2.4) and (2.5). To prove the second assertion, consider the involution $*$ of the Jordan algebra $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}(1)$:

$$X^* = B_\tau(o_{p,r-p}, X, o_{p,r-p}), \quad X \in \mathfrak{g}_{-1}. \tag{2.16}$$

Then B_τ can be reconstructed as follows ([**21**]):

$$B_\tau(X, Y, U) = (X\square_p Y^*)\square_p U + X\square_p(Y^*\square_p U) - Y^*\square_p(X\square_p U). \tag{2.17}$$

Let W be an ideal of the Jordan algebra $\mathfrak{g}_{-1}$. Then, by using (2.17), we have that $B_\tau(W, \mathfrak{g}_{-1}, \mathfrak{g}_{-1}) + B_\tau(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}, W) \subset W$. This means that W is a K-ideal (cf. [**13**]) of the JTS $\mathfrak{B}$. $\mathfrak{B}$ is compact simple, and hence by a result of [**1**], it is K-simple. Therefore $W = (0)$ or $W = \mathfrak{g}_{-1}$. Thus the Jordan algebra $\mathfrak{g}_{-1}$ is simple. $\square$

The simple Jordan algebra $(\mathfrak{g}_{-1}, \square_p)$ is denoted by $\mathfrak{A}_p$.

§ 3. Generalities on the orbit decomposition of $\mathfrak{g}_{-1}$.

We retain the notation in the previous sections. We will consider exclusively a simple GLA (2.1) : $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$. We denote by $\mathrm{Aut}\,\mathfrak{g}$ the automorphism group of the Lie algebra $\mathfrak{g}$, and denote by G^0 the identity component of $\mathrm{Aut}\,\mathfrak{g}$. Let G_0 be the subgroup of $\mathrm{Aut}\,\mathfrak{g}$ consisting of all grade-preserving automorphisms of the GLA $\mathfrak{g}$. We need the following subgroups of $\mathrm{Aut}\,\mathfrak{g}$:
$G := G_0 G^0$, which is an open subgroup of $\mathrm{Aut}\,\mathfrak{g}$,
G' the Zariski connected component of $\mathrm{Aut}\,\mathfrak{g}$, which is a subgroup of G,
$G_0' := G_0 \cap G'$, which is the Zariski connected component of G_0,
G_0^0 the (topological) identity component of G_0,
$K := \{g \in G : g\tau = \tau g\}$, which is the maximal compact subgroup of G with $\mathrm{Lie}\,K = \mathfrak{k}$.
$K_0 = G_0 \cap K$,
K_0^0 the identity component of K_0.

Let Δ be a root system of $\mathfrak{g}$ compatible with the gradation and $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ be a fundamental system of Δ with respect to an order satisfying (1.8). Let $\{Z_1, \dots, Z_\ell\}$ be the basis of $\mathfrak{a}$ dual to Π with respect to $(\, , \,)$. Consider the involutive automorphisms of $\mathfrak{g}$:

$$\varepsilon_k = \operatorname{Ad} \exp \pi i Z_k, \quad 1 \le k \le \ell. \tag{3.1}$$

LEMMA 3.1 (Matsumoto [**19**]). *Let Q_1 be the free abelian subgroup of* $\operatorname{Aut} \mathfrak{g}$ *generated by $\varepsilon_1, \dots, \varepsilon_\ell$, and let $Q_0 := Q_1 \cap G^0$. Then Q_1 is a subgroup of G', and*

$$G'/G^0 \simeq Q_1/Q_0, \tag{3.2}$$

in particular,

$$G' = Q_1 G^0. \tag{3.3}$$

Since ε_k is $+1$ or -1 on each root space $\mathfrak{g}^\alpha, \alpha \in \Delta \cup (0)$, it follows from (1.7) that ε_k is grade-preserving for any gradation of $\mathfrak{g}$. This implies, in particular, that Q_1 is a subgroup of G_0, and hence we have

$$Q_1 G_0^0 \subset G_0'. \tag{3.4}$$

Look at the (σ, τ)-decomposition (1.5) for the GLA $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$. It is easy to see that $\mathfrak{g}^* := \mathfrak{k}_0 + \mathfrak{m}_p$ is a reductive subalgebra of $\mathfrak{g}$. The center of $\mathfrak{g}^*$ is at most one-dimensional and the semisimple part of $\mathfrak{g}^*$ is simple ([**7**]). The triple $(\mathfrak{g}^*, \mathfrak{k}_0, \tau)$ is a Riemannian symmetric triple, the noncompact dual of $(\mathfrak{k}, \mathfrak{k}_0, \sigma)$. Let G^* be the connected Lie subgroup of G corresponding to $\mathfrak{g}^*$. Then K_0^0 is a maximal compact subgroup of G^*. $M^* = G^*/K_0^0$ is the symmetric space corresponding to $(\mathfrak{g}^*, \mathfrak{k}_0, \tau)$. We have the Cartan decomposition

$$G^* = K_0^0 \exp \mathfrak{m}_\mathfrak{p}. \tag{3.5}$$

Since $\mathfrak{c}$ is a maximal abelian subspace of $\mathfrak{m}_\mathfrak{p}$, one can consider the root system Δ^* for the pair $(\mathfrak{g}^*, \mathfrak{c})$ (or for the symmetric space M^*). In Table I, we give a list of real simple GLA's of the first kind and the corresponding subset Π_1 of Π ([**13, 12, 14, 18**]). In Table II, we give the root systems $\Delta(\mathfrak{g}, \mathfrak{c})$ and Δ^* for each simple GLA's of the first kind ([**20, 25, 18**]). The following notations are used in Table I: $\boldsymbol{H}$ the quaternion algebra over $\boldsymbol{R}, \boldsymbol{O}$ (resp. $\boldsymbol{O}'$) the Cayley (resp. the split Cayley) algebra over $\boldsymbol{R}$, and $\boldsymbol{O}^{\boldsymbol{C}} = \boldsymbol{O} \otimes_{\boldsymbol{R}} \boldsymbol{C}$. $M_{p,q}(\boldsymbol{K})$ the vector space of $p \times q$ matrices with entries in $\boldsymbol{K}$, where $\boldsymbol{K} = \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}, \boldsymbol{O}, \boldsymbol{O}'$ or $\boldsymbol{O}^{\boldsymbol{C}}$; $H_n(\boldsymbol{K})$ the vector space of hermitian matrices of degree n with entries in $\boldsymbol{K}$; $SH_n(\boldsymbol{H})$ the vector space of skew-hermitian quaternion matrices of degree n; $\operatorname{Alt}_n(\boldsymbol{K})$ the vector space of skew-symmetric matrices of degree n with entries in $\boldsymbol{K}$; $\operatorname{Sym}_n(\boldsymbol{C})$ the vector space of complex symmetric matrices of degree n. We employ the numbering of simple roots used in Bourbaki [**2**].

By the property $[\mathfrak{k}_0, \mathfrak{m}] \subset [\mathfrak{g}_0, \mathfrak{m}] \subset \mathfrak{m}$, the group K_0^0 acts on $\mathfrak{m}$ by the adjoint representation. Moreover, since $[\mathfrak{k}_0, \mathfrak{m}_\mathfrak{p}] \subset \mathfrak{m}_\mathfrak{p}$ and $[\mathfrak{k}_0, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{-1}$, it follows that this K_0^0-action on $\mathfrak{m}$ leaves both $\mathfrak{m}_\mathfrak{p}$ and $\mathfrak{g}_{-1}$ stable.

Table I

$(\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_{-1})$		Π	Π_1
I1 $(\mathfrak{sl}(n,\boldsymbol{R}), \mathfrak{sl}(p,\boldsymbol{R}) + \mathfrak{sl}(n-p,\boldsymbol{R}) + \boldsymbol{R}, M_{p,n-p}(\boldsymbol{R}))$,	$n \geq 3,\ 1 \leq p \leq [n/2]$	A_{n-1}	$\{\alpha_p\}$
I2 $(\mathfrak{sl}(n,\boldsymbol{H}), \mathfrak{sl}(p,\boldsymbol{H}) + \mathfrak{sl}(n-p,\boldsymbol{H}) + \boldsymbol{R}, M_{p,n-p}(\boldsymbol{H}))$,	$n \geq 3,\ 1 \leq p \leq [n/2]$	A_{n-1}	$\{\alpha_p\}$
I3 $(\mathfrak{su}(n,n), \mathfrak{sl}(n,\boldsymbol{C}) + \boldsymbol{R}, H_n(\boldsymbol{C}))$,	$n \geq 3$	C_n	$\{\alpha_n\}$
I4 $(\mathfrak{sp}(n,\boldsymbol{R}), \mathfrak{sl}(n,\boldsymbol{R}) + \boldsymbol{R}, H_n(\boldsymbol{R}))$,	$n \geq 3$	C_n	$\{\alpha_n\}$
I5 $(\mathfrak{sp}(n,n), \mathfrak{sl}(n,\boldsymbol{H}) + \boldsymbol{R}, SH_n(\boldsymbol{H}))$,	$n \geq 2$	C_n	$\{\alpha_n\}$
I6 $(\mathfrak{so}(p+1,q+1), \mathfrak{so}(p,q) + \boldsymbol{R}, M_{1,p+q}(\boldsymbol{R}))$,	$0 \leq p < q$ or $3 \leq p = q$	$B_{p+1}(p<q)$ $D_{p+1}(p=q)$	$\{\alpha_1\}$ $\{\alpha_1\}$
I7 $(\mathfrak{so}^*(4n), \mathfrak{sl}(n,\boldsymbol{H}) + \boldsymbol{R}, H_n(\boldsymbol{H}))$,	$n \geq 3$	C_n	$\{\alpha_n\}$
I8 $(\mathfrak{so}(n,n), \mathfrak{sl}(n,\boldsymbol{R}) + \boldsymbol{R}, \mathrm{Alt}_n(\boldsymbol{R}))$,	$n \geq 4$	D_n	$\{\alpha_n\}$
I9 $(E_{6(6)}, \mathfrak{so}(5,5) + \boldsymbol{R}, M_{1,2}(\boldsymbol{O}'))$		E_6	$\{\alpha_1\}$
I10 $(E_{6(-26)}, \mathfrak{so}(1,9) + \boldsymbol{R}, M_{1,2}(\boldsymbol{O}))$		A_2	$\{\alpha_1\}$
I11 $(E_{7(7)}, E_{6(6)} + \boldsymbol{R}, H_3(\boldsymbol{O}'))$		E_7	$\{\alpha_7\}$
I12 $(E_{7(-25)}, E_{6(-26)} + \boldsymbol{R}, H_3(\boldsymbol{O}))$		C_3	$\{\alpha_3\}$
I13 $(\mathfrak{sl}(n,\boldsymbol{C}), \mathfrak{sl}(p,\boldsymbol{C}) + \mathfrak{sl}(n-p,\boldsymbol{C}) + \boldsymbol{C}, M_{p,n-p}(\boldsymbol{C}))$,	$n \geq 3,\ 1 \leq p \leq [n/2]$	A_{n-1}	$\{\alpha_p\}$
I14 $(\mathfrak{sp}(n,\boldsymbol{C}), \mathfrak{sl}(n,\boldsymbol{C}) + \boldsymbol{C}, \mathrm{Sym}_n(\boldsymbol{C}))$	$n \geq 3$	C_n	$\{\alpha_n\}$
I15 $(\mathfrak{so}(n+2,\boldsymbol{C}), \mathfrak{so}(n,\boldsymbol{C}) + \boldsymbol{C}, M_{1,n}(\boldsymbol{C}))$	$n \geq 3,\ n \neq 4$	$B_{[(n+2)/2]}$ $D_{(n+2)/2}$	$\{\alpha_1\}$ $\{\alpha_1\}$
I16 $(\mathfrak{so}(2n,\boldsymbol{C}), \mathfrak{sl}(n,\boldsymbol{C}) + \boldsymbol{C}, \mathrm{Alt}_n(\boldsymbol{C}))$	$n \geq 4$	D_n	$\{\alpha_n\}$
I17 $(E_6^{\boldsymbol{C}}, \mathfrak{so}(10,\boldsymbol{C}) + \boldsymbol{C}, M_{1,2}(\boldsymbol{O}^{\boldsymbol{C}}))$		E_6	$\{\alpha_1\}$
I18 $(E_7^{\boldsymbol{C}}, E_6^{\boldsymbol{C}} + \boldsymbol{C}, H_3(\boldsymbol{O}^{\boldsymbol{C}}))$		E_7	$\{\alpha_7\}$

LEMMA 3.2. *Let us define a linear endomorphism φ on $\mathfrak{m}$ by*

$$\varphi(X) = \frac{1}{2}(X - IX), \quad X \in \mathfrak{m}, \tag{3.6}$$

where $I = \mathrm{ad}_{\mathfrak{m}} Z$. Then φ is a K_0^0-isomorphism of $\mathfrak{m}_{\mathfrak{p}}$ onto $\mathfrak{g}_{-1}$.

PROOF. The inclusion $\varphi(\mathfrak{m}_{\mathfrak{p}}) \subset \mathfrak{g}_{-1}$ follows from the fact $I^2 = 1$. Since I interchanges $\mathfrak{m}_{\mathfrak{p}}$ with $\mathfrak{m}_{\mathfrak{k}}$, φ sends $\mathfrak{m}_{\mathfrak{p}}$ to $\mathfrak{g}_{-1}$ isomorphically. Since K_0^0 acts on $\mathfrak{g}$ as grade-preserving automorphisms, the element Z is left fixed by K_0^0. Hence we have $[\mathrm{Ad}_{\mathfrak{m}} K_0^0, I] = 0$, which implies that φ commutes with the K_0^0-action. □

Let $\mathfrak{a}_{-1} := \varphi(\mathfrak{c}) \subset \mathfrak{g}_{-1}$. Then $\mathfrak{a}_{-1}$ is spanned by $e_1, \ldots, e_r$, since $\varphi(X_i) = e_i$. Let $W(\Delta^*)$ be the Weyl group for the root system Δ^* (or, for the symmetric space M^*). Then we have

$$W(\Delta^*) \simeq N_{K_0^0}(\mathfrak{c})/C_{K_0^0}(\mathfrak{c}), \tag{3.7}$$

where $N_{K_0^0}(\mathfrak{c})$ (resp. $C_{K_0^0}(\mathfrak{c})$) is the normalizer (resp. centralizer) of $\mathfrak{c}$ in K_0^0. $W(\Delta^*)$ acts on $\mathfrak{c}$ as signed permutations:

$$X_i \mapsto \pm X_{\rho(i)}, \quad \rho \in \mathfrak{S}_r, \tag{3.8}$$

where $\mathfrak{S}_r$ is the permutation group of $\{1, \ldots, r\}$. By Lemma 3.2, this action of $W(\Delta^*)$ is transferred onto $\mathfrak{a}_{-1}$ via φ as the signed permutations:

$$e_i \mapsto \pm e_{\rho(i)}, \quad \rho \in \mathfrak{S}_r. \tag{3.9}$$

Table II

$(\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_{-1})$		$\Delta(\mathfrak{g}, \mathfrak{c})$	Δ^*
I1	$p = n/2 = 1$	A_1	A_0
	$p = n/2 > 1$	C_p	D_p
	$1 \leq p < n-p$	BC_p	B_p
I2	$p = n/2$	C_p	C_p
	$1 \leq p < n-p$	BC_p	BC_p
I3		C_n	A_{n-1}
I4		C_n	A_{n-1}
I5		C_n	C_n
I6	$p = 0$ & $3 \leq q \neq 4$	C_1	C_1
	$p = 1$ & $2 \leq q (\neq 3)$	C_2	A_1
	$2 \leq p \leq q$	C_2	D_2
I7		C_n	A_{n-1}
I8	n even	$C_{n/2}$	$D_{n/2}$
	n odd	$BC_{[n/2]}$	$B_{[n/2]}$
I9		BC_2	B_2
I10		BC_1	BC_1
I11		C_3	D_3
I12		C_3	A_2
I13	$p = n/2$	C_p	C_p
	$1 \leq p < n-p$	BC_p	BC_p
I14		C_n	C_n
I15		C_2	C_2
I16	n even	$C_{n/2}$	$C_{n/2}$
	n odd	$BC_{[n/2]}$	$BC_{[n/2]}$
I17		BC_2	BC_2
I18		C_3	C_3

Recall the quadratic representation P of the compact simple JTS $\mathfrak{B} = (\mathfrak{g}_{-1}, B_\tau)$:

$$P(X)Y = (XYX), \quad X, Y \in \mathfrak{g}_{-1}. \tag{3.10}$$

The structure group $\operatorname{Str}\mathfrak{B}$ of the JTS $\mathfrak{B}$ is, by definition, the totality of the elements $g \in \mathrm{GL}(\mathfrak{g}_{-1})$ satisfying the condition:

$$g(XYU) = ((gX)(g^{*-1}Y)(gU)), \quad X, Y, U \in \mathfrak{g}_{-1}, \tag{3.11}$$

where g^* is the adjoint operator of g with respect to the trace form of $\mathfrak{B}$. A computation shows that

$$\operatorname{Str}\mathfrak{B} = \{g \in GL(\mathfrak{g}_{-1}) : P(gX) = gP(X)g^*, X \in \mathfrak{g}_{-1}\}. \tag{3.12}$$

Noting that the GLA $\mathfrak{g}$ is isomorphic to the Kantor-Tits-Koecher construction for B_τ, we conclude from Satake [**21**] that the group G_0 is isomorphic to $\operatorname{Str}\mathfrak{B}$ and that this isomorphism is given by taking the restriction of the G_0-action on $\mathfrak{g}$ to $\mathfrak{g}_{-1}$. As a result, the rank of the operator $P(X)$ is constant on each G_0-orbit in $\mathfrak{g}_{-1}$, when X varies through that orbit. Let V_k $(0 \leq k \leq r)$ be the union of G_0^0-orbits through the points $o_{p,q}$

with $p+q=k$, that is,

$$V_k = \bigcup_{p+q=k} G_0^0 \cdot o_{p,q} \subset \mathfrak{g}_{-1}, \quad 0 \le k \le r. \tag{3.13}$$

THEOREM 3.3 (Gindikin-Kaneyuki [6]). *Let $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ be a real simple* GLA *and r be the split rank of the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$. Then* (1) *$V_k$ is expressed as*

$$V_k = \{X \in \mathfrak{g}_{-1} : \operatorname{rk} P(X) = i_k\}, \quad 0 \le k \le r, \tag{3.14}$$

where rk *denotes the rank and $i_k = \operatorname{rk} P(o_{k,0})$. The closure $\bar{V}_k$ of V_k is given by*

$$\bar{V}_k = \{X \in \mathfrak{g}_{-1} : \operatorname{rk} P(X) \le i_k\}, \quad 0 \le k \le r. \tag{3.15}$$

(2) *Each V_k is G_0-stable and*

$$\mathfrak{g}_{-1} = V_0 \coprod V_1 \coprod \cdots \coprod V_r. \tag{3.16}$$

(3) *An orbit $G_0^0 \cdot o_{p,q}$ is open if and only if it is contained in V_r, or equivalently, $p+q=r$.*

The assertion (2) was obtained also by Takeuchi [27] by a different method.

LEMMA 3.4. *Let* $\operatorname{Aut}\mathfrak{B}$ *denote the automorphism group of the* JTS $\mathfrak{B}$. *Then*

$$\operatorname{Aut}\mathfrak{B} = K_0. \tag{3.17}$$

PROOF. The trace form $\gamma_{\mathfrak{B}}$ of $\mathfrak{B}$ is positive definite, since $\mathfrak{B}$ is compact. Aut $\mathfrak{B}$ is, by definition, the subgroup of $\operatorname{Str}\mathfrak{B} = G_0$ consisting of all elements $g \in \operatorname{Str}\mathfrak{B}$ satisfying the condition

$$\gamma_{\mathfrak{B}}(gX, gY) = \gamma_{\mathfrak{B}}(X, Y), \quad X, Y \in \mathfrak{g}_{-1}. \tag{3.18}$$

On the other hand, we have (cf. [1] and Lemma 3.10 [13])

$$\gamma_{\mathfrak{B}}(X, Y) = -\frac{1}{2}(X, \tau Y), \quad X, Y \in \mathfrak{g}_{-1}. \tag{3.19}$$

Now let $g \in K_0$. Then, since g commutes with τ, we have that g satisfies (3.18), which implies that $K_0 \subset \operatorname{Aut}\mathfrak{B}$. By the definition, $\operatorname{Aut}\mathfrak{B}$ is a compact subgroup of $\operatorname{Str}\mathfrak{B}$. But K_0 is a maximal compact subgroup of G_0. Hence we have that $K_0 = \operatorname{Aut}\mathfrak{B}$. □

§ 4. The orbit decompositions of $\mathfrak{g}_{-1}$.

THEOREM 4.1. *Let $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ be a real simple* GLA, *and r be the split rank of the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$. Suppose that Δ^* is of type A. Then the orbit decompositions of $\mathfrak{g}_{-1}$ under the groups G_0^0 and G_0 are given by*

$$\mathfrak{g}_{-1} = \coprod_{p+q \le r} G_0^0 \cdot o_{p,q} = \coprod_{\substack{p+q \le r \\ p \ge q}} G_0 \cdot o_{p,q}. \tag{4.1}$$

PROOF. Since Δ^* is of type A, it follows (Tables I and II) that $\mathfrak{A}_r = (\mathfrak{g}_{-1}, \Box_r)$ is a compact simple Jordan algebra. In this case, the JTS $\mathfrak{B}$ comes from the Jordan algebra $\mathfrak{A}_r$. As a result, G_0, identified with the structure group Str $\mathfrak{B}$, coincides with the structure group of $\mathfrak{A}_r$. Therefore the first equality in (4.1) is the one proved by Kaneyuki [**9**, **10**] and Satake [**23**]. Since $\mathfrak{A}_r$ is compact simple, it is known (Koecher [**15**], Vinberg [**29**]) that $V_{r,0} := G_0^0 \cdot o_{r,0}$ is a homogeneous irreducible self-dual convex cone in $\mathfrak{g}_{-1}$. Let $G(V_{r,0})$ be the automorphism group of the cone $V_{r,0}$. By Satake [**21**], we have

$$G_0|_{\mathfrak{g}_{-1}} = \operatorname{Str}\mathfrak{B} = G(V_{r,0}) \times \{\pm 1\}. \tag{4.2}$$

As was shown in [**10**], any $G(V_{r,0})$-orbit in $\mathfrak{g}_{-1}$ coincides with a G_0^0-orbit in $\mathfrak{g}_{-1}$. Therefore the second equality in (4.1) follows from (4.2). □

Now let

$$\begin{aligned} \Gamma_k &= \left\{ \begin{array}{r} \sum_{\ell=1}^{k} \delta_{i_\ell} e_{i_\ell} \in \mathfrak{a}_{-1} : \delta_{i_1}, \dots, \delta_{i_k} = \pm 1, \\ 1 \le i_1, \dots, i_k \le r \end{array} \right\}, \quad 1 \le k \le r, \\ \Gamma_0 &= \{0\}. \end{aligned} \tag{4.3}$$

Then the Weyl group $W(\Delta^*)$ acts on Γ_k by (3.9) and we have

$$\Gamma_k = \bigcup_{p+q=k} W(\Delta^*) \cdot o_{p,q}, \quad 0 \le k \le r. \tag{4.4}$$

Therefore it follows from (3.7) and (3.13) that

$$V_k = G_0^0 \Gamma_k, \quad 0 \le k \le r. \tag{4.5}$$

THEOREM 4.2. *Let $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ and r be the same as in Theorem* 4.1. *Suppose that Δ^* is of type B, BC or C. Then the orbit decompositions of $\mathfrak{g}_{-1}$ under G_0^0 and G_0 are given by*

$$\mathfrak{g}_{-1} = \coprod_{k=0}^{r} G_0^0 \cdot o_{k,0} = \coprod_{k=0}^{r} G_0 \cdot o_{k,0}. \tag{4.6}$$

In particular, there is a single open orbit $G_0^0 \cdot o_{r,0} = G_0 \cdot o_{r,0}$.

PROOF. In view of (3.16), it suffices to show that

$$V_k = G_0^0 \cdot o_{k,0} = G_0 \cdot o_{k,0}, \quad 0 \le k \le r. \tag{4.7}$$

By the assumption for Δ^*, the Weyl group $W(\Delta^*)$ consists of all signed permutations of the form (3.9). Consequently, $W(\Delta^*)$ acts on Γ_k transitively, i.e., $\Gamma_k = W(\Delta^*) \cdot o_{k,0}$. Hence (4.5) implies the first equality in (4.7). The second equality in (4.7) follows from the fact that V_k is G_0-stable (Theorem 3.3). □

REMARK. The second equality in (4.7) was obtained also by Takeuchi [**27**].

In the following we will be concerned exclusively with the case where Δ^* is of type *D*.

LEMMA 4.3. *Suppose that Δ^* is of type D_r. Then*

$$V_r = G_0^0 \cdot o_{r,0} \cup G_0^0 \cdot o_{r-1,1}. \tag{4.8}$$

$$V_k = G_0^0 \cdot o_{k,0} = G_0 \cdot o_{k,0}, \quad 0 \le k \le r-1. \tag{4.9}$$

PROOF. In view of (4.5), it suffices to prove that

$$\Gamma_r = W(\Delta^*) \cdot o_{r,0} \coprod W(\Delta^*) \cdot o_{r-1,1}, \quad \Gamma_k = W(\Delta^*) \cdot o_{k,0}, \qquad 0 \le k \le r-1. \tag{4.10}$$

By the assumption for Δ^*, a signed permutation $e_i \mapsto \delta_i e_i$, $\delta_i = \pm 1$ $(1 \le i \le r)$ lies in $W(\Delta^*)$ if and only if $\prod_{i=1}^r \delta_i = 1$. Therefore $o_{p,q}$ with q even (resp. odd) is conjugate to $o_{r,0}$ (resp. $o_{r-1,1}$) under $W(\Delta^*)$. Hence $(4.10)_1$ follows from (4.4). Let us next consider $o_{p,q}$ with $p+q=k$, $0 \le k \le r-1$. If q is even, then $o_{p,q}$ is conjugate to $o_{k,0}$ under $W(\Delta^*)$. Suppose q is odd. Let μ be the signed permutation defined by $\mu(e_\ell) = \delta_\ell e_\ell$ $(1 \le \ell \le r)$, where $\delta_\ell = -1$ for $p+1 \le \ell \le p+q+1$, otherwise $\delta_\ell = 1$. Then μ belongs to $W(\Delta^*)$ and $\mu(o_{p,q}) = o_{k,0}$. This implies $(4.10)_2$. □

Back to the situation in §2, suppose that $\Delta(\mathfrak{g}, \mathfrak{c})$ is of type C, and consider the Jordan algebra $\mathfrak{A}_p = (\mathfrak{g}_{-1}, \Box_p)$, $0 \le p \le r$. Let $P_p : \mathfrak{g}_{-1} \to \operatorname{End} \mathfrak{g}_{-1}$ be the quadratic representation of $\mathfrak{A}_p$. Then we have

LEMMA 4.4. *Let* $0 \le p \le r$. *Then*

$$P(X) = P_p(X) P(o_{p,r-p}), \quad X \in \mathfrak{g}_{-1}. \tag{4.11}$$

Moreover the operator $P(o_{p,r-p})$ *is nondegenerate on* $\mathfrak{g}_{-1}$.

PROOF. Let $Y \in \mathfrak{g}_{-1}$. By using (2.16) and (2.17), we have

$$\begin{aligned} P(X)Y = (XYX) &= (X \Box_p Y^*) \Box_p X + X \Box_p (Y^* \Box_p X) - Y^* \Box_p (X \Box_p X) \\ &= 2X \Box_p (X \Box_p Y^*) - (X \Box_p X) \Box_p Y^* \\ &= P_p(X) Y^* = P_p(X) P(o_{p,r-p}) Y. \end{aligned} \tag{4.12}$$

Since $\Delta(\mathfrak{g}, \mathfrak{c})$ is of type C, we have that $\mathfrak{g}_{-1}(1) = \mathfrak{g}_{-1}$ (cf. §2). On the other hand, by Satake [**21**], ± 1 are the only eigenvalues of $P(o_{p,r-p})$ on $\mathfrak{g}_{-1}(1)$, which yields the second assertion. □

Consider the JTS $(\ )_p$ coming from $\mathfrak{A}_p$ $(0 \le p \le r)$:

$$(XYU)_p = (X \Box_p Y) \Box_p U + X \Box_p (Y \Box_p U) - Y \Box_p (X \Box_p U), \tag{4.13}$$

where $X, Y, U \in \mathfrak{g}_{-1}$, and define the linear operator $L_p(X, Y)$ by

$$L_p(X, Y) U = (XYU)_p. \tag{4.14}$$

LEMMA 4.5. *Let* $X, Y \in \mathfrak{g}_{-1}$. *Then*

$$L_p(X, Y) = L(X, P(o_{p,r-p}) Y). \tag{4.15}$$

PROOF. For simplicity we write f_p for $o_{p,r-p}$. By the definition of a JTS, we have

$$\begin{aligned} L(X, P(f_p) Y) U &= (X (f_p Y f_p) U) \\ &= ((Y f_p X) f_p U) + (X f_p (Y f_p U)) - (Y f_p (X f_p U)) \\ &= (X \Box_p Y) \Box_p U + X \Box_p (Y \Box_p U) - Y \Box_p (X \Box_p U) \\ &= (XYU)_P = L_p(X, Y) U. \end{aligned} \tag{4.16}$$

□

Proposition 4.6. *Suppose that $\Delta(\mathfrak{g}, \mathfrak{c})$ is of type C. Let* $(\mathrm{Str}\,\mathfrak{A}_p)^0$ *and* $(\mathrm{Str}\,\mathfrak{B})^0$ *denote the identity components of the structure groups* $\mathrm{Str}\,\mathfrak{A}_p$ *and* $\mathrm{Str}\,\mathfrak{B}$, *respectively. Then we have*

$$(\mathrm{Str}\,\mathfrak{A}_p)^0 = (\mathrm{Str}\,\mathfrak{B})^0 = G_0^0. \tag{4.17}$$

Proof. Lie $\mathrm{Str}\,\mathfrak{A}_p$ (resp. Lie $\mathrm{Str}\,\mathfrak{B}$) is generated by $L_p(X, Y)$ (resp. $L(X, Y)$), when X and Y vary through $\mathfrak{g}_{-1}$. Therefore the proposition follows from Lemma 4.5 and the non-degeneracy of $P(o_{p,r-p})$. □

Table II tells us that if Δ^* is of type D_r, then $\Delta(\mathfrak{g}, \mathfrak{c})$ is of type C. In this case one has the Jordan algebra $\mathfrak{A}_r = (\mathfrak{g}_{-1}, \square_r)$ (Proposition 2.2).

Proposition 4.7. *Let* $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ *be a real simple* GLA. *Suppose that* Δ^* *is of type* D_r. *Let N be the reduced norm of the Jordan algebra* $\mathfrak{A}_r = (\mathfrak{g}_{-1}, \square_r)$. *Suppose* $N(o_{r,0})N(o_{r-1,1}) < 0$. *Then*

$$V_r = G_0^0 \cdot o_{r,0} \coprod G_0^0 \cdot o_{r-1,1}. \tag{4.18}$$

In particular, there are exactly two open G_0^0-orbits in $\mathfrak{g}_{-1}$.

Proof. By the assumption, $\Delta(\mathfrak{g}, \mathfrak{c})$ is of type C. Therefore, by Corollary 2.11 [**6**], we have that $V_r = \{X \in \mathfrak{g}_{-1} : \det P(X) \neq 0\}$. Lemma 4.4 implies that $X \in V_r$ if and only if $\det P_r(X) \neq 0$ if and only if $N(X) \neq 0$. We have thus

$$V_r = \{X \in \mathfrak{g}_{-1} : N(X) \neq 0\}. \tag{4.19}$$

Let V_r^+ (resp. V_r^-) be the totality of elements $X \in \mathfrak{g}_{-1}$ satisfying $N(X) > 0$ (resp. < 0). Then

$$V_r = V_r^+ \coprod V_r^-. \tag{4.20}$$

Suppose for simplicity that $N(o_{r,0}) > 0$. Then $N(o_{r-1,1}) < o$. We have $o_{r,0} \in V_r^+$ and $o_{r-1,1} \in V_r^-$. The reduced norm N is a relative invariant polynomial on $\mathfrak{g}_{-1}$, that is,

$$N(gX) = \chi(g)N(X), \quad X \in \mathfrak{g}_{-1}, \quad g \in \mathrm{Str}\,\mathfrak{A}_r, \tag{4.21}$$

where χ is an $\boldsymbol{R}^*$-valued character of $\mathrm{Str}\,\mathfrak{A}_r$. Suppose now that $g \in G_0^0 = (\mathrm{Str}\,\mathfrak{A}_r)^0$ (cf. Proposition 4.6). Then we have $N(go_{r,0}) = \chi(g)N(o_{r,0}) > 0$, and hence $G_0^0 \cdot o_{r,0} \subset V_r^+$. Similarly $G_0^0 \cdot o_{r-1,1} \subset V_r^-$. These two imply (4.18). □

Corollary 4.8. *Under the situation in Proposition* 4.7, *suppose that* $N(o_{r,0}) > 0$ (resp. < 0) *and* $N(o_{r-1,1}) < 0$ (resp. > 0). *Then*

$$\begin{aligned} G_0^0 \cdot o_{r,0} &= \{X \in \mathfrak{g}_{-1} : N(X) > 0 \text{ (resp. } < 0)\}, \\ G_0^0 \cdot o_{r-1,1} &= \{X \in \mathfrak{g}_{-1} : N(X) < 0 \text{ (resp. } > 0)\}. \end{aligned} \tag{4.22}$$

§ 5. The orbit decompositions of $\mathfrak{g}_{-1}$ (continued).

In this section we consider the case where Δ^* is of type D.

5.1.

THEOREM 5.1. *Let* $(\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_{-1}) = (\mathfrak{sl}(2p, \boldsymbol{R}), \mathfrak{sl}(p, \boldsymbol{R}) + \mathfrak{sl}(p, \boldsymbol{R}) + \boldsymbol{R}, M_p(\boldsymbol{R}))$. *Then the orbit decompositions of* $\mathfrak{g}_{-1}$ *under the groups* G_0^0 *and* G_0 *are given by*

$$\mathfrak{g}_{-1} = \coprod_{k=0}^{p-1} G_0^0 \cdot o_{k,0} \coprod G_0^0 \cdot o_{p,0} \coprod G_0^0 \cdot o_{p-1,1}, \tag{5.1}$$

$$\mathfrak{g}_{-1} = \coprod_{k=0}^{p} G_0 \cdot o_{k,0}. \tag{5.2}$$

There are exactly two open orbits $G_0^0 \cdot o_{p,0}$ *and* $G_0^0 \cdot o_{p-1,1}$ *which are mutually diffeomorphic.*

PROOF. In this case, Δ is of type A_{2p-1} and is given by

$$\Delta = \{\pm(\lambda_i - \lambda_j) : 1 \le i < j \le 2p\}. \tag{5.3}$$

The simple root system Π is given by

$$\Pi = \{\alpha_i = \lambda_i - \lambda_{i+1} : 1 \le i \le 2p-1\}. \tag{5.4}$$

Since $\Pi_1 = \{\alpha_p\}$ (cf. Table I), we have

$$\Delta_1 = \{\lambda_i - \lambda_{p+j} : 1 \le i, j \le p\}. \tag{5.5}$$

The corresponding gradation of $\mathfrak{g} = \mathfrak{sl}(2p, \boldsymbol{R})$ is

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 \\ &\quad \overset{p}{\leftrightarrow}\ \overset{p}{\leftrightarrow} \\ &= \left\{ \left(\begin{array}{c|c} 0 & 0 \\ \hline * & 0 \end{array} \right) \begin{array}{l} \updownarrow p \\ \updownarrow p \end{array} \right\} + \left\{ \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array} \right) \right\} + \left\{ \left(\begin{array}{c|c} 0 & * \\ \hline 0 & 0 \end{array} \right) \right\}. \end{aligned} \tag{5.6}$$

Let

$$\Gamma = \{\beta_i = \lambda_i - \lambda_{p+i} : 1 \le i \le p\}. \tag{5.7}$$

Then Γ is a maximal system of strongly orthogonal roots in Δ_1. Let $E_{ij} \in \mathfrak{g}_{-1} = M_p(\boldsymbol{R})$ $(1 \le i, j \le p)$ be the matrix whose (k, ℓ)-entry is $\delta_{ik}\delta_{j\ell}$. It can be seen that the root vector $E_{-i} \in \mathfrak{g}^{-\beta_i}$ $(1 \le i \le p)$ is given by the matrix $E_{ii} \in M_p(\boldsymbol{R}) = \mathfrak{g}_{-1}$. Therefore

$$\begin{aligned} o_{k,0} &= \sum_{i=1}^{k} E_{ii} \in M_p(\boldsymbol{R}), \quad 1 \le k \le p, \\ o_{p-1,1} &= \sum_{i=1}^{p-1} E_{ii} - E_{pp}. \end{aligned} \tag{5.8}$$

The reduced norm N of the Jordan algebra $\mathfrak{A}_p = M_p(\boldsymbol{R})$ is given by $N(X) = \det X$, $X \in M_p(R)$. Hence $N(o_{p,0}) = 1$ and $N(o_{p-1,1}) = -1$. Consequently, by Proposition 4.7, we have that $V_p = G_0^0 \cdot o_{p,0} \coprod G_0^0 \cdot o_{p-1,1}$. Combining this with (4.9) and (3.16), we get (5.1).

Let us next consider the G_0-orbit decomposition of $\mathfrak{g}_{-1}$. For $\mathfrak{g} = \mathfrak{sl}(2p, \boldsymbol{R})$, it is known (Matsumoto [**19**]) that Q_1 mod Q_0 is generated by ε_1. Since ε_1 is not in G^0, we have $\varepsilon_1 \in G_0 - G_0^0$ (cf. (3.4)). Choose the subset $\Pi_1' = \{\alpha_1\}$ of Π. Then

$h_{\Pi'_1}(\vartheta) = 1$. Let

$$\mathfrak{g} = \mathfrak{g}'_{-1} + \mathfrak{g}'_0 + \mathfrak{g}'_1 \tag{5.9}$$

be the gradation of $\mathfrak{g}$ corresponding to Π'_1 (cf. § 1), and let

$$\Delta = \coprod_{k=-1}^{1} \Delta'_k \tag{5.10}$$

be the corresponding partition of Δ. Since $\beta_1 \in \Delta'_1$ and $\beta_k \in \Delta'_0$ for $k \geq 2$, we have that E_{-1} lies in $\mathfrak{g}'_{-1}$ and E_{-k} $(k \geq 2)$ lies in $\mathfrak{g}'_0$. On the other hand $\varepsilon_1 = 1$ on $\mathfrak{g}'_0$ and $\varepsilon_1 = -1$ on $\mathfrak{g}'_{-1} + \mathfrak{g}'_1$ (cf. (3.1), (1.7), (1.11), (1.12)). Hence ε_1 sends E_{-1} to $-E_{-1}$ and leaves E_k $(k \geq 2)$ fixed. Consequently $\varepsilon_1(o_{p,0}) = -E_{-1} + \sum_{i=2}^{p} E_{-i}$. Let $a \in W(\Delta^*)$ be the element interchanging E_{-1} with E_{-p} and leaving all other E_{-k} $(k \neq 1, p)$ fixed. Then it follows that $a\varepsilon_1(o_{p,0}) = o_{p-1,1}$, and hence $G_0 \cdot o_{p-1,1} = G_0 a\varepsilon_1(o_{p,0}) = G_0 \cdot o_{p,0}$, which proves (5.2). Since ε_1 normalizes G_0^0, it is easily seen that ε_1 sends $G_0^0 \cdot o_{p,0}$ to $G_0^0 \cdot o_{p-1,1}$. □

5.2.

THEOREM 5.2. *Let* $(\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_{-1}) = (\mathfrak{so}(2n, 2n), \mathfrak{gl}(2n, \boldsymbol{R}), \mathrm{Alt}_{2n}(\boldsymbol{R}))$. *Then the orbit decompositions of* $\mathfrak{g}_{-1}$ *under the groups* G_0^0 *and* G_0 *are given by* (5.1) *and* (5.2) *with* p *replaced by* n.

PROOF. The Lie algebra $\mathfrak{g} = \mathfrak{so}(2n, 2n)$ is realized as

$$\begin{aligned} \mathfrak{so}(2n, 2n) &= \{A \in \mathfrak{gl}(4n, \boldsymbol{R}) : {}^t\!AS + SA = 0\} \\ &= \left\{ \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} : {}^t\!A_1 + A_4 = 0, \quad A_2, A_3 \in \mathrm{Alt}_{2n}(\boldsymbol{R}) \right\}, \end{aligned} \tag{5.11}$$

where $S = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$. The root system Δ is of type D_{2n}.

$$\begin{aligned} \Delta &= \{\pm(\lambda_i \pm \lambda_j) : 1 \leq i < j \leq 2n\}, \\ \Pi &= \{\alpha_i = \lambda_i - \lambda_{i+1} \ (1 \leq i \leq 2n-1), \alpha_{2n} = \lambda_{2n-1} + \lambda_{2n}\}. \end{aligned} \tag{5.12}$$

Since $\Pi_1 = \{\alpha_{2n}\}$ (cf. Table I), we have

$$\Delta_1 = \{\lambda_i + \lambda_j : 1 \leq i < j \leq 2n\}. \tag{5.13}$$

The gradation $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ corresponding to Π_1 is given by (5.6) with p replaced by $2n$. Put

$$\Gamma = \{\beta_i = \lambda_{2i-1} + \lambda_{2i} : 1 \leq i \leq n\}. \tag{5.14}$$

Then Γ is a maximal system of strongly orthogonal roots in Δ_1. It can be seen that the root vector $E_{-i} \in \mathfrak{g}^{-\beta_i}$ $(1 \leq i \leq n)$ is given by the matrix $-E_{2i-1,2i} + E_{2i,2i-1} \in \mathrm{Alt}_{2n}(\boldsymbol{R}) = \mathfrak{g}_{-1}$. If we denote by $\mathrm{Pff}(X)$ the Pfaffian of an alternating matrix X, then the above matrix realization of E_{-i} shows that $\mathrm{Pff}(o_{n,0}) = (-1)^n$ and $\mathrm{Pff}(o_{n-1,1}) = (-1)^{n-1}$. Since the Pfaffian is the reduced norm of the Jordan algebra $\mathfrak{A}_n = \mathrm{Alt}_{2n}(\boldsymbol{R})$, it follows from Proposition 4.7 that $V_n = G_0^0 \cdot o_{n,0} \coprod G_0^0 \cdot o_{n-1,1}$. Therefore we get (5.1) with p replaced by n.

Let us next study the open G_0-orbits. For $\mathfrak{g} = \mathfrak{so}(2n, 2n)$, it is known (Matsumoto [**19**]) that ε_1 is one of representatives of $Q_1 \bmod Q_0$. Similarly as before, we have $\varepsilon_1 \in G_0 - G_0^0$. Choose a subset $\Pi_1' = \{\alpha_1\}$ of Π. Then $h_{\Pi_1'}(\vartheta) = 1$.

Consider the gradation (5.9) of $\mathfrak{g} = \mathfrak{so}(2n, 2n)$ corresponding to Π_1' and the partition (5.10) of Δ. Since $h_{\Pi_1'}(\beta_1) = 1 \neq 0$ and $h_{\Pi_1'}(\beta_k) = 0$ for $k \geq 2$, we have that $E_{-1} \in \mathfrak{g}_{-1}'$ and $E_{-k} \in \mathfrak{g}_0'$ for $k \geq 2$. On the other hand $\varepsilon_1 = 1$ on $\mathfrak{g}_0'$ and $= -1$ on $\mathfrak{g}_{-1}' + \mathfrak{g}_1'$. Hence ε_1 sends E_{-1} to $-E_{-1}$ and leaves E_{-k} $(k \geq 2)$ fixed. Let $a \in W(\Delta^*)$ be the element interchanging E_{-1} with E_{-n} and leaving all other elements E_{-k} $(k \neq 1, n)$ fixed. Then we have that $a\varepsilon_1(o_{n,0}) = o_{n-1,1}$, and hence $G_0 \cdot o_{n-1,1} = G_0 \cdot o_{n,0}$, which proves (5.2) with p replaced by n. Since ε_1 normalizes G_0^0, we see that $\varepsilon_1(G_0^0 \cdot o_{n,0}) = G_0^0 \cdot o_{n-1,1}$. □

5.3 Let us now consider the case $(\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_{-1}) = (E_{7(7)}, E_{6(6)} + \boldsymbol{R}, H_3(\boldsymbol{O}'))$. There is only one possibility of gradations of the first kind for $\mathfrak{g} = E_{7(7)}$. That gradation corresponds to $\Pi_1 = \{\alpha_7\}$. Let $\Gamma = \{\beta_1, \beta_2, \beta_3\}$, where

$$(5.15)\qquad \begin{cases} \beta_1 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7, \\ \beta_2 = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \\ \beta_3 = \alpha_7. \end{cases}$$

It can be checked that Γ is a maximal system of strongly orthogonal roots in Δ_1. As was shown in [**6**], $\{e_1, e_2, e_3\}, e_i = E_{-i}$, is a frame (= a maximal system of orthogonal primitive idempotents) of $\mathfrak{B}$. In the present case, the triple product B_τ of $\mathfrak{B}$ comes from the natural Jordan algebra structure $\mathfrak{A}$ of $\mathfrak{g}_{-1} = H_3(\boldsymbol{O}')$ (cf. Loos [**18**]), that is,

$$(5.16)\qquad B_\tau(X, U, Y) = X \circ (U \circ Y) + (X \circ U) \circ Y - U \circ (X \circ Y),$$

where $\circ$ denotes the Jordan multiplication in $\mathfrak{A}$. Therefore the two structure groups coincide:

$$(5.17)\qquad \operatorname{Str} \mathfrak{A} = \operatorname{Str} \mathfrak{B}.$$

Let e_{ii} $(i = 1, 2, 3)$ be the diagonal matrix $\operatorname{diag}(\delta_{1i}, \delta_{2i}, \delta_{3i}) \in H_3(\boldsymbol{O}')$. Then $\{e_{11}, e_{22}, e_{33}\}$ is a frame in $H_3(\boldsymbol{O}')$.

Lemma 5.3. *$o_{3,0}$ is an invertible element in the Jordan algebra $\mathfrak{A} := H_3(\boldsymbol{O}')$.*

Proof. Let $P_{\mathfrak{A}}$ be the quadratic representation of $\mathfrak{A}$. Then (5.16) implies that $P_{\mathfrak{A}}(X) = P(X)$ for $X \in \mathfrak{g}_{-1} = H_3(\boldsymbol{O}')$, and hence $P_{\mathfrak{A}}(o_{3,0}) = P(o_{3,0})$. The operator $P(o_{3,0})$ is nondegenerate, by Lemma 4.4. Therefore $o_{3,0}$ is an invertible element in $\mathfrak{A}$. □

Recall the Jordan algebra $\mathfrak{A}_3 = (\mathfrak{g}_{-1}, \square_3)$ in §2. By (5.16), $\mathfrak{A}_3$ is a mutant of $\mathfrak{A}$ by the invertible element $o_{3,0}$.

Lemma 5.4. $N(o_{3,0})N(o_{2,1}) < 0$.

Proof. Let $N_{\mathfrak{A}}$ be the reduced norm of $\mathfrak{A}$. Then we have (Braun-Koecher [**3**])

$$(5.18)\qquad N(X) = N_{\mathfrak{A}}(X) N_{\mathfrak{A}}(o_{3,0}), \quad X \in \mathfrak{g}_{-1}.$$

Since $o_{3,0}$ is invertible in $\mathfrak{A}$, we have $N_{\mathfrak{A}}(o_{3,0}) \neq 0$. Now consider the two frames $\{e_1, e_2, e_3\}$ and $\{e_{11}, e_{22}, e_{33}\}$ in $\mathfrak{B}$. By Proposition 11.8 in Loos [**18**] and Lemma 3.4 here, there exists an element $k \in K_0^0$ such that

$$ke_{3,0} = \sum_{i=1}^{3} \delta_i e_{ii}, \tag{5.19}$$

where $\delta_i = \pm 1$. $N_{\mathfrak{A}}$ is a relative invariant polynomial for the group $\operatorname{Str} \mathfrak{A}$. Therefore there exists an $\boldsymbol{R}^*$-valued character χ of $\operatorname{Str} \mathfrak{A} = \operatorname{Str} \mathfrak{B} = G_0$ such that

$$N_{\mathfrak{A}}(gX) = \chi(g) N_{\mathfrak{A}}(X), \quad X \in \mathfrak{g}_{-1}, g \in G_0. \tag{5.20}$$

Since K_0 is contained in the commutator subgroup $[G_0, G_0]$, $\chi(K_0) = 1$. Therefore we have

$$N_{\mathfrak{A}}(o_{3,0}) = N_{\mathfrak{A}}(ko_{3,0}) = N_{\mathfrak{A}}\left(\sum_{i=1}^{3} \delta_i e_{ii}\right) = \delta_1\delta_2\delta_3. \tag{5.21}$$

Similarly we have $N_{\mathfrak{A}}(o_{2,1}) = -\delta_1\delta_2\delta_3$. Therefore, in view of (5.18), we have $N(o_{3,0})N(o_{2,1}) < 0$. □

THEOREM 5.5. *Let* $(\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_{-1}) = (E_{7(7)}, E_{6(6)} + \boldsymbol{R}, H_3(\boldsymbol{O}'))$. *Then the orbit decompositions of* $\mathfrak{g}_{-1}$ *under the groups* G_0^0 *and* G_0 *are given by*

$$\mathfrak{g}_{-1} = \coprod_{k=0}^{2} G_0^0 \cdot o_{k,0} \coprod G_0^0 \cdot o_{3,0} \coprod G_0^0 \cdot o_{2,1}, \tag{5.22}$$

$$\mathfrak{g}_{-1} = \coprod_{k=0}^{3} G_0 \cdot o_{k,0}. \tag{5.23}$$

There are exactly two open orbits $G_0^0 \cdot o_{3,0}$ *and* $G_0^0 \cdot o_{2,1}$ *which are mutually diffeomorphic. There is a single open* G_0*-orbit in* $\mathfrak{g}_{-1}$.

PROOF. (5.22) follows from Lemmas 4.3 and 5.4 and Proposition 4.7. Let us consider the G_0-orbit decomposition of $\mathfrak{g}_{-1}$. In the present case $\mathfrak{g} = E_{7(7)}, Q_1 \bmod Q_0$ is generated by ε_2 (Matsumoto [**19**]), and hence $\varepsilon_2 \in G_0 - G_0^0$. Consider the subset $\Pi_1' = \{\alpha_2\}$ of Π_1. Then $h_{\Pi_1'}(\vartheta) = 2$. Let $\mathfrak{g} = \sum_{k=-2}^{2} \mathfrak{g}_k'$ be the gradation of $\mathfrak{g}$ corresponding to Π_1' and let $\Delta = \coprod_{k=-2}^{2} \Delta_k'$ be the corresponding partition of Δ. By the same reason as for $\mathfrak{g} = \mathfrak{sl}(2p, \boldsymbol{R})$, we have that $\varepsilon_2 = 1$ on $\mathfrak{g}_{-2}' + \mathfrak{g}_0' + \mathfrak{g}_2'$ and $\varepsilon_2 = -1$ on $\mathfrak{g}_{-1}' + \mathfrak{g}_1'$. On the other hand, we have $\beta_1 \in \Delta_2'$, $\beta_2 \in \Delta_1'$ and $\beta_3 \in \Delta_0'$ (cf. (5.15)). Consequently $\varepsilon_2(o_{3,0}) = e_1 - e_2 + e_3$. Let $a \in W(\Delta^*)$ be the element interchanging e_2 with e_3 and leaving e_1 fixed. Then it follows that $a\varepsilon_2(o_{3,0}) = o_{2,1}$, which implies $\varepsilon_2(G_0^0 \cdot o_{3,0}) = G_0^0 \cdot o_{2,1}$. This proves (5.23). □

5.4. Let us consider the final case $(\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_{-1}) = (\mathfrak{so}(p+1, q+1), \mathfrak{so}(p, q) + \boldsymbol{R}, \boldsymbol{R}^{p+q})$, $2 \leq p \leq q$, in which case $r = 2$ (cf. Table II). The root system Δ of $\mathfrak{g}$ is of type B_{p+1} or D_{p+1}, according as $p < q$ or $p = q$, respectively. Δ is given by

$$\Delta = \{\pm(\lambda_i \pm \lambda_j)\ (1 \leq i < j \leq p+1); \lambda_i\ (1 \leq i \leq p+1)\}, \quad p < q, \tag{5.24}$$

or

$$\Delta = \{\pm(\lambda_i \pm \lambda_j) : 1 \leq i < j \leq p+1\}, \quad p = q.$$

The gradation of $\mathfrak{g}$ corresponds to the subset $\Pi_1 = \{\alpha_1 = \lambda_1 - \lambda_2\}$ of Π. Δ_1 is given by

$$\Delta_1 = \{\lambda_1 \pm \lambda_i \ (2 \leq i \leq p+1); \lambda_1\}, \tag{5.25}$$

where λ_1 occurs only when $p < q$. The subset of Δ_1

$$\Gamma = \{\beta_1 = \lambda_1 + \lambda_2, \beta_2 = \lambda_1 - \lambda_2\} \tag{5.26}$$

is a maximal system of strongly orthogonal roots in Δ_1. In this situation we get the simple Jordan algebra $\mathfrak{A}_2 = (\mathfrak{g}_{-1}, \square_2)$ of rank 2 with unit element $e := o_{2,0}$ (cf. § 2). We need some results on simple Jordan algebras of rank 2 due to Braun-Koecher [**3**]: The reduced norm N of $\mathfrak{A}_2$ is of signature (p,q), and the multiplication $\square_2$ can be expressed as

$$x \square_2 y = N(e,x)y + N(e,y)x - N(x,y)e, \quad x, y \in \mathfrak{g}_{-1}, \tag{5.27}$$

where $N(x,y) = (1/2)(N(x+y) - N(x) - N(y))$. From this it follows that

$$N(e_1,e_1) = N(e_2,e_2) = 0, \quad N(e_1,e_2) = \frac{1}{2}. \tag{5.28}$$

THEOREM 5.6. *Let* $(\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_{-1}) = (\mathfrak{so}(p+1, q+1), \ \mathfrak{so}(p,q) + \boldsymbol{R}, \boldsymbol{R}^{p+q}), \ 2 \leq p \leq q$. *Then the* G_0^0*-orbit decomposition of* $\mathfrak{g}_{-1}$ *is given by*

$$\mathfrak{g}_{-1} = \coprod_{k=0}^{1} G_0^0 \cdot o_{k,0} \coprod G_0^0 \cdot o_{2,0} \coprod G_0^0 \cdot o_{1,1}. \tag{5.29}$$

PROOF. By using (5.28), we see that $N(o_{2,0}) = 1$ and $N(o_{1,1}) = -1$. Therefore, from Lemma 4.3 and Proposition 4.7, the assertion follows. □

THEOREM 5.7. *Under the same assumption as in Theorem* 5.6, *the* G_0*-orbit decomposition of* $\mathfrak{g}_{-1}$ *is given as follows*:

$$\mathfrak{g}_{-1} = \coprod_{k=0}^{2} G_0 \cdot o_{k,0} \quad \textit{for } p = q, \tag{5.30}$$

$$\mathfrak{g}_{-1} = \coprod_{k=0}^{1} G_0 \cdot o_{k,0} \coprod G_0 \cdot o_{2,0} \coprod G_0 \cdot o_{1,1} \quad \textit{for } p < q. \tag{5.31}$$

PROOF. Suppose first $p = q$. In this case, one of generators of $Q_1 \bmod Q_0$ is ε_{p+1} (Matsumoto [**19**]). Note that $\varepsilon_{p+1} \in G_0 - G_0^0$. Choose the subset $\Pi_1' = \{\alpha_{p+1}\}$ of Π. Then $h_{\Pi_1'}(\vartheta) = 1$. Let $\mathfrak{g} = \sum_{k=-1}^{1} \mathfrak{g}_k'$ be the gradation of $\mathfrak{g}$ corresponding to Π_1', and let $\Delta = \coprod_{k=-1}^{1} \Delta_k'$ be the corresponding partition of Δ. We have $\varepsilon_{p+1} = 1$ on $\mathfrak{g}_0'$, and $\varepsilon_{p+1} = -1$ on $\mathfrak{g}_{-1}' + \mathfrak{g}_1'$. We also have $\beta_1 \in \Delta_1'$ and $\beta_2 \in \Delta_0'$, since $h_{\Pi_1'}(\beta_1) = 1$ and $h_{\Pi_1'}(\beta_2) = 0$. As a result, $\varepsilon_{p+1}(o_{2,0}) = -e_1 + e_2$. Choose an element $a \in W(\Delta^*)$ interchanging e_1 with e_2. Then $a\varepsilon_{p+1}(o_{2,0}) = o_{1,1}$, which implies that $\varepsilon_{p+1}(G_0^0 \cdot o_{2,0}) = G_0^0 \cdot o_{1,1}$. This, together with Lemma 4.3, proves (5.30).

Next consider the case $p < q$. Put $C_{pq}^+ = G_0^0 \cdot o_{2,0}$ and $C_{pq}^- = G_0^0 \cdot o_{1,1}$ for simplicity. Choose a coordinate system (x_i) in $\mathfrak{g}_{-1} = \boldsymbol{R}^{p+q}$ such that the reduced norm $N(X)$ is expressed as the canonical form $x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$. Then

$$C_{pq}^{\pm} = \left\{ (x_i) \in \boldsymbol{R}^{p+q} : \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2 \gtrless 0 \right\}. \tag{5.32}$$

Let $S_{pq}^{\pm}$ be the level surfaces of N, that is,

$$S_{pq}^{\pm} = \{(x_i) \in C_{pq}^{\pm} : N(X) = \pm 1\}. \tag{5.33}$$

Then $C_{pq}^{\pm}$ are diffeomorphic to $S_{pq}^{\pm} \times \boldsymbol{R}^+$, respectively. An easy argument shows that S_{pq}^+ (resp. S_{pq}^-) is diffeomorphic to $S^{p-1} \times \boldsymbol{R}^q$ (resp. $S^{q-1} \times \boldsymbol{R}^p$), where S^k denotes a k-sphere. Consider the i-th homology groups $H_i(C_{pq}^{\pm}, \boldsymbol{Z}), 0 \le i \le p+q$. Then the above argument shows that $H_i(C_{pq}^+, \boldsymbol{Z}) \simeq H_i(S^{p-1}, \boldsymbol{Z})$ and $H_i(C_{pq}^-, \boldsymbol{Z}) \simeq H_i(S^{q-1}, \boldsymbol{Z})$. Suppose that $C_{pq}^{\pm}$ are homeomorphic to each other. Then we have $H_i(S^{p-1}, \boldsymbol{Z}) \simeq H_i(S^{q-1}, \boldsymbol{Z})$ for any $i, 0 \le i \le p+q$, which implies $p = q$. This contradicts the hypothesis $p < q$. Therefore C_{pq}^+ is not homeomorphic to C_{pq}^-. Suppose now that there exists only one open G_0-orbit in $\mathfrak{g}_{-1}$. Then there exists $a \in G_0 - G_0^0$ such that $a o_{2,0} = o_{1,1}$. We then have $a(C_{pq}^+) = C_{pq}^-$, and hence C_{pq}^+ is homeomorphic to C_{pq}^-, which is a contradiction. Therefore there are exactly two open G_0-orbits. □

6. Open G_0^0-orbits

Let $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ be a real simple GLA. Suppose that the split root system $\Delta(\mathfrak{g}, \mathfrak{c})$ of the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$ is of type C_r. Then we have the simple Jordan algebras $\mathfrak{A}_p = (\mathfrak{g}_{-1}, \square_p)$ with unit element $o_{p,r-p} (0 \le p \le r)$ (cf. §2). For an element $g \in \operatorname{Str} \mathfrak{A}_p$, we define

$$\theta(g) := (g^*)^{-1}, \tag{6.1}$$

where g^* is the adjoint operator of g with respect to the trace form γ_p of $\mathfrak{A}_p$. Then θ is an involutive automorphism of $\operatorname{Str} \mathfrak{A}_p$. We denote by $\operatorname{Aut}_{\mathrm{JTS}} \mathfrak{A}_p$ the automorphism group of the JTS (4.13) coming from the Jordan algebra $\mathfrak{A}_p$, and we denote by $(\operatorname{Str} \mathfrak{A}_p)_\theta$ the subgroup of θ-fixed elements of $\operatorname{Str} \mathfrak{A}_p$. Then, by the definition of $\operatorname{Aut}_{\mathrm{JTS}} \mathfrak{A}_p$, we have

$$(\operatorname{Str} \mathfrak{A}_p)_\theta = \operatorname{Aut}_{\mathrm{JTS}} \mathfrak{A}_p, \tag{6.2}$$

Proposition 6.1. *Suppose that the split root system $\Delta(\mathfrak{g}, \mathfrak{c})$ of the symmetric pair $(\mathfrak{g}, \mathfrak{g}_0)$ is of type C_r. Then the open orbit $G_0^0 \cdot o_{p,r-p}$ $(0 \le p \le r)$ is expressed as a symmetric coset space*:

$$G_0^0 \cdot o_{p,r-p} = (\operatorname{Str} \mathfrak{A}_p)^0 / (\operatorname{Str} \mathfrak{A}_p)^0 \cap \operatorname{Aut} \mathfrak{A}_p, \tag{6.3}$$

where $\operatorname{Aut} \mathfrak{A}_p$ *denotes the automorphism group of the Jordan algebra* $\mathfrak{A}_p$. (*Note that* $G_0^0 = (\operatorname{Str} \mathfrak{A}_p)^0$ *by* (4.17)).

PROOF. $\mathrm{Aut}\,\mathfrak{A}_p$ is an open subgroup of $\mathrm{Aut}_{\mathrm{JTS}}\,\mathfrak{A}_p$ (cf. Satake [**21**]). Consequently, noting (6.2), we have the inclusions

$$((\mathrm{Str}\,\mathfrak{A}_p)_\theta)^0 \subset \mathrm{Aut}\,\mathfrak{A}_p \subset (\mathrm{Str}\,\mathfrak{A}_p)_\theta. \tag{6.4}$$

By taking the intersection of each term in (6.4) with $(\mathrm{Str}\,\mathfrak{A}_p)^0$, it follows that

$$(((\mathrm{Str}\,\mathfrak{A}_p)^0)_\theta)^0 \subset (\mathrm{Str}\,\mathfrak{A}_p)^0 \cap \mathrm{Aut}\,\mathfrak{A}_p \subset ((\mathrm{Str}\,\mathfrak{A}_p)^0)_\theta, \tag{6.5}$$

which implies that the coset space in the right-hand side of (6.3) is a symmetric coset space. Since $\mathrm{Aut}\,\mathfrak{A}_p$ is the isotropy subgroup of $\mathrm{Str}\,\mathfrak{A}_p$ at the unit element $o_{p,r-p}$, $G_0^0 \cdot o_{p,r-p}$ has the coset space expression (6.3). □

Every open orbit $G_0^0 \cdot o_{p,r-p}$ is an ω-domain in the sense of Koecher [**16**], since that orbit is a connected component of V_r (note that V_r coincides with the totality of invertible elements in $\mathfrak{A}_p$, by Lemma 4.4). As a result, open G_0^0-orbits exhaust all ω-domains in real simple Jordan algebras. The results similar to Proposition 6.1 were obtained also by Faraut-Gindikin [**5**] and Vinberg [**29**].

REMARK 6.2. Assuming that $\Delta(\mathfrak{g}, \mathfrak{c})$ is of type C, let us consider the quadratic representation $P(X)$ of the JTS $\mathfrak{B}$. Then $P(X)$ is nondegenerate for $X \in V_r$ ([**6**]). $\det P(X)$ has a constant sign on each connected component of V_r. Put

$$\Phi(X) = \log|\det P(X)|, \quad X \in V_r. \tag{6.6}$$

Then, by Koecher [**16**] together with Lemma 4.4, the Hessian $\mathrm{Hess}(\Phi(X))$ is nondegenerate on each open G_0^0-orbit. Hence $\mathrm{Hess}(\Phi(X))$ is a G_0^0-invariant pseudo-riemannian metric on it. As a conclusion, an open G_0^0-orbit provides with an example of pseudo-Hessian symmetric space (For the definition of a Hessian symmetric space, see Shima [**24**]).

In the following, we give the explicit forms of open G_0^0-orbits and their coset space expression (6.3) for each simple $\mathrm{GLA}(\mathfrak{g}, \mathfrak{g}_0, \mathfrak{g}_{-1})$ with split root system of type C. Partial results have been obtained by Kaneyuki [**11**] and d'Atri-Gindikin [**4**].

(I1) with $p = n/2$,

$$\{X \in M_p(\boldsymbol{R}) : \det X > 0\}, \quad \{X \in M_p(\boldsymbol{R}) : \det X < 0\}.$$

Both are expressed as $GL(p, \boldsymbol{R})^0 \times GL(p, \boldsymbol{R})^0/$ diagonal.

(I2) with $p = n/2$,

$$\{X \in M_p(\boldsymbol{H}) : \det X \neq 0\} = GL(p, \boldsymbol{H}) \times GL(p, \boldsymbol{H})/\text{diagonal}.$$

$$H_{n-i,i}(\boldsymbol{C}) = GL(n, \boldsymbol{C})/U(n-i, i), \quad 0 \le i \le n. \tag{I3}$$

$$H_{n-i,i}(\boldsymbol{R}) = GL(n, \boldsymbol{R})^0/SO(n-i, i), \quad 0 \le i \le n. \tag{I4}$$

$$\{X \in SH_n(\boldsymbol{H}) : \det X \neq 0\} = GL(n, \boldsymbol{H})/SO^*(2n). \tag{I5}$$

(I6) i) $p = 0$,

$$\{(x_i) \in \boldsymbol{R}^q : x_1^2 + \cdots + x_q^2 \neq 0\} = \boldsymbol{R}^+ \times SO(q)/SO(q-1).$$

ii) $p = 1$,

$$\{(x_i) \in \boldsymbol{R}^{q+1} : x_1^2 - x_2^2 - \cdots - x_{q+1}^2 > 0, x_1 > 0\},$$
$$\{(x_i) \in \boldsymbol{R}^{q+1} : x_1^2 - x_2^2 - \cdots - x_{q+1}^2 > 0, x_1 < 0\},$$
$$\{(x_i) \in \boldsymbol{R}^{q+1} : x_1^2 - x_2^2 - \cdots - x_{q+1}^2 < 0\},$$

The first two are expressed as $\boldsymbol{R}^+ \times SO(1,q)^0/SO(q)$. The third one is expressed as $\boldsymbol{R}^+ \times SO(1,q)^0/SO(1,q-1)^0$.

iii) $p \geq 2$,

$$\left\{(x_i) \in \boldsymbol{R}^{q+p} : \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2 > 0\right\} = \boldsymbol{R}^+ \times SO(p,q)^0/SO(p-1,q)^0,$$
$$\left\{(x_i) \in \boldsymbol{R}^{q+p} : \sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2 < 0\right\} = \boldsymbol{R}^+ \times SO(p,q)^0/SO(p,q-1)^0.$$

(I7) $$H_{n-i,i}(\boldsymbol{H}) = GL(n,\boldsymbol{H})/Sp(n-i,i), \quad 0 \leq i \leq n.$$

(I8) $$\{X \in \mathrm{Alt}_{2n}(\boldsymbol{R}) : \mathrm{Pff}(X) > 0\}, \quad \{X \in \mathrm{Alt}_{2n}(\boldsymbol{R}) : \mathrm{Pff}(X) < 0\}.$$

Both are expressed as $GL(2n,\boldsymbol{R})^0/Sp(n,\boldsymbol{R})$.

(I11) $$\{X \in H_3(\boldsymbol{O}') : N(X) > 0\}, \quad \{X \in H_3(\boldsymbol{O}') : N(X) < 0\},$$

where N denotes the reduced norm of $H_3(\boldsymbol{O}')$. Both are expressed as $\boldsymbol{R}^+ \times E_{6(6)}/F_{4(4)}$.

(I12) $H_{3-i,i}(\boldsymbol{O}), \quad i = 0,1,2,3.$

$H_{3,0}(\boldsymbol{O})$ and $H_{0,3}(\boldsymbol{O})$ are expressed as $\boldsymbol{R}^+ \times E_{6(-26)}/F_4$.

$H_{2,1}(\boldsymbol{O})$ and $H_{1,2}(\boldsymbol{O})$ are expressed as $\boldsymbol{R}^+ \times E_{6(-26)}/F_{4(-20)}$.

(I13) with $p = n/2$,

$$\{X \in M_p(\boldsymbol{C}) : \det X \neq 0\} = GL(p,\boldsymbol{C}) \times GL(p,\boldsymbol{C})/\text{diagonal}.$$

(I14) $$\{X \in \mathrm{Sym}_n(\boldsymbol{C}) : \det X \neq 0\} = GL(p,\boldsymbol{C})/SO(n,\boldsymbol{C}).$$

(I15) $$\{(z_i) \in \boldsymbol{C}^n : z_1^2 + \cdots + z_n^2 \neq 0\} = \boldsymbol{C}^* \times SO(n,\boldsymbol{C})/SO(n-1,\boldsymbol{C}).$$

(I16) $$\{X \in \mathrm{Alt}_{2n}(\boldsymbol{C}) : \mathrm{Pff}(X) \neq 0\} = GL(2n,\boldsymbol{C})/Sp(n,\boldsymbol{C}).$$

(I18) $$\{X \in H_3(\boldsymbol{O}^{\boldsymbol{C}}) : N(X) \neq 0\} = \boldsymbol{C}^* \times E_6^{\boldsymbol{C}}/F_4^{\boldsymbol{C}},$$

where N denotes the reduced norm of the Jordan algebra $H_3(\boldsymbol{O}^{\boldsymbol{C}})$.

In the above list, $H_{n-i,i}(\boldsymbol{K})$ denotes the set of $n \times n$ $\boldsymbol{K}$-hermitian matrices of signature $(n-i,i)$, where $\boldsymbol{K} = \boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}, \boldsymbol{O}$.

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